JUMP BEHAVIOR OF CIRCUITS AND SYSTEMS

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Research sponsored by the National Science Foundation Grant ENG-78-09032-A01.

ABSTRACT

With particular reference to circuits we study the jump behavior, that is, the seemingly discontinuous change in state of systems driven by constrained (or implicitly defined) dynamics; i.e. $\dot{x} = f(x,y)$, $0 = g(x,y)$. To be specific, dynamics of a circuit are defined implicitly by specifying the velocities (time-derivatives) of capacitor voltages and inductor currents as well as the non-linear resistive and Kirchhoff constraints that the branch voltages and currents must satisfy. These constraints represent a constraint manifold over the base space of capacitor voltages and inductor currents. The process of integrating the circuit dynamics to obtain the transient response of the circuit consists of "lifting" the specified velocities to a vector field on the constraint manifold ("lifting" is the inverse operation of projecting). Lifting may not, however, be possible at points of singularity of the projection map, from the constraint manifold to the base space. We propose a way of <u>resolving these singularities</u>, consistent with the interpretation that the constraint manifold is a degeneration of very fast or singularly-perturbed dynamics. The physical meaning of this degeneration is the neglect of certain parasitic elements in the course of modelling. To resolve the singularities, we augment the base space as well as the configuration manifold by introducing sufficiently many additional ε-linear positive inductors and capacitors so that the projection map from the <u>augmented</u> constraint manifold to the <u>augmented</u> base space has no singularities. To obtain a notion of the qualitative behavior of the circuit, we take the (degenerate) limits as $\varepsilon \searrow 0$ of the trajectories of the augmented system (provided they exist). The limit trajectories may be discontinuous and these discontinuities are referred to as jump behavior. The detailed development is as follows:

(i) We propose a (discontinuous) solution concept for constrained differential equations consistent with the physical interpretation of the constraints. The mathematical tools used here to characterize jump behavior, precisely, are some recently developed bifurcation and geometric singular perturbation theory.

(ii) We apply the machinery of (i) to the study of time-invariant, non-linear circuit equations in two ways -- first viewing the resistive sub-network from the extrinsic (or port) viewpoint and second viewing it from the intrinsic (or circuit-topological) viewpoint. We give circuit-theoretic interpretation to the perturbations and assumptions introduced in (i) using suitably defined notions of resistive dissipativeness as a mechanism for energy loss in the circuit.

(iii) We study critical elements of constrained differential equations with particular reference to persistent relaxation oscillations. Existence of relaxation oscillations in the plane for Liénard-type systems is proven.

Finally, we give two simple examples of the application of our ideas to circuits, and indicate several important directions in which our work may be extended.

Key words: jump behavior, singularities, bifurcation, singular perturbation, non-linear circuit dynamics, dissipativeness, relaxation oscillations.

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Chapter I. INTRODUCTION

We study the jump behavior, that is, the seemingly discontinuous changes in state of systems driven by constrained (or implicitly defined) dynamics, with particular reference to circuits. To be precise, dynamics of a circuit are specified implicitly by specifying the "velocities" (time derivatives) of capacitor voltages and inductor currents, as well as the non-linear resistive and Kirchhoff constraints that the branch voltages and currents must satisfy. These constraints represent a constraint manifold over the base space of capacitor voltages and inductor currents. (Usually this manifold is of the same dimension as the base space.) The process of integrating the circuits dynamics to obtain the transient response of the circuit consists of "lifting" the specified velocities to a vector field on the constraint manifold (lifting is the inverse operation of projecting). Lifting may not however be possible at points where the projection map (restricted to the tangent space of the constraint manifold) has singularities. We suggest in this paper a way of resolving these singularities. We enlarge the base space as well as the configuration manifold by introducing additional dynamic elements in the circuit, ε-linear inductances and capacitances representing parasitics neglected in the course of modelling so that the projection map from the augmented manifold to the augmented base space has no singularities. Then, for each $\varepsilon > 0$ the dynamics of the augmented system are well defined. To obtain a qualitative understanding of the behavior of the original system, which presumably is close to the behavior of the augmented system for small ε, we take the (degenerate) limits as $\varepsilon \searrow 0$ of the trajectories of the augmented system (provided they exist). The limit trajectories may be discontinuous and these discontinuities are referred to as jump behavior of circuits. Another important conclusion that is obtained, incidentally, is that certain segments of the original configuration manifold are unstable under the parasitic dynamics and so are not physically observable.

The layout of this paper, where we carry out in detail the program outlined above, and its connection with previous work is as follows:

In Chapter II we re-examine, with the intention of motivating the generalization, the well-studied "degenerate" van-der Pol equation (recent references are [9], [37] and [43]) arising from a simple RC-circuit. Our aim here is to highlight some features of this example which are prototypical in the subsequent generalization.

The mathematical machinery required to formalize the description of jump behavior is the study of (discontinuous) solution concepts for constrained differential equations consistent with the interpretation that the constraints arise from singularly perturbed dynamics. We develop this in Chapter III. We use as tools two important recent advances in mathematics:

(i) The work of Fenichel [12] in geometric singular perturbation theory from an invariant manifold standpoint which includes and generalizes (greatly) the work of Hoppensteadt [20,21], Levin and Levinson [25,26], Tihonov, Pontryagin, Misçenko [see [20], [25], [28] for the relevant references).

(ii) The work of Hale [16] in bifurcation or the study of singularities -- this uses elementary analysis and calculus to study the unfolding of singularities.

Our solution concept for constrained differential equations inlcudes discontinuous trajectories, explains and lays down rigorous mathematical conditions for jump behavior. Our work includes and generalizes the solution concept proposed by Takens [40,41] in his study of gradient constrained systems. We also prove a restricted completeness theorem (i.e. that solutions may either be extended to $t = \infty$ or escape in finite time) for our solution concept.

In Chapters IV-V we apply this machinery to the problem of determining circuit dynamics outlined earlier. In Chapter IV our approach to the resistive network is from an extrinsic or port standpoint while in Chapter V the approach is from an intrinsic or graph topological standpoint. We explain and give physical interpretation to the various assumptions and perturbations that needed to be introduced in the theory of Chapter III using notions of resistive dissipativeness as the mechanism of energy loss. The intrinsic or graph topological standpoint is shown to be more revealing in certain instances,for example in explicating the connection between non-monotonicity and jump.

In Chapter VI, we study briefly critical elements-equilibrium points, closed orbits and closed relaxation oscillations. We prove an existence theorem for relaxation oscillations in Liénard type systems in the plane, and demonstrate the existence of a Poincarè or first return map for a class of persistent (defined in Chapter VI) relaxation oscillations also studied by Miščenko [28].

Chapter VII contains two elementary circuit examples of our ideas.

In Appendix I, we analyse further a class of non-gradient systems studied using transform methods by Popov [31,32] to show that they satisfy the assumptions introduced in Chapter III.

In Appendix II, we present complete stability theorems for non-linear, non-reciprocal circuits using a generalization of resistor passivity as the mechanism for dissipation. These theorems are similar in spirit to those of Chua and Suwannukul [8].

Important extensions to our work are possible. For instance, to constrained differential equations with a manifold rather than vector space as base space (a practical example is in power systems where the base space is the tangent bundle of an n-torus [36]). Other possible extensions are collected in Chapter VIII.

Chapter II. MOTIVATION

For the purposes of motivating and using as example in what follows we study in some detail a degenerate form of the van der Pol oscillator equation, arising from the RC-circuit shown in Figure 1. The circuit equations, with v and i labelled as shown are given by

$$\dot{v} = i \tag{II.1}$$

$$0 = -v - i^3 + i \tag{II.2}$$

we interpret these equations to mean an implicitly defined vector-field on a 1-dimensional manifold M in (v,i) space given by

$$M = \{(v,i) : v = i^3 - i\} \quad .$$

Difficulties arise in this interpretation at points $(\frac{-2}{3\sqrt{3}}, \frac{-1}{\sqrt{3}})$, $(\frac{2}{3\sqrt{3}}, \frac{1}{\sqrt{3}})$ where the projection of the tangent plane TM on the v-space is only a point so that the positive $\dot{v}$ at $(\frac{2}{3\sqrt{3}}, \frac{1}{\sqrt{3}})$ and the negative $\dot{v}$ at $(\frac{-2}{3\sqrt{3}}, \frac{-1}{\sqrt{3}})$ specified by (II.1) cannot be "lifted" to a vector field on M. This is shown in Figure 2. In an attempt to explain the observed behavior of systems modelled by these equations, we make the assumption that the trajectories of (II.1), (II.2) are the singularly perturbed limit as $\varepsilon \downarrow 0$ (provided they exist) of the trajectories of

$$\dot{v} = i \tag{II.3}$$

$$\varepsilon\dot{i} = -v - i^3 + i \tag*{(II.4)$_\varepsilon$}$$

This regularization has been suggested by several authors in the literature (recent references are [9], [37]) as also in the context of models for heart beat by Zeeman [43]. Equations (II.3), (II.4)$_\varepsilon$

represent a well defined dynamical system on $\mathbb{R}^2$ for each $\varepsilon > 0$. The corresponding phase portrait is plotted in Figure 3 for fixed small $\varepsilon > 0$. On M, $\dot{i} = 0$ so that the vector field is vertical, whereas outside a region (a "boundary layer" of M) where $- v + i^3 - i$ of order ε, the flow is largely horizontal. We further require that the flow of equations (II.3), $(II.4)_\varepsilon$ resemble that of equations (II.1), (II.2) in regions where the solution to (II.1), (II.2) is well defined. Roughly speaking, we are asking for solutions of (II.1), (II.2) on certain portions of the manifold M to attract solutions of (II.3), (II.4) starting from initial conditions close to the manifold M. This is a kind of stability requirement referred to as consistency in the theory of singular perturbations [4,5]. The condition for consistency is the local asymptotic stability of an equilibrium point (v_0, i_0) of the "boundary-layer" system $(II.4)_\varepsilon$ with v frozen at v_0. This is in turn guaranteed by

$$\frac{\partial}{\partial i}(-v-i^3+i)\bigg|_{(v_0,i_0)} = -3i_0^2 + 1 < 0 \qquad \text{(II.5)}$$

The regions of M satisfying this condition are two disconnected subsets of M, labeled M_a in Figure 4. We note also that by the implicit function theorem equations (II.1), (II.2) restricted to M_a are well defined. We contend that M_a is the only physically observable portion of the configuration-space M of the circuit dynamics. To see this we visualize equation $(II.4)_\varepsilon$ as arising from adding a small linear inductor (ε inductance) in series with the resistor and capacitor arising from parasitics neglected in the course of modelling the circuit. Now, it is conceivable that noise and other disturbances will force (v,i) slightly off M so that the "boundary layer" parasitic dynamics, in the parasitic's time scale, will determine if the flow will return toward M or be repelled away from it. The interpretation then of (II.5) in the language of circuit theory is strict local passivity of the resistor.

What remains to be specified is the behavior of (II.1), (II.2) on the boundary of M_a, i.e. at points m_2 and m_3 in Figure 4. For this purpose, we examine the trajectories of (II.3), (II.4) starting from m_2 and m_3 and take the limit of these as $\varepsilon \downarrow 0$. The resulting limit trajectories indicate a jump in zero-time from one component of M_a to the other as shown by the arrowed lines in Figure 4 (from m_2 to m_1 and m_3 to m_4 respectively). This particular example thus admits in the steady state of a relaxation oscillation -- a closed orbit with two extremely fast (0 time) transitions. It is the burden of Chapter III to give a systematic mathematical development of jump behavior in constrained differential equations. Critical elements of constrained differential equations are discussed in Chapter VI.

NOTATION

(i) If $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is a smooth function $(x,y) \mapsto g(x,y)$, we mean by $D_1g(x,y)(D_2g(x,y))$ the Jacobian matrix representing the derivative of g with respect to its first (second) argument.

(ii) $\sigma(D_2g(x,y) \subset \mathbb{C}$ is the set of m eigenvalues of the matrix $D_2g(x,y)$.

(iii) $\mathbb{C}_-(\mathbb{C}_+,\mathring{\mathbb{C}}_-,\mathring{\mathbb{C}}_+)$ is the left (right, open left, open right) half complex plane.

(iv) $D_1^k g(x,y)[u_1][u_2]\cdots[u_k]$ is the kth derivative of g with respect to its first argument (a k-linear map from $\mathbb{R}^n \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n$ (k times) $\to \mathbb{R}^n$) evaluated at $u_1, u_2, \ldots, u_k$.

(v) The (degenerate) system Σ is

$$\begin{aligned} \dot{x} &= f(x,y) \qquad (\Sigma) \\ 0 &= g(x,y) \end{aligned}$$

(vi) The (augmented) system Σ_ε is

$$\begin{aligned} \dot{x} &= f(x,y) \qquad (\Sigma_\varepsilon) \\ \varepsilon\dot{y} &= g(x,y) \end{aligned}$$

(vii) The "frozen boundary layer" system $\mathcal{B}_{x_0}$ is

$$\frac{dy}{d\tau} = g(x_0,y) \qquad (\mathcal{B}_{x_0})$$

(viii) $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the natural projection $(x,y) \mapsto x$.

(ix) $M = \{(x,y) \in \mathbb{R}^{n+m} : g(x,y) = 0\}$ is the configuration manifold of Σ.

$$M_a = \{(x,y) \in \mathbb{R}^{n+m} : g(x,y) = 0, \ \ \sigma(D_2g(x,y)) \subset \mathring{\mathbb{C}}_-\}$$

is the "attracting portion" of the configuration manifold M.

$$M_h : \{(x,y) \in \mathbb{R}^{n+m} : g(x,y) = 0, \ \ \sigma(D_2g(x,y)) \cap]-j\infty, j\infty[\ = \phi\}$$

is the "hyperbolic portion" of the configuration manifold M.

$$M_0 = \{(x,y) \in \mathbb{R}^{n+m} : g(x,y) = 0, \;\; \det D_2 g(x,y) = 0\}$$

is the "singular portion" of the configuration manifold M.

(x) TM(x,y) is the tangent space to M at (x,y) - a vector space of dimension n with its origin translated to (x,y)

$$\pi|_{TM(x,y)} =: \pi(x,y) \text{ is the restriction of } \pi \text{ to } RM(x,y).$$

(xi) A^T is the transpose of $A \in \mathbb{R}^{n\times m}$ and $|\cdot|$ stands for the Euclidean norm on $\mathbb{R}^n$.

(xii) $\mathcal{R}(A)$ is the range-space of a vector $A \in \mathbb{R}^{n\times m}$ and $\mathcal{N}(A)$ is its null space.

Chapter III. SOLUTION CONCEPTS FOR CONSTRAINED DIFFERENTIAL EQUATIONS

We study here solution concepts (including discontinuous or jump behavior) for constrained differential equations of the form:

$$\left.\begin{aligned} \dot{x} &= f(x,y) \qquad (III.1) \\ 0 &= g(x,y) \qquad (III.2) \end{aligned}\right\} (\Sigma)$$

For the purposes of this paper we have $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ smooth functions (smooth means C^r for r sufficiently large so that all derivatives used in the sequel are continuous). Further assume that 0 is a regular value of g (i.e. that $\forall x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ such that $g(x,y) = 0$, $\text{rank}[D_1 g(x,y) \vdots D_2 g(x,y)] = m$). (More generally, the ideas in our paper should be valid for E a vector bundle over a smooth manifold B, with $\pi : E \to B$ a smooth projection, $\dot{x}$, a specified vector field on B and $g : E \to \mathbb{R}^m$ a smooth map whose zero set is the constraint set.)

Equations (III.1) and (III.2) need to be interpreted. The most naive interpretation is that (III.2) is an algebraic equation which is to be "solved" for y, given x, -- possibly many solutions (or none) exist -- and these "solutions" are to be substituted in (III.1), i.e., if

$$A_{x_0} = \{y : g(x_0,y) = 0\} \neq \phi \ .$$

then $\dot{x}$ is understood to belong to

$$B_{x_0} = \{f(x_0,y_0) : y_0 \in A_{x_0}\} \ .$$

so that we have the following differential inclusion in $\mathbb{R}^n$:

$$\dot{x} \in B_x \qquad (III.3)$$

In general, solutions to differential inclusions are not smooth -- absolutely continuous trajectories satisfying (III.3) exist in the sense of Caratheodory (see Hale [17]). For example, if $g(x,y) = x-y^2$ and $f(x,y) = y$ with $x,y \in \mathbb{R}$, then (III.3) takes the form

$$\dot{x} = \pm \sqrt{x} \quad \text{for} \quad x \geq 0, \quad \text{undefined for } x < 0 \quad .$$

Nowhere differentiable trajectories of this equation starting from $x = 1$ can be shown to exist. Further, there is no hope of unique trajectories for the initial value problem for (III.3). For intuitive physical reasons, our aim is to obtain "maximally smooth," solutions to (III.1), (III.2) keeping in mind the physical circumstances in which these equations arise.

III.1. Constrained Differential Equations as Dynamical Systems with Singularities

We try to interpret (III.1), (III.2) as describing *implicitly* a dynamical system on the n-dimensional configuration manifold for Σ:

$$M = \{(x,y) : g(x,y) = 0\} \subset \mathbb{R}^{n+m}$$

(M is a manifold since 0 is a regular value of g [2]). The vector field $X(x,y)$ on M is specified by specifying its projection along the x-axis, namely,

$$\pi X(x,y) = f(x,y) \quad . \tag{III.4}$$

(Here $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the projection map $(x,y) \to x$.) We will choose to identify the tangent space to M at (x,y) with a (real) vector space $TM(x,y)$ of dimension n attached at its origin to the manifold M at (x,y); $X(x,y)$ then is (in coordinates) a vector belonging to $TM(x,y)$. At points at which $\pi TM(x,y) = \mathbb{R}^n$ it is clear that $f(x,y)$ uniquely specifies $X(x,y)$. Difficulties however arise when $\pi\, TM(x,y) \subsetneq \mathbb{R}^n$ and $f(x,y)$ is a vector transverse to $\pi\, TM(x,y)$ i.e. $f(x,y)$ points out of $\pi\, TM(x,y)$ so that it is not possible to lift a specified $f(x,y)$ along the x-axis onto $TM(x,y)$. As specimens, two different kinds of behavior are illustrated in Figure 5 at a point where M has a "fold:"

(i) (Figure (5a)) $f(x,y)$ points out of the manifold M at (x_0,y_0) so that it would seem that the trajectory would jump off the manifold i.e. the y-coordinate changes discontinuously (x is constrained to vary absolutely continuously by (III.1)) to appear on some other portion of M, say (x_0,y_1) where $\pi\, TM(x_0,y_1) = \mathbb{R}^n$.

(ii) (Figure (5b)) $f(x,y)$ points into the manifold M at (x_0,y_0) so that trajectories starting away from (x_0,y_0) do not tend towards (x_0,y_0).

In case (i) above, (x_0,y_0) is referred to as a point at which M is <u>overflowing</u>. The preceding discussion then motivates the definition.

<u>Definition III.1</u>. (Singularities of Σ)

The set of points $(x,y) \in M$ at which the range of $\pi|_{TM(x,y)} \stackrel{\Delta}{=}: \pi(x,y) \subsetneq \mathbb{R}^n$ are called the <u>singular points</u> of Σ. ¤

A more concrete characterization of the set of singular points can be given for the manifold M characterized by (III.2) since $TM(x,y)$ is then characterized by the vector space $\{v \in \mathbb{R}^{n+m} : Dg(x,y)v = 0\}$ with its origin translated to (x,y).

<u>Proposition III.1</u>. (Characterization of Singular points)

The set of singular points of Σ is precisely the set

$$M_0 :\stackrel{\Delta}{=} \{(x,y) : g(x,y) = 0, \ \det D_2 g(x,y) = 0\}$$

Proof. $TM(x,y) = \{(v_1,v_2) \in \mathbb{R}^{n+m} : D_1g(x,y)v_1 + D_2g(x,y)v_2 = 0,$

$$v_1 \in \mathbb{R}^n,\ v_2 \in \mathbb{R}^m \}$$

If $D_2g(x,y)$ is non-singular we can rewrite this as

$$TM(x,y) = \{(v_1, D_2g^{-1}(x,y)D_1g(x,y)v_1) : v_1 \in \mathbb{R}^n\}$$

$\pi(TM(x,y))$ is then $\{v_1 : v_1 \in \mathbb{R}^n\} = \mathbb{R}^n$ and (x,y) is not a singular point. Conversely, if $D_2g(x,y)$ is singular, there exists $\tilde{v}_1 \in \mathbb{R}^n$ such that $D_1g(x,y)\tilde{v}_1 \notin \mathcal{R}(D_2g(x,y))$ since the rank $[D_1g(x,y) : D_2g(x,y)] = m$ (by assumption, 0 is a regular value of g). Hence, span $(\tilde{v}_1) \notin$ Range of $\pi(TM(x,y))$ so that (x,y) is a singular point. ¤

Note that it may not always be necessary to admit of a jump or discontinuous behavior at a singularity. Consider, for example

$$\dot{x} = f(y)$$

$$0 = y^3 - x \quad .$$

The point (0,0) is a singular point of this system. However, a trajectory of this system tending towards (0,0) may be extended continuously by the following prescription:

$$\dot{x} = f(x^{1/3}) \qquad x \neq 0$$

$$y = x^{1/3}$$

The point $x = 0$ is excluded since $f(x^{1/3})$ loses smoothness at that point. This point is treated in detail in Section III.5.

In the light of this example it is clear that the study of the trajectories at singular points for the purpose of extension involves checking for the existence of solutions y of the equation (III.2) in the "direction in which x is varying." Clearly, continuous extension is possible iff such solutions y can be obtained as continuous (but not necessarily even Lipschitz) functions of x. The tools used for checking this are clearly the tools of bifurcation theory [16,27]. If infact continuous extension is not possible from a singular point (x_0,y_0) we do not wish to restart the integral curve at (x_0,y_1) for some y_1 solution of (III.2) chosen in some ad-hoc fashion. In order to give a physically meaningful way of restricting the number of discontinuous extensions of Σ from singular points we introduce the notions of singular perturbations for the system Σ.

III.2. Consistency with an augmented system

Empirical and physical evidence leads us to postulate that in practical problems (III.2) is the degenerate limit as $\varepsilon \downarrow 0$ of

$$\varepsilon\dot{y} = g(x,y) \tag{III.6}$$

where $\varepsilon > 0$ is a small parameter representing parasitics neglected in the course of modelling. The system Σ is referred to as the degenerate system and the system (III.1), (III.6) for $\varepsilon > 0$, is referred to as the augmented system. We denote the augmented system Σ_ε for $\varepsilon > 0$. We define solution concepts for (III.1), (III.2) consistent with this interpretation. Before we recall some facts from singular perturbation theory we give insight to the qualitative behavior of Σ_ε by rescaling time t to $\tau = t/\varepsilon$. This yields for the suspended flow

$$\left.\begin{aligned} \frac{dx}{d\tau} &= \varepsilon f(x,y) \\ \frac{dy}{d\tau} &= g(x,y) \\ \frac{d\varepsilon}{d\tau} &= 0 \end{aligned}\right\} \quad \mathcal{S} \tag{III.7}$$

Note now that the set of equilibria for this speeded up system $\mathcal{S}$ is precisely M so that the "fast dynamics" of (III.6) describes whether the configuration manifold M is attracting or not to the parasitic dynamics. Intuition then leads us to believe that trajectories of Σ_ε converge to those of Σ in those regions of M which are attracting to the parasitic dynamics. We make this precise:

Define

$$\begin{aligned} M_1 &= \{(x,y) : g(x,y) = 0, \quad \det D_2 g(x,y) \neq 0\} \\ &= M \backslash M_0 . \end{aligned} \tag{III.8}$$

Let $(x_0,y_0) \in M_1$. Then, by the implicit function theorem there is a unique integral curve $\gamma(t) = (x(t),y(t))$ of (III.1), (III.2) through (x_0,y_0) defined on $[0,\alpha[$ for some $\alpha > 0$. In coordinates,

$$y = \psi(x) \text{ in a neighborhood } U \text{ of } x_0 \text{ in } \mathbb{R}^n$$

and on U (III.1), (III.2) may be written equivalently as

$$\begin{aligned} \dot{x} &= f(x,\psi(x)), \qquad x(0) = x_0 \qquad x \in U \\ y &= \psi(x) . \end{aligned} \tag{III.9}$$

Trajectories of the augmented system exist, originating from arbitrary $(x_0,y_0) \in \mathbb{R}^{n+m}$ for all $\varepsilon > 0$. Some of these tend uniformly (consistently) to $\gamma(t)$ as $\varepsilon \downarrow 0$ on all closed intervals of $]0, \alpha[$ as follows (see also Figure 6):

<u>Theorem III.1</u> [25] (Consistency over bounded time intervals)

Given that a unique solution curve $(x(t),y(t)) = \gamma(t)$ starting from $(x_0,y_0) \in M_1$ of Σ exists on $[0,\alpha[$, then $\exists \delta > 0$ such that solutions of Σ_ε starting from $x(0)$, $y(0)$ with $|x(0)-x_0| + |y(0)-y_0| < \delta$ converge <u>uniformly</u> to $\gamma(t)$ on all closed subintervals of $]0,\alpha[$ as $\varepsilon \downarrow 0$ provided that the spectrum of $D_2 g(x(t),y(t))$ over the trajectory $\gamma(t)$ of Σ lies in the left-half complex plane bounded away from the $j\omega$-axis. ¤

Remarks: (i) If the pointwise consistency condition $D_2g(x_0,y_0) \subset \mathring{\mathbb{C}}_-$ holds for $(x_0,y_0) \in M$ it holds in a neighborhood of (x_0,y_0). Thus we label the subset M_a of M given by

$$M_a = \{(x,y) : g(x,y) = 0, \quad \sigma(D_2g(x,y)) \subset \mathring{\mathbb{C}}_-\} \qquad \text{(III.10)}$$

the "attracting" portion of the manifold M.

(ii) Some weaker consistency conditions hold if $(x_0,y_0) \in M_1$ but $\sigma(D_2g(x_0,y_0)) \not\subset \mathbb{C}_-$. First some notation:

Let $P(t) \in \mathbb{R}^{m \times m}$ $(0 \leqq t < \alpha)$ be a family of non-singular matrices such that for $(x(t),y(t)) = \gamma(t)$:

$$P^{-1}(t)D_2g(x(t),y(t))\ P(t) = \begin{pmatrix} B(t) & 0 \\ 0 & C(t) \end{pmatrix} \qquad 0 \leqq t < \alpha \qquad \text{(III.11)}$$

where $B(t) \in \mathbb{R}^{m_1 \times m_1}$ has spectrum in $\mathring{\mathbb{C}}_-$ bounded away from the $j\omega$-axis and $C(t)$ has spectrum in $\mathring{\mathbb{C}}_+$ bounded away from the $j\omega$-axis.

Theorem III.2 (Consistency from stable initial manifolds over bounded time intervals)[26]

Given that a unique solution $(x(t),y(t)) \equiv \gamma(t)$ starting from $(x_0,y_0) \in M$ of Σ exists on $[0,\alpha[$, satisfying (III.11) above then $\exists$ an m_1-dimensional manifold $S(\varepsilon) \subset \mathbb{R}^m$ depending smoothly on ε for $0 < \varepsilon \leqq \varepsilon_0$ such that if the initial vector for Σ_ε is $(x_0,y_\varepsilon(0))$ with $y_\varepsilon(0) \in S(\varepsilon)$. then the solution to (III.1), (III.6) satisfies the inequalities (with $b \in \mathbb{R}^{m_1}$ representing coordinates for $S(\varepsilon)$ in a neighborhood of y_0)

$$|x_\varepsilon(t)-x(t)| \leqq K(\varepsilon|b|+\omega(\varepsilon)) \qquad 0 \leqq t < \alpha$$

$$|y_\varepsilon(t)-y(t)| \leqq K(\varepsilon|b|+|b|e^{-\sigma t/\varepsilon}+\omega(\varepsilon))$$

where K and σ are positive constants independent of ε and $\omega(\varepsilon) \in C^0$ with $\omega(0) = 0$. ¤

Remarks: (i) The manifold $S(\varepsilon)$ is defined by

$$S(\varepsilon) = \{y : y = y_0 + P(0)\begin{pmatrix} b \\ z(b,\varepsilon)\end{pmatrix}, \; b \in \mathbb{R}^{m_1}\}$$

where $z : \mathbb{R}^{m_1} \times]0,\varepsilon_0] \to \mathbb{R}^{m_2}$ is a continuous function.

(ii) $\lim_{\varepsilon \downarrow 0} S(\varepsilon)$ exists and is an m_1-dimensional manifold (see Levin [26]) which is the local-stable manifold of the equilibrium y_0 of the "frozen" (i.e. x is fixed at x_0) boundary-layer system $\mathcal{B}_{x_0}$

$$\frac{dy}{d\tau} = g(x_0,y) \; . \qquad \mathcal{B}_{x_0} \qquad \text{(III.12)}$$

This is visualized in Figure 7.

(iii) It is important to note that consistency is more delicate in this case than in the case of Theorem III.1, since it is not generally the case that if the initial vector of Σ_ε belongs to S(0) then the trajectory of Σ_ε denoted $(x_\varepsilon(t),y_\varepsilon(t))$ satisfies

$$\lim_{t\downarrow 0}\lim_{\varepsilon\downarrow 0} (x_\varepsilon(t),y_\varepsilon(t)) = (x_0,y_0) = \gamma(0) \; .$$

what is true is that if

$$(x_\varepsilon(0) = x_0, y_\varepsilon(0) = y^\varepsilon) \in S(\varepsilon) \quad .$$

with $\lim_{\varepsilon\downarrow 0} (x_0,y^\varepsilon) = (x_0,y) \in S(0)$.

that

$$\lim_{t\downarrow 0}\lim_{\varepsilon\downarrow 0}(x_\varepsilon(t),y_\varepsilon(t)) = \gamma(0)$$

(iv) Theorems (III.1), (III.2) as stated are "local" -- i.e., the initial condition of Σ_ε lies close to M. Using a global center manifold technique of Fenichel for (III.7) (see [12] and the references there in) they can be made global: so that consistency is obtained for initial conditions belonging to a smooth (C^r) family of globally defined manifolds $S_{y_0}^{x_0}(\varepsilon)$ with $S_{y_0}^{x_0}(0)$, hence forth referred to as $S_{y_0}^{x_0}$, being the global stable manifold of the equilibrium y_0 of the frozen boundary layer system $\mathcal{B}_{x_0}$ (equation (III.13)). Rate estimates exactly as before can also be given, namely

$$\left.\begin{aligned} |x_\varepsilon(t)-x(t)| &\leqq K(\omega(\varepsilon)+\varepsilon|y_0-y_\varepsilon(0)|) \\ |y_\varepsilon(t)-y(t)| &\leqq K(\omega(\varepsilon)+e^{-\sigma t/2\varepsilon}|y_0-y_\varepsilon(0)|) \end{aligned}\right\} \qquad \text{(III.13)}$$

(v) Theorem (III.2) leads us to expect that any solution concepts for Σ interpreted as the limit of Σ_ε must allow for jump from points of M which are not "atracting." We take this up next.

III.3. Jump behavior from non-singular points

We aggregate the geometric picture obtained from Theorems (III.1), (III.2): for a hyperbolic equilibrium point y_0 of the frozen boundary layer system $\mathcal{B}_{x_0}$ we attach the stable manifold $S_{y_0}^{x_0}$ transversally to M at (x_0,y_0). When the attached manifold $S_{y_0}^{x_0}$ is of full codimension (i.e., m, since M is an n-dimensional manifold $\subset \mathbb{R}^{n+m}$) then disturbances and noise will not cause the "state" (x,y) of Σ to slip off M. If the attached manifold is not of full codimension the state can indeed slip off M and transit rapidly to another portion of M driven by the parasitic dynamics. Our definition of solution concept for Σ in Section (III.6) will infact allow for jump from those $(x_0,y_0) \in M$ for which

$$\sigma(D_2 g(x_0,y_0)) \cap C_+ \neq \phi \quad . \tag{III.14}$$

However to make this intuitive picture correct we need certain assumptions:

<u>Assumption 1</u> (Finitely many equilibrium points for $\mathcal{B}_{x_0}$).

For each $x_0 \in \pi M$, the system $\mathcal{B}_{x_0}$ described by (III.12) has finitely many equilibrium points.

<u>Assumption 2</u> (Complete stability of $\mathcal{B}_{x_0}$)

For each $x_0 \in \pi M$, the system $\mathcal{B}_{x_0}$ is <u>completely stable</u> i.e. if $\xi(\tau,\tilde{y})$ is the trajectory of

$$\frac{dy}{d\tau} = g(x_0,y) \qquad y(0) = \tilde{y}$$

Then, $\lim_{\tau\to\infty} \xi(\tau,\tilde{y})$ exists and $\in \{\bar{y} : g(x_0,\bar{y}) = 0\} \neq \phi$. Equivalently, $\xi(\tau,\tilde{y})$ converges to an equilibrium point of $\mathcal{B}_{x_0}$ for each $\tilde{y}$; so that if

$$S_{y_0}^{x_0} = \{\tilde{y} : \lim_{\tau\to\infty} \xi(\tau,\tilde{y}) = y_0\} \quad .$$

(i.e. $S_{y_0}^{x_0}$ is the set of initial conditions from which an element of $\mathcal{B}_{x_0}$ converges to y_0) and $C(x_0) = \{\bar{y} : g(x_0,\bar{y}) = 0\}$, then we assume that

$$\mathbb{R}^m = \bigcup_{y_0 \in C(x_0)} S_{y_0}^{x_0} \qquad \text{for each } x_0 \in \pi M \; . \qquad \square$$

<u>Remarks</u>: (i) The above assumption is obviously satisfied for a class of gradient systems -- i.e. $\exists\; F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ such that

$$g(x,y) = D_2F(x,y) \qquad \text{(III.15)}$$

and F has finitely many critical points and is proper and bounded above for each $x \in \pi M$ (on each fibre).

(ii) The class of systems admitted by our assumptions may be called a family of gradient-like (or dissipative) systems (see [39]). For a concrete example of non-gradient systems satisfying this assumption, see Appendix 1. Define M_h to be the set of hyperbolic equilibrium points of $\mathcal{B}_{x_o}$ i.e.,

$$M_h = \{(x,y) : g(x,y) = 0, \sigma(D_2g(x,y)) \cap \;]-j\infty, j\infty[\; = \phi\}$$

We now visualize $\pi M \times \mathbb{R}^m$ as being comprised of a family of foliations i.e.,

$$\{x_o\} \times \mathbb{R}^m = \bigcup_{y_o \in C(x_o)} S_{y_o}^{x_o} =: \mathcal{F}_{x_o}$$

$$\pi M \times \mathbb{R}^m = \bigcup_{x_o \in \pi M} \{x_o\} \times \mathbb{R}^m =: \bigcup_{x_o \in \pi M} \mathcal{F}_{x_o} \qquad \text{(III.16)}$$

(for a precise definition of foliations refer to [1] -- the foliations $\mathcal{F}_{x_o}$ are singular since the leaves $S_{y_o}^{x_o}$ may have varying dimension). We visualize M as a "stem" manifold to which "supporting" leaves of varying dimension $S_{y_o}^{x_o}$ are attached. These leaves represent the degenerate (infinitely fast) "parasitic" dynamics which will result in the state tending to (x_o,y_o). Thus if a disturbance, say noise, causes the state (x_o,y) to slip off the leaf $S_{y_o}^{x_o}$ it falls onto some other leaf, by the assumption 2 above, and makes an infinitely fast transition to some other $(x_o,y_1) \in M$. This situation is liable to happen precisely when (III.14) is satisfied so that $S_{y_o}^{x_o}$ is of dimension less than m .

Note that if $(x_0,y_0) \in M$ is a hyperbolic equilibrium of $\mathcal{B}_{x_0}$, $S_{y_0}^{x_0}$ is a manifold of dimension equal in magnitude to the number of eigenvalues in the left half plane of $D_2g(x_0,y_0)$ by the stable-manifold theorem [23]. Note that (x_0,y_0) belongs to the relative interior of $S_{y_0}^{x_0}$. Further, by Assumptions 1 and 2, for all $(x_0,y_0) \in M_0$; $S_{y_0}^{x_0}$ is a stratified set [42], with (x_0,y_0) belonging to its boundary. This means that $S_{y_0}^{x_0}$ is "almost" a manifold-an open dense set of it is a manifold (of dimension k say), an open dense set of the residual is a manifold of dimension $k-\ell$ and so on. This rather cumbersome statement is visualized by noting that if $(x_0,y_0) \in M_0$ it is an equilibrium of $\mathcal{B}_{x_0}$ formed by the fusion of several equilibria (parametrized by x) so that its attracting set (or basin in the terminology of [42]) is formed from the fusion of the stable manifolds of varying dimension of several equilibria. This point becomes clear in Section III.4 .

Remark: This same picture has been used by us to explain the "post-switching" behavior of constrained equations arising from dynamics of power-systems in [36]. This is also visualized in Figure 8.

Example: A complete picture of the foliation for the example of Section II is given in Figure 9.

A precise definition of jump from nonsingular points is postponed to Section (III.5) where it is dealt on par with jump behavior at singularities.

First, however we restrict the class of non-hyperbolic equilibrium points to M_0. We call this the no-dynamic bifurcation assumption since eigenvalues of the linearization of $\mathcal{B}_{x_0}$ crossing the $j\omega$-axis is symptomatic of the appearance of various kinds of non-trivial invariant sets (for example, orbits by the Hopf bifurcation) bifurcating from the equilibrium point of y_0 of $\mathcal{B}_{x_0}$ (see for a good discussion of possible bifurcations Guckenheimer [14]). Note that the occurrence of these non-trivial invariant sets would contradict the Assumption 2 of complete stability.

Assumption 3 (no dynamic bifurcations)

$$M = M_0 \cup M_h$$

i.e. as (x_0,y_0) moves over M, the eigenvalues of $D_2g(x_0,y_0)$ cross the $j\omega$-axis only at the origin.

III.4. Behavior at Singularities

In this section we undertake the program outlined in Section (III.2) utilizing mainly the techniques of Hale [16] to study the bifurcations of equation (III.2) at singularities. We also study the flow in a neighborhood of the singularity points. First we introduce the method of Lyapunov Schmidt to obtain the bifurcation function.

III.4.1. Bifurcation function from the method of Lyapunov-Schmidt

We study the solutions "y" of equation (III.2), namely

$$g(x,y) = 0 \qquad \text{(III.2)}$$

with x as parameter in a neighborhood of $(x_0,y_0) \in M_0$ i.e.

$$g(x_0,y_0) = 0 \quad \text{and} \quad \det D_2g(x_0,y_0) = 0$$

Notation: Let p be the rank defect of $D_2g(x_0,y_0)$. Then one can choose non-singular matrices $[P_U : U]$ and $[P_V : V]$ with $U,V \in \mathbb{R}^{m\times p}$ such that

$$P_V^T D_2 g(x_o,y_o) P_U \text{ is non-singular}$$

$$V^T D_2 g(x_o,y_o) = 0$$

$$D_2 g(x_o,y_o) U = 0 \ .$$

Then, (III.2) may be "decomposed" as

$$P_V^T g(x,y) = 0 \ , \quad V^T g(x,y) = 0 \tag{III.17a}$$

Also, represent y in $\mathbb{R}^n$ by

$$y := P_U w + Uq \tag{III.17b}$$

where $w \in \mathbb{R}^{m-p}$ and $q \in \mathbb{R}^p$. With this choice of coordinates (III.17) may be rewritten as

$$P_V^T g(x, P_U w + U_q) = 0 \tag{III.18}$$

$$V^T g(x, P_U w + U_q) = 0 \tag{III.19}$$

Let w_o, q_o(necessarily unique) be such that $y_o = P_U w_o + Uq_o$. Equation (III.18) is well behaved since by the Implicit function theory there is a neighborhood N_o of (x_o, w_o, q_o) and a function $w^*(x,q)$ such that

$$w_o = w^*(x_o,q_o)$$

$$P_V^T g(x, P_U w + Uq) = 0 \text{ in } N_o \Leftrightarrow w = w^*(x,q):$$

Use this function in (III.19) to obtain

$$N(x,q) := V^T g(x, P_U w^*(x,q) + Uq) = 0 \ . \tag{III.20}$$

Since $V^T D_2 g(x_o,y_o) = 0$ it is clear that

$$N(x_o,q_o) = 0 \quad \text{and} \quad D_2 N(x_o,q_o) = 0 \tag{III.21}$$

Since both N and its first derivative vanish at (x_0,q_0) it is referred to as the <u>bifurcation function</u>. The nature of the solution set of (III.2) in the vicinity of (x_0,y_0) is determined from the knowledge of the higher order derivatives of N(x,q) at (x_0,q_0). Note that the bifurcation equation (III.20) consists of only <u>p-equations</u> to be solved for <u>p-variables</u> (q) as a function of the parameter x. Thus, the foregoing procedure has reduced the size of the problem to the dimension of the null-space of $D_2g(x_0,y_0)$ (referred to as the <u>co-dimension of the bifurcation</u>).

III.4.2. Codimension-one bifurcations

In this case $D_2g(x_0,y_0)$ has precisely one zero eigenvalue so that the bifurcation function N(x,q) is a scalar function of $x \in \mathbb{R}^n$ and $q \in \mathbb{R}^1$. We pick out for detailed study two instances when the bifurcation function N(x,q) is quadratic or cubic in $q-q_0$ in a neighborhood of q_0. The quadratic or fold case is particularly important since it is the building block for more complicated bifurcations.

III.4.2.1. The quadratic or fold-case

Assume that

$$\frac{\partial^2}{\partial q^2} N(x_0,q_0) \neq 0 \qquad \text{(III.22)}$$

Then, in a sufficiently small neighborhood of (x_0,q_0) there is a unique $q^*(x)$ such that

$$\frac{\partial}{\partial q} N(x,q) = 0 \text{ at } q = q^*(x)$$

that is, N(x,q) has a local minimum or maximum at $(x,q^*(x))$ (depending on the sign of $\frac{\partial^2}{\partial q^2} N(x_0,q_0)$). If we define $\xi(x) := N(x,q^*(x))$ we have the following theorem . (Stated, assuming $\xi(x)$ to be a minimum -- if it is a maximum, replace ξ by $-\xi$).

Theorem III.3 [16] (Fold bifurcation for many parameters)

Consider equation (III.2). Let (x_0,y_0) be a solution of (III.2), such that $D_2g(x_0,y_0)$ has rank m-1. Let N(x,q) be defined as in (III.20).

Further, let $\frac{\partial^2}{\partial q^2} N(x_0,q_0) > 0$. Then, there is a neighborhood U_0 of (x_0,y_0) and a real valued-function $\xi(x)$, $\xi(x_0) = 0$ such that in the neighborhood U_0:

(i) (III.2) has no solution if $\xi(x) > 0$

(ii) (III.2) has one solution if $\xi(x) = 0$

(iii) (III.2) has two solution if $\xi(x) < 0$ ¤

Comments (i) The condition $\frac{\partial^2}{\partial q^2} N(x_0,q_0) \neq 0$ can be checked directly from the given function g(x,y) and its derivatives, since calculation using (III.20) verifies that

$$\frac{\partial^2}{\partial q^2} N(x_0,q_0) = v^T D_2^2 g(x_0,y_0)\ [u][u]$$

where (because the rank of $D_2g(x_0,y_0)$ is assumed to be (m-1), the matrix U(V) in (III.17b) becomes a column vector $u(v) \in \mathbb{R}^m$). $D_2^2g(x_0,y_0)[u][u]$ is the second derivative at (x_0,y_0) of g with respect to y evaluated at u, u.
(ii) The statement of the theorem is visualized in Figure 10. The theorem is a generalization to many parameters $(x \in \mathbb{R}^n)$ of the fold behavior associated with the scalar parabolic equation $p^2+q = 0$ in the neighborhood of $q = 0$.
(iii) $\{x \in U : \xi(x) = 0\}$ is an (n-1) dimensional manifold embedded in $\mathbb{R}^n$ with normal vector at x_0 given by $\nabla\xi(x^0) = D_1g(x_0,y_0)^T v$. The fact that $D_1g(x_0,y_0)^T v \neq 0$ follows readily from the assumption that 0 is a regular value of g ($Dg(x_0,y_0)$ has rank m. Hence, $v^T Dg(x_0,y_0) = [v^T D_1 g(x_0,y_0),0] \neq 0$).

(iv) Of the two distinct solutions in (iii) of the theorem one of the solutions has index +1 and the other index -1, when considered as equilibria of the boundary layer system $\mathcal{B}_x$. We visualize these two equilibria of $\mathcal{B}_x$ as annihilating each other at the fold boundary.

Three qualitative different kinds of behavior of the trajectories of Σ near a fold boundary are possible depending on the sign of $\frac{d}{dt}\,\xi(x)$, given by

$$D\xi(x_0)\dot{x} = v^T D_1 g(x_0,y_0)\ f(x_0,y_0)$$

(i) $v^T D_1 g(x_0,y_0)\ f(x_0,y_0) > 0$ (Figure 11(a)) implies that x is varying in a direction in which the equation (III.2) is losing solutions y (locally). The trajectory then needs to be continued from a point (x_0,y_1) which lies outside the neighborhood U of Theorem III.3. Thus, one is forced to admit jump or discontinuous behavior at such fold boundaries. The formalism is developed in Section (III.5).

(ii) $v^T D_1 g(x_0,y_0)\ f(x_0,y_0) < 0$ (Figure 11(b)). Trajectories of Σ, defined in M seem to point away from the fold boundary, and into the region of two solutions, specified in Theorem (III.3). One could then extend trajectories starting from (x_0,y_0) continuously (but not uniquely -- since we have a choice of two solutions $y_1(x)$, $y_2(x)$ of (III.2) with $y_1(x_0) = y_2(x_0) = y_0$) into M . It is, however, intuitive that trajectories starting inside M would not tend to such fold boundaries:

<u>Theorem III.4</u> (Repelling fold singularities)

Let $(x_0,y_0) \in M_0$ be such that $D_2 g(x_0,y_0)$ has rank (m-1). Further, let (x_0,y_0) satisfy the conditions of Theorem (III.3) (i.e. (x_0,y_0) is a fold singularity). Then, C^1 trajectories of Σ starting in M will not tend to (x_0,y_0) if

$$v^T D_1 g(x_0,y_0)\; f(x_0,y_0) < 0 \tag{III.23}$$

Proof: By contradiction. Let $\gamma(t) : [0,\alpha[\to M$ be a C^1 trajectory of Σ, with $\gamma(0) = (x,y) \in M$ and $\lim_{t \uparrow \alpha} \gamma(t) = (x_0,y_0)$. Then for t sufficiently closed to α, $\pi\gamma(t) \in U$, the neighborhood of x_0 in Theorem III.3 with $\xi(\pi\gamma(t))$ increasing along the trajectory so that

$$D\xi(\pi\gamma(t))\cdot f(\gamma(t)) > 0 \quad .$$

By continuity then we obtain $D\xi(x_0).\; f(x_0,y_0) \geqq 0$ which contradicts (III.23) above. ¤

(iii) $v^T D_1 g(x_0,y_0)\; f(x_0,y_0) = 0$. (Figure 11(c)) This is the case when either $f(x_0,y_0) = 0$ or the trajectories of Σ in M tend to touch the fold boundary at M_0 tangentially i.e., for some $\gamma(t) = [0,\alpha[\to M$, $\lim_{t \uparrow \alpha} \pi\gamma(t)$ is tangent to the manifold $U \cap \{x : \xi(x) = 0\}$. Precisely, we have

Proposition III.2 (Well defined dynamical system at some fold points)

If at a fold point (x_0,y_0) of M, i.e., at a point (x_0,y_0) satisfying the conditions of Theorem III.3

$$v^T D_1 g(x_0,y_0)\; f(x_0,y_0) = 0 \tag{III.24}$$

then there exist (several) $X(x_0,y_0) \in TM(x_0,y_0)$ such that $\pi X(x_0,y_0) = f(x_0,y_0)$.

Proof: The vector field X(x,y) at non-singular points is given by

$$X(x,y) = \pi^{-1}(x,y)\; f(x,y)$$

or in coordinates

$$X(x,y) = [f(x,y),\ D_2g^{-1}(x,y)D_1g(x,y)\ f(x,y)] \in TM(x,y) \qquad \text{(III.25)}$$

At fold points where (III.24) holds, (III.25) is still true since

$$v^T D_2 g(x_0,y_0) = 0 \quad \text{and} \quad v^T D_1 g(x_0,y_0)\ f(x_0,y_0) = 0$$

together imply that $D_1g(x_0,y_0)\ f(x_0,y_0) \in \mathcal{R}(D_2g(x_0,y_0))$. This completes the proof. Note, however, that the $X(x,y)$ defined by (III.25) is not uniquely specified since $D_2g(x,y)$ is not one-to one. ¤

<u>Comment</u>: In the sequel we will obtain uniqueness of $X(x,y)$ by requiring that $D_2g^{-1}(x,y)\ D_1g(x,y)\ f(x,y) \in \mathcal{R}(D_2g(x,y)^T)$ (i.e., no component in the null space of $D_2g(x,y)$).

Points where $v^T D_1g(x_0,y_0)\ f(x_0,y_0) = 0$ form the transition between jump points on a fold boundary (case (i) above) and repelling fold boundaries (case (ii) above). It is of interest to determine the nature of this transition -- we attempt to study the qualitative characteristics of the flow of Σ (i.e. upto homeomorphism) near such points. We use two tricks:

(i) Define a new vector field $\bar{X}(x,y)$ by

$$\bar{X}(x,y) = \det \pi(x,y)\ \pi^{-1}(x,y)\ f(x,y)$$

or in coordinates

$$\bar{X}(x,y) = [\det D_2g(x,y)\ f(x,y),\ \det D_2g(x,y)\ D_2g^{-1}(x,y)\ D_1g(x,y)\ f(x,y)]$$

The map $\det D_2g(x,y)\ D_2g^{-1}(x,y)$ can be extended to a smooth map on all of M so that $\bar{X}(x,y)$ is a vector field without singularities so long as

$$D_1g(x,y)\ f(x,y) \in \mathcal{R}(D_2g(x,y)) \quad .$$

The study of the integral curves of $\bar{X}(x,y)$ is of interest since they are the same as those of $X(x,y)$ except in regions where $\det D_2g(x,y) = 0$, only the parametrization (and perhaps direction, if $\det D_2g(x,y) < 0$) is different (see [2]).

(ii) Since $x \in \mathbb{R}^n$ is not a parametrization for M near fold points choose a different parametrization: namely $z := P_z x \in \mathbb{R}^{n-1}$ and $q \in \mathbb{R}$ where $P_z \in \mathbb{R}^{n-1 \times n}$ is a matrix of norm 1 whose span is orthogonal to the normal of the fold boundary at (x_0,y_0), namely $D_1g^T(x_0,y_0)v$ and q is the scalar variable from the bifurcation function (III.20). It may be checked that this is indeed a parametrization for M in a neighborhood of (x_0,y_0). $\bar{X}(x,y)$ can now be rewritten as $\bar{X}(z,q)$.

Now note that $(x_0,y_0) \cong (z_0,q_0)$ is a critical point of $\bar{X}(z,q)$. A basic theorem of Hartman [18] says that qualitative behavior (i.e., upto homeomorphism) of the integral curves of $\bar{X}$ in a neighborhood of (z_0,q_0) is determined by the eigenvalues of its linearization at that point (provided they are not on the imaginary axis). It may be verified after somewhat tedious computation that the linearization of $\bar{X}(z,q)$ at (z_0,q_0) has at most rank 2. Consider the case when the base space (x-space) has dimension 2. Then, we have the four possibilities:

(i) Both eigenvalues of the linearization real and positive (source),

(ii) Both eigenvalues of the linearization real and negative (sink),

(iii) Both eigenvalues of the linearization real and of opposite sign (saddle).

(iv) Both eigenvalues imaginary.

These possibilities can also be viewed near the fold boundary in the x-space as shown in Figures 12(i)-12(iv). Corresponding to the four cases above note the jump-fold points on one side of the point (x_0,y_0) and the repelling-fold points on the other side of the point

(x_0,y_0). In the instance that the base space has dimension >2; the transition is more complicated except in the two dimensions corresponding to the eigenspaces where the linearization of $\bar{X}$ is nonzero.

III.4.2.2 The cubic or cusp case

We now assume that the bifurcation function $N(x,q)$ is cubic in $q-q_0$, i.e.

$$\frac{\partial^2}{\partial q^2} N(x_0,q_0) = 0 \quad \text{and} \quad \frac{\partial^3}{\partial q^3} N(x_0,q_0) \neq 0 \qquad \text{(III.27)}$$

we assume that $\frac{\partial^3}{\partial q^3} N(x_0,q_0) > 0$ for definiteness. Then, $\frac{\partial N}{\partial q}(x,q)$ has a smooth minimum $\gamma_0(x)$ in a neighborhood of (x_0,q_0). If this minimum is negative for some values of x close to x_0 then $N(x,q)$ has at these x a unique local maximum $\gamma_1(x)$ and a unique local minimum $\gamma_2(x)$ near (x_0,q_0). Define $\gamma(x) = \gamma_1(x)\ \gamma_2(x)$. Then, we have

Theorem III.5 [16] (Cusp bifurcation for many parameters)

Consider equation (III.2). Let (x_0,y_0) be a solution of (III.2) such that $D_2 g(x_0,y_0)$ has rank m-1. Let $N(x,q)$ be the bifurcation function defined as in (III.20). Further, let $\frac{\partial^2}{\partial q^2} N(x_0,q_0) = 0$ and $\frac{\partial^3}{\partial q^3} N(x_0,q_0) > 0$. Then, there is a neighborhood U of (x_0,y_0) and two functions $\gamma_0(x)$, $\gamma(x)$ with $\gamma_0(x_0) = \gamma(x_0) = 0$ such that in the neighborhood U:

(i) If $\gamma_0(x) \geq 0$ there is a unique solution to (III.2).

(ii) If $\gamma_0(x) < 0$ then $\gamma(x)$ is defined and

(a) $\gamma(x) < 0$ implies one simple solution to (III.2),

(b) $\gamma(x) = 0$ implies one simple and one double solution to (III.2),

(c) $\gamma(x) < 0$ implies three simple solutions to (III.2). ¤

Comments: (i) The statement of the theorem is visualized in Figure 13. The theorem is a generalization to many parameters of the behavior of the number of real solutions of the cubic scalar equation $p^3+ap+b = 0$ in the vicinity of $(a,b) = (0,0)$. Recall from elementary algebra [22] that this cubic scalar equation has three real solutions if $4a^3+27b^2 > 0$ and one double and one simple real solution if $4a^3+27b^2 = 0$ and one simple solution if $4a^3+ 27b^2 < 0$.

(ii) Though at the point (x_0,y_0) of the cusp $\pi TM(x_0,y_0) \underset{\neq}{\subset} \mathbb{R}^n$, trajectories of Σ can indeed be extended continuously through the cusp since (III.2) does not "lose solutions" y as x varies through the cusp.

(iii) The study of the full unfolding of the cusp as revealed by the theorem shows it to be formed by the tangential (non-traversal) intersection of two fold surfaces as shown in Figure 13. Typical phase portraits for the three sheets of the cusp assuming that either the case of Figure 11(a) or that of Figure 11(b) occur at the fold surfaces is shown in Figure 14. Note the continuous extension through the cusp point.

III.4.2.3 Higher order codimension one singularities

These are the singularities at which the first nonvanishing derivative of $N(x_0,q)$ with respect to q is at least of order 3. The key to the study of these singularities is the following normal form theorem of Thom ([42], see also Hale [16]).

Theorem III.6 [42] (Normal form)

Suppose $N(x,q) = \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a C^∞ map with

$$N(x_0,q) = c(q-q_0)^k + O(|q-q_0|^{k+1}) \quad c \neq 0 \qquad \text{(III.28)}$$

$$\text{Rank } (a_{ij}) = k\text{-}1 \text{ where } \frac{\partial N(x_0,q)}{\partial x_i} = \sum_{j=0}^{k-2} \frac{a_{ij}}{j!} (q-q_0)^j + O(|q-q_0|^{k-1}) \tag{III.29}$$

Then, there is a C^∞ change of coordinates $\bar{x} = \eta(x) \in \mathbb{R}^{k-1}$, $\bar{\xi} = \xi(x,q) \in \mathbb{R}$ near $(x,q) = (x_0,q_0)$ such that in these coordinates, N has the form

$$\bar{N}(\bar{x},\bar{q}) = \bar{q}^k + \sum_{i=0}^{k-1} \bar{x}_{i+1}\bar{q}^i \tag{III.30}$$ ¤

Remarks: (1) The function $\bar{N}(\bar{x},\bar{q})$ is called the normal form of $N(x,q)$ or the universal unfolding of N. Note that in a specific example some of the parameters $\bar{x}_i$ may be zero so that the actual unfolding or bifurcation of the singularity of (III.2) may only be a projection of the universal unfolding.

(2) Theorem (III.6) is very powerful -- it contains as special cases versions of Theorem (III.3), (III.5). However, to find the explicit bifurcation functions ($\xi(x)$ in Theorem (III.3) and $\gamma_0(x)$, $\gamma(x)$ in Theorem (III.5)) is more involved. However $\eta(x)$ of Theorem III.6 can be determined approximately (upto $O(|x|^k)$)) from the matrix a_{ij}.

We show one example of the application of these ideas. The basic message is: at an even order codimension one singular point solutions to (III.2) may be lost in the direction in which x is varying (as in the fold case) and at odd-order singular points, solutions are not lost locally (as in the cusp case). The unfolding of the singularities on the other hand gets increasingly complicated.

III.4.2.4. The quartic or swallow tail singularity

We will assume here that the bifurcation function has already been reduced to normal form, i.e.

$$N(x,q) = q^4 + \bar{x}_1 q^2 + \bar{x}_2 q + \bar{x}_3 \quad . \tag{III.31}$$

The projection of the surface $N(x,q) = 0$ on the $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$ coordinates is shown in Figure 15(a). The fold lines comprising the bifurcation with two symmetric cusps are shown in Figure 15(b). The flow near folds and cusps is as discussed in III.4.2.1 and III.4.2.2.

Jump is infact predicted if the flow tends to (x_o, y_o), a swallow tail singularity point, since four solutions of (III.2) annihilate each other at (x_o, y_o).

III.4.3. Higher codimension bifurcations

If more than one eigenvalue of $D_2g(x_o, y_o)$ is 0 at $(x_o, y_o) \in M_o$ the bifurcation function $N(x,q)$ has higher dimension. The difficulty here is that unless $N(x,q)$ is a gradient function, i.e. there exists $V(x,q) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ such that

$$N(x,q) = (D_2V(x,q))^T \quad ,$$

there is no theory of normal forms (no extension of Theorem (III.6)). Thus, specific bifurcations need to be checked, often by laborious numerical work. In the special case that $N(x,q) = D_2V(x,q)$ the singularities involved are called umbilics (see Thom [42]). One example of what is called a hyperbolic umbilic is shown unfolded in Figure 16.

For the study of more general (non-gradient) $N(x,q)$ the scaling techniques of Chow, et al. [3] are of interest.

III.5. Systematic mathematical formulation of jump behavior

Definition III.2 (Jump behavior)

The solution of the system Σ described by equations (III.1), (III.2) is said to <u>jump from $(x_0,y_0) \in M$ to $(x_0,y_1) \in M_h$</u> if given $\delta > 0$, $\exists \varepsilon_0 > 0$, $t_0 > 0$ such that $\forall \varepsilon \in]0,\varepsilon_0]$,

$$|x_\varepsilon - x_0| + |y_\varepsilon - y_0| < \delta \tag{III.32}$$

and for $t \in [\varepsilon t_0, \alpha[$

$$|x(t,\varepsilon) - \tilde{x}(t)| + |y(t,\varepsilon) - \tilde{y}(t)| < \delta \tag{III.33}$$

where $x(t,\varepsilon)$, $y(t,\varepsilon)$ is the trajectory of Σ_ε starting from $(x_\varepsilon, y_\varepsilon)$ at $t = 0$; $\tilde{x}(t), \tilde{y}(t)$ is the trajectory of Σ starting from $(x_0,y_1) \in M$ at $t = 0$ and defined on $[0,\alpha[$. ¤

<u>Comment</u>: The intuitive content of our definition is that trajectories of the augmented system start close to one equilibrium of $\mathcal{B}_{x_0}$ and tend <u>increasingly rapidly</u> towards trajectories starting from a different equilibrium of $\mathcal{B}_{x_0}$. We first characterize how our definition admits of jump from points belonging to M_h/M_a.

<u>Theorem III.7</u> (Jump from certain non-singular points)

Let $(x_0,y_0) \in M_h$ and further let $\sigma(D_2 g(x_0,y_0)) \cap \mathbb{C}_+ \neq \phi$. Further, let all sufficiently small neighborhoods V of y_0 in $\{x_0\} \times \mathbb{R}^m$ be decomposed as

$$V = (V \cap S_{y_0}^{x_0}) \cup (V \cap S_{y_1}^{x_0}) \cup \ldots \cup (V \cap S_{y_p}^{x_0}) \tag{III.34}$$

where $V \cap S_{y_i}^{x_0} \neq \phi$ for $i = 1,\ldots,p$ and the $S_{y_i}^{x_0}$ are the stable manifolds of the (hyperbolic) equilibria y_i of $\mathcal{B}_{x_0}$. Then, the system Σ admits of jump from (x_0,y_0) to (x_0,y_1), (x_0,y_0) to (x_0,y_2), ..., (x_0,y_0) to (x_0,y_p).

<u>Proof</u>: From Assumption 2 (the complete stability assumption) it is clear that neighborhoods V of y_0 can be decomposed as in (III.34). From the

assumption 1 (finitely many equilibria) for $\mathcal{B}_{x_0}$ we have that $y_1, y_2, \ldots, y_p$ are fixed for sufficiently small neighborhoods V in (III.34). From the assumption that the y_i are hyperbolic equilibrium points of $\mathcal{B}_{x_0}$, the sets $S_{y_i}^{x_0}$ are the stable manifolds of the y_i. (Thus, Theorem III.2 may be applied). From the assumption that $\sigma(D_2 g(x_0,y_0)) \cap C_+ \neq \phi$ it is clear that $V \cap S_{y_0}^{x_0} \not\subset V$.

Now from the results of Fenichel [12] stated in comment (iv) after Theorem (III.2) $S_{y_k}^{x_0}(\varepsilon)$ tends smoothly to $S_{y_k}^{x_0}$ as $\varepsilon \downarrow 0$. Further, from (III.34) for sufficiently small V, $V \cap S_{y_k}^{x_0} \neq \phi$ so that given $\delta > 0$ we can find ε_1 small such that for $\varepsilon \in]0,\varepsilon_1]$, $\exists\ (x_\varepsilon,y_\varepsilon) \in S_{y_k}^{x_0}(\varepsilon)$ and $|x_\varepsilon - x_0| + |y_\varepsilon - y_0| < \delta$. Using the rate estimate (III.13) we have that for some K, $\sigma > 0$

$$|x_\varepsilon(t)-\tilde{x}(t)| \leqq K(\omega(\varepsilon)+\varepsilon|y_k-y_\varepsilon(0)|)$$

$$|y_\varepsilon(t)-\tilde{y}(t)| \leqq K(\omega(\varepsilon)+e^{-\sigma t/2\varepsilon}|y_k-y_\varepsilon(0)|)$$

provided $x_\varepsilon(0) = x_\varepsilon$, $y_\varepsilon(0) = y_\varepsilon \in S_{y_k}^{x_0}(\varepsilon)$. Since $\omega(\varepsilon)$ is a C^0 function with $\omega(0) = 0$ and K,σ are independent of ε, we can choose t_0, ε_0 such that for $\varepsilon \in]0,\varepsilon_0]$

$$|x_\varepsilon(t)-\tilde{x}(t)| + |y_\varepsilon(t)-\tilde{y}(t)| < \delta \qquad \forall t \geqq t_0\varepsilon$$

with $(x_\varepsilon(0) = x_\varepsilon, y_\varepsilon(0) = y_\varepsilon) \in S_{y_k}^{x_0}(\varepsilon)$.

Thus jump is admissible from (x_0,y_0) to (x_0,y_k). ¤

<u>Comments</u>. (i) The content of the theorem is visualized in Figure 17.

(ii) It is intuitive that a subset of M that does not admit of jumps as per our definition is

$$M_a = \{(x,y) : g(x,y) = 0,\ \sigma(D_2 g(x,y)) \subset \mathring{\mathbb{C}}_-\} \ .$$

We next describe the application of our definition to jump at singularities: (x_0,y_0) is a singularity of Σ if y_0 is an equilibrium of $\mathcal{B}_{x_0}$ formed by the coalescence of two or more hyperbolic equilibria; for example, a fold is formed by the coalescence of two equilibria of index +1 and -1; the cusp by the coalescence of three equilibria and so on. The attracting set of such a non-hyperbolic equilibrium $S_{y_0}^{x_0}$ does not necessarily contain y_0 in its interior. For instance, the attracting set $S_{y_0}^{x_0}$ for (x_0,y_0) on a fold boundary of M_a is an m-dimensional manifold with boundary, with $y_0 \in$ boundary $S_{y_0}^{x_0}$ (see Figure 16). Then, for any neighborhood V of y_0 in $\{y_0\} \times \mathbb{R}^m$, $V \subsetneq S_{y_0}^{x_0}$ so that it is intuitive to verify that the techniques of Theorem III.7 can be used to verify that jump is permissible from (x_0,y_0) to (x_0,y_1), (x_0,y_2) or (x_0,y_3). In general, we have as a corollary to Theorem III.7.

<u>Corollary III.8</u> (Jump from singular points)

Let $(x_0,y_0) \in M_0$. Then, the system Σ admits of jump from (x_0,y_0) if for all neighborhoods V of (x_0,y_0) in $\{x_0\} \times \mathbb{R}^m$

$$V \subsetneq S_{y_0}^{x_0} \quad .$$

Further, let all sufficiently small neighborhoods V of (x_0,y_0) in $\{x_0\} \times \mathbb{R}^m$ be decomposed as

$$V = (V \cap S_{y_0}^{x_0}) \cup (V \cap S_{y_1}^{x_0}) \cup \ldots (V \cap S_{y_p}^{x_0})$$

where $V \cap S_{y_i}^{x_0} \neq \phi$ for i = 1,...,p, and the $S_{y_i}^{x_0}$ are the stable manifolds of the hyperbolic equilibria y_i of $\mathcal{B}_{x_0}$. Then, Σ admits of jump from (x_0,y_0) to (x_0,y_1), ..., (x_0,y_p). ¤

Remarks: (i) It may be shown that if (x_0,y_0) is a cusp point, with $\sigma(D_2g(x_0,y_0)) \subset \mathbb{C}_-$ then $S_{y_0}^{x_0}$ is infact the center stable manifold of $\mathcal{B}_{x_0}$; so that y_0 belongs to the interior of $S_{y_0}^{x_0}$ (this is derivable from

Kelley [24]) and no jump is predicted at such cusp points. This is in keeping with the conclusion of III.4.2.2.

It is clear that jump is "optional" at non-singular points belonging to M_h/M_a i.e., solutions may infact be continuously extended (see also Comment (iv) below). Furthermore, continuous extension is infact possible at some singular points, for instance recall the example of Section III.1 and Case (iii) of III.4.2.2 with a "cubic" singularity. By continuous extension of solution of Σ through a singularity (x_0,y_0) at t = 0 we mean an integral curve of Σ, $\gamma(t) :]0,\alpha[\to M_h$ with $\lim_{t \downarrow 0} \gamma(0) = (x_0,y_0)$. Then, we assume

<u>Assumption 4</u> (Continuous extension from some singular points)

Let $(x_0,y_0) \in M_0$. If $S_{y_0}^{x_0}$ contains (x_0,y_0) in its relative interior in $\{x_0\} \times \mathbb{R}^m$, then there is a continuous extension of solutions of Σ through (x_0,y_0).

<u>Comment</u>: The validity of Assumption 4 for gradient systems seems tacit in Thom [42].

(ii) In the statement of Theorem (III.7) and Corollary (III.8) we insisted that the jump end points (x_0,y_1) , ..., (x_0,y_p) be <u>hyperbolic</u> thereby ruling out situations of the sort shown in Figure 19 where

$$V = (V \cap S_{y_0}^{x_0}) \cup (V \cap S_{y_1}^{x_0})$$

and (x_0,y_0) and (x_0,y_1) are both non-hyperbolic. In such a case it is clear that jump should infact be to (x_0,y_2). It is clear that Theorem (III.7) does not predict jump to (x_0,y_1) in this case since estimates of the form (III.33) required by Definition (III.2) will not hold in this instance. Details of the characterization of jump in this instance are more involved. For instance,

Theorem (III.9) (Jump characterization with multiple singularities)

Let $(x_0,y_0) \in M_0$. Further, let for all neighborhoods V of y_0 in $\{x_0\} \times \mathbb{R}^m$, $V \subsetneq S_{y_0}^{x_0}$ and let all small V be decomposed as

$$V = (V \cap S_{y_0}^{x_0}) \cup (V \cap S_{y_1}^{x_0}) \cup \ldots \cup (V \cap S_{y_p}^{x_0})$$

where $V \cap S_{y_i}^{x_0} \neq \phi$ for $i = 1,\ldots,p$ and $S_{y_i}^{x_0}$ are the stable manifolds of the hyperbolic equilibria y_i for $i = 2, \ldots, p$ of $\mathcal{B}_{x_0}$ and $(x_0,y_1) \in M_0$. Decompose small enough neighborhoods U of (x_0,y_1) in $\{x_0\} \times \mathbb{R}^m$ as

$$U = (U \cap S_{y_1}^{x_0}) \cup (U \cap S_{y_{p+1}}^{x_0}) \cup \ldots \cup (U \cap S_{y_{p+q}}^{x_0}) \quad .$$

with $y_{p+1}, \ldots, y_{p+q}$ hyperbolic equilibria of $\mathcal{B}_{x_0}$. Then, Σ admits of jump from (x_0,y_0) to (x_0,y_2) , ..., (x_0,y_{p+q}).

Proof: Exactly as in Theorem (III.7).

Note: Jump is not predicted from (x_0,y_0) to (x_0,y_1). ¤

(iii) In the instance that $g(x,y) = D_2F(x,y)$ (gradient constraints) our definition of jump specializes to a very simple criterion proposed by Takens [40,41] as follows:

Theorem III.10 (Jump characterization for gradient constraints)

In the instance that $g(x,y) = D_2F(x,y)$ where $F(x,y) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is for each x a proper function bounded above with finitely many critical points, the system Σ admits of jump from (x_0,y_0) to (x_0,y_1) if there exists a path in $\{x_0\} \times \mathbb{R}^m$ from (x_0,y_0) to (x_0,y_1) such that $F(x_0,y)$ increases continuously along this path.

Proof: In the special case of gradient constraints the boundary layer system $\mathcal{B}_{x_0}$ is a gradient system with

$$\frac{dy}{d\tau} = D_2F(x_0,y) \quad . \qquad \text{(III.35)}$$

The critical points of $F(x_0,y)$ are then the equilibrium points of $\mathcal{B}_{x_0}$. The specialization of Theorem (III.7) Corollary (III.8) and Theorem (III.9) to this system are immediate since $(x_0,\tilde{y}) \in S_{y_0}^{x_0}$ iff there exists a path from $\tilde{y}$ to y_0 along which $F(x_0,y)$ increases continuously. ¤

(iv) Theorem (III.7), Corollary (III.8) and Theorem (III.9) specify conditions when the system Σ admits of a jump. Jump may infact not be necessary for continuous extension of the trajectories of Σ at (x_0,y_0), for instance at $(x_0,y_0) \in M_h \backslash M_a$; or in cases (i) and (iii) of Section III.4.2.1 when $(x_0,y_0) \in M_0$ is a fold point. But we will still permit jump since such points are sensitive to noise and disturbances.

III.6. Complete solutions of constrained differential equations

Motivated by the discussion so far we define our solution concept for Σ:

Definition (III.3)(Solution Concept)

A possibly discontinuous function $\gamma(t) :]0,\alpha[\to M$ $(\gamma(t) = (x(t),y(t))$ is a solution to Σ if

(i) At given $t_1 > 0$, the right and left hand limits of $\gamma(t)$, $\dot{x}(t)$ exist. Further,

$$\dot{x}(t_1^-) = f(\gamma(t_1^-)) \ , \quad \dot{x}(t_1^+) = f(\gamma(t_1^+))$$

(ii) When $\gamma(t_1^+) \neq \gamma(t_1^-)$ then the jump satisfies the conditions of definition (III.2). ¤

Comments: (i) The solution is maximally smooth in that we do not allow for discontinuous changes in y unless it is consistent with the physical interpretation of Σ.

(ii) Since $\gamma(0)$ need not belong to M, the solution can be chosen to begin with a jump onto M. When we refer to a solution $\gamma(t) : [0,\alpha[$ of Σ we will infact allow this possibility.

(iii) The model we have for the solutions of the dynamical system Σ is a continuous-discrete system in the following fashion:

At $x = x_0$ we label the solutions to (III.2) 1, 2, 3, ..., p. Then we proceed to draw a bifurcation diagram for (III.2) by continuation methods, showing how new solutions are born or how solutions coalesce as for instance in Fig. 20(a). These labels are the states for the discrete dynamics which consist of jump information between different labels as shown in Figure 20(b). Figure 20(b) is what is referred to as a labelled diagram in the theory of dynamical systems [38]; with vertices representing equilibria of $\mathcal{B}_{x_0}$ and oriented segments representing possible jumps from saddle or unstable equilibria.

It is clear that jumps are possible in the labelled diagram till one reaches vertices from which there are no outward pointing segments. These are precisely points of M_a.

Note that in general the labelled diagram admits of cycles so that there may be a set of equilibria of $\mathcal{B}_{x_0}$, $y_1, y_2, \ldots, y_p$ such that jumps are possible from (x_0,y_1) to (x_0,y_2), ..., (x_0,y_p) to (x_0,y_1). (called a cyclic saddle loop [1].) These cycles are however not possible if

$$g(x,y) = D_2F(x,y) \quad .$$

and $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ satisfies the assumptions of Theorem (III.10) for then we would have (by Theorem (III.10))

$$F(x_0,y_1) < F(x_0,y_2) < \dots < F(x_0,y_p) < F(x_0,y_1)$$

which is a contradiction. Moreover, we have

Proposition III.3 (Jump always possible to M_a)

Given that $S_{y_0}^{x_0} \cap V \subset V$ for some neighborhood V of $(x_0,y_0) \in M$ in $\{x_0\} \times \mathbb{R}^m$ and $\pi M_a \underset{\neq}{=} \mathbb{R}^n$, then Σ admits of jump to $(x_0,y_1) \in M_a$, provided there is no cycle involving points of M_o.

Proof: We consider two cases:

(i) $(x_0,y_0) \in M_h$. Then, $S_{y_0}^{x_0}$ is a manifold of dimension < m. Further, since

$$V = (V \cap S_{y_0}^{x_0}) \cup (V \cap S_{y_1}^{x_0}) \cup \dots \cup (V \cap S_{y_p}^{x_0})$$

by the Baire category theorem, atleast one of the $S_{y_i}^{x_0}$ (say $S_{y_1}^{x_0}$) has interior; so that $(x_0,y_1) \in M_a \cup M_o$. If $(x_0,y_1) \in M_a$, we are done; if $(x_0,y_1) \in M_o$ the situation is treated as in (ii) below.

(ii) If $(x_0,y_0) \in M_o$, $S_{y_0}^{x_0}$ is not a manifold but is almost one; technically it is a stratified set (see Thom [42]), i.e., an open dense set of it is a manifold of dimension k (say), an open dense set of the resuidal set is a manifold of dimension (k-1) and so on (see also the discussion after Assumption 2). Thus, $(x_0,y_0) \in$ boundary $S_{y_0}^{x_0}$ and $S_{y_0}^{x_0}$ is a closed set so that if $S_{y_0}^{x_0} \cap V \underset{\neq}{\subset} V$, then $V \setminus (S_{y_0}^{x_0} \cap V)$ must be open. By the Baire category theorem as before there must exist $(x_0,y_1) \in M_a \cup M_o$ with $S_{y_1}^{x_0} \cap V \neq \phi$. If $(x_0,y_1) \in M_o$ we repeat this step. The process will end in finitely many steps since the number of equilibria is finite, $\pi M_a = \mathbb{R}^n$ and there are no loops involving points of M_o. ¤

Definition (III.4) (Complete solution)

Given a solution $\gamma(t):[0,\alpha[$ of Σ, it is said to be complete if either $\alpha = \infty$ or $\gamma(t)$ can be extended to $[0,\alpha]$ or $\overline{\{\gamma(t) : t \in [0,\alpha[\}}$ is not compact. ⌑

Comments: (i) $\overline{\{\gamma(t), t \in [0,\alpha[\}}$ not compact is the condition referred to in the theory of ordinary differential equations [17] as finite escape time.

(ii) It is clear that if the solutions of Σ are complete and stay bounded they can be extended till they are defined on $[0,\infty[$.

From a little reflection it is clear that there is no hope for proving a completeness theorem for trajectories of Σ, taking values on all of M, since as we have noted above a cycle of jumps is possible. Thus we could possibly have a situation in which we have a sequence of jumps at times $\{t_i\}_{i=1}^{\infty}$ with $\lim_{i\to\infty} t_i = \alpha$ with the lim infimum of jump distances non-zero. However, we will show that if we restrict ourselves to the attracting portion of M,

$$M_a = \{(x,y) = g(x,y) = 0, \quad \sigma(D_2 g(x,y)) \subset \mathring{\mathbb{C}}_-\}$$

we do have complete solutions. For reasons that will become clear in Section IV, we refer to M_a as the physical measurable portion of the configuration space M.

Theorem III.11 (Completeness of solutions on physically measurable portion of the configuration space)

Let $\pi M_a = \mathbb{R}^n$; and $\gamma(t):[0,\alpha[\to M_a$ be a solution of Σ. Then, either $\overline{\{\gamma(t), t \in [0,\alpha[\}}$ is not compact or $\alpha = \infty$ or the solution $\gamma(t)$ can be extended to take values in M_a on $[0,\alpha]$, provided there are no cycles involving points in M_o. ⌑

Proof: All we need to show is that if $(x_0,y_0) = \lim_{t\uparrow\alpha} \gamma(t) \in M_0$, then a continuation (possibly jump) can be prescribed to belong to M_a. Two possibilities arise: (a) If $V \cap S_{y_0}^{x_0} \not\subset V$ for some neighborhood V -- then by Proposition (III.3), we can choose a jump from (x_0,y_0) to $(x_0,y_1) \in M_a$. (b) If $V \subset S_{y_0}^{x_0}$ for small enough neighborhoods V then by assumption 3, a continuous extension possibly not belonging to M_a exists. But if such an extension exists then by Proposition (III.3) a (jump) extension ending in some $(x_0,y_1) \in M_a$ exists.

Thus in any case $\exists (x_0,y_1) \in M_a$ such that we may choose $(x_0,y_1) = \gamma(\alpha)$ and $\gamma(t) : [0,\alpha] \to M_a$ satisfying Definition III.3. ¤

Chapter IV: JUMP BEHAVIOR IN CIRCUITS AND PHYSICALLY MEASURABLE OPERATING POINTS

We consider the application of the theory of Chapter III to a fairly general class of non-linear, time-invariant networks shown in Figure 21. The batteries, constant sources and other non-dynamic elements are assumed to be lumped into the resistive time-invariant n-port.

IV.1. Circuit Equations

(C)&(L) We assume the capacitors to be time-invariant charge controlled and the inductors to be time-invariant flux-controlled. Let $z \in \mathbb{R}^{n_c+c_\ell}$ be the vector of charges on the capacitors ($z_1 \in \mathbb{R}^{n_c}$), fluxes in the inductors ($z_2 \in \mathbb{R}^{n_\ell}$) and $x \in \mathbb{R}^{n_c+n_\ell}$ be the vector of capacitor voltages ($x_1 \in \mathbb{R}^{n_c}$) and inductor currents ($x_2 \in \mathbb{R}^{n_\ell}$). Further, assume that the capacitors and inductors are reciprocal, that is, their constitutive relations are given by

$$x = h(z) = \nabla H(z) \qquad \text{(IV.1)}$$

where $H(z)$ is the stored energy of the capacitors and inductors. To prevent bifurcation of (IV.1) and hence discontinuous changes in capacitor voltages and inductor currents for continuous changes in capacitor charges and inductor fluxes we assume that h is a C^1 diffeomorphism on $\mathbb{R}^{n_c+n_\ell}$.

(R) The constitutive relations of the $n_c + n_\ell =: n$ port are given by

$$g(x,y) = 0 \qquad \text{(IV.2)}$$

where y is the hybrid vector of capacitor port currents (y_1) and inductor port-voltages (y_2) with reference directions so chosen that $x^T y$ represents power into the n-port and g is a C^1 function:

$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with 0 as a regular value (in the terminology of [7], the n-port is <u>strongly regular</u>).

Then the circuit equations are:

$$\dot{z} = -y \qquad \text{(IV.3)}$$

$$x = h(z) = \nabla H(z); \qquad g(x,y) = 0 \qquad \text{(IV.4)}$$

These could be written as

$$\dot{x} = -Dh(h^{-1}(x))y \qquad \text{(IV.5)}$$

$$0 = g(x,y) \qquad \text{(IV.6)}$$

If there exists a <u>global</u> hybrid representation for the n-port involving capacitor port voltages and inductor port currents as dependent variables i.e., a function $\psi : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$0 = g(x,y) \Leftrightarrow y = \psi(x) \quad .$$

the above equations (IV.5) and (IV.6) can be reduced to differential equations in the normal form, namely

$$\dot{x} = -Dh(h^{-1}(x)) \; \psi(x) \quad .$$

It is precisely the absence of global hybrid representations involving capacitor port voltages and inductor port currents as dependent variables that may lead to jump behavior. We give now a physical interpretation in <u>a circuit context</u> to the perturbations and assumptions introduced in the previous section. For this purpose, we first specialize (IV.6) to <u>assume</u> that the n-port admits of some <u>global hybrid representation</u>.

There exists a partition $A \cup B$ of $\{1,\ldots,n\}$ such that

$$x_A = f_A(y_A, x_B) \qquad \text{(IV.7)}$$

$$y_B = f_B(y_A, x_B) \qquad \text{(IV.8)}$$

are equivalent to (IV.6). (Here again $x^T y$ is the power into the n-port.) We use these equations in (IV.5) to obtain

$$\dot{x} = -Dh(h^{-1}(x)) \begin{bmatrix} y_A \\ f_B(y_A, x_B) \end{bmatrix} \qquad \text{(IV.9)}$$

with

$$0 = x_A - f_A(y_A, x_B) \quad . \qquad \text{(IV.7)}$$

Jump behavior may occur when there exist $(x, y_A) \in \mathbb{R}^{n+n_A}$ such that $D_1 f_A(y_A, x_B)$ is singular.

IV.2. Physically measurable operating points

The first task is to define the circuit dynamics on the n-dimensional manifold $M \subset \mathbb{R}^{n+n_A}$ given by

$$M = \{x \in \mathbb{R}^n; y_A \in \mathbb{R}^{n_A} : x_A - f_A(y_A, x_B) = 0\} \qquad \text{(IV.10)}$$

We assume that 0 is a regular value of the map $x_A - f_A(y_A, x_B) : \mathbb{R}^{n+n_A} \to \mathbb{R}^{n_A}$.

The understanding is that equation (IV.7) is the singularly perturbed limit as $\varepsilon \downarrow 0$ of

$$\varepsilon \dot{y}_A = x_A - f_A(y_A, x_B) \qquad \text{(IV.11)}$$

Physical significance of singular perturbation assumption

The assumption (IV.11) has the interpretation of introducing small (parasitic) linear inductances and capacitances at the A-ports of the n-port as shown in Figure 22. Note that (IV.11) requires that at the A-ports the parasitics to be introduced at the capacitor ports are ε-inductors in series and at the inductor ports are ε-capacitors in parallel. This interpretation does not of course mean that the parasitics introduced are the only ones associated with the circuit -- they do, however, represent, in a sense, the "net" effect of parasitics as seen at the A-ports which cannot be completely discarded in the qualitative analysis of the slow-speed dynamics of (IV.7), (IV.9), particularly in the neighborhood of operating points where $D_1 f_A$ is singular. Further justification arises as follows:

A current-controlled resistor is envisioned as the singularly perturbed limit as $\varepsilon \downarrow 0$ of the resistor in series with a small linear parasitic inductor because the current is the controlling variable. The dual is true for a voltage-controlled resistor.

The multiport generalization of this notion is the addition of parasitic inductances at ports where the current is the controlling (independent) variable and the dual for the voltage-controlled ports. From this standpoint, then, appropriate parasitics must be added both at A and B ports -- however at the B ports the just added parasitic capacitances are in parallel with the large capacitors and the parasitic inductors in series with large inductors so that their perturbation is negligible (regular perturbation). This is suggested by the dotted lines in Figure 22. At the A-ports the parasitics are paired with elements of the opposite kind and cannot be neglected as we have seen in the relaxation-oscillation example of Chapter II.

The seemingly ad-hoc assumption of having all parasitics of equal magnitude is made inconsequential to our development by assuming that any required scaling has already been incorporated into the algebraic equations (IV.7).

From Theorem III.8 the attracting portion of M is precisely

$$M_a = \{(x,y_A) : x_A = f_A(y_A,x_B) \; ; \; \sigma(D_1 f_A(y_A,x_B)) \subset \mathring{\mathbb{C}}_+\} \quad . \qquad \text{(IV.12)}$$

(We need $\mathring{\mathbb{C}}_+$ in (IV.12) from the minus sign in (IV.7)). From the discussion in Chapter III it is clear that this is the only portion of M stable to noise and modelling inaccuracies and the only portion which is robustly physically measurable using perfect instruments.

We now relate M_a and the strict local dissipativeness of an operating point (x,y_A).

Consider an n_A-port with $p_A \in \mathbb{R}^{n_A}$ standing for a hybrid list of port voltages and currents and $q_A \in \mathbb{R}^{n_A}$ standing for the complementary list of port voltages and currents with reference directions chosen so that $p_A^T q_A$ stands for the power into the n_A-port. Let the n_A port have global hybrid constitutive relation $q_A := \hat{q}(p_A)$. Then,

Definition IV.1 (Strict local dissipativeness)

An operating point (p_A^o, q_A^o) of the n_A-port described above is said to be strictly locally dissipative if there exists a positive definite Q such that $QD\hat{q}(p_A^o)$ is positive definite.

Remarks (i) The definition above is a generalization of strict-local passivity which is defined by setting Q = I.

(ii) The definition allows for scalings in the measurements of currents and voltages by choice of suitable diagonal Q.

Consider now the n_A-port of Figure 23 with hybrid constitutive relation

$$q_A = x_A^o - f_A(p_A, x_B^o) \quad . \tag{IV.13}$$

Here x^o stand for the voltage and current values of independent voltage and current sources (frozen-large capacitors and inductors). Since the dynamics of the n_A-port are given by

$$\varepsilon \dot{p}_A = x_A^o - f_A(p_A, x_B^o) \tag{IV.14}$$

it is clear that the set of equilibria of (IV.13) parameterized by x is precisely

$$M = \{(x, p_A) \in \mathbb{R}^{n+n_A};\ x_A - f_A(p_A, x_B) = 0\} \quad . \tag{IV.10}$$

Then, we have

Theorem IV.1 (Physically measurable operating points)

$$M_a = \{(x, y_A) \in \mathbb{R}^{n+n_A} : x_A - f_A(y_A, x_B) = 0,\ \sigma(D_1 f_A(y_A, x_B) \subset \mathring{\mathbb{C}}_+\}$$

is precisely that subset of M, at which the operating points parameterized by x of the hybrid n_A port of Figure 23 with constitutive relation (IV.13) are strictly locally dissipative.

Proof: Follows from the Lyapunov lemma, i.e.,

$$\sigma(D_1 f_A(y_A, x_B)) \subset \mathring{\mathbb{C}}_+ \Leftrightarrow \exists Q > 0 \text{ such that } D_1 f_A(y_A, x_B)Q + QD_1 f_A(y_A, x_B)^T > 0 \ .$$

¤

IV.3. Complete stability of the circuit equations associated with the parasitics

Recall from Section III that the key assumption (Assumption 2) for the description of Σ as the singularly perturbed limit of Σ_ε was the complete stability of the "frozen" boundary layer system $\mathcal{B}_{x_0}$ for all x_0. In the circuit context this assumption has the physical interpretation of no-high frequency or parasitic ringing. More explicitly, Assumption 2 is equivalent to the complete stability of the n_A-port shown in Figure 23 with capacitors, inductors "frozen" (constant voltage, current sources).

We draw on the notions of dissipativeness introduced in Definition (IV.1) to give conditions on the n_A-port to guarantee complete stability of the equation (IV.11) with x frozen at x^o. Our conditions use techniques similar to those of Chua and Suwannukul [8]. We however use dissipativeness rather than passivity. Recall that the dynamics of the n_A-port of Figure 23 are

$$\varepsilon\dot{p}_A = x_A^o - f_A(p_A, x_B^o) \qquad \text{(IV.14)}$$

with the x's frozen.

As is Assumption 1, Section III we assume that (IV.13) has finitely many equilibria, say $\{p_A^i\}_{i\in J}$.

Definition IV.2 (Dissipativeness relative to p_A^i)

The n_A-port with hybrid constitutive relation $q_A = \hat{q}(p_A)$ is said to be dissipative with respect to an equilibrium point of (IV.14), p_A^i in a set $\mathcal{C}(p_A^i) \subset \mathbb{R}^{n_A}$ with $p_A^i \in \mathcal{C}(p_A^i)$ if there exists a matrix $Q_{p_A^i} > 0$ such that

$$(p_A - p_A^i)^T Q_{p_A^i}\, \hat{q}(p_A) \geqq 0 \qquad \forall p_A \in \mathcal{C}(p_A^i) \qquad \text{(IV.15)}$$

with equality holding iff $p_A = p_A^i$. ¤

Remarks: (i) If $D\hat{q}(p_A^i)$ has eigenvalues in the open right half plane, then $\mathcal{C}(p_A^i)$ can be chosen to include an open neighborhood of p_A^i.

(ii) As before, if $Q_{p_A^i}$ can be chosen to be the identity matrix in (IV.15) dissipative may be replaced by passive.

We now state a special case of a Theorem proved in Appendix 2 for complete stability of circuits using Definition (IV.2) for the circuit of Figure 23.

Theorem IV.2 (Physical basis for complete stability)

Every trajectory of (IV.14) is bounded and converges to the discrete set $\{p_A^i\}_{i \in J}$ if:

(i) there exists $I \subset J$ such that $\hat{q}(p_A)$ is dissipative with respect to p_A^i, $i \in I$ in the sets $\mathcal{C}(p_A^i)$ defined by

$$\mathcal{C}(p_A^i) = \{p_A : |Q_{p_A^i}(p_A - p_A^i) + p_A^i|^2 - p_A^{iT} Q_{p_A^i}(p_A - p_A^i) \leq a(p_A^i)\} \qquad \text{(IV.16)}$$

where $Q_{p_A^i}$ is obtained from the definition of dissipativeness and $a(p_A^i)$ is suitably chosen such that

(ii) there exists $R > 0$ such that

$$z_A^T R \hat{q}(z_A) \geq 0 \qquad \forall z_A \notin \bigcup_{i \in J} \mathcal{C}(p_A^i)$$

with equality holding iff $\hat{q}(z_A) = 0$. ¤

Comments: (i) The above theorem is to be interpreted to mean using global R-dissipativeness (with respect to the origin) to stitch together local $Q_{p_A^i}$ dissipativeness to yield physical mechanisms for energy loss in the entire state space.

(ii) If an equilibrium p_A^i is not asymptotically stable it is clear that the inequality of (IV.15) can not hold in an open neighborhood of p_A^i so that $\mathcal{E}(p_A^i)$ can at best be a lower dimensional set.

Proof: As presented in Appendix 2 and the statement of the theorem visualized in Figure 25.

IV.4. Representation invariant property of singular points

Non-linear resistive n-ports often have several global hybrid representations, with different port voltages and currents as controlling variables. It is of obvious importance to assert that if one hybrid representation predicts the existence of singularities, other hybrid representations (provided they exist) also predict this. To be specific, let the hybrid n-port admit of two hybrid representations, one of the form (IV.7) and (IV.8) and the other of the form

$$x_{\tilde{A}} = f_{\tilde{A}}(y_{\tilde{A}}, x_{\tilde{B}}) \tag{IV.17}$$

$$y_{\tilde{B}} = f_{\tilde{B}}(y_{\tilde{A}}, x_{\tilde{B}}) \tag{IV.18}$$

Equations (IV.17), (IV.18) combined with equations (IV.5) yield equations of the form (IV.7), (IV.8) with A, B replaced by $\tilde{A}$, $\tilde{B}$ respectively. Then, our first conclusion is

Proposition IV. 3 (Invariance of singular points)

$$\{(x,y) : x_A = f_A(y_A . x_B);\ y_B = f_B(y_A, x_B),\ \det D_1 f_A(y_A, x_B) = 0\}$$

$$= \{(x,y) : x_{\tilde{A}} = f_{\tilde{A}}(y_{\tilde{A}}, x_{\tilde{B}});\ y_{\tilde{B}} = f_{\tilde{B}}(y_{\tilde{A}}, x_{\tilde{B}}),\ \det D_1 f_{\tilde{A}}(y_{\tilde{A}}, x_{\tilde{B}}) = 0\}. \tag{IV.19}$$

Proof: It is clear that the resistive constraint manifolds are the same in both representations i.e.,

$$\{(x,y) : x_A = f_A(y_A, x_B);\ y_B = f_B(y_A, x_B)\}$$

$$= \{(x,y) : x_{\tilde{A}} = f_{\tilde{A}}(y_{\tilde{A}}, x_{\tilde{B}});\ y_{\tilde{B}} = f_{\tilde{B}}(y_{\tilde{A}}, x_{\tilde{B}})\}$$

Recall from Proposition III.1 that the set of singular points of the system Σ defined on $\{(x,y) : g(x,y) = 0\}$ is $\{(x,y) : g(x,y) = 0,\ \det D_2 g(x,y) = 0\}$. We compute this set for the circuit dynamics in two different ways using the two representations. In the instance of the A, B hybrid representation

$$D_2 g(x,y) = \left[\begin{array}{c|c} D_1 f_A(y_A, x_B) & 0 \\ \hline D_1 f_B(y_A, y_B) & -I \end{array}\right]$$

so that the set of points at which $\det D_2 g(x,y) = 0$ is the set of points where $\det D_1 f_A(y_A, x_B) = 0$. Similarly computing using the second representation , we obtain that the set of points at which $\det D_2 g(x,y) = 0$ to be the set of points where $\det D_1 f_{\tilde{A}}(y_{\tilde{A}}, x_{\tilde{B}}) = 0$. This completes the proof. ¤

With proposition IV.3 at hand, it is largely a matter of checking for instance that a fold in one representation is equivalent to a fold in the other and the three cases of Chapter III, Section 4.2.2 arise equivalently in both hybrid representations. These are intuitively clear but the algebraic details are messy and omitted.

There are certain other <u>global</u>, physically-reasonable circuit invariants like the jumps which we are not yet able to show explicitly. The difficulty seems to arise from the different spaces (dimensionally) in which the dynmaics are embedded for different representations.

IV.5. The case of no-hybrid representation

Resistive n-ports can easily fail to have global hybrid representations (see [11] and [7] for numerous examples). To explicate the nature of the difficulties involved here we appeal to the example of Section II with a different (v-i) characteristic for the resistor as shown in Figure 26, say

$$\psi(i,v) = 0 \quad . \tag{IV.20}$$

For, $i \in]-\infty, i_3[$ the implicit function theorem may be applied to (IV.20) to yield $v = g(i)$ (current controlled). On augmenting with a parasitic inductor we obtain

$$\varepsilon \dot{i} = v - g(i) \quad i \in]-\infty, i_3[\quad . \tag{IV.21}$$

Trajectories of this equation with initial condition, $i = i_0$ and $v > v_0$ fixed, tend to i_3, when the augmenting equation (IV.21) fails to be defined. In keeping with our earlier comments, we note that the resistor characteristic changes at i_3 from being current controlled to being voltage controlled, so that the augmenting parasitic ought to be a capacitor rather than an inductor. Our standpoint here is that the resistor characteristic is not the relation shown in Figure 26 but the hysteretic characteristic shown in Figure 27 obtained when a variable current source is applied to the terminals of the non-linear resistor. Then, the resistor characteristic is given by

$$\left.\begin{array}{ll} v = g(i) & i \in]-\infty, i_3[\\ v = g_1(i) & i \in [i_3, \infty[\end{array}\right\} \quad \text{for } i \uparrow \tag{IV.22}$$

$$\left.\begin{array}{ll} v = g_1(i) & i \in \,]i_2,\infty[\\ v = g(i) & i \in \,]-\infty,i_2] \end{array}\right\} \quad \text{for } i\downarrow \tag{IV.23}$$

These equations (only piecewise continuous) are used in the augmented dynamics of (IV.21) with (IV.22) or (IV.23) being used depending on whether i is increasing or decreasing.

In the general n-port case, non-existence of any hybrid representation is symptomatic of neglected parasitics inside the n-port whose effect cannot be reflected to the n-port terminals. In analogy to the previous case hysteretic characteristics can be obtained for various combinations of port-excitations (↑ and ↓); but perhaps what is needed in such instances is a detailed study of the network inside the n-port to determine the circuit dynamics and jump behavior. This is done in the next chapter.

Chapter V. GRAPH TOPOLOGICAL DESCRIPTION OF JUMP BEHAVIOR IN CIRCUITS

We have limited our discussion of jump behavior in circuits to a port-description of the resistive network. This description may be inadequate to establish circuit behavior when the n-port has no hybrid representation. Also, certain global representation invariant properties of jump behavior seemed difficult to deduce. For this reason we give a circuit topological description.

V.1. Topological formulation of circuit equations as constrained equations

The components of the networks we consider (see Fig. 28) are non-linear resistors, inductors and capacitors, independent constant voltage and current sources and linear controlled sources. The network satisfies the usual consistency requirements: no E-loop, no J-cutset, no C-E loop, no L-J cutset* and the network admits of a normal tree (see [6] for a detailed description of assumptions). Let v_E, v_C, v_{RT} be the voltage vectors of voltage sources, capacitors and non-linear tree resistors and i_J, i_L, i_{RL} the current vectors of current sources, inductors and non-linear cotree resistors. Then under mild conditions (see [6]) on the resistive network we have, from Kirchhoff laws,

$$\begin{bmatrix} i_{RT} \\ v_{RL} \end{bmatrix} = \left[\begin{array}{c|c} H_{11} & H_{12} \\ \hline H_{21} & H_{22} \end{array}\right] \left[\begin{array}{c} v_{RT} \\ \hline i_{RL} \end{array}\right] + G_1 \left[\begin{array}{c} v_C \\ \hline i_L \end{array}\right] + G_2 \left[\begin{array}{c} v_E \\ \hline i_J \end{array}\right] \tag{V.1}$$

for suitably dimensioned matrices H_{11}, H_{12}, H_{21}, H_{22}, G_1 and G_2. Furthermore,

$$\begin{bmatrix} i_C \\ v_L \end{bmatrix} = G_3 \begin{bmatrix} v_C \\ i_L \end{bmatrix} + G_4 \begin{bmatrix} v_{RT} \\ i_{RL} \end{bmatrix} + G_5 \begin{bmatrix} v_E \\ i_J \end{bmatrix} \tag{V.2}$$

*The last two assumptions are for simplicity alone -- there is no loss of generality (see [5], [33]).

so that if $C(v_C)$ and $L(i_L)$ are the incremental nonsingular capacitance and inductance matrices then we have from (V.2)

$$\frac{d}{dt}\begin{bmatrix} v_C \\ i_L \end{bmatrix} = \begin{bmatrix} C(v_C) & 0 \\ 0 & L(i_L) \end{bmatrix}^{-1} \left\{ G_3 \begin{bmatrix} v_C \\ i_L \end{bmatrix} + G_4 \begin{bmatrix} v_{RT} \\ i_{RL} \end{bmatrix} + G_5 \begin{bmatrix} v_E \\ i_J \end{bmatrix} \right\} \qquad (V.3)$$

Assume for simplicity that the non-linear tree resistors are voltage controlled and non-linear cotree resistors are current controlled (other cases can be treated similarly) with characteristics

$$i_{RT} = -g(v_{RT}) = \begin{bmatrix} -g_1(v_{RT1}) \\ \vdots \\ -g_1(v_{RTp}) \end{bmatrix} \quad \text{and} \quad v_{RL} = -f(i_{RL}) = \begin{bmatrix} -f_1(i_{RL1}) \\ \vdots \\ -f_q(i_{RLq}) \end{bmatrix} \qquad (V.4)$$

so that we get as constraint equation for (V.3)

$$0 = \psi(v_{RT}, i_{RL}, v_C, i_L) = \begin{bmatrix} g(v_{RT}) \\ f(i_{RL}) \end{bmatrix} + \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} v_{RT} \\ i_{RL} \end{bmatrix} + G_1 \begin{bmatrix} v_C \\ i_L \end{bmatrix} + G_2 \begin{bmatrix} v_E \\ i_J \end{bmatrix} \qquad (V.5)$$

Our conclusion (also proved as a local rather than global perturbation result in [29], Theorem 2) is

Theorem V.1 (Choice of Parasitics for circuit dynamics)

There exists a choice of linear parasitic capacitors and inductors for the circuit described above such that circuits equations for the augmented system are in normal form.

Proof: We consider only the case when non-linear tree resistors are voltage controlled and non-linear cotree resistors are current controlled (the same results hold for other cases so long as each resistor is controlled by its own voltage or current).

Associate an ε-linear parasitic capacitance in parallel (soldering-iron entry) with each voltage controlled resistor and a ε-linear inductor in series (pliers entry) with each current controlled resistor. From the discussion so far it is clear that the augmented system has dynamics given by (V.3) and the (p+q)-equations

$$\varepsilon \begin{bmatrix} \dot{v}_{RT} \\ \dot{i}_{RL} \end{bmatrix} = - \begin{bmatrix} g(v_{RT}) \\ f(i_{RL}) \end{bmatrix} - \left[\begin{array}{c|c} H_{11} & H_{12} \\ \hline H_{21} & H_{22} \end{array}\right] \begin{bmatrix} v_{RT} \\ i_{RL} \end{bmatrix} - G_1 \begin{bmatrix} v_C \\ i_L \end{bmatrix} - G_2 \begin{bmatrix} v_E \\ i_J \end{bmatrix} \qquad (V.6)$$

This completes the proof. ¤

<u>Comments</u>: (i) The perturbations in this theorem bear resemblance to the perturbations of the previous section (see Figure 29).

(ii) In several practically important cases resistors may be either current or voltage controlled and the parasitics may be dictated by either layout or device physics (for instance in the Ebers-Moll model of the transistors, the parasitic capacitances for the base emitter and base-collector are determined from device-physics). Our prescription for handling such systems is to augment the given set of parasitics till a set of equations in normal form is obtained. Instead of integrating the resulting stiff differential equation we prescribe taking the limit -- which is a constrained differential equation whose solutions are discussed in Section IV. Parasitic dynamics reenter the picture only when jump is permitted i.e. on M_o and $M_h \backslash M_a$.

(iii) When the network has non-linear controlled sources -- one replaces them by a combination of linear controlled sources and non-linear resistors (this is always possible) and then applies Theorem (V.1).

For more general circuit elements, the above theorem may still be used to introduce perturbations to obtain locally well defined dynamics.

V.2. Connections between non-monotonicity and jump behavior

Folklore in circuit theory has it that jump-behavior arises from non-monotonicity of resistor characteristics. The origin of such folklore can be checked from a study of the solutions of the (p+q) equations (V.5) as arising from the intersection of (p+q) manifolds, (each of dimension p+q-1) each arising from one constraint (equation). The qualitative nature of the jth such manifold (with v_C, v_E, i_L, i_J fixed), say, is visualized easily as being the intersection of a load line and the jth resistor characteristic as shown in Figures 30(i), (ii). One sees that for monotone resistor characteristics and load lines of negative slope (corresponding to passivity) there is atmost one solution, whereas for non-monotone characteristics different numbers of solutions are possible depending on the values of (v_C and i_J) thereby showing precisely the possibility of jump behavior through the loss of solutions (locally) to (V.5). For instance we have indicated in the example of Fig. 30(ii) instances of 1, 3 or 1 solutions for different load lines. To make this precise, we have

Proposition V.1 (No jump for monotone circuits)

If the hybrid equations for the (p+q)-port associated with non-linear resistors is locally strictly passive and the non-linear resistors are monotone then the circuit equations (V.3), (V.5) have no singularities and in particular do not exhibit jump behavior.

Proof: We only need to check that for all v_E, i_J, v_C, i_L for which a solution v_{RT}, i_{RL} to (V.5) exists the Jacobian $[D_{v_{RT},i_{RL}}\psi(v_{RT},i_{RL},v_C,i_L)$ has eigenvalues in the open right half plane (so that the eigenvalues of the linearization of (V.6) lie in the left half plane). This Jacobian has the form $\left[\begin{array}{c|c} Dg(v_{RT}) & 0 \\ \hline 0 & Df(i_{RL}) \end{array}\right] + \left[\begin{array}{c|c} H_{11} & H_{12} \\ \hline H_{21} & H_{22} \end{array}\right]$. Further, by strict local passivity

$$\left[\begin{array}{c|c} H_{11} & H_{12} \\ \hline H_{21} & H_{22} \end{array}\right] > 0$$

and $\left[\begin{array}{c|c} Dg(v_{RT}) & 0 \\ \hline 0 & Df(i_{RL}) \end{array}\right]$ is a non-negative definite diagonal matrix. Hence

$$\left(\left[\begin{array}{c|c} H_{11} & H_{12} \\ \hline H_{21} & H_{22} \end{array}\right] + \left[\begin{array}{c|c} Dg(v_{RT}) & 0 \\ \hline 0 & Df(i_{RL}) \end{array}\right]\right) > 0$$ proving the proposition.

¤

Comment. The above proposition guarantees no singularities and jumps only if a solution to (V.5) exists. The existence of solutions to (V.5) for all v_E, i_J, v_C, i_L is more delicate and conditions for this are derived in [34], [35]-namely that H be a matrix of Class P_0. Note however that a matrix of Class P_0 is dissipative iff it is positive definite. For detailed proofs of the existence of normal form equations for monotone networks see [10].

In the instance that some of the algebraic equations (V.5) are known to have unique solution the alternative method [4] can be used to simplify computation. A trivial instance is

Proposition V.2 (Reducing the dimension of (V.5))

If $H_{22} \in$ Class P_0 and each $\{f_i\}_{i=1}^{q}$ is monotone increasing and C', there exists a C' function $h : v_{RT} \to i_{RL}$ such that (V.5) is equivalent to

$$0 = g(v_{RT}) + H_{11}v_{RT} + H_{12}h(v_{RT}) + \tilde{G}_1\begin{bmatrix} v_C \\ i_L \end{bmatrix} + \tilde{G}_2\begin{bmatrix} v_E \\ i_T \end{bmatrix} \qquad \text{(V.7)}$$

$$i_{RL} = h(v_{RT})$$

where $\tilde{G}_1$, $\tilde{G}_2$ stand for the first p rows of G_1, G_2 respectively.

Proof: See for instance [34].

Comment. One needs of course to still check the local-dissipativeness of the operating point before accepting the solution.

Chapter VI. CRITICAL ELEMENTS OF CONSTRAINED DIFFERENTIAL EQUATIONS, ESPECIALLY RELAXATION OSCILLATIONS

We return here to the notation and terminology of Chapter III. By a critical element of a dynamical system is meant an equilibrium point or a closed orbit. Constrained differential equations admit of two kinds of closed orbits: closed orbits which are continuous and closed orbits which are discontinuous.

Definition VI.1 (Relaxation oscillations)

A (necessarily) discontinuous curve $\gamma : [0,\alpha] \to M$ $(\alpha>0)$ is called a relaxation oscillation if

(i) $\gamma(t)$ is a solution of Σ

(ii) $\gamma(\alpha) = \gamma(0)$. ¤

Remark. (i) Consider a class of gradient differential equations with gradient constraints given by

$$\begin{aligned} \dot{x} &= -D_1F(x,y) \\ 0 &= D_2F(x,y) \end{aligned} \qquad \text{(VI.1)}$$

Here $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. Then, we have

Proposition VI.1 (Possibility of relaxation oscillations)

Along continuous solutions of (VI.1) not passing through singularities F decreases, and if the system exhibits a jump from (x_0,y_0) to (x_0,y_1)

$$F(x_0,y_1) > F(x_0,y_0) \quad .$$

Proof: The second part of the proposition is a restatement of Theorem (III.10). The first part follows from the fact that if the solution is continuous and does not pass through singularities it is differentiable so that

$$\frac{d}{dt} F(x,y) = \langle D_1F(x,y),\dot{x}\rangle + \langle D_2F(x,y),\dot{y}\rangle$$

$$= -|D_1F(x,y)|^2 \leqq 0 \quad . \qquad ¤$$

Inorder to have relaxation oscillations, we need a combination of continuous relaxation (F decreasing) interspersed with jump regeneration (F increasing).

VI. 1. Relaxation oscillations in the plane

We use the intuition of Proposition (VI.1) to generalize the example of Section II.

Theorem VI.1 (Liénard type relaxation oscillation)

Consider the constrained differential equation in the plane ($x,y \in \mathbb{R}$) given by

$$\dot{x} = -g(y) \tag{VI.2}$$

$$0 = x - f(y) \tag{VI.3}$$

where: (i) $g(-x) = -g(x)$ and $xg(x) > 0 \quad \forall x \neq 0$,

(ii) $f(-y) = -f(y)$ and $f(y) < 0$ for $0 < y < a$,

(iii) for $y > a$, $f(y)$ is positive and increasing ,

(iv) $f(y) \to \infty$ as $y \to \infty$,

(v) for each x, there are only finitely many y satisfying (VI.3).

Then, the system Σ admits of a unique relaxation oscillation. Further, trajectories starting from all $(x,y) \neq (0,0)$ converge to this orbit. ¤

Proof: (i) Construction of the orbit.

For $y \in [0,a]$ $f(y)$ reaches a minimum (at most at finitely many points by assumption (v) above). Consider the value y_0 closest to 0 at which the minimum is reached. At this point the system Σ allows of a jump from $(x_0=f(y_0),y_0)$ to (x_0,y_1), a non singular point (from assumptions (ii) and (iii) above) with $y_1 < 0$. Further, for all $y > 0$, $\dot{x}$ is negative, so that the solutions of Σ, starting from (x,y) with $y > 0$ must tend to the minimum permissible value of x in the half plane $\{(x,y) = y > 0\}$ i.e., x_0. If there are several values of y at which this minimum is reached, jump is permissible from these points to (x_0,y_0) as discussed in remark (ii) after corollary (III.8).

For $y < 0$, x is positive and since $f(-y) = -f(y)$ an exactly analogous argument serves to establish jump from $(-x_0 = f(-y_0),-y_0)$ to $(-x_0,y_1)$. Puting these pieces together one completes the construction of the orbit as shown in Figure 31.

(ii) The rest of the proof (uniqueness and all non-zero trajectories tending to the orbit) is straightforward and is omitted. ⌑

Remark: (i) Using Theorem (VI.2) and the comments in Section VI.2 we may conclude that the augmented system

$$\begin{aligned} \dot{x} &= -g(y) \\ \varepsilon\dot{y} &= x - f(y) \end{aligned} \tag{VI.4}$$

admits of a unique periodic orbit for $\varepsilon \in]0,\varepsilon_0]$ for some $\varepsilon_0 > 0$. Note the resemblance of this result to a result of Hartmann [18] showing the existence of unique periodic solutions to (VI.4) with $\varepsilon = 1$ (see also [19]).

VI.1. Persistence of equilibrium points and closed orbits

The set of equilibrium points for Σ is precisely the same as the set of equilibrium points for Σ_ε. Further from the results of Hoppensteadt [21] we have that if (x_0,y_0) is an equilibrium of Σ, Σ_ε then the eigenvalues of the linearization of the vector field of Σ_ε are of the form

$$\{\sigma(D_1f(x_0,y_0) - D_2f(x_0,y_0)\, D_2g(x_0,y_0)^{-1}\, D_1g(x_0,y_0)) + O(\varepsilon),$$

$$\frac{1}{\varepsilon}\,\sigma(D_2g(x_0,y_0)) + O(\varepsilon^0)\} \ .$$

In particular if $(x_0,y_0) \in M_a$ and it a stable equilibrium point of Σ, it is a stable equilibrium point of Σ_ε for $\varepsilon \in]0,\varepsilon_0]$ for some $\varepsilon_0 > 0$.

Further, for closed orbits we have the following result from Fenichel [12]:

Let C be a closed orbit not passing through any singular points. By assumption 2 then for each $(x,y) \in C$, $D_2g(x,y)$ has the same number of eigenvalues in the open left (right) half plane, say p(m-p). The stability of the orbit m under the flow of Σ is determined by checking if the eigenvalues of the linearization of the first-return or Poincarè map lie inside the unit disc with the exception of one eigenvalue which is always 1 (see [1] for details of the definition of the Poincarè map). The orbit is said to be hyperbolic if none of the eigenvalues of the linearization of the Poincarè map (except one) lie on the unit disc. Then, we have

Theorem VI.2 [12] (Persistence of closed orbits)

Let C be a closed hyperbolic orbit of Σ passing through no singular points, with q,(n-2-q),eigenvalues of the linearization of the Poincaré map inside,(outside),the open unit disc. Further, let for each $(x,y) \in C$, p,(m-p), eigenvalues of $D_2 g(x,y)$ lie in the open left,(right) half plane. Then, for $\varepsilon \in]0,\varepsilon_0]$. Σ_ε has a closed hyperbolic orbit C_ε with C_ε tending smoothly to C (the period also depends smoothly on ε). Further, p+q,(m+n-2-p-q), eigenvalues of the linearization of the Poincaré map for Σ_ε lie inside (outside) the open unit disc. ⌑

VI.2. Poincaré map for persistent relaxation oscillations

A relaxation oscillation $\gamma : [0,T] \to M$ is persistent if $\exists \varepsilon_0$ such that $\forall \varepsilon \in]0,\varepsilon_0]$, Σ_ε admits of a closed orbit $\gamma_\varepsilon : [0,T_\varepsilon] \to \mathbb{R}^n \times \mathbb{R}^m$ with γ_ε converging to γ and T_ε tending to T as $\varepsilon \downarrow 0$.

Consider $\gamma : [0,T] \to M_a \cup M_0$ a relaxation oscillation with two jumps from fold singularities -- from $(x_1,y_1) \in M_0$ to $(x_1,\bar{y}_1) \in M_a$ and from $(x_2,y_2) \in M_0$ to $(x_2,\bar{y}_2) \in M_a$ (see Figure 32). Further for sufficiently small neighborhoods U_1 and U_2 of (x_1,y_1) and (x_2,y_2) in $\{x_1\} \times \mathbb{R}^m$ and $\{x_2\} \times \mathbb{R}^m$ respectively let

$$U_1 = (U_1 \cap S_{y_1}^{x_1}) \cup (U_1 \cap S_{\bar{y}_1}^{x_1}) \qquad \text{(VI.5)}$$

$$U_2 = (U_2 \cap S_{y_2}^{x_2}) \cup (U_2 \cap S_{\bar{y}_2}^{x_2}) \ . \qquad \text{(VI.6)}$$

} Unique jump

Further let

$$v_1^T D_1 g(x_1,y_1)\ f(x_1,y_1) > 0 \qquad \text{(VI.7)}$$

$$v_2^T D_2 g(x_2,y_2)\ f(x_2,y_2) > 0 \qquad \text{(VI.8)}$$

where v_1, v_2 are bases for the null-space of $D_1 g(x_1,y_1)$ and $D_2 g(x_2,y_2)$ respectively. Miŝĉenko [28] has shown that such γ are persistent. A simpler proof is possible by our methods. However, our aim here is to demonstrate the existence of first return maps for γ: Let N be a submanifold of M_a of codimension 1 transverse to $\gamma(t)$ at $\gamma(0) = (x_0,y_0)$ $\in M_a$ between $(x_1,\bar{y}_1)$ and (x_2,y_2). From Theorem III.3, it follows that the singular points in a neighborhood of (x_1,y_1), (x_2,y_2) are fold points. Further inequalities like (VI.7), (VI.8) hold at such points and conditions like (VI.5), (VI.6) are valid at such points so that jump is still uniquely defined. Thus, under the flow of Σ (in the sense of Definition (IV.3)), a small neighborhood of (x_0,y_0) in N is mapped (smoothly) back into itself. This is the Poincaré map for γ. Conjugacy of Poincaré maps [1] defined using different submanifolds N is easy to show. We conjecture that the eigenvalues of the linearization of this Poincaré map contains the same stability information as in the case of regular closed orbits.

<u>Remarks</u>: (i) The extension of this theorem to relaxation oscillations with more than two jumps from fold boundaries (as in Theorem (VI.2) for instance) is trivial provided conditions like (VI.5), (VI.7) are satisfied at each fold point.

(ii) Relaxation oscillations with their continuous segments not in M_a are generally not persistent.

Chapter VII. EXAMPLES

VII.1 An example of Chua et al [7]

We analyze by our methods an example from [7] shown in Fig. 33(i). Clearly, the resulting equation has singularities, as shown in Fig. 33 (ii). To obtain equations in the normal form, we add parasitics as shown, to get

$$\left.\begin{aligned} \dot{i}_\rho &= -v_{R1} - v_{R2} \\ \varepsilon\dot{v}_{R1} &= -f_{R1}(v_{R1}) + i_\rho \\ \varepsilon\dot{v}_{R2} &= -f_{R2}(v_{R2}) + i_\rho \end{aligned}\right\} \qquad \text{(VII.1)}$$

Clearly, M_a, the observable portion of the configuration space, is the subset of $\left\{(i_\rho, v_{R1}, v_{R2}) : i_\rho = f_{R1}(v_{R1}) = f_{R2}(v_{R2})\right\}$ where $\frac{d}{dv_{R1}} f_{R1}(v_{R1}) > 0$ and $\frac{d}{dv_{R2}} f_{R2}(v_{R2}) > 0$. These conditions can be shown on individual resistor characteristics as in Fig.33(ii) (solid lines) and for the composite dynamics in Fig.33 (iii) with $v_\rho = v_{R1} + v_{R2}$. Some possible jumps are also shown in Figure 33(iii) - they are all from fold singularities.

VII.2 An astable multivibrator

We consider the simplest astable multivibrator consisting of two npn transistors with symmetric cross coupling between the base of one transistor and the collector of the other through capacitors as as shown in Fig. 34 (a).

The description of the resulting relaxation oscillation in this circuit is found in elementary books (eq. [45]). Jump takes place at points where the solutions to the Ebers Moll equations change discontinuously - qualitatively, the jumps are described by transition from Q1 off, Q2 saturated to Q1 saturated, Q2 off and vice versa. The discrete system whose states describe the jump transition has labeled diagram (see Section III) as shown in Fig. 34(b). The state with Q1 and Q2 both "on" is unstable to the parasitic dynamics and so is not observed. This state alternately annihilates the other two stable states, namely, Q1 saturated, Q2 off; and Q1 off, Q2 saturated in fold bifurcations. For a more detailed quantitative analysis of multi-vibrator equations using the techniques of Mišcenko, see [30].

Chapter VIII. CONCLUDING REMARKS

Important extensions to our work are possible in several directions; from a mathematical standpoint, some areas are:

(i) The description of jump behavior for implicitly defined or constrained dynamics defined over a manifold, rather than a vector-space, as the base space.

(ii) The detailed description of the flow near singularities of codimension larger than one.

(iii) The development of solution concepts for constrained partial differential equations.

(iv) Stability and persistence of relaxation oscillations, as well as the description of more complicated invariant sets than critical points and orbits for constrained systems. Also, the study of bifurcations of these invariant sets.

From an applications viewpoint, open problems lie in:

(i) The detailed study of the dynamics of special classes of circuits, for instance, transistor circuits.

(ii) The description of jump behavior in non-linear distributed circuits.

(iii) Variational calculus and optimal control for constrained systems.

Appendix 1. A CLASS OF COMPLETELY STABLE SYSTEMS SATISFYING ASSUMPTION 2

We consider here a class of system studied by Popov [31,32]. They are of the standard-control form with set point u_0

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= SCx \qquad u = h(y) + u_0 \end{aligned}\right\} \tag{A1.1}$$

We assume that A, B, C have the following special structure

$$A = \text{diag}(A_1, A_2, \ldots, A_p) \in \mathbb{R}^{n \times n} \text{ with } A_i \in \mathbb{R}^{n_i \times n_i}$$

$$B = \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & b_p \end{bmatrix} \in \mathbb{R}^{n \times p} \quad \text{with} \quad b_i \in \mathbb{R}^{n_i \times 1}$$

$$C = \begin{bmatrix} c_1^T & 0_T & \cdots & 0 \\ 0 & c_2^T & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & c_p^T \end{bmatrix} \in \mathbb{R}^{p \times n} \quad \text{with} \quad c_i^t \in \mathbb{R}^{n_i \times 1}$$

$h(\cdot) : \mathbb{R}^p \to \mathbb{R}^p$ is of the form $y \mapsto \begin{pmatrix} h_1(y_1) \\ \vdots \\ h_p(y_p) \end{pmatrix}$. $S \in \mathbb{R}^{p \times p}$ is symmetric and nonsingular. Further, we assume

Monotonicity Assumption

(i) The matrices A_i are strictly Hurwitz and the impulse response functions $g_i(t) = c_i^T e^{A_i t} b_i$ are completely monotonically decreasing for $t \geq 0$. Further, $g_i(0)$ is normalized to 1 for $\forall i$.

A large subset of the class of C^{∞} functions satisfying this assumption are characterized by their Laplace transforms having poles on the negative real axis with positive residues at these poles.

(ii) The functions h_i are uniformly Lipschitzian and strictly increasing, that is $\exists L, \delta > 0$ such that

$$\delta\mu < h_i(\rho+\mu) - h_i(\rho) < L\mu \qquad \forall i \tag{A1.2}$$

for all $\mu > 0$ and $\rho \in \mathbb{R}$.

Using transform methods, Popov [31] proves

Theorem A1.1 [31] (Complete stability of (A1.1))

Under the monotonicity assumption every solution x(t) of (A1.1) is bounded and approaches the set of stationary solutions of (A1.1) as $t \to \infty$, i.e.

$$\lim_{t\to\infty} \{\inf|x(t)-z_0| : z_0 \in \{z : Az + Bh(SCz) = Bu_0\} = 0 \qquad \square$$

Thus, the system of (A1.1) with the monotonicity assumption is a completely stable non-gradient system. We use the alternative method [4] to study the set of stationary solutions of (A1.1) i.e.

$$\{z_0 \in \mathbb{R}^n : Az_0 + Bh(SCz_0) + Bu_0 = 0\} \quad . \tag{A1.3}$$

Let $P_B \in \mathbb{R}^{pxn}$ be the left inverse of B and $P_{SC} \in \mathbb{R}^{nxp}$ the right inverse of SC. Choose $\tilde{P}_B \in \mathbb{R}^{n-pxn}$ and $\tilde{P}_{SC} \in \mathbb{R}^{nx(n-p)}$ such that

$\begin{bmatrix} P_B \\ \hline \tilde{P}_B \end{bmatrix}$ and $[P_{SC} \vdots \tilde{P}_{SC}]$ are non-singular and

$$\tilde{P}_B \cdot B = 0 \tag{A1.4}$$

$$SC \cdot \tilde{P}_{SC} = 0 \tag{A1.5}$$

Decomposing z as $z = P_{SC}q + \tilde{P}_{SC}r$, we rewrite $Az + Bh(SCz) = Bu_o$ as

$$P_B A P_{SC} q + P_B A \tilde{P}_{SC} r + h(q) = u_o \tag{A1.6}$$

$$\tilde{P}_B A P_{SC} q + \tilde{P}_B A \tilde{P}_{SC} r = 0 \tag{A1.7}$$

Note that (A1.7) is a linear equation. We assume that it can be solved uniquely for r (i.e. $\tilde{P}_B A \tilde{P}_{SC}$ is non-singular) so that we have a low-order nonlinear equation in q (p equations in p variables) from (A1.6), namely

$$\hat{P}_B A P_{SC} q + h(q) = u_o \tag{A1.8}$$

where

$$\hat{P}_B := P_B[I - A\tilde{P}_{SC}(\tilde{P}_B A \tilde{P}_{SC})^{-1}\tilde{P}_B] \quad . \tag{A1.9}$$

Equation (A1.8) is well studied in circuit theory (see Sandberg and Wilson [34,35]) and there are known sufficient conditions for (A1.7) to have multiple solutions, which is the situation of interest to us. For instance, we have

<u>Theorem A1.2</u> [34] (Multiple equilibria of (A1.1))

If $\hat{P}_B A P_{SC}$ is not of Class P_o and $h(\cdot)$ satisfies (A1.2) then given any $\delta > 0$, $\exists q_1$, q_2 and $u_1 \in \mathbb{R}^p$ such that

$$\hat{P}_B A P_{SC} q_1 + h(q_1) = \hat{P}_B A P_{SC} q_2 + h(q_2) = u_1 \quad .$$

and $|q_1 - q_2| = \delta$. ¤

With these theorems in mind we can now state a class of functions g(x,y) which satisfy the absolute assumption (Assumption 2) and has multiple solutions for $g(x_o,y) = 0$; namely,

$$g(x,y) = Ay + Bh(SCy) + Bx \tag{A1.10}$$

where A, B, C and h satisfy the assumptions made earlier. The class can obviously be generalized to

$$g(x,y) = Ay + Bh(x,SCy) \tag{A1.11}$$

where $h(x_0))$ satisfies the general assumptions for fixed x_0. The study of the solutions is now a more subtle matter.

We next prove that the class of systems (A1.11) satisfies our other main assumption -- that eigenvalues of $D_2g(x,y)$ cross the $j\omega$ axis at the origin as (x,y) moves on M. We infact prove that all eigenvlaues of $D_2g(x_0,y_0)$ in the right half plane are real slightly modifying Theorem 8.3 from [31].

<u>Theorem A1.3</u> (<u>no dynamic bifurcations for A1.11</u>)

Let (x_0,y_0) be a solution of

$$Ay + Bh(x,SCy) = 0$$

Then all $\mathbb{C}_+$ eigenvalues of $A + BD_2h(x_0,SCy_0)SC$ are real. ¤

<u>Proof</u>: Since each A_i is strictly Hurwitz, the $\mathbb{C}_+$ eigenvalues of $A + BD_2h(x_0,SCy_0)SC$ coincide with the $\mathbb{C}_+$ solutions s of

$$\det(I-SC(sI-A)^{-1}BD_2h(x_0,SCy_0)) = 0 \quad . \tag{A1.12}$$

Now $Ce^{At}B = g(t)$ is a diagonal matrix of completely monotone decreasing functions (i.e. with derivatives of alternating sign). We have by Bernstein's theorem [14] that

$$g(t) = \int_0^\infty e^{-\alpha t}dQ(\alpha)$$

where $Q(\cdot)$ is a diagonal matrix with all its elements bounded, increasing functions.

By Fubini's theorem

$$G(s) = C(sI-A)^{-1}B = \int_0^\infty \frac{1}{s+\alpha}\, dQ(\alpha) \tag{A1.13}$$

This is an analytic function for every $s \in \mathring{\mathbb{C}}_+$. Now, denote $D_2h(x_0,SCy_0)$ by K. K is a positive definite diagonal matrix. Choose $z \in \mathbb{C}^n$ such that for $s_0 \in \mathring{\mathbb{C}}_+$ a solution of (A1.12),

$$z - SG(s_0)Kz = 0 \quad .$$

Since S is symmetric and non-singular

$$\langle z,G(s_0)Kz\rangle = \langle z,S^{-1}z\rangle \in \mathbb{R} \quad . \tag{A1.14}$$

However from (A1.13)

$$\text{Im } G(s_0) = -\text{Im } s_0 \int_0^\infty \frac{1}{|s_0+\alpha|^2}\, dQ(\alpha) \quad .$$

so that the diagonal elements of Im $G(s_0)K$ are different from 0 and of the same sign if Im $s_0 \neq 0$. This contradicts (A1.14) unless Im $s_0 = 0$. This establishes the theroem. Further it establishes that eigenvalues of $A + BD_2h(x_0,SCy_0)SC$ cross the $j\omega$-axis at the origin. ⌑

Appendix 2. COMPLETE STABILITY THEOREMS

We begin by globalizing the basic theorem of the first method of Lyapunov for C^1 dynamical systems with well defined flow (no finite escape time) on $\mathbb{R}^n$ having finitely many equilibrium points

$$\dot{x} = f(x) \tag{A2.1}$$

Recall that a set is ω-complete [1] if the ω-limit points of every trajectory originating in that set belongs to the set.

Theorem A2.1 [8] (Globalization of first method of Lyapunov)

The system (A2.1) is completely stable, i.e. all bounded trajectories converge to equilibrium points if:

(i) There is a finite collection of ω-complete sets $\{K_i\}_{i=1}^{\ell}$ with associated C^1 functions $V_i : K_i \to \mathbb{R}_+$ such that

$$\nabla V_i(x)^T f(x) \leqq 0 \quad \forall x \in K_i \tag{A2.2}$$

with equality holding iff $f(x) = 0$.

(ii) There is a C^1 (stitching) function $V_0 : \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\nabla V_0(x)^T f(x) \leqq 0 \quad \forall x \in \mathbb{R}^n \setminus \bigcup_{i=1}^{n} K_i \tag{A2.3}$$

With equality hold iff $f(x) = 0$. ¤

Proof: It is easy to check from (A2.2) that every bounded trajectory of (A2.1) starting in K_i at $t = 0$ converges to an equilibrium point in K_i. Further any trajectory that intersects one of the K_i, say K_j, at some finite time converges to an equilibrium point in K_j.

We are left with the case when a bounded trajectory labelled $x(t,x_0) \in \mathbb{R}^n \setminus \bigcup_{i=1}^{\ell} K_i \ \forall t \in [0,\infty[$. Let $\omega(x_0)$ be the (compact) ω-limit set of $x(t,x_0)$. ($p \in \omega(x_0)$ iff $\exists\, t_k \uparrow \infty$ such that $x(t_k,x_0) \to p$). We now have two possibilities:

(i) $\omega(x_0) \cap \bigcup_{i=1}^{\ell} K_i \neq \phi$ and $\omega(x_0)$ does not consist of equilibrium points of (A2.1). Restart (A2.1) from a point in $\omega(x_0) \cap K_j$ for some j. This converges to an equilibrium point in K_j; so that by uniqueness of solutions of (A2.1) $x(t,x_0)$ also converges to that equilibrium point.

(ii) $\omega(x_0) \subset \mathbb{R}^n \setminus \bigcup_{i=1}^{\ell} K_i$. But then one notes that V_0 is a C^1 function bounded below decreasing along trajectories of (A2.1) and defined on the ω-complete set $\{x(t,x_0), t \in [0,\infty]\} \cup \omega(x_0)$. Thus, $x(t,x_0)$ must converge to an equilibrium point in $\mathbb{R}^n \setminus \bigcup_{i=1}^{\ell} K_i$. ¤

We use definition (IV.2) to apply this theorem to the study of complete stability of circuit equations in the normal form described by

$$\dot{x} = -g(\nabla H(x)) \tag{A2.4}$$

where $H : \mathbb{R}^n \to \mathbb{R}_+$ is proper, $g : \mathbb{R}^n \to \mathbb{R}^n$ are smooth and $h := \nabla H : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism.

Here, x is the vector of charges on capacitors, fluxes in inductors, H is the energy storage function of the "reciprocal" capacitors and inductors, and $\nabla H(x)$ is the function of constitutive relationships between capacitor charges, inductor fluxes and capacitors voltages, inductor currents. g is the global hybrid representation of the resistive n-port with capacitor port voltages and inductor port currents as independent variables (assumed to exist). The minus sign in (A2.4) arises from physical considerations.

We <u>assume</u> again that (A2.4) has finitely many equilibria.

The dynamics of capacitor voltages and inductor currents $z = \nabla H(x) = h(x)$ are given by

$$\dot{z} = -Dh(h^{-1}(z))\, g(z) \tag{A2.5}$$

where $Dh(\cdot) \in \mathbb{R}^{n \times n}$ is the Hessian of H and so is a symmetric matrix.

We first consider the case when the inductors and capacitors are linear so that

$$H(x) = \frac{1}{2} x^T P x \text{ for some positive definite } P.$$

Then, (A2.5) may be rewritten as

$$\dot{z} = -P\, g(z)\ . \tag{A2.6}$$

Then, we have

<u>Theorem A2.2</u> (Complete stability of circuit equations with linear capacitors and inductors)

Every trajectory of (A2.6) is bounded and converges to an equilibrium of (A2.6) if:

(i) there exists a finite set of equilibrium points of (A2.6) $\{z_i\}_{i=1}^p$ such that Pg is dissipative with respect to z_i in the set $\mathcal{A}(z_i)$ given by

$$(z_i) = \{z \in \mathbb{R}^n : |z_i + Q_{z_i}(z-z_i)|^2 - |z_i|^2 - (z-z_i)^T Q_{z_i}(z-z_i) \leqq z_i\}$$

where $a_{z_i} \in \mathbb{R}_+$ is suitably chosen and $Q_{z_i} > 0$ arises from the definition of dissipativeness, i.e.

$$(z-z_i) Q_{z_i}\, g(z) \geqq 0 \qquad \forall z \in \mathcal{A}(z_i) \tag{A2.7}$$

with equality holding iff $z = z_i$.

(ii) there exists R > 0 such that $\forall z \notin \bigcup_{i=1}^{p} \mathcal{A}(z_i)$

$$z^T R g(z) \geqq 0 \tag{A2.8}$$

with equality holding iff $g(z) = 0$. ¤

Proof: It may be verified using (A2.7) and (A2.8) that the sets $\mathcal{Q}(z_i)$ with the corresponding functions

$$V_i(z) = |z_i + Q_{z_i}^{1/2}(z-z_i)|^2 - |z_i|^2 - z_i^T Q_{z_i}(z-z_i)$$

and the function

$$V_0(z) = |R^{1/2}z|^2$$

satisfy the conditions of Theorem (A2.1). Thus, all bounded trajectories of (A2.6) converge to equilibrium points of (A2.6). All we need to show then is that all trajectories are bounded. For this choose $\mathcal{K}$ a large closed ball so that all the equilibrium points and $\bigcup_{i=1}^{p} \mathcal{Q}(z_i) \subset \mathcal{K}$. Such a choice is possible since the $\mathcal{Q}(z_i)$ are compact and the number of equilibria are finite. Further, by (A2.8)

$$\dot{V}_0 < 0 \quad \forall z \notin \mathcal{K}.$$

Since V_0 is proper, there exists c > 0 so that

$$V_0(z) \leq c \Rightarrow z \in \mathcal{K} \quad .$$

Thus every trajectory wandering outside $\mathcal{K}$ is eventually attracted back to $\mathcal{K}$; and so is bounded.

This completes the proof. ¤

Remark: Theorem (IV.2) is a restatement of Theorem (A2.2) with P = I. For the case when the capacitors and inductors are non-linear the absolute stability is more delicate and it is more convenient to work with equation (A2.4) with its built-in energy function H(x). We have

Theorem A2.3 (Complete stability of normal form circuit equations)

Every trajectory of (A2.4) is bounded and converges to an equilibrium point of (A2.4) if

(i) there exists a finite set of equilibrium points of (A2.4) $\{x_i\}_{i=1}^{p}$ such that g is passive with respect to $z_i = h(x_i)$ in the sets $\mathcal{A}(z_i) = h(\mathcal{A}(x_i))$ where

$$\mathcal{A}(x_i) = \{x \in \mathbb{R}^n : H(x) - H(x_i) - h(x_i)^T(x-x_i) \leqq a_{x_i}\}$$

for suitably chosen a_{x_i}, i.e.

$$(h(x)-h(x_i))^T\ g(h(x)) \geqq 0 \qquad \forall x \in \mathcal{A}(x_i) \tag{A2.9}$$

with equality holding iff $x = x_i$

(ii) there exists $R > 0$ such that $\forall x \notin \bigcup_{i=1}^{p} \mathcal{A}(x_i)$

$$h(Rx)^T\ Rg(h(x)) \geqq 0 \tag{A2.10}$$

with equality holding iff $g(h(x)) = 0$.

<u>Remark</u>: The theorem is visualized in Figure 19.

<u>Proof</u>: It is easy to verify using (A2.9), (A2.10) that the functions $V_i(x)$ defined on $\mathcal{A}(x_i)$ by

$$V_i(x) = H(x) - H(x_i) - h(x_i)^T\ (x-x_i)$$

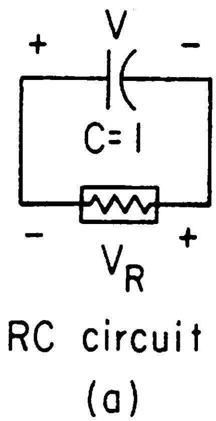
and $\quad V_0(x) = H(Rx)$

satisfy the conditions of Theorem (A2.1). All trajectories are bounded from an argument exactly like that in Theorem A2.2 above using the assumption that $H(\cdot)$ is proper. ¤

<u>Note</u>: A more general version of Theorem (A2.3) involving dissipativeness runs into difficulties, because of the non-linearity associated with the

constitutive relations between the capacitors charges, inductor fluxes and capacitor voltages inductor currents. In particular, if H is strictly convex, then it induces a Riemannian metric on $\mathbf{R}^n$, i.e. its Hessian is used to define the inner product. (See [2] for details). Dissipativeness then defined relative to this metric, obtains the required stability theorem.

FIGURE CAPTIONS



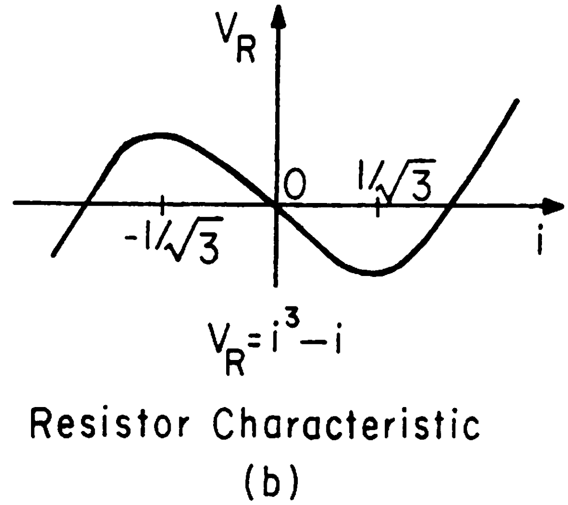


Figure 1. Degenerate van der Pol oscillator circuit

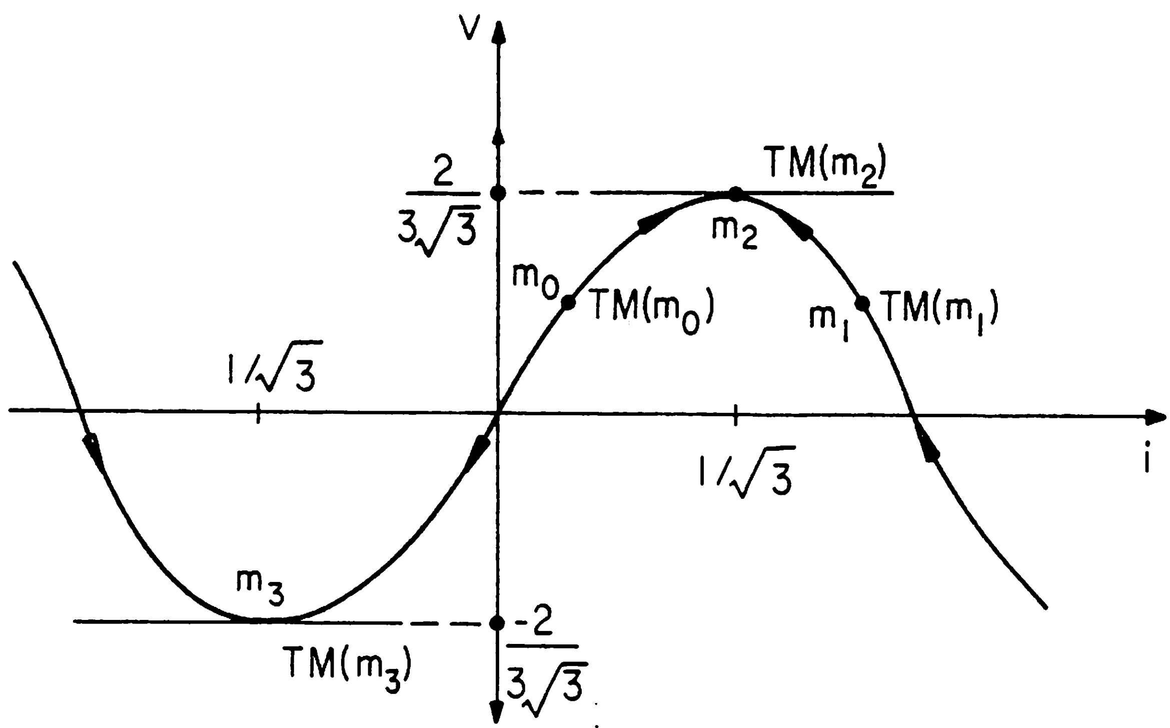


Figure 2. The dynamical system of (II.1), (II.2).

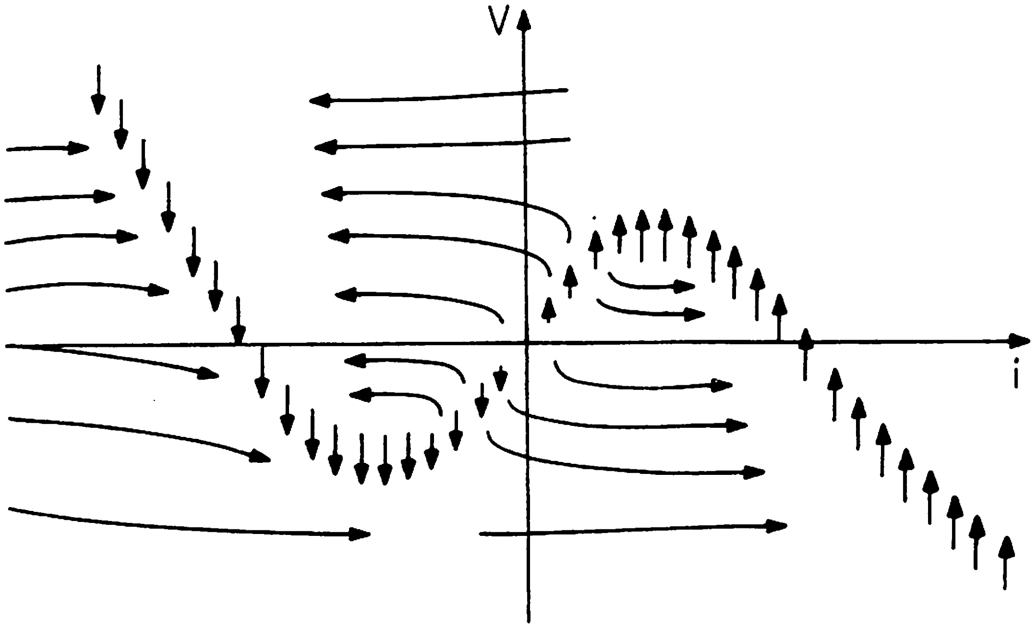


Figure 3. Phase portrait of 'regularized circuit equations' for $\varepsilon > 0$.

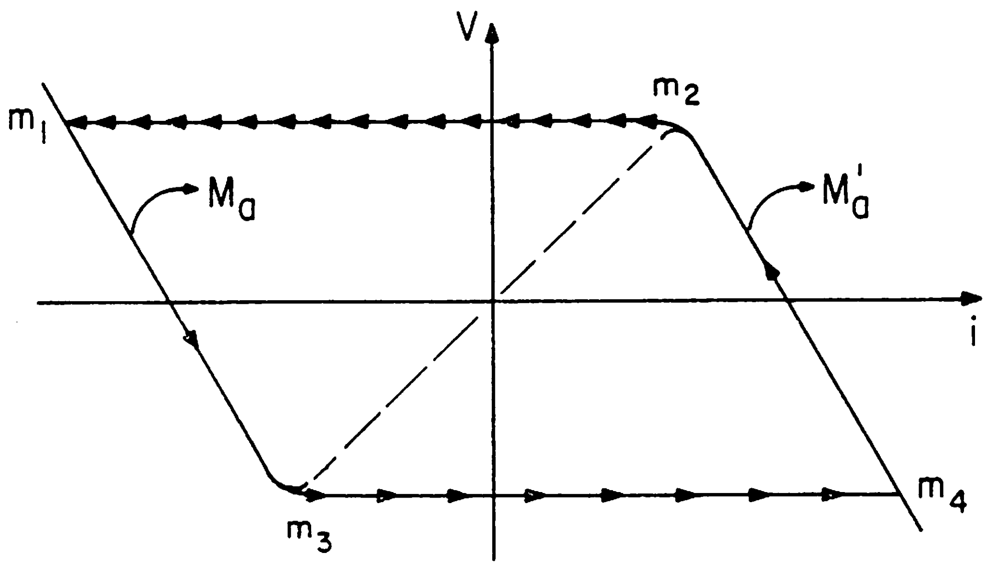


Figure 4. Choice of configuration space of (II.1), (II.2) consistent with (II.6), (II.7).

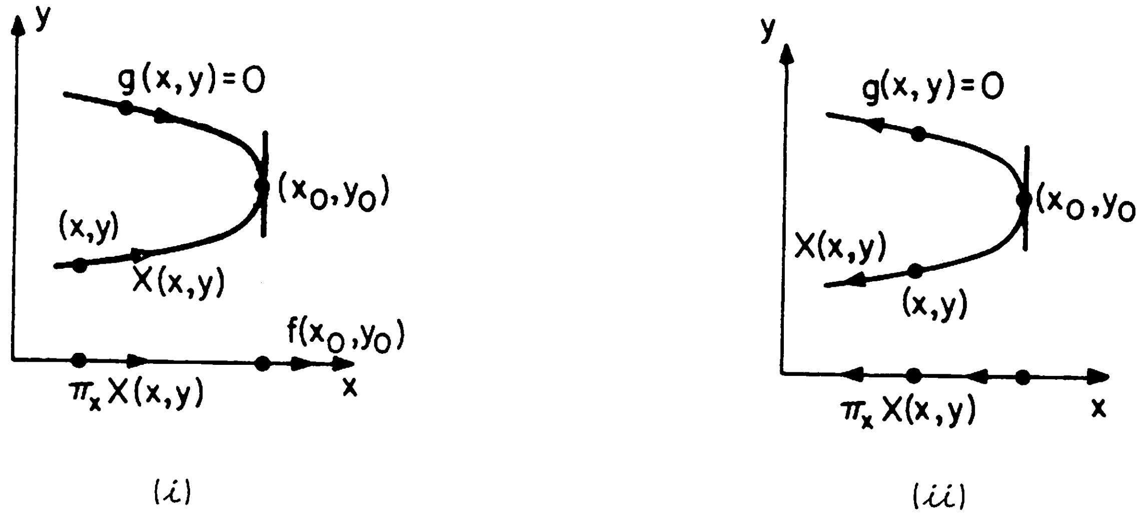


Figure 5. Illustrating the nature of the difficulty in obtaining X(x,y) from f(x,y).

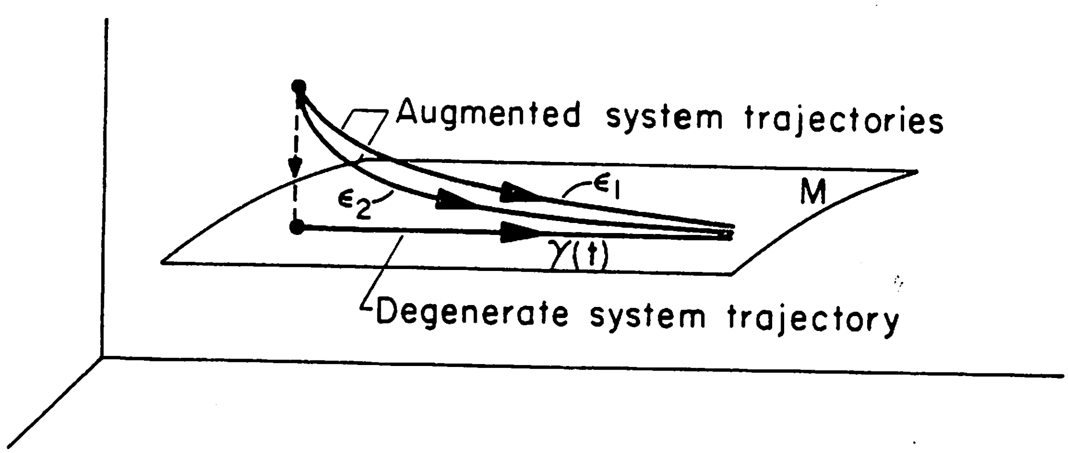


Figure 6. Illustrating the consistency requirement.

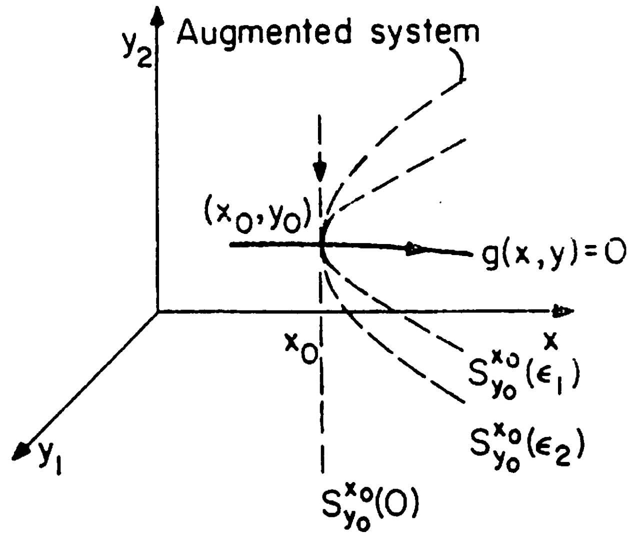


Figure 7. Visualization of stable initial manifoled for augmented system.

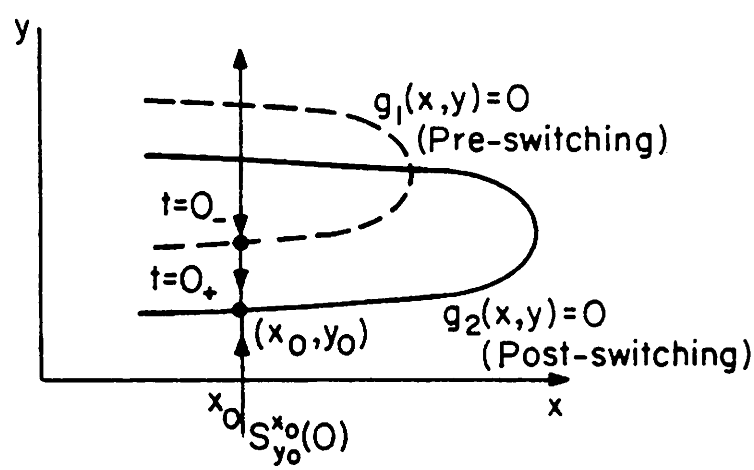


Figure 8. Post-switching or post-fault behavior.

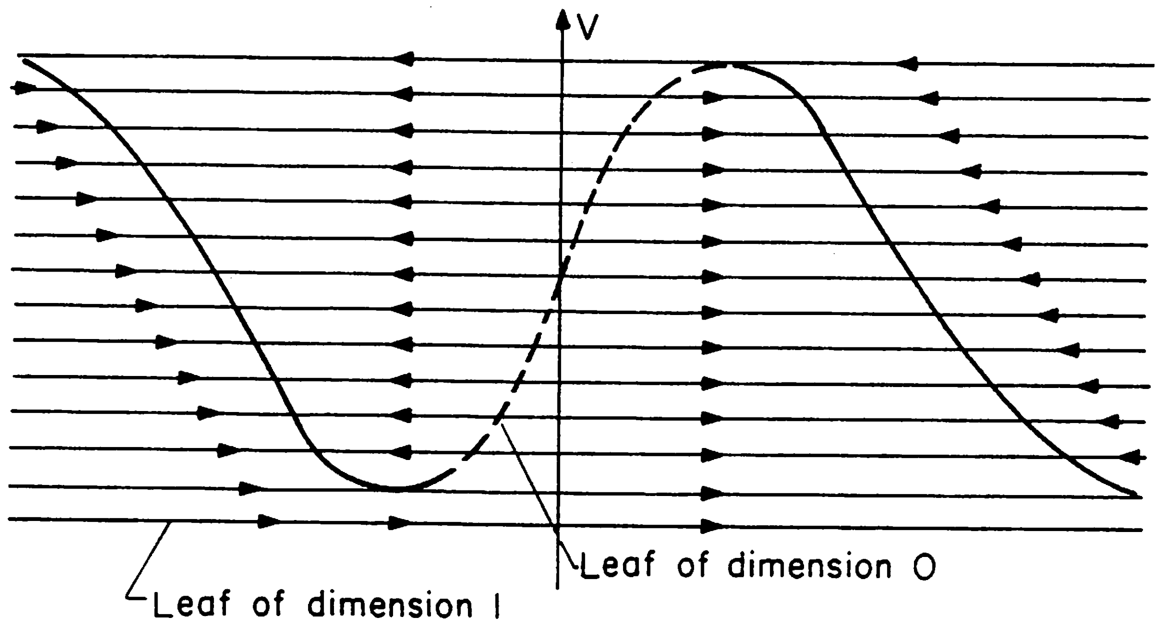


Figure 9. Foliation of $\mathbb{R}^2$ corresponding to (II.1), (II.2).

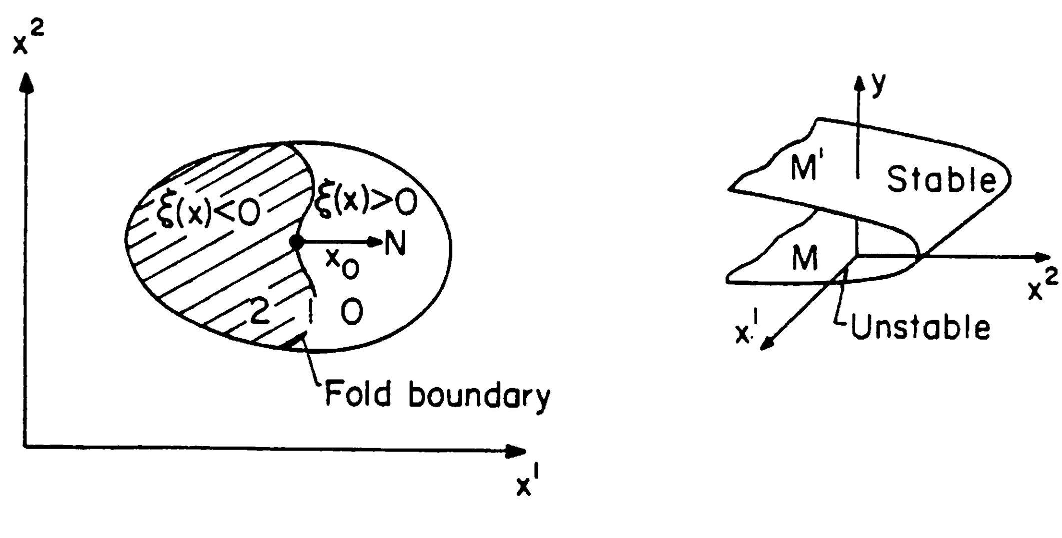


Figure 10. Visualization of fold bifurcation.

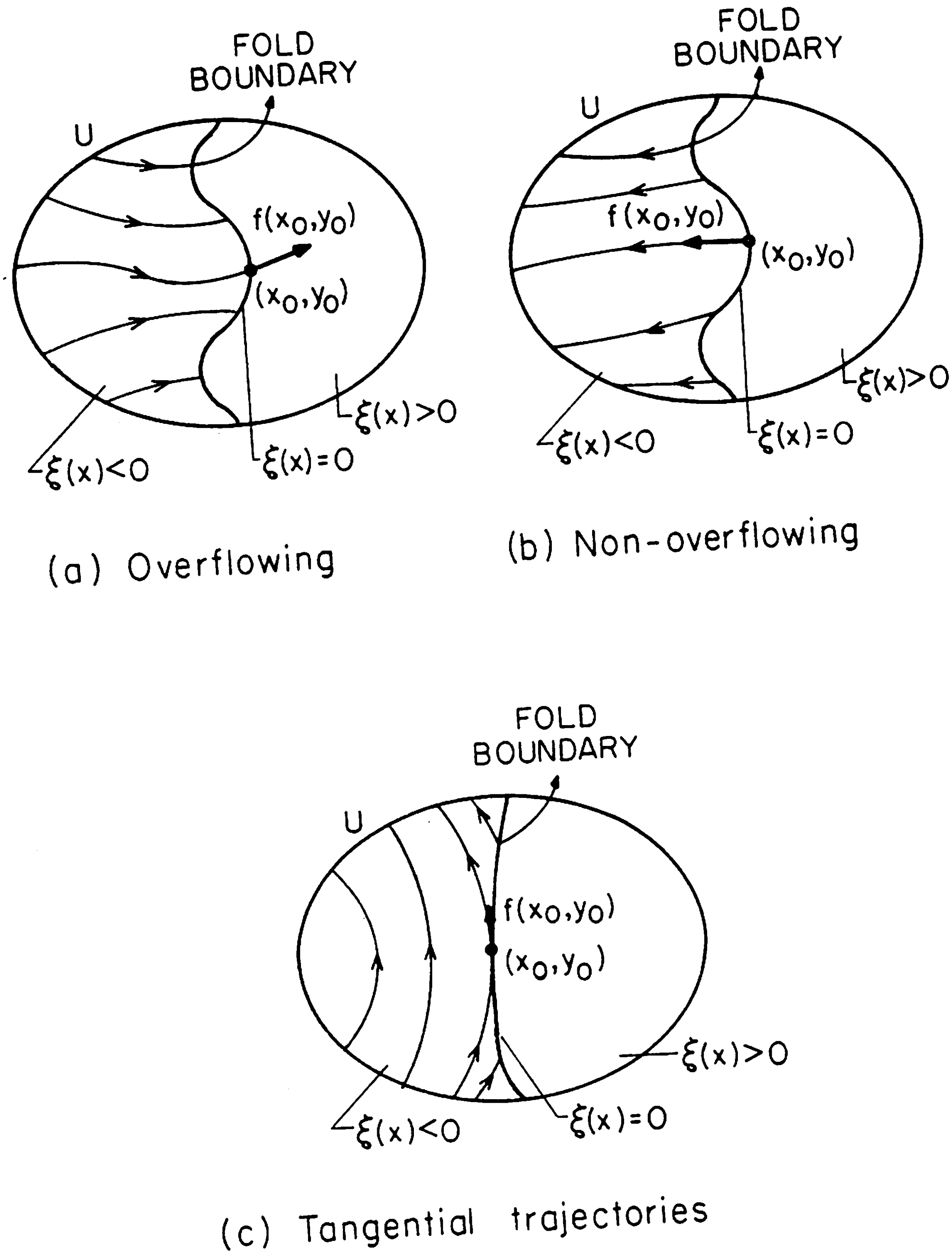


Figure 11. Behavior of vector field X near fold boundary.

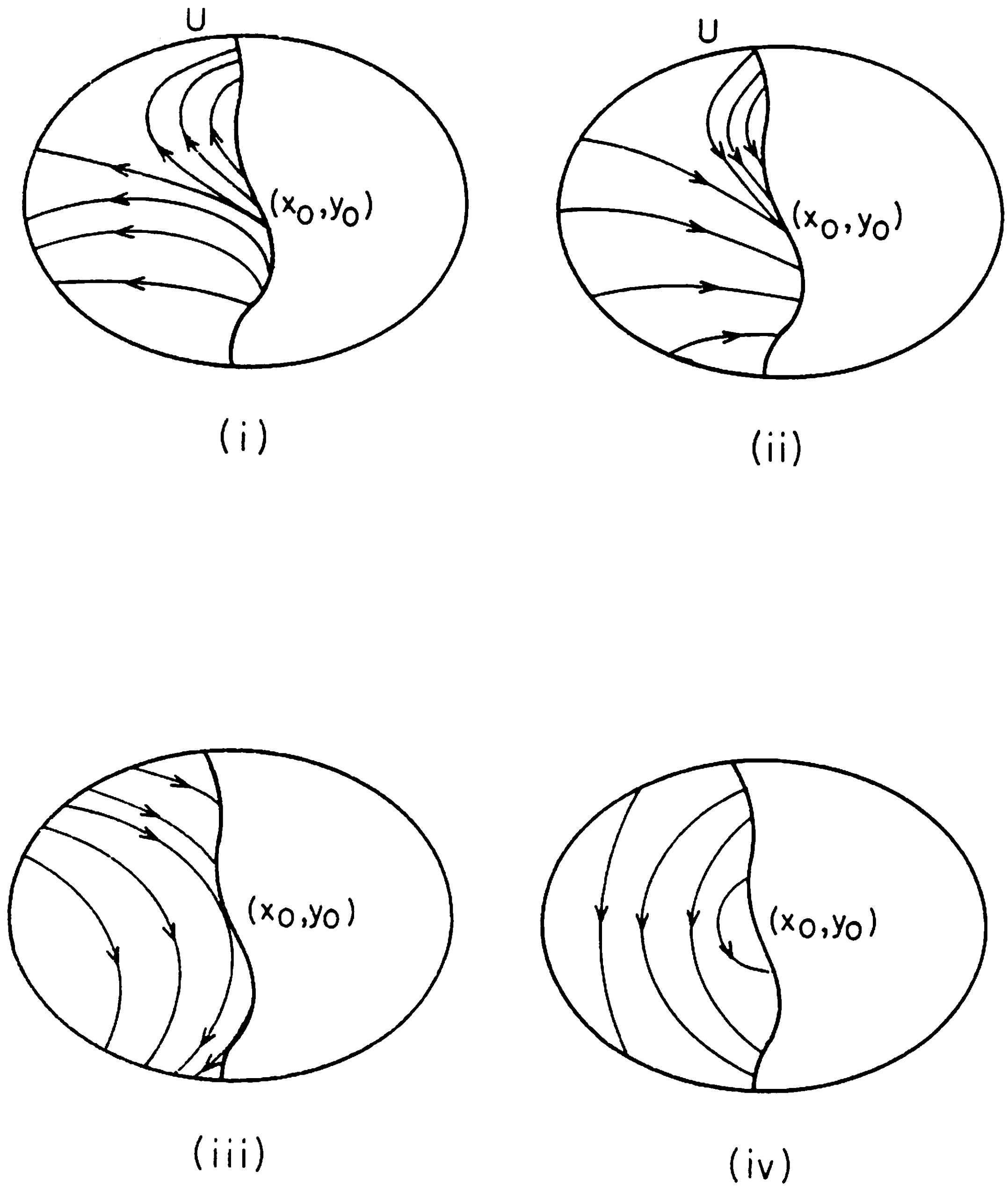


Figure 12. Behavior of $X(x,y)$ in the neighborhood of points where $v^T D_1 g(x_0,y_0) f(x_0,y_0) = 0$.

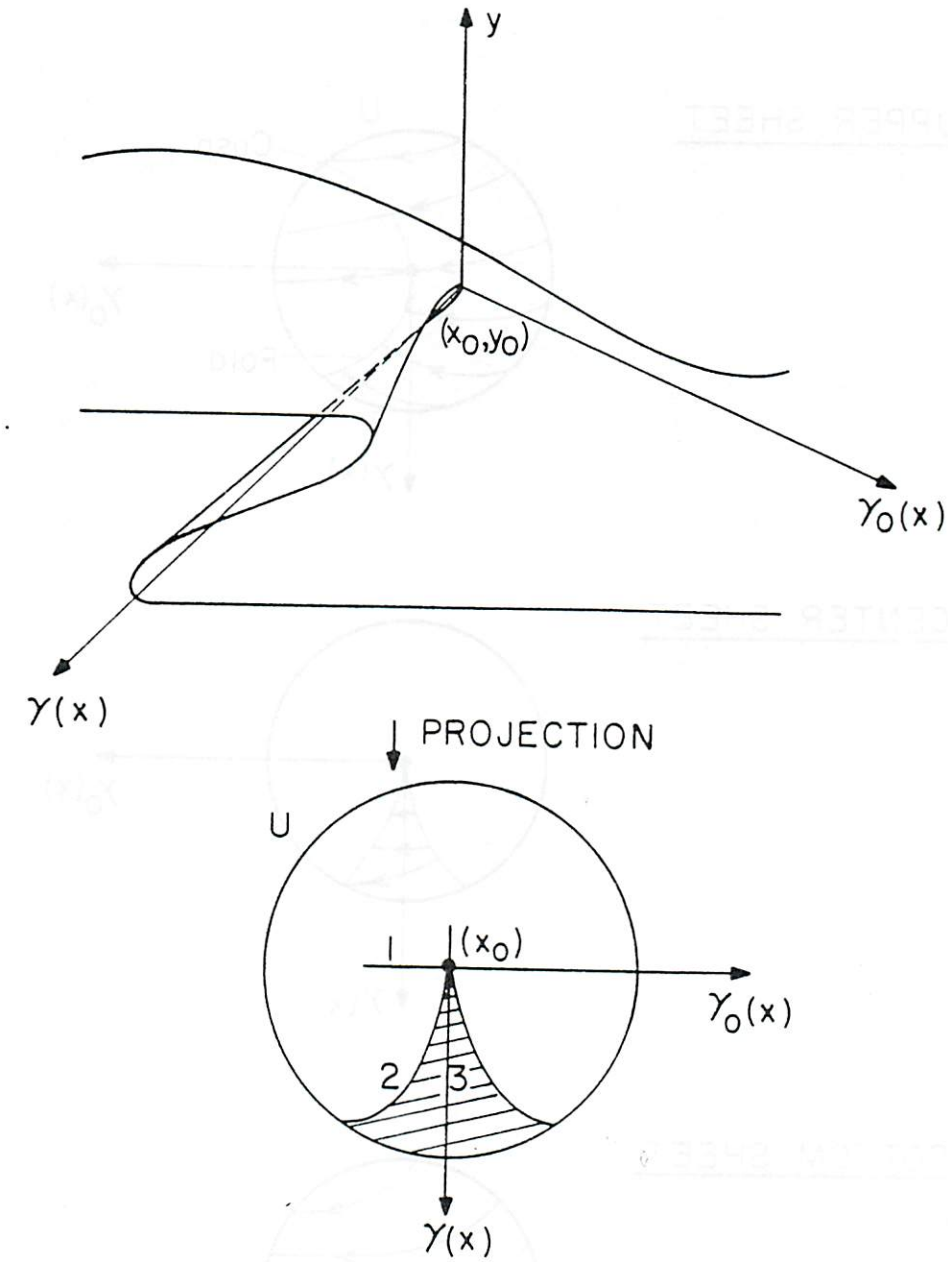


Figure 13. Visualisation of cusp bifurcation.

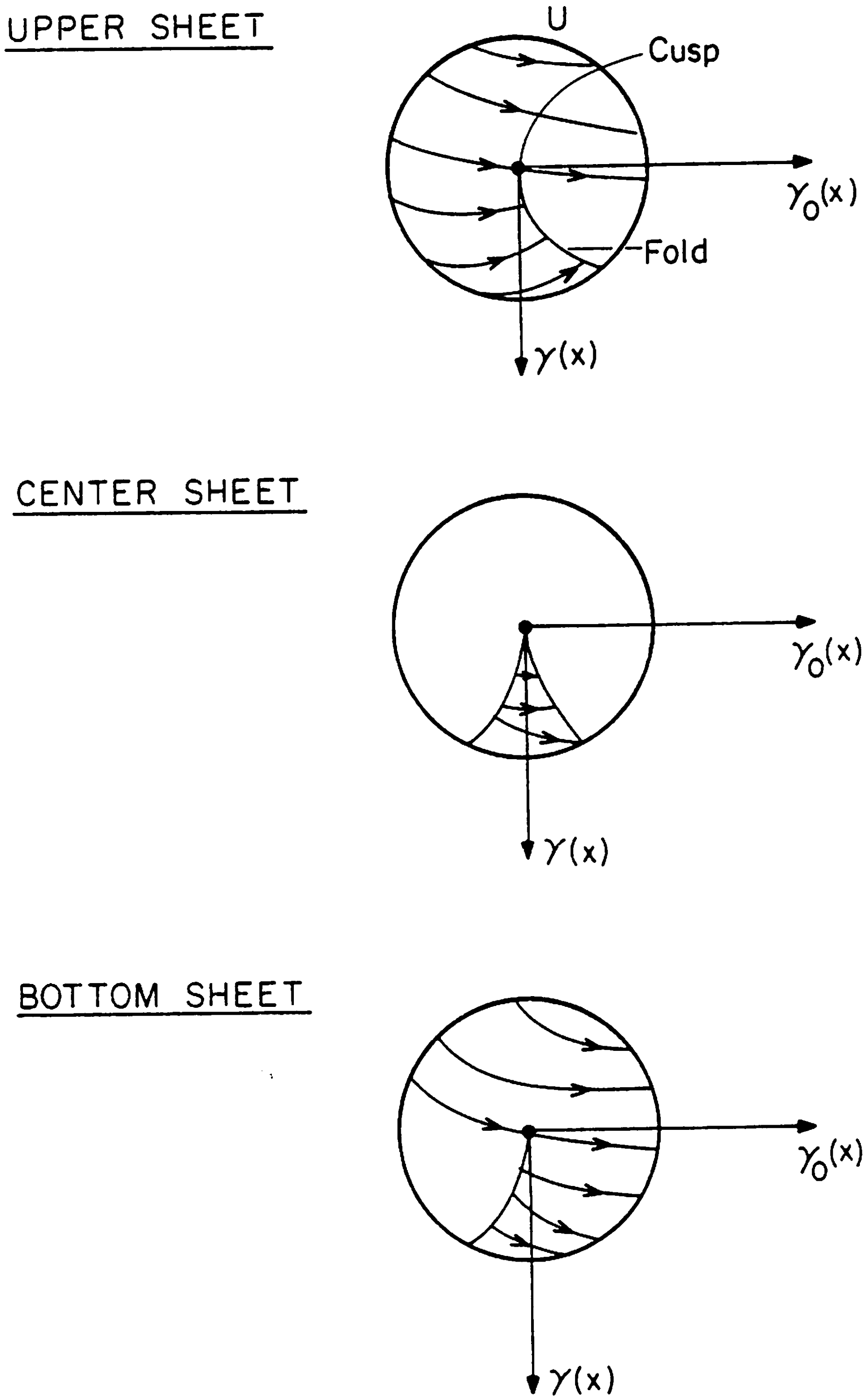
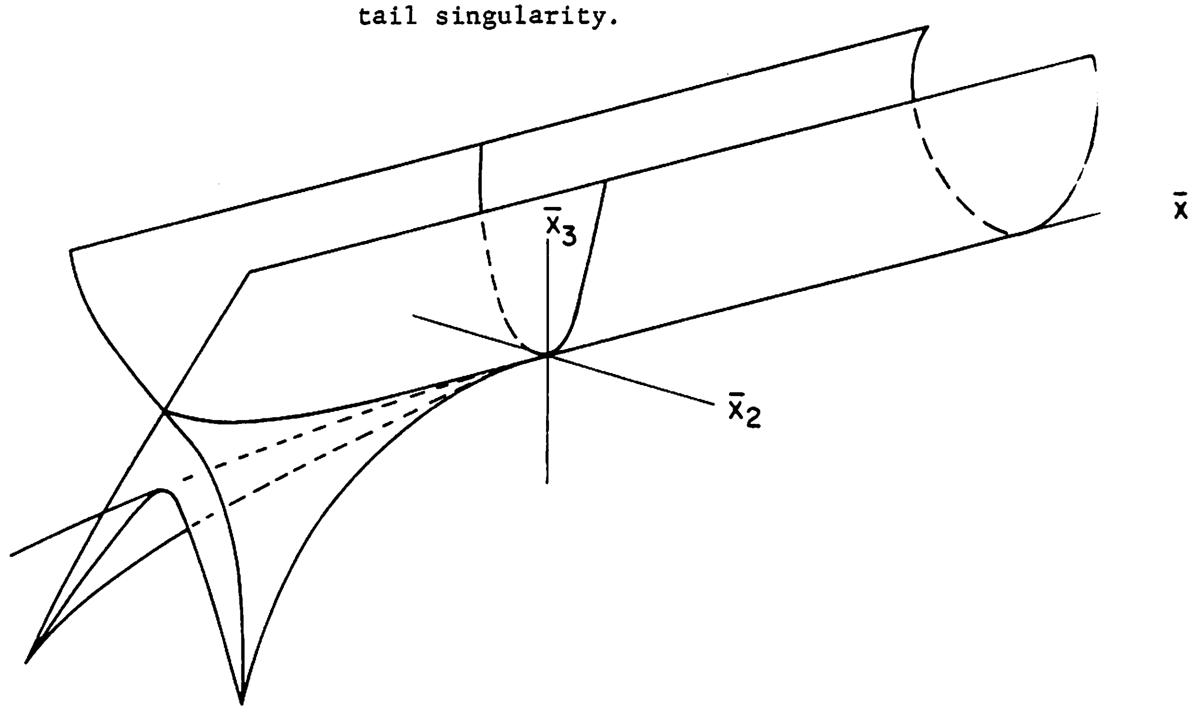


Figure 14. Typical flow of X in the neighborhood of a cusp.

Figure 15 (a) Complete unfolding of the quartic or swallow tail singularity.



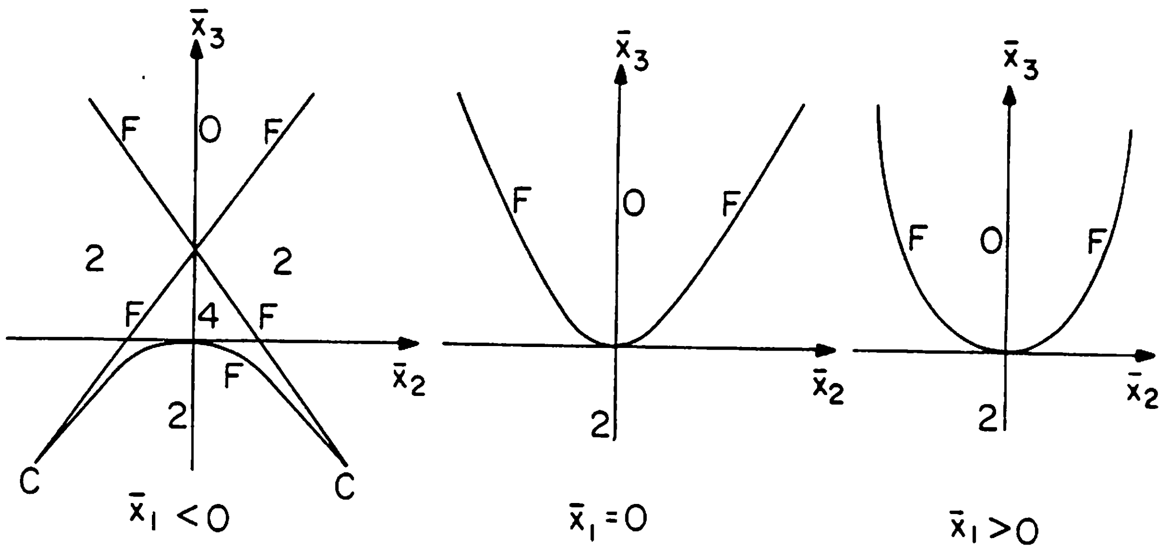


Figure 15 (b) Sections of the unfolding showing number of solutions.

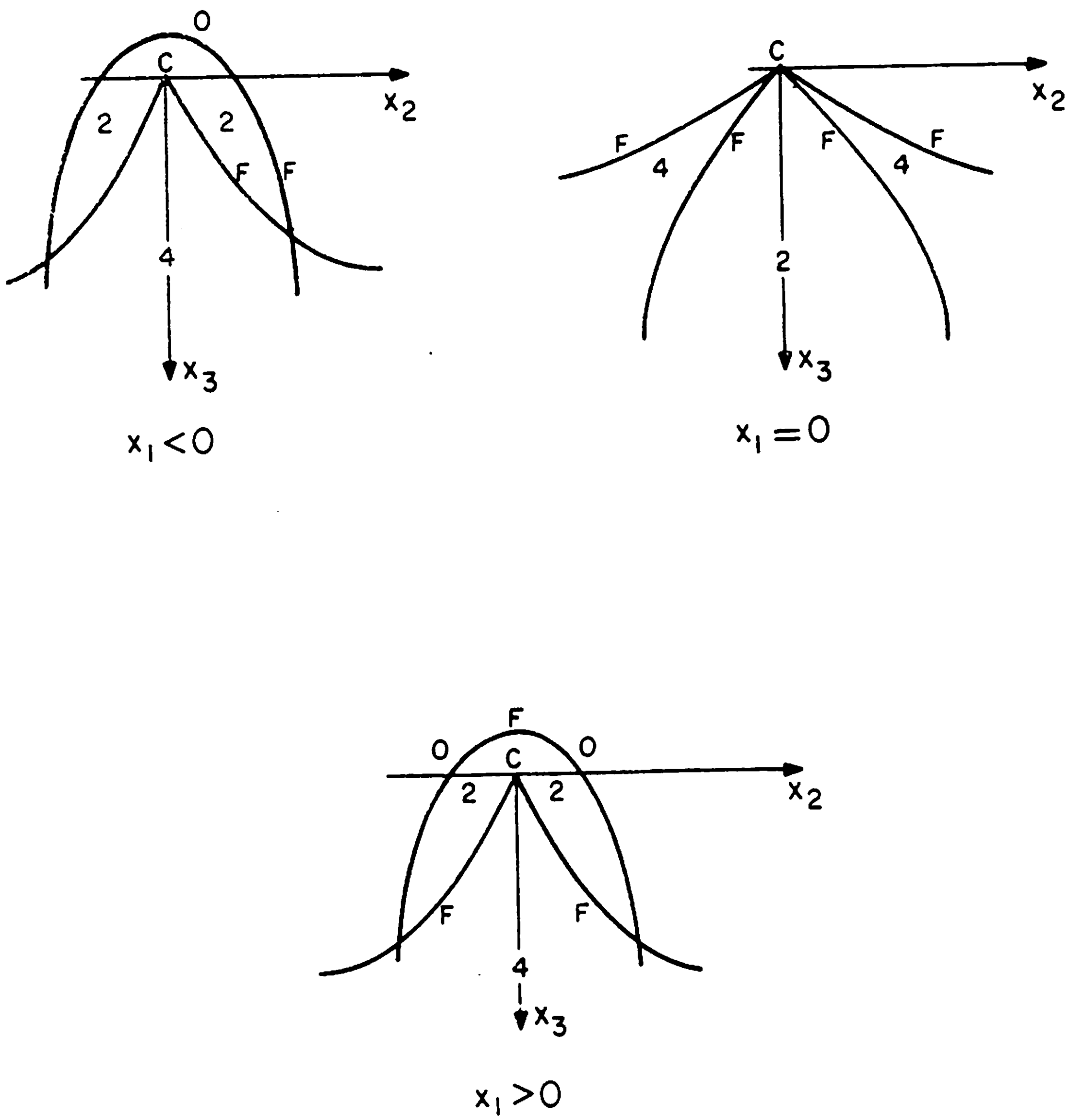
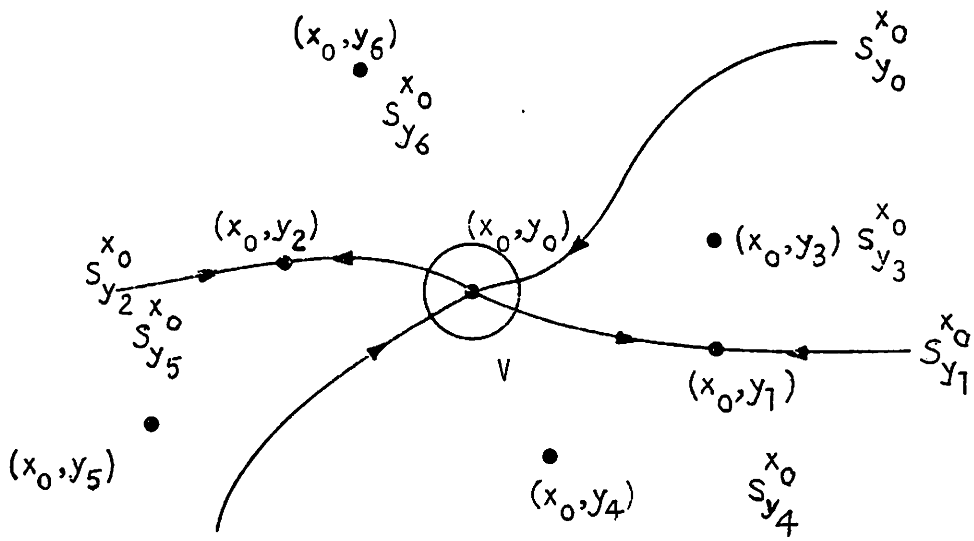


Figure 16. Complete unfolding of the hyperbolic umbilic.



Picture drawn in $\{x_0\}$ x

Figure 17. Jump from non-singular points.

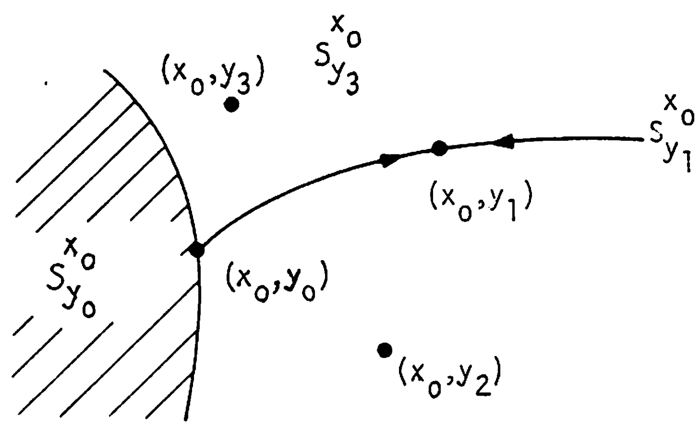


Figure 18. Jump from a (fold) singularity.

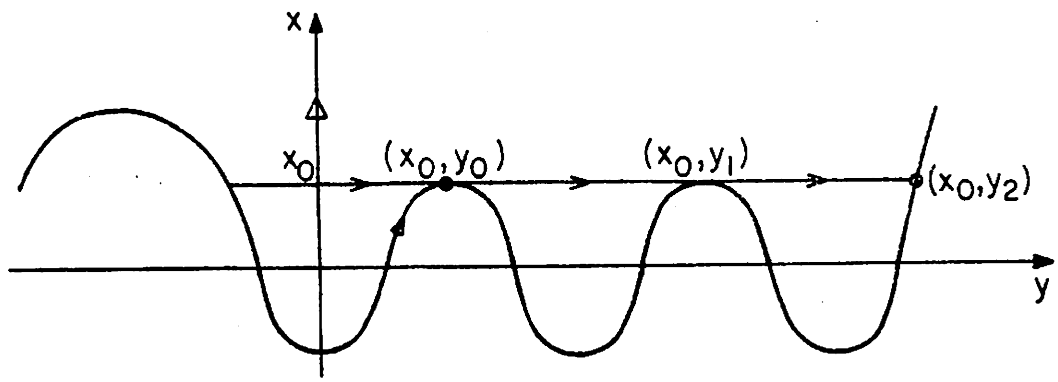
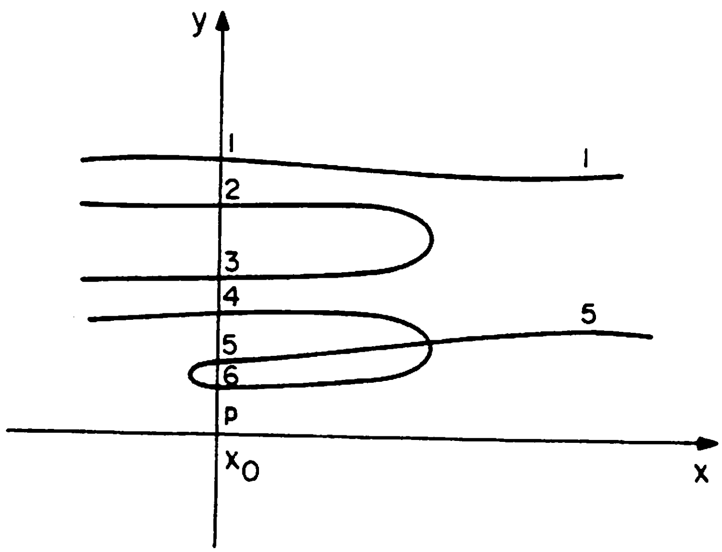
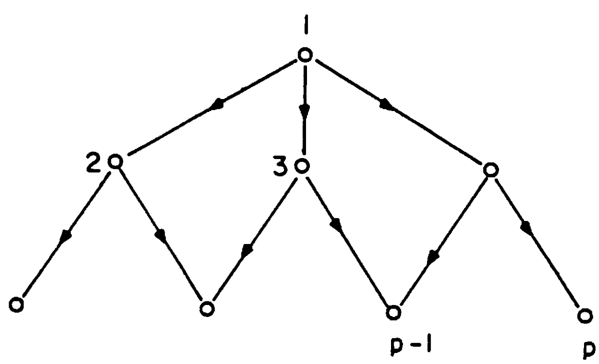


Figure 19. Showing multiple singularities at x_0.



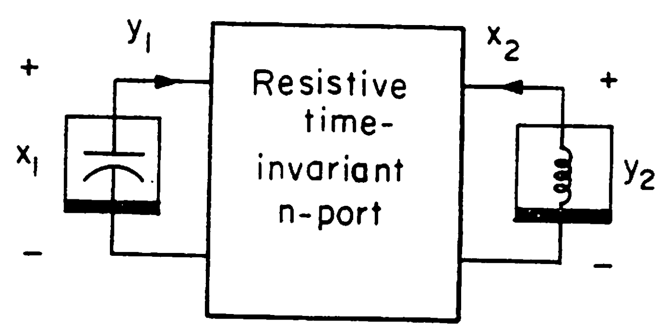
(a) Defining the states of the discrete system.



(b) Labeled diagram showing jumps between states of the discrete system.

Figure 20. Continuous-discrete system model for Σ.

Figure 21. Nonlinear circuit from a port standpoint.



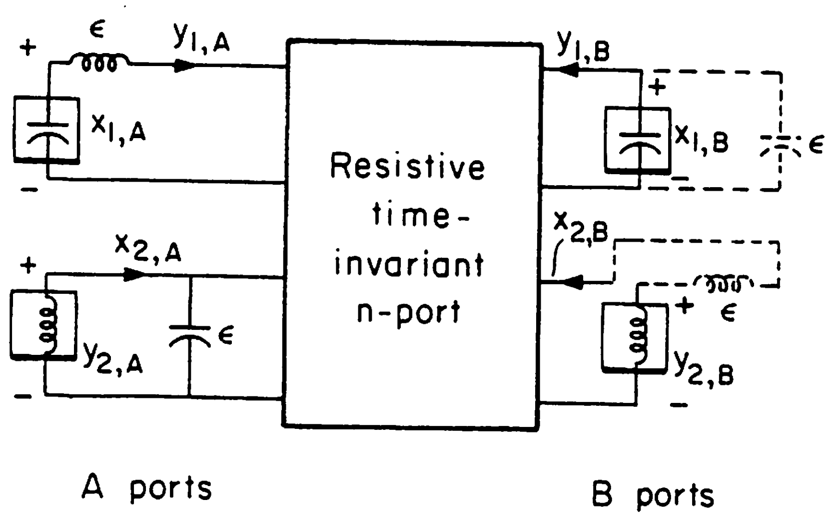


Figure 22. Nonlinear circuit with parasitics introduced.

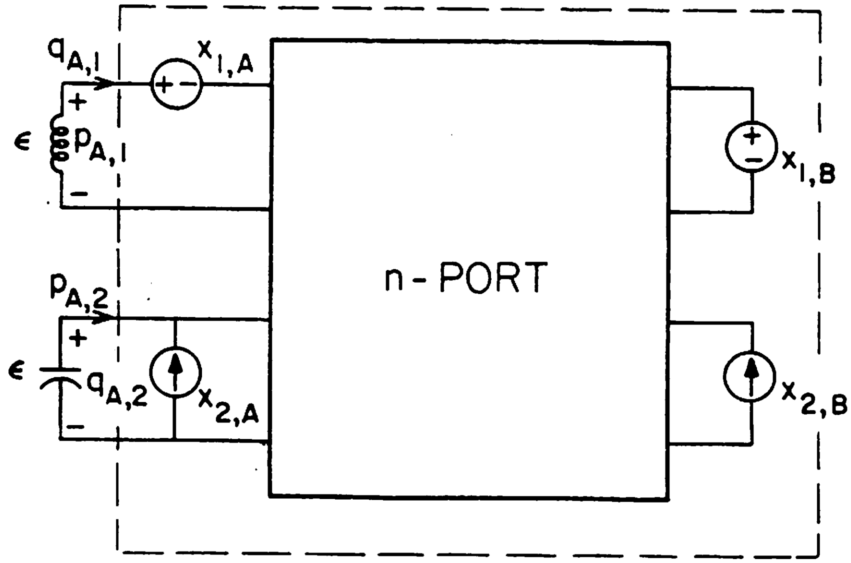


Figure 23. The n_A port associated with the parasitics with the larger capacitors and inductors frozen.

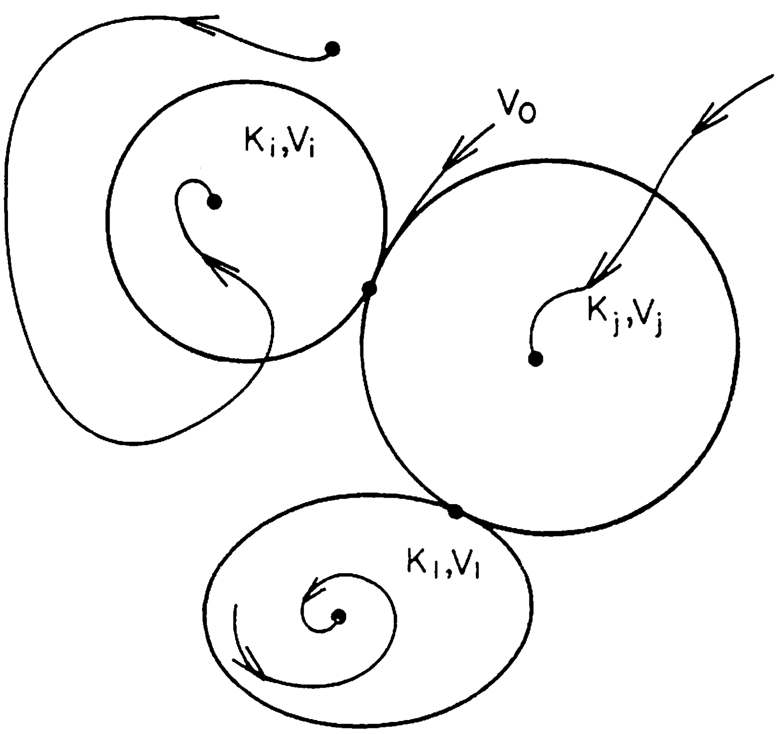


Figure 24. Visualizing the globalization of the Lyapunov theorem.

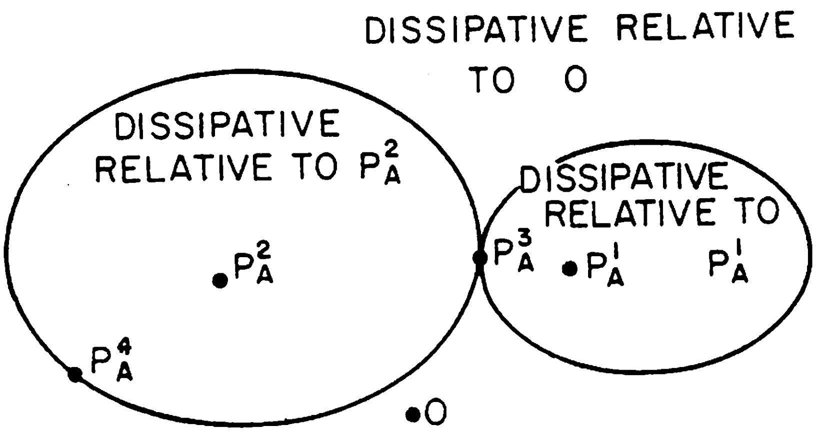


Figure 25. Depicting the theorem on absolute stability.

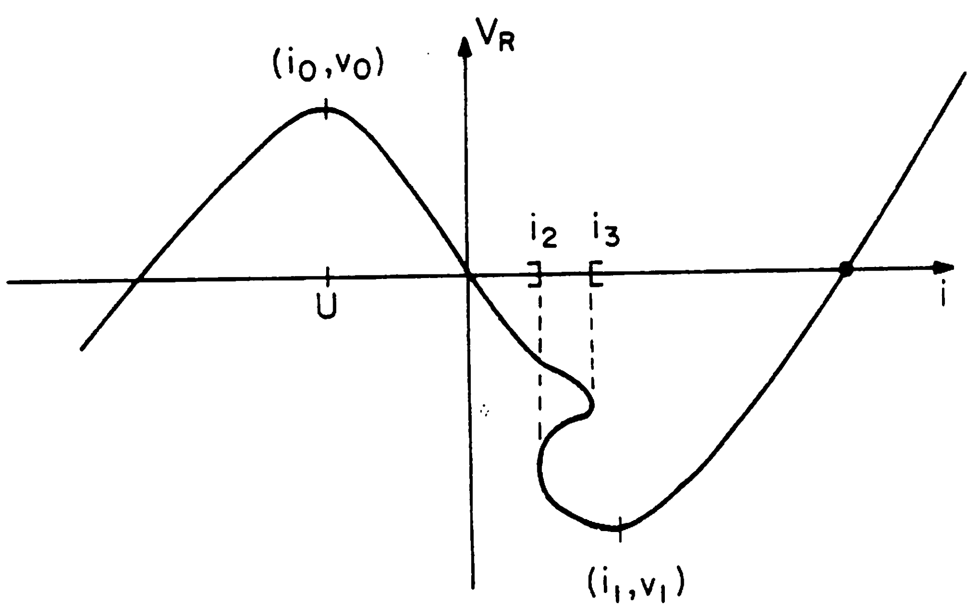


Figure 26. Resistor characteristic for circuit of Figure 1.

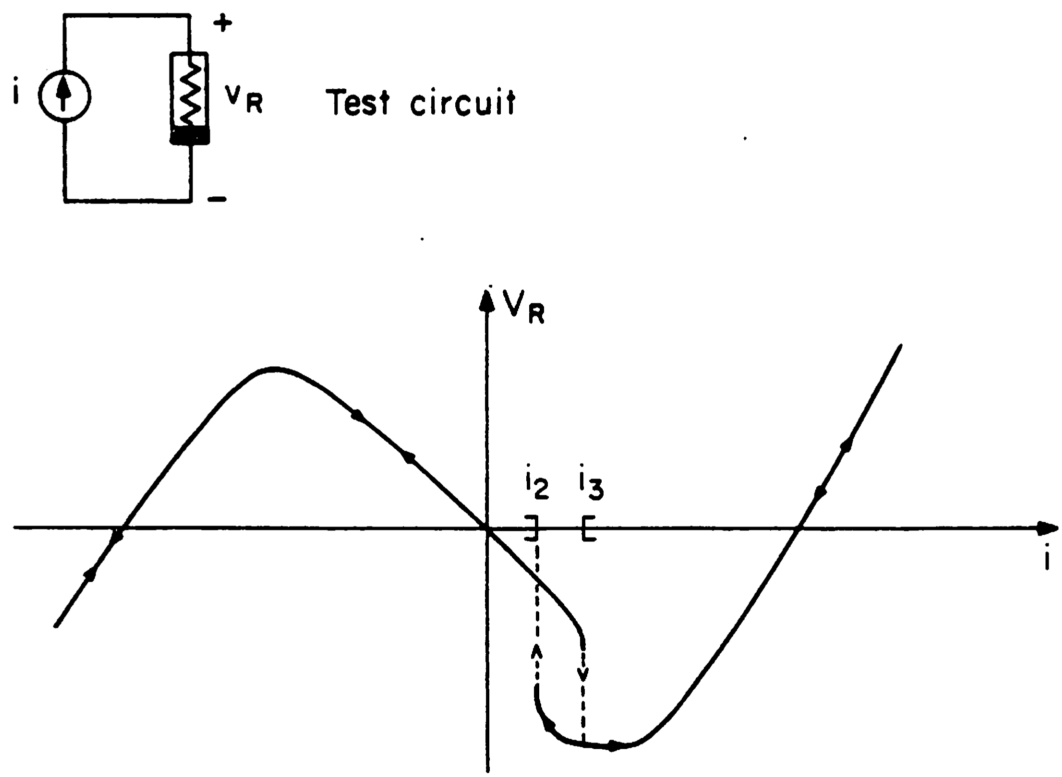


Figure 27. Hysteretic characteristic of resistor shown in Figure 26.

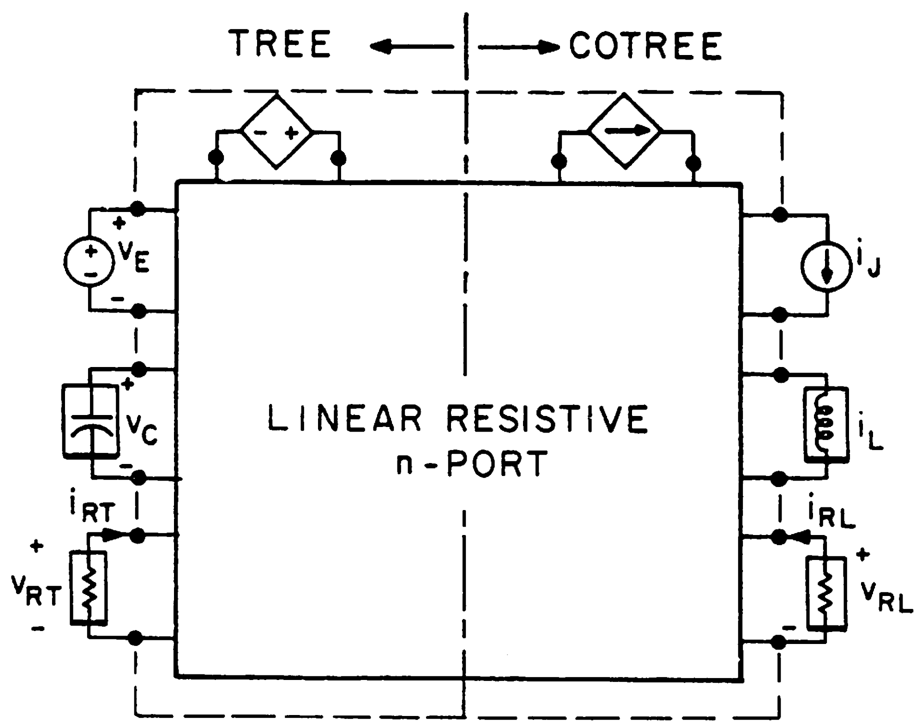


Figure 28. An n-port N created by extracting all independent sources and nonlinear elements.

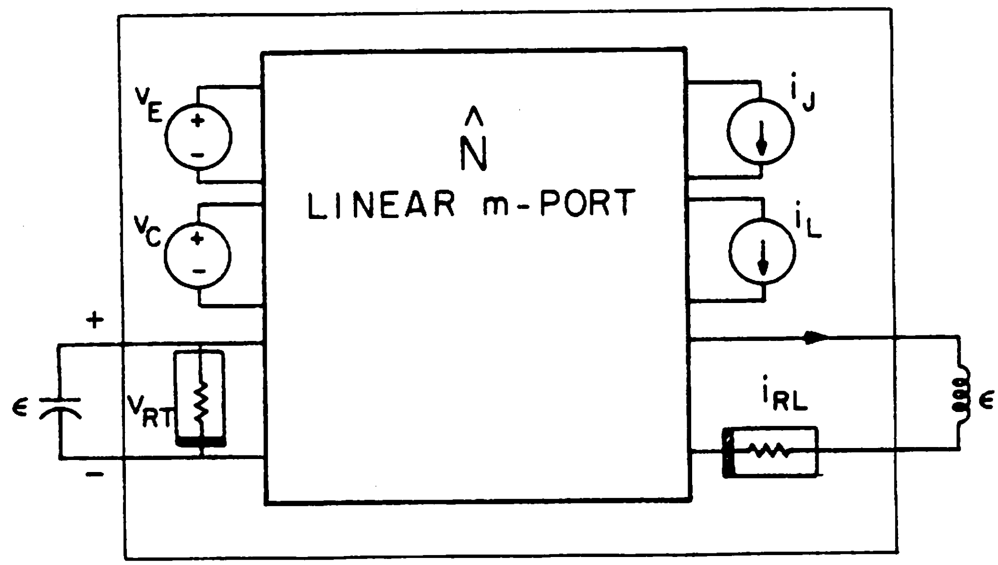


Figure 29. Parasitic augmentation to get circuit equations in normal form.

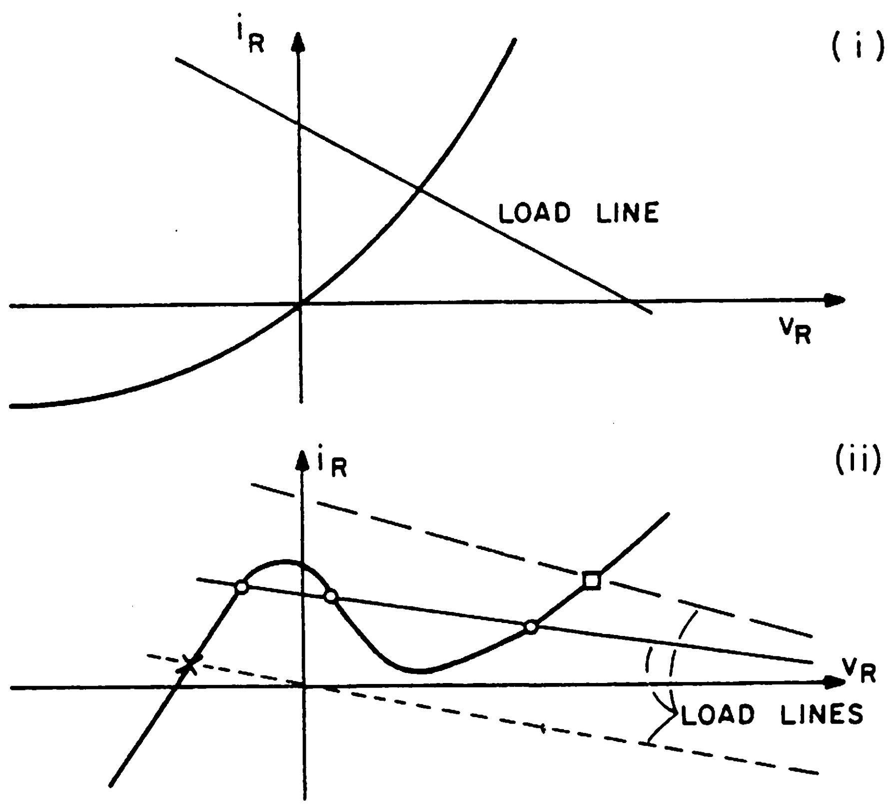


Figure 30. Illustrating multiple solutions arising from nonmonotone resistor characteristics.

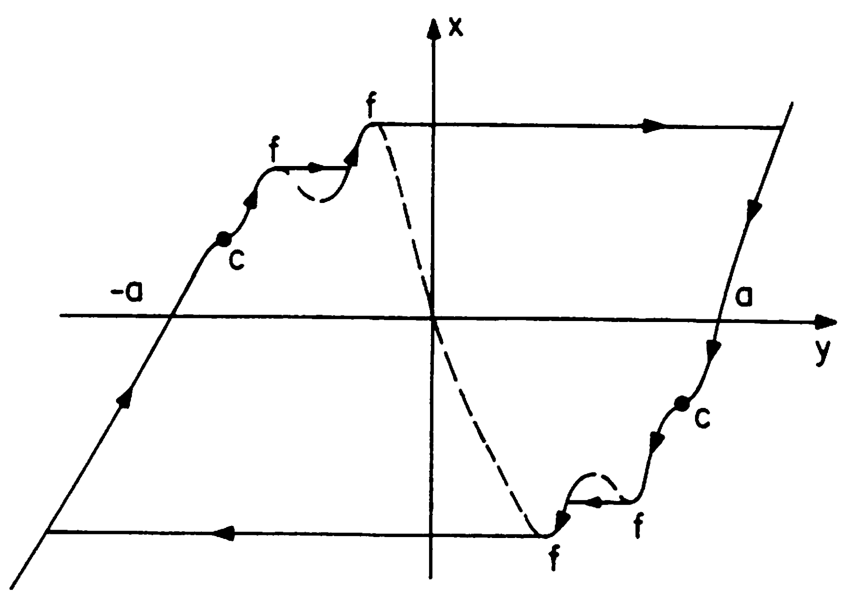


Figure 31. Constructing the relaxation oscillation for Theorem VI.1.

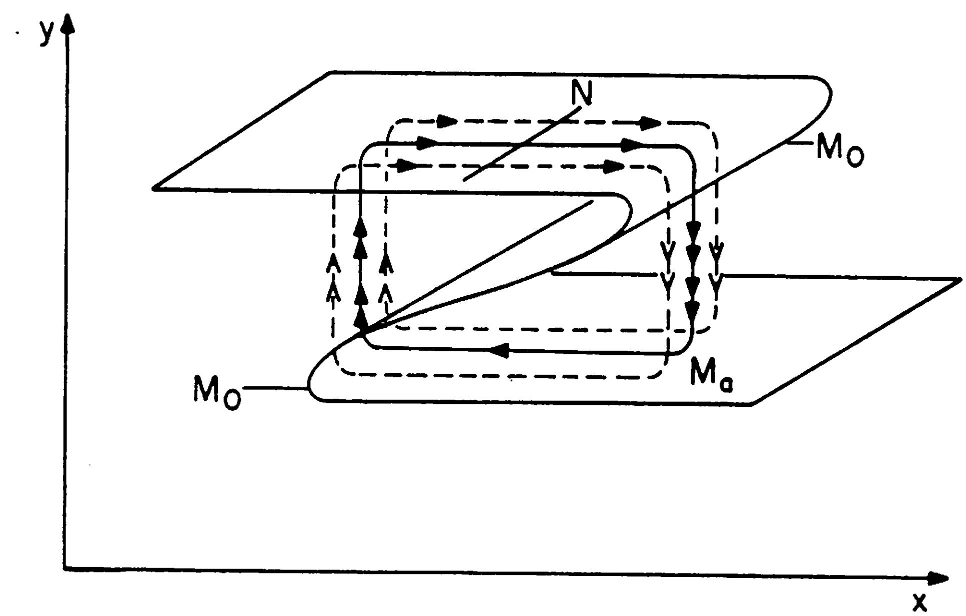


Figure 32. Relaxation oscillation of Section VI.3.

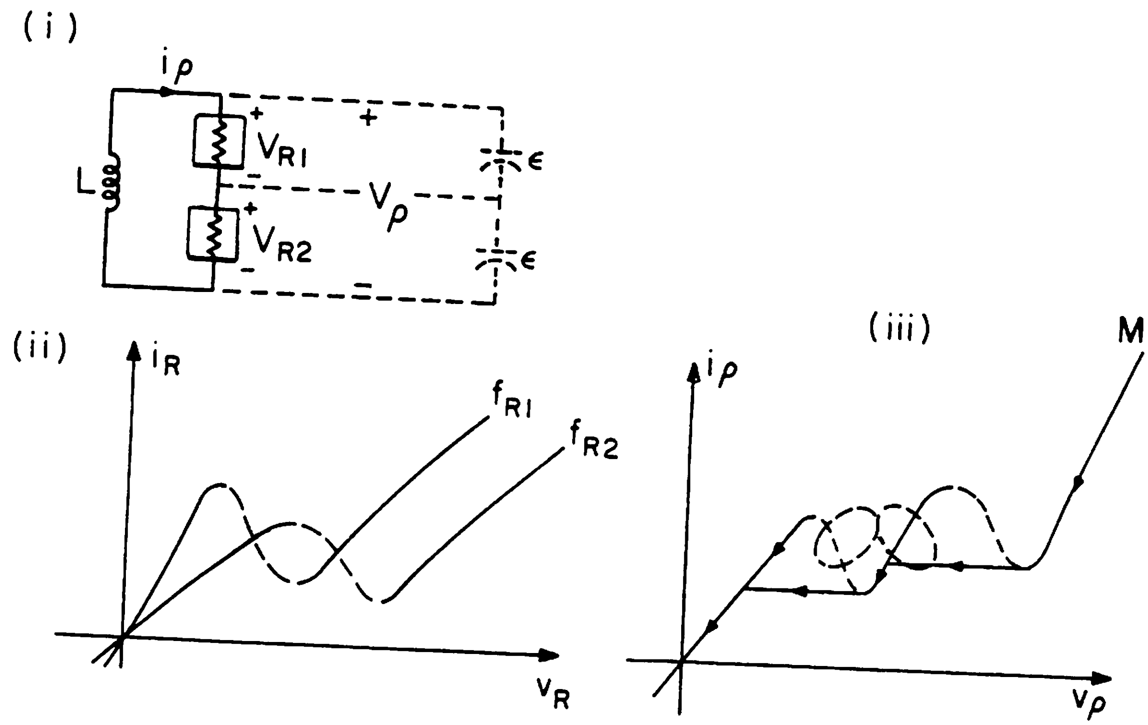


Figure 33. Circuit example (after Chua et al. [7]).

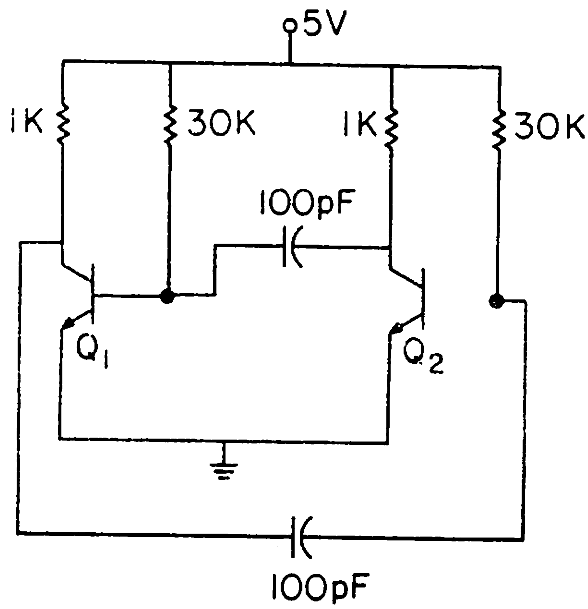


Figure 34 (a). Astable multivibrator circuit.

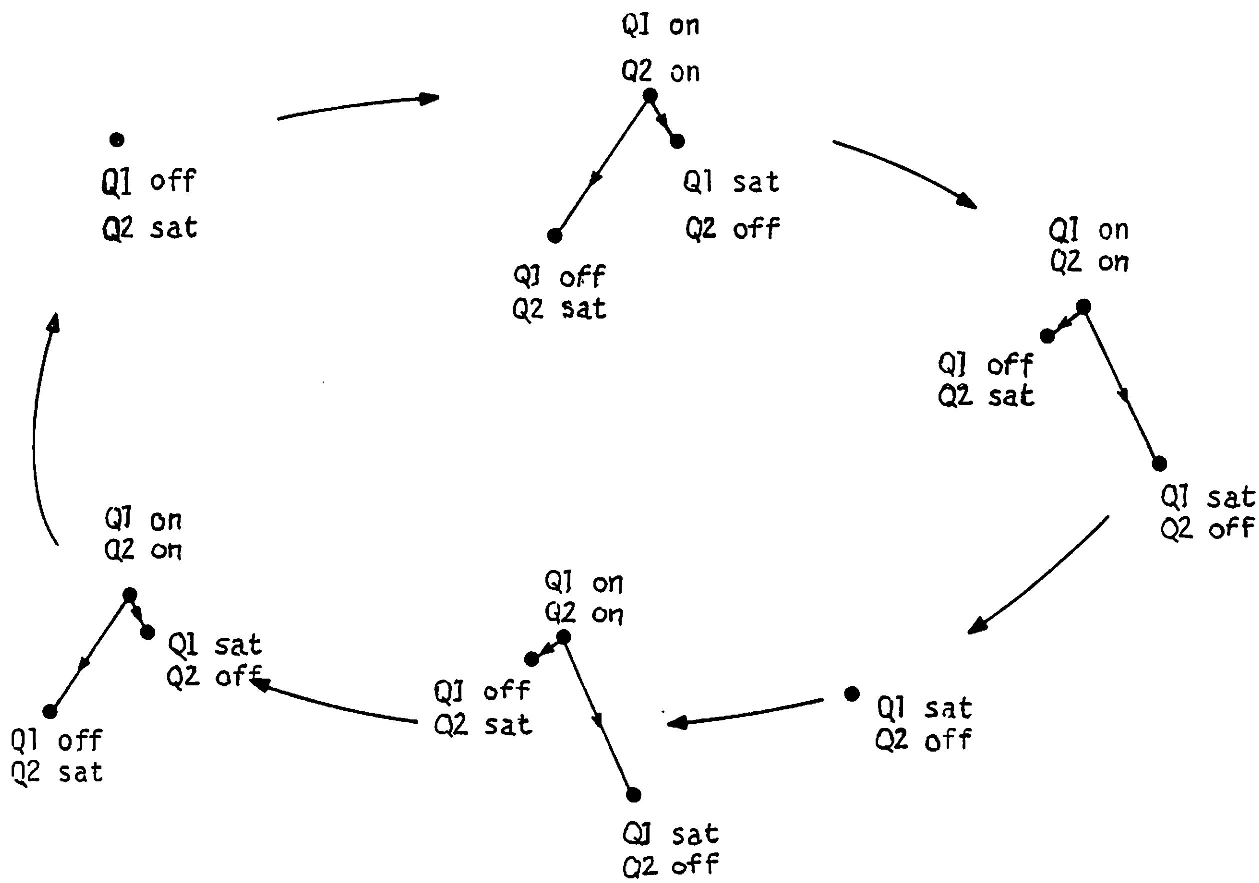


Figure 34 (b). Labeled diagrams associated with discrete dynamics of the circuit of Figure 34 (a).

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