STRONG STRUCTURAL STABILITY OF RESISTIVE NONLINEAR n-PORTS

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ABSTRACT

The paper gives several fundamental results on strong structural stability of nonlinear resistive n-ports. A nonlinear resistive n-port consists of n_R (coupled) internal resistors and n external ports. Intersection of the internal resistor constitutive relations and the Kirchhoff space is called the configuration space. The projected image of the configuration space onto the port space is called the <u>constitutive relation of the composite n-port</u>. Strong structural stability means qualitative persistence of the constitutive relation of composite n-port under small perturbations of internal resistor constitutive relations. <u>Theorem</u> 1 asserts that a nonlinear resistive n-port is strongly structurally stable <u>if and only if</u> (i) Kirchhoff space is <u>transversal</u> to the internal resistor constitutive relations and (ii) the projection map of the configuration space onto port space is a <u>nice immersion</u>. There is, however, an underlying assumption for this fact to be true; there are no port-only loops and no port-only cut sets. (<u>Condition P</u>). <u>Theorem 2</u> says that there are "many" strongly structurally stable n-ports, <u>Theorem 3</u> gives a strong structural stabilization result via network perturbation, and <u>Theorem 4</u> and <u>Theorem 5</u> give results for special class of internal resistor constituitve relations.

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I. Introduction

This paper gives several fundamental results on strong structural stability of nonlinear resistive n-ports. A nonlinear resistive n-port consists of n_R (coupled) internal resistors and n external ports. Intersection of the internal resistor constitutive relations and Kirchhoff space (space where Kirchhoff laws are satisfied) is called the configuration space. The projected image of the configuration space onto port space is called the constitutive relation of the composite n-port.[1] In this paper, strong structural stability means qualitative persistence of the constitutive relation of composite n-port under small perturbations of internal resistor constitutive relations. This is a reasonable concept because circuit elements (e.g. resistors, transistors etc.) are subject to small perturbations of parameters (e.g. temperature), and we would like a circuit to operate in a qualitatively presistent manner under these perturbations.

Structural stability discussed in [1] is the qualitative persistence of configuration space under small perturbations of internal resistor constitutive relations. It is shown in [1] that structural stability is equivalent to transversality of Kirchhoff space and internal resistor constitutive relations.

Sometimes strong structural stability is more appropriate than structural stability, because the former guarantees persistence of constitutive relation of the composite n-port. Theorem 1 (characterization result) asserts that a nonlinear resistive n-port is strongly structurally stable if and only if (i) Kirchhoff space is transversal to internal resistor constitutive relations and (ii) the projection map of configuration space onto port space is a nice immersion. (This will be explained in Section III) There is, however, an underlying assumption for this fact to be true; there are no port-only loops and no port-only cut sets. (Condition P) It is interesting to see that this purely graph-theoretic condition is required in order to obtain strong structural stability results. This condition is crucial for the validity of a version of Whitney Immersion Theorem which plays an important role for proving the results. (Section III). The result is best illustrated by the following examples:

[1] An n-port made of all interconnection of elements is called composite n-port.

Example 1. Consider the 1-port of Fig. 1(a), where constitutive relations of internal resistors R_1 and R_2 are given by Fig. 1(b), $i_{R_k} = f_{R_k}(v_{R_k})$, $k = 1,2$. Tunnel diode is a typical element which has constitutive relation of this type. It is not difficult to show that the constitutive relation of the composite 1-port is given by the set $\mathcal{R}$ of Fig. 1(c)[2]. If one perturbs f_{R_1} in such a way that the local maximum at $v_{R_{10}}$ is slightly higher than i_R^* of Fig. 1(b), then constitutive relation of the composite 1-port is given by the set $\mathcal{R}'$ of Fig. 1(c). On the other hand, if one perturbs f_{R_1} in such a way that the local maximum at $v_{R_{10}}$ is slightly lower than i_R^*, then constitutive relation of the composite 1-port is given by $\mathcal{R}''$ of Fig. 1(c). The sets $\mathcal{R}$, $\mathcal{R}'$ and $\mathcal{R}''$ are qualitatively different from each other, because $\mathcal{R}$ has an isolated point, $\mathcal{R}'$ has a bow-tie shape loops, whereas $\mathcal{R}''$ has nothing in the center. In other words, $\mathcal{R}$, $\mathcal{R}'$ and $\mathcal{R}''$ are <u>not</u> homeomorphic to each other. Similar phenomena occur if one perturbs f_{R_2} slightly in a neighborhood of $v_{R_{20}}$. Therefore, the set $\mathcal{R}$ does not persist qualitatively under small perturbations of internal resistor constitutive relations. This means that the 1-port of Fig. 1(a) and (b) is <u>not</u> strongly structurally stable.

Example 2. Consider the 1-port of Fig. 1(a) with constitutive relations of R_1 and R_2 given by Fig. 2(a). Then the constitutive relation of composite 1-port is given by $\mathcal{R}$ of Fig. 2(b). It is easy to see that slight perturbations of f_{R_1} and f_{R_2} give rise to $\mathcal{R}'$ and $\mathcal{R}''$ of Fig. 2(b). In $\mathcal{R}$, the bow-tie shape loops intersect the main curve at one point, in $\mathcal{R}'$ they intersect the main curve at two points whereas in $\mathcal{R}''$ they never intersect the main curve. Since $\mathcal{R}$, $\mathcal{R}'$ and $\mathcal{R}''$ are qualitatively different from each other, i.e., they are not homeomorphic to each other, the 1-port of Fig. 1(a) and Fig. 2(a) is <u>not</u> strongly structurally stable.

Example 3. Consider the 1-port of Fig. 1(a) with constitutive relations of R_1 and R_2 given by Fig. 3(a). Then the constitutive relation of composite 1-port is given by $\mathcal{R}$ of Fig. 3(b). This set persists qualitatively

[2] As in [1], the polarity of v_P is chosen opposite to the usual convention in order to simplify several hypotheses of the paper.

under small perturbations of f_{R_1} and f_{R_2}, i.e., $\mathcal{R}$, $\mathcal{R}'$ and $\mathcal{R}''$ of Fig. 3(b) are homeomorphic to each other. Therefore the 1-port of Fig. 1(a) and Fig. 3(a) <u>is</u> strongly structurally stable.

Now, let us explain the difference between strong structural stability and structural stability discussed in [1]. First look at the configuration space Σ of the 1-port of Example 2. Observe that the internal resistor constitutive relations, say Λ, is described by $i_{R_1} - f_{R_1}(v_{R_1}) = 0$, $i_{R_2} - f_{R_2}(v_{R_2}) = 0$, and that Kirchhoff space K is described by $v_{R_2} + v_{R_1} + v_p = 0$, $i_{R_2} - i_{R_1} = 0$, $i_{R_2} = i_p = 0$. By eliminating i_{R_1} and i_{R_2}, we see that the configuration space $\Sigma = \Lambda \cap K$ is described by

$$i_p = f_{R_1}(v_{R_1}) = 0 \qquad \text{(i)}$$

$$i_p - f_{R_2}(v_{R_2}) = 0 \qquad \text{(ii)}$$

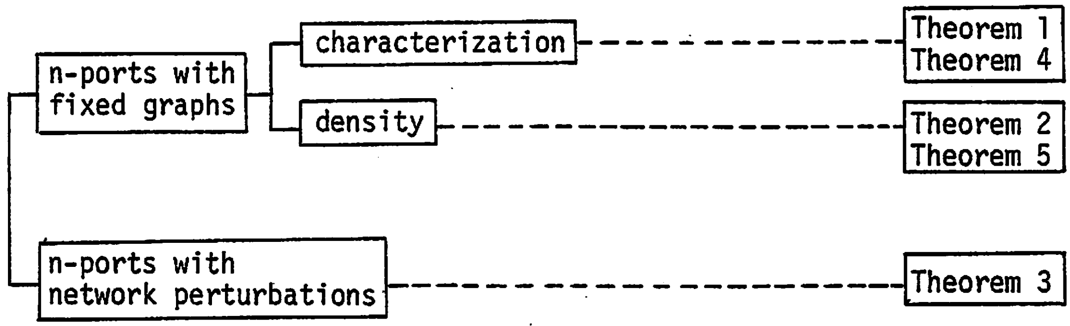
$$v_{R_2} + v_{R_1} + v_p = 0 \quad . \qquad \text{(iii)}$$

Notice that intersection of 2-dimensional surfaces defined by (i) and (ii) in the (i_p, v_{R_1}, v_{R_2}) - space gives the 1-dimensional submanifold Σ of Fig. 4. Since (iii) does not contain i_p, it does not give rise to any further constraint on Σ and hence this set Σ is the configuration space embedded in $\mathbb{R}^3$. Using a result in [1] one can show that this 1-port is structurally stable, i.e., Σ does not exhibit abrupt qualitative changes under small perturbations of f_{R_1} and f_{R_2}. This is illustrated in Fig. 4 where Σ' and Σ'' are perturbed configuration spaces. The projected image $\mathcal{R}$ of Σ, however, <u>does</u> exhibit abrupt qualitative changes under small perturbations of f_{R_1} and f_{R_2}. The sets $\mathcal{R}$, $\mathcal{R}'$ and $\mathcal{R}''$ are the same as those of Fig. 2. The maps $\underset{\sim}{\pi}_p$, $\underset{\sim}{\pi}'_p$ and $\underset{\sim}{\pi}''_p$ are the projection maps of the configuration spaces Σ, Σ' and Σ'' respectively, onto the port space. Notice that for a given $-v_p$, (iii) defines an affine submanifold $v_{R_1} = -v_{R_2} - v_p$. Intersection of this affine submanifold with Σ gives points of the set $\mathcal{R}$. Therefore, $\mathcal{R}$ is the projection of Σ onto the $(-v_p, i_p)$ - space of Fig. 4. Similar statements apply to $\mathcal{R}'$ and $\mathcal{R}''$. Hence this 1-port is <u>structurally stable but not strongly structurally stable</u>. Fig. 5 shows how Σ and $\mathcal{R}$ of the 1-port of Fig. 1(a) and (b) change under small perturbations of f_{R_1} and f_{R_2}. Again, $\mathcal{R}$, $\mathcal{R}'$ and $\mathcal{R}''$ are the same as those of Fig. 1(c). It is clear that this 1-port is neither structurally stable nor strongly structurally stable. Next, consider the 1-port of Example 3. Figure 6 shows how Σ and $\mathcal{R}$ change under small perturbations of f_{R_1} and f_{R_2}. It is clear that this 1-port is structurally stable and strongly structurally stable, i.e., Σ and $\mathcal{R}$ qualitatively persist under small perturbations of f_{R_1} and f_{R_2}. There is, however, a crucial distinction between qualitative persistence of Σ and that of $\mathcal{R}$, because Σ is a submanifold while $\mathcal{R}$ is not. (The latter has self intersection points.) For small perturbations of f_{R_1} and f_{R_2}, the perturbed configuration spaces Σ' and Σ'' are diffeomorphic to the old one Σ. The perturbed constitutive relations $\mathcal{R}'$ and $\mathcal{R}''$ of the composite 1-port, however, are not diffeomorphic to the original $\mathcal{R}$ because of the self intersection points, i.e., the derivative of a function cannot be defined at self intersection points. The sets $\mathcal{R}$, $\mathcal{R}'$ and $\mathcal{R}''$ are only homeomorphic to each other. This naturally forces us to define strong structural stability by using homeomorphism rather than diffeomorphism.

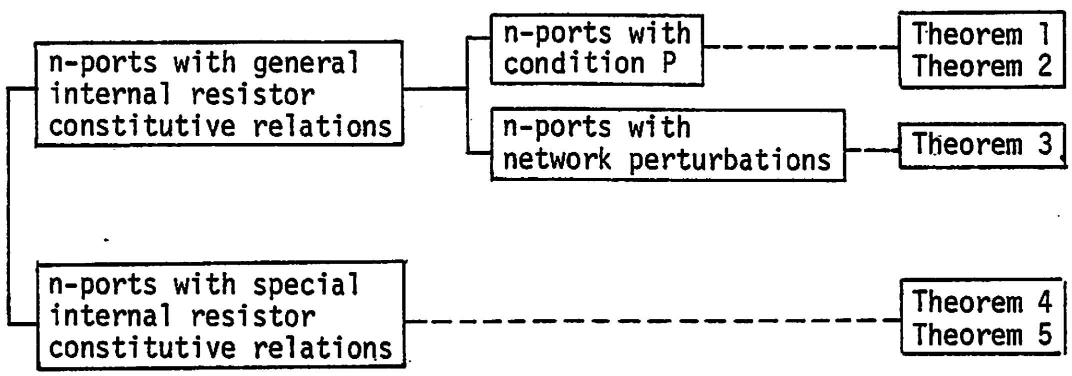
In Secion II, we will give an important preliminary result (Proposition 1) which is necessary for the proofs of Theorem 1 and Theorem 2 in Section III. As a by product of this, we will prove a conjecture in [1] which gives a sharpened version of a result in [1]. In Section III, after giving the characterization result (Theorem 1), we will give a density result (Theorem 2) which asserts that there are "many" strongly structurally stable n-ports. In Section IV, we will describe two strong structural stabilization methods. One is by element perturbation which amounts to perturbing the existing internal resistor constitutive relations. The other is by network perturbation (Theorem 3) which amounts to creating extra ports by plier's-type entry and/or soldering-iron entry. In Section V, we will relax Condition P but deal with a slightly restricted class of n-ports where internal resistor constitutive relations do not impose coupling between resistor variables and port variables. Theorem 4 and Theorem 5 are the characterization result and density result, respectively for such n-ports.

In order to help the reader to grasp main results of the paper we will give the following two different classifications:

Classification I



Classification II



Also, various mathematical concepts are explained as they are needed.

Throughout the paper, we will take full advantage of the coordinate-free property of the geometric approach.

General Remarks: For simplicity's sake, we will sometimes abuse our notation with regards to the transpose of a vector or a matrix. In order to avoid wordiness, we will usually refer to the constitutive relation of an n-port instead of the constitutive relation of a "composite" n-port.

II. A Preliminary Result.

A resistive n-port N is an interconnection of "n_R" internal coupled 2-terminal resistors and "n" external terminal pairs which are called ports. Let $\underset{\sim}{v}_R$ and $\underset{\sim}{v}_P$ denote the voltages of the internal resistors and the external ports, respectively, and let $\underset{\sim}{i}_R$ and $\underset{\sim}{i}_P$ be the currents of the internal resistors and the external ports, respectively. Then $(\underset{\sim}{v}_R,\underset{\sim}{i}_R) \in \mathbb{R}^{2n_R}$ and $(\underset{\sim}{v}_P,\underset{\sim}{i}_P) \in \mathbb{R}^{2n}$. Let $\underset{\sim}{v} = (\underset{\sim}{v}_R,\underset{\sim}{v}_P)$, $\underset{\sim}{i} = (\underset{\sim}{i}_R,\underset{\sim}{i}_P)$ and

$b = n_R + n$ so that $(\underset{\sim}{v},\underset{\sim}{i}) \in \mathbb{R}^{2b}$. The following are the standing hypotheses of this paper:

(a) The graph defining the topology of N is connected.

(b) N is time-invariant.

(c) The internal resistor constitutive relations are characterized by

$$(\underset{\sim}{v},\underset{\sim}{i}) \in \Lambda \subset \mathbb{R}^{2b} \tag{1}$$

where Λ is a $(2b-n_R)$-dimensional C^1 submanifold.

It is explained in [1] that (a)-(c) are very general conditions. Sometimes, we consider the situation where Λ is given by

$$\Lambda = \{(\underset{\sim}{v},\underset{\sim}{i}) \in \mathbb{R}^{2b} \mid (\underset{\sim}{v}_R, \underset{\sim}{i}_R) \in \Lambda_R\} \tag{2}$$

where Λ_R is an n_R-dimensional C^1 submanifold. In Examples 1-3, Λ is of the form (2) and $\Lambda_R = \{(\underset{\sim}{v}_R,\underset{\sim}{i}_R) \in \mathbb{R}^4 \mid i_{R_k} = f_{R_k}(v_{R_k}),\ k = 1,2\}$. Let K be the Kirchhoff space, i.e., the set of all $(\underset{\sim}{v},\underset{\sim}{i})$ satisfying Kirchhoff laws and let

$$\Sigma \triangleq \Lambda \cap K \tag{3}$$

be the configuration space. We will sometimes denote a point $(\underset{\sim}{v},\underset{\sim}{i}) \in \mathbb{R}^{2b}$ by $\underset{\sim}{x}$. From time to time, we view an n-port N as a network $\mathcal{N}$ by terminating the ports of N by norators[3] [1].

Let $M \subset \mathbb{R}^m$ be a differentiable submanifold and let $\underset{\sim}{F}$ and $\underset{\sim}{G} : M \to \mathbb{R}^m$ be C^1 functions. Then the C^1 distance between $\underset{\sim}{F}$ and $\underset{\sim}{G}$ at $\underset{\sim}{x} \in M$ is given by

$$d_1(\underset{\sim}{F},\underset{\sim}{G})(\underset{\sim}{x}) \triangleq \|\underset{\sim}{F}(\underset{\sim}{x}) - \underset{\sim}{G}(\underset{\sim}{x})\| + \|(d\underset{\sim}{F})_{\underset{\sim}{x}} - (d\underset{\sim}{G})_{\underset{\sim}{x}}\|. \tag{4}$$

Recall that $\underset{\sim}{G} : M \to \mathbb{R}^m$ is called an embedding if it is an immersion and if it maps M diffeomorphically onto its image. The set of all positive numbers is denoted by $\mathbb{R}^+$. The strong or Whitney C^1 topology for the set of all C^1 functions $C^1(M;\mathbb{R}^m)$ from M into $\mathbb{R}^m$ is generated by sets of the form

$$\mathcal{U}(\underset{\sim}{F};\varepsilon(\cdot)) \triangleq \{\underset{\sim}{G} : M \to \mathbb{R}^m \mid d_1(\underset{\sim}{F},\underset{\sim}{G})(\underset{\sim}{x}) < \varepsilon(\underset{\sim}{x}) \text{ for all } \underset{\sim}{x} \in M\} \tag{5}$$

where $\underset{\sim}{F} \in C^1(M;\mathbb{R}^m)$ and $\varepsilon : M \to \mathbb{R}^+$ is an arbitrary continuous function. In this paper, topology for set of functions is always with respect to

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[3] A norator is a 2-terminal element whose constitutive relation is given by $\Lambda_R = \mathbb{R}^2$.

this topology.

Let $\underset{\sim}{\imath}_\Lambda : \Lambda \to \mathbb{R}^{2b}$ be the inclusion map. Then there is a neighborhood $V(\underset{\sim}{\imath}_\Lambda)$ of $\underset{\sim}{\imath}_\Lambda$ with respect to the above topology such that every element of $V(\underset{\sim}{\imath}_\Lambda)$ is an embedding of Λ[1],[2]. A strong or Whitney C^1 perturbation $\hat{\Lambda}$ of Λ is defined by

$$\hat{\Lambda} \triangleq \underset{\sim}{F}(\Lambda),\ \underset{\sim}{F} \in V(\underset{\sim}{\imath}_\Lambda) .$$

It is explained in [1] that this is an appropriate perturbation for electrical networks. In what follows, whenever we say a perturbation $\hat{\Lambda}$, it will always mean a strong C^1 perturbation.

Recall, now, that if $\Lambda \cap K \neq \emptyset$ and $\Lambda \pitchfork K$, i.e., Λ is transversal to K, then [1] Σ is an n-dimensional submanifold. The following proposition plays an important role in this paper.

<u>Proposition 1</u>. Suppose that $\Lambda \cap K \neq \emptyset$ and $\Lambda \pitchfork K$. Given any continuous function $\varepsilon : \Sigma \to \mathbb{R}^+$, there is a continuous function $\delta : \Lambda \to \mathbb{R}^+$ with the following property: For a C^1 embedding $\underset{\sim}{F} : \Lambda \to \mathbb{R}^{2b}$ with $d_1(\underset{\sim}{F},\underset{\sim}{\imath}_\Lambda)(\underset{\sim}{x}) < \delta(\underset{\sim}{x})$ for all $\underset{\sim}{x} \in \Lambda$, there is a C^1 diffeomorphism $\underset{\sim}{G} : \Sigma \to \hat{\Sigma}$ such that $d_1(\underset{\sim}{G}, \underset{\sim}{i}_\Sigma)(\underset{\sim}{x}) < \varepsilon(\underset{\sim}{x})$ for all $\underset{\sim}{x} \in \Sigma$, where

$$\hat{\Sigma} \triangleq \hat{\Lambda} \cap K,\ \hat{\Lambda} = \underset{\sim}{F}(\Lambda). \tag{6}$$

The proof is nontrivial and is technically involved. It is given in Appendix I.

As a by product of this proposition, we can sharpen the structural stability result in [1]. To state our sharpening result, we need the following definition:

<u>Definition 1</u>. A resistive n-port N is said to be structurally stable if for any small C^1 perturbation $\hat{\Lambda}$ of Λ, the new configuration space $\hat{\Sigma} \triangleq \hat{\Lambda} \cap K$ is homeomorphic to the old configuration space $\Sigma = \Lambda \cap K$.

<u>Proposition 2</u>. (<u>Characterization of Structural Stability</u>)

Given a resistive n-port N assume that $\Lambda \cap K \neq \emptyset$.

(i) If $\Lambda \pitchfork K$, then Σ is an n-dimensional C^1 submanifold and for any small C^1 perturbation $\hat{\Lambda}$ of Λ, there is a diffeomorphism of Σ onto $\hat{\Sigma}$ which is close to inclusion map : $\Sigma \to K$ in the strong C^1 topology. Therefore N is structurally stable.

(ii) If $\Lambda \not\pitchfork K$, then there are small C^1 perturbations Λ' and Λ'' such that $\Sigma' \triangleq \Lambda' \cap K$ is an n-dimensional C^1 submanifold and $\Sigma'' \triangleq \Lambda'' \cap K$ contains an (n+k)-dimensional submanifold, $k > 0$. Therefore N is not structurally stable.

Proof. (i) is a direct consequence of Proposition 1. (ii) is proved in [1]. ¤

Remarks. 1) Observe that in Definition 1, homeomorphism is used instead of diffeomorphism, because, *a priori* we don't know if Σ is a differentiable submanifold. Proposition 2, however, tells us that one can replace homeomorphism with diffeomorphism, a sharper property.

2) In [1], a diffeomorphism between Σ and $\hat{\Sigma}$ is constructed under the assumption that Λ is a C^2 submanifold. Proposition 1 and Proposition 2 assume that Λ is only a C^1 submanifold. In this paper everything is handled within the C^1 category.

We will next show that there are "many" structurally stable n-ports.

Proposition 3 (Density of Structural Stability)

Given any n-port N with $\Lambda \cap K \neq \emptyset$, there is an arbitrarily small C^1 perturbation $\hat{\Lambda}$ of Λ such that $\hat{\Lambda} \cap K \neq \emptyset$ and $\hat{\Lambda} \pitchfork K$, i.e. the perturbed n-port $\hat{N}$ is structurally stable. If Λ is given by (2), i.e. if $\Lambda = \Lambda_R \times \mathbb{R}^{2n}$, then $\hat{\Lambda}$ can be obtained in the form $\hat{\Lambda} = \hat{\Lambda}_R \times \mathbb{R}^{2n}$, where $\hat{\Lambda}_R$ is a perturbation of Λ_R in $\mathbb{R}^{2n_R}$.

Proof. Proof for a general Λ is given in [1]. To prove the case $\Lambda = \Lambda_R \times \mathbb{R}^{2n}$, let $\underset{\sim}{\pi}'_R : \mathbb{R}^{2b} \to \mathbb{R}^{2n_R}$ be the projection map defined by

$$\underset{\sim}{\pi}'_R(\underset{\sim}{v},\underset{\sim}{i}) = (\underset{\sim}{v}_R,\underset{\sim}{i}_R) \tag{7}$$

and set

$$\underset{\sim}{\chi}_R \triangleq \underset{\sim}{\pi}'_R \circ \underset{\sim}{\iota}_K \tag{8}$$

where $\underset{\sim}{\iota}_k$ is the inclusion map

$$\underset{\sim}{\iota}_k : K \to \mathbb{R}^{2b} \quad . \tag{9}$$

By Lemma 1 which will be given below, we know that (i) $\Lambda \pitchfork K$ if and only if $\Lambda_R \pitchfork \underset{\sim}{\chi}_R(K)$ and (ii) $\Sigma = \underset{\sim}{\chi}_R^{-1}(\Lambda_R)$. Then using an argument similar to that of Theorem 3 of [1], one can obtain the result. ¤

Lemma 1. Let $\Lambda = \Lambda_R \times \mathbb{R}^{2n}$. Then

(i) $\Lambda \pitchfork K$ if and only if $\Lambda_R \pitchfork \underset{\sim}{\chi}_R(K)$.

(ii) $\Sigma = \underset{\sim}{\chi}_R^{-1}(\Lambda_R)$.

Proof. (i) To prove necessity, observe that for any $(\underset{\sim}{v}_R,\underset{\sim}{i}_R) \in \Lambda_R \cap \underset{\sim}{\chi}_R(K)$, there is a $(\underset{\sim}{v},\underset{\sim}{i}) \in \Lambda \cap K$ such that $\chi_R(\underset{\sim}{v},\underset{\sim}{i}) = (\underset{\sim}{v}_R,\underset{\sim}{i}_R)$. Let $\underset{\sim}{\pi}'_R$ be the projection map defined by (7). Then

$$d\underset{\sim\sim}{\pi}'_R(T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda) = T_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)}\Lambda_R \tag{10}$$

$$d\underset{\sim\sim}{\pi}'_R(T_{(\underset{\sim}{v},\underset{\sim}{i})}K) = T_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)}\underset{\sim}{\chi}_R(K) \tag{11}$$

$$d\underset{\sim\sim}{\pi}'_R(T_{(\underset{\sim}{v},\underset{\sim}{i})}\mathbb{R}^{2b}) = T_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)}\mathbb{R}^{2n_R} \quad . \tag{12}$$

By hypothesis

$$T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda + T_{(\underset{\sim}{v},\underset{\sim}{i})}K = T_{(\underset{\sim}{v},\underset{\sim}{i})}\mathbb{R}^{2b} \ .$$

which together with (10) - (12) implies

$$\begin{aligned} T_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)}\Lambda_R + T_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)}\underset{\sim}{\chi}_R(K) &= d\underset{\sim\sim}{\pi}'_R(T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda + T_{(\underset{\sim}{v},\underset{\sim}{i})}K) \\ &= d\underset{\sim\sim}{\pi}'_R(T_{(\underset{\sim}{v},\underset{\sim}{i})}\mathbb{R}^{2b}) = T_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)}\mathbb{R}^{2\tilde{n}_R} \ . \end{aligned}$$

In order to prove sufficiency, set $(\underset{\sim}{v}_R,\underset{\sim}{i}_R) = \underset{\sim}{\pi}'_R(\underset{\sim}{v},\underset{\sim}{i})$, where $(\underset{\sim}{v},\underset{\sim}{i}) \in \Lambda \cap K$. By assumption

$$T_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)}\Lambda_R + T_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)}\underset{\sim}{\chi}_R(K) = T_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)}\mathbb{R}^{2n_R} \quad . \tag{13}$$

It follows from $\Lambda = \Lambda_R \times \mathbb{R}^{2n}$ that

$$T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda = T_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)}\Lambda_R \oplus T_{(\underset{\sim}{v}_p,\underset{\sim}{i}_p)}\mathbb{R}^{2n} \ . \tag{14}$$

Using (13) we have

$$\begin{aligned} d\underset{\sim\sim}{\pi}'_R(T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda + T_{(\underset{\sim}{v},\underset{\sim}{i})}K) &= T_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)}\Lambda_R + T_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)}\underset{\sim}{\chi}_R(K) \\ &= T_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)}\mathbb{R}^{2n_R} \ . \end{aligned} \tag{15}$$

It follows from (14) that

$$\mathrm{Ker}\ d\underset{\sim\sim}{\pi}'_R | T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda = T_{(\underset{\sim}{v}_p,\underset{\sim}{i}_p)}\mathbb{R}^{2n} \ . \tag{16}$$

Next let

$$T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda + T_{(\underset{\sim}{v},\underset{\sim}{i})}K = (\ker d\underset{\sim\sim}{\pi}'_R)^{\perp} \oplus \mathrm{Ker}\ d\underset{\sim\sim}{\pi}'_R \ . \tag{17}$$

Since $\mathrm{Im}\ d\underset{\sim\sim}{\pi}'_R$ is isomorphic to $(\mathrm{Ker}\ d\underset{\sim\sim}{\pi}'_R)^{\perp}$, we have from (15) that

$$T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda + T_{(\underset{\sim}{v},\underset{\sim}{i})}K = \underset{\sim}{A}(T_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)}\mathbb{R}^{2n_R}) \oplus \mathrm{Ker}\ d\underset{\sim\sim}{\pi}'_R \tag{18}$$

where

$$\underset{\sim}{A} : T_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)}\mathbb{R}^{2n_R} \to (\mathrm{Ker}\ d\underset{\sim\sim}{\pi}'_R)^{\perp}$$

is the isomorphism. It follows from (16) and (18) that

$$T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda + T_{(\underset{\sim}{v},\underset{\sim}{i})}K = \underset{\sim}{A}(T_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)}\mathbb{R}^{2n_R}) \oplus \mathrm{Ker}\ d\underset{\sim\sim}{\pi}'_R$$

$$= \underset{\sim}{A}(T_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)}\mathbb{R}^{2n_R}) \oplus T_{(\underset{\sim}{v}_P,\underset{\sim}{i}_P)}\mathbb{R}^{2n} .$$

Since $\underset{\sim}{A}$ is an isomorphism, we conclude that

$$T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda + T_{(\underset{\sim}{v},\underset{\sim}{i})}K = T_{(\underset{\sim}{v},\underset{\sim}{i})}\mathbb{R}^{2b} .$$

(ii) is straightforward. ¤

III. Strong Structural Stability

Let $\underset{\sim}{\pi}'_P : \mathbb{R}^{2b} \to \mathbb{R}^{2n}$ be the projection map defined by

$$\underset{\sim}{\pi}'_P(\underset{\sim}{v},\underset{\sim}{i}) = (\underset{\sim}{v}_P,\underset{\sim}{i}_P) \tag{19}$$

and let

$$\underset{\sim}{\pi}_P \triangleq \underset{\sim}{\pi}'_P \circ \underset{\sim}{\iota} \tag{20}$$

where $\underset{\sim}{\iota}$ is the inclusion map

$$\underset{\sim}{\iota} : \Sigma \to \mathbb{R}^{2b} . \tag{21}$$

Then the set

$$\mathcal{R} \triangleq \underset{\sim}{\pi}_P(\Sigma) \tag{22}$$

is called the constitutive relation of the n-port N. This is the object shown in Figs. 1-6 for 1-ports. Let $\hat{\Lambda}$ be a C^1 perturbation of Λ and let

$$\hat{\mathcal{R}} \triangleq \hat{\underset{\sim}{\pi}}_P(\hat{\Sigma}) \tag{23}$$

where $\hat{\Sigma}$ is defined by (3) and $\hat{\underset{\sim}{\pi}}_P$ is the corresponding projection map for $\hat{\Sigma}$. We are now ready to give a formal definition of strong structural stability.

<u>Definition 2</u>. A resistive n-port N is said to be <u>strongly structurally stable</u> if for any small C^1 perturbation $\hat{\Lambda}$ of Λ, the new constitutive relation $\hat{\mathcal{R}}$ is <u>homeomorphic</u> to the original $\mathcal{R}$.

In order to state our results on strong structural stability, we will need a new concept which we call a nice immersion and a graph theoretic condition which is called Condition P.

Definition 3. Suppose that Σ is an n-dimensional C^1 submanifold. Then the map π_P defined by (20) is said to be a nice immersion if

(i) π_P is an immersion, i.e., rank $(d\pi_P)_x = n$ for all $x \in \Sigma$.

(ii) π_P is transversal to itself, i.e., $\pi_P(x_1) = \pi_P(x_2)$ and $x_1 \neq x_2$ imply

$$(d\pi_P)_{x_1}(T_{x_1}\Sigma) + (d\pi_P)_{x_2}(T_{x_2}\Sigma) = \mathbb{R}^{2n} . \tag{24}$$

(iii) There is no family of three points $\{x_1,x_2,x_3\} \subset \Sigma$, such that $x_i \neq x_j$ $(i \neq j)$ and $\pi_P(x_1) = \pi_P(x_2) = \pi_P(x_3)$.

Remark. Since $\dim T_{x_1}\Sigma = \dim T_{x_2}\Sigma = n$, condition (24) forces the sum + to be the direct sum $\oplus$.

Let us give several examples to explain nice immersions. Consider Fig. 7(a), where we assume that Σ is a 1-dimensional submanifold and that π_P is an immersion. This map π_P is not a nice immersion because at y, condition (ii) is not satisfied while (i) and (iii) are satisfied. Consider Fig. 7(b). Again, assume that Σ is a 1-dimensional submanifold and that π_P is an immersion. This map π_P fails to be a nice immersion because the point y has three preimages and condition (iii) is violated although (i) and (ii) are satisfied. In Fig. 7(c), the map π_P is a nice immersion provided that Σ is a 1-dimensional submanifold and that π_P is an immersion.

Definition 4. An n-port N is said to satisfy condition P if there are no port-only loops and no port-only cut sets.

This graph theoretic condition is going to play one of the crucial roles in our main results. It will turn out that Condition P is closely related to the validity of a version of Whitney Immersion Theorem (Lemma F of Appendix II) which will be used in the proof of our main results in a crucial manner (see Proofs of Theorem 1 and Theorem 2). Let us observe that the following facts are true.

Proposition 4. The following conditions are equivalent:

(i) There are no port-only loops.

(ii) Each port forms a cut set exclusively with resistors.

(iii) There is a tree containing all the ports.

(iv) There is a cotree consisting only of resistors.

Proposition 5. The following conditions are equivalent:

(i) There are no port-only cut sets.

(ii) Each port forms a loop exclusively with resistors.

(iii) There is a tree consisting only of resistors.

(iv) There is a cotree containing all ports.

Now, recall $\underset{\sim}{\pi}'_P$ of (19) and set

$$\underset{\sim}{\chi}_P \triangleq \underset{\sim}{\pi}'_P \circ \underset{\sim}{\iota}_K \tag{25}$$

where $\underset{\sim}{\iota}_K$ is defined by (9). Notice that $\underset{\sim}{\pi}_P$ of (20) can be written as

$$\underset{\sim}{\pi}_P = \underset{\sim}{\chi}_P|\Sigma , \tag{26}$$

i.e, the restriction of $\underset{\sim}{\chi}_P$ to Σ. Recall $\underset{\sim}{\chi}_R$ defined by (8). Let ρ (resp. μ) be the number of independent port-only loops (resp. port-only cut sets). Let T_1 (resp. T_2) be a tree containing maximum number of ports (resp. resistors) and let L_1 (resp. L_2) be its associated cotree. Let ρ_1 (resp. μ_2) be the number of ports in L_1 (resp. T_2).

Proposition 6

(i) codim Im $\underset{\sim}{\chi}_P = \rho + \mu =$ dim Ker $\underset{\sim}{\chi}_R$

where codim means the complementary dimension and Im (resp. Ker) means the image (resp. kernel) of a linear map.

(ii) $\rho = \rho_1$, $\mu = \mu_2$.

Proof. The fundamental loop matrix [1] with respect to T_1 is given by

$$\underset{\sim}{B}_1 \triangleq [\underset{\sim}{1} \mid \underset{\sim}{B}_{T_1}]$$

$$= \begin{array}{c} \begin{array}{cccc} \underset{\sim}{v}_{R_{L_1}} & \underset{\sim}{v}_{P_{L_1}} & \underset{\sim}{v}_{R_{T_1}} & \underset{\sim}{v}_{P_{T_1}} \end{array} \\ \left[\begin{array}{cc|cc} \underset{\sim}{1} & \cdot & \underset{\sim}{B}^1_{RR} & \underset{\sim}{B}^1_{RP} \\ \cdot & \underset{\sim}{1} & \cdot & \underset{\sim}{B}^1_{PP} \end{array}\right] \end{array} \tag{27}$$

where R_{L_1} denotes resistors in L_1 and other symbols have similar meanings. A dot denotes a zero submatrix of appropriate size. The fundamental loop matrix with respect to T_2 is given by

$$\underset{\sim}{B}_2 \triangleq [\underset{\sim}{1} \mid \underset{\sim}{B}_{T_2}]$$

$$= \begin{array}{cccc} \underset{\sim}{v}_{R_{L_2}} & \underset{\sim}{v}_{P_{L_2}} & \underset{\sim}{v}_{R_{T_2}} & \underset{\sim}{v}_{P_{T_2}} \\ \end{array}$$

$$= \left[\begin{array}{cc|cc} \underset{\sim}{1} & \cdot & \underset{\sim}{B}^2_{RR} & \cdot \\ \cdot & \underset{\sim}{1} & \underset{\sim}{B}^2_{PR} & \underset{\sim}{B}^2_{PP} \end{array}\right] . \tag{28}$$

It follows from (27) and (28) that $\rho = \rho_1$ and $\mu = \mu_2$. Let $\underset{\sim}{Q}_1$ (resp. $\underset{\sim}{Q}_2$) be the fundamental cut set matrix with respect to T_1 (resp. T_2). Then $\underset{\sim}{Q}_1 = [-\underset{\sim}{B}^T_{T_1} \mid \underset{\sim}{1}]$ (resp. $\underset{\sim}{Q}_2 = [-\underset{\sim}{B}^T_{T_2} \mid \underset{\sim}{1}]$) [1]. Since

$$\underset{\sim}{v} = \underset{\sim}{Q}^T_1 \underset{\sim}{v}_{T_1}, \quad \underset{\sim}{i} = \underset{\sim}{B}^T_2 \underset{\sim}{i}_{L_2} \tag{29}$$

for $(\underset{\sim}{v},\underset{\sim}{i}) \in K$ [1], one has

$$K = \left\{ (\underset{\sim}{v},\underset{\sim}{i}) \;\middle|\; \begin{array}{l} \underset{\sim}{v} = \begin{bmatrix} \underset{\sim}{v}_{R_{L_1}} \\ \underset{\sim}{v}_{P_{L_1}} \\ \underset{\sim}{v}_{R_{T_1}} \\ \underset{\sim}{v}_{P_{T_1}} \end{bmatrix} = \begin{bmatrix} -\underset{\sim}{B}^1_{RR} & -\underset{\sim}{B}^1_{RP} \\ \cdot & -\underset{\sim}{B}^1_{PP} \\ \underset{\sim}{1} & \cdot \\ \cdot & \underset{\sim}{1} \end{bmatrix} \begin{bmatrix} \underset{\sim}{v}_{R_{T_1}} \\ \underset{\sim}{v}_{P_{T_1}} \end{bmatrix} \\ \\ \underset{\sim}{i} = \begin{bmatrix} \underset{\sim}{i}_{R_{L_2}} \\ \underset{\sim}{i}_{P_{L_2}} \\ \underset{\sim}{i}_{R_{T_2}} \\ \underset{\sim}{i}_{P_{T_2}} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{1} & \cdot \\ \cdot & \underset{\sim}{1} \\ \underset{\sim}{B}^{2T}_{RR} & \underset{\sim}{B}^{2T}_{PR} \\ \cdot & \underset{\sim}{B}^{2T}_{PP} \end{bmatrix} \begin{bmatrix} \underset{\sim}{i}_{R_{L_2}} \\ \underset{\sim}{i}_{P_{L_2}} \end{bmatrix} \end{array} \right\} . \tag{30}$$

Hence we have the parametrization $\psi_1 : \mathbb{R}^b \to K$ defined by

$$\begin{bmatrix} \mathbf{v}_{R_{T_1}} \\ \mathbf{v}_{P_{T_1}} \\ \mathbf{i}_{R_{L_2}} \\ \mathbf{i}_{P_{L_2}} \end{bmatrix} \mapsto \begin{bmatrix} \mathbf{v} \\ \\ \mathbf{i} \end{bmatrix} .$$

Then

$$\chi_p \circ \psi_1 = \begin{array}{c} \begin{array}{cccc} \mathbf{v}_{R_{T_1}} & \mathbf{v}_{P_{T_1}} & \mathbf{i}_{R_{L_2}} & \mathbf{i}_{P_{L_2}} \end{array} \\ \left[\begin{array}{cc|cc} \cdot & -\mathbf{B}^1_{PP} & & \cdot \\ \cdot & \mathbf{1} & & \\ \hline & & \cdot & \mathbf{1} \\ \cdot & & \cdot & \mathbf{B}^{2T}_{PP} \end{array}\right] \end{array} \tag{31}$$

and hence

$$\begin{aligned} \text{codim Im } \chi_p &= 2n - \text{rank } \chi_p \circ \psi_1^{-1} \\ &= (n-n_{P_{T_1}}) + (n-n_{P_{L_2}}) \\ &= n_{P_{L_1}} + n_{P_{T_2}} \end{aligned} \tag{32}$$

where $n_{P_{T_1}}$ is the number of ports in T_1 and other symbols have similar meanings. Next, we choose another parametrization ψ_2 for K defined by

$$\underset{\sim}{v} = \begin{bmatrix} \underset{\sim}{v}_{R_{L_2}} \\ \underset{\sim}{v}_{P_{L_2}} \\ \underset{\sim}{v}_{R_{T_2}} \\ \underset{\sim}{v}_{P_{T_2}} \end{bmatrix} = \begin{bmatrix} -\underset{\sim}{B}^2_{RR} & \cdot \\ -\underset{\sim}{B}^2_{PR} & -\underset{\sim}{B}^2_{PP} \\ \underset{\sim}{1} & \cdot \\ \cdot & \underset{\sim}{1} \end{bmatrix} \begin{bmatrix} \underset{\sim}{v}_{R_{T_2}} \\ \underset{\sim}{v}_{P_{T_2}} \end{bmatrix} \tag{33}$$

$$\underset{\sim}{i} = \begin{bmatrix} \underset{\sim}{i}_{R_{L_1}} \\ \underset{\sim}{i}_{P_{L_1}} \\ \underset{\sim}{i}_{R_{T_1}} \\ \underset{\sim}{i}_{P_{T_1}} \end{bmatrix} \begin{bmatrix} \underset{\sim}{1} & \cdot \\ \cdot & \underset{\sim}{1} \\ \underset{\sim}{B}^{1T}_{RR} & \cdot \\ \underset{\sim}{B}^{1T}_{RP} & \underset{\sim}{B}^{1T}_{PP} \end{bmatrix} \begin{bmatrix} \underset{\sim}{i}_{R_{L_1}} \\ \underset{\sim}{i}_{P_{L_1}} \end{bmatrix} . \tag{34}$$

Then

$$\underset{\sim}{\chi}_R \circ \underset{\sim}{\psi}_2 = \begin{array}{c} \begin{array}{cccc} \underset{\sim}{v}_{R_{T_2}} & \underset{\sim}{v}_{P_{T_2}} & \underset{\sim}{i}_{R_{L_1}} & \underset{\sim}{i}_{P_{L_1}} \end{array} \\ \left[\begin{array}{cc|cc} -\underset{\sim}{B}^2_{RR} & \cdot & & \\ \underset{\sim}{1} & \cdot & & \cdot \\ \hline & & \underset{\sim}{1} & \cdot \\ & \cdot & \underset{\sim}{B}^{1T}_{RR} & \cdot \end{array}\right] \end{array} \tag{35}$$

which implies that

$$\text{Ker } \underset{\sim}{\chi}_R \circ \underset{\sim}{\psi}_2 = \{(\underset{\sim}{v}_{R_{T_2}}, \underset{\sim}{v}_{P_{T_2}}, \underset{\sim}{i}_{R_{L_1}}, \underset{\sim}{i}_{P_{L_1}}) \mid \underset{\sim}{v}_{R_{T_2}} = \underset{\sim}{0},\ \underset{\sim}{i}_{R_{L_1}} = \underset{\sim}{0}\} .$$

Therefore

$$\operatorname{Ker} \underset{\sim}{\chi}_R = \left\{ (\underset{\sim}{v},\underset{\sim}{i}) \;\middle|\; \begin{bmatrix} \underset{\sim}{v}_{R_{L_2}} \\ \underset{\sim}{v}_{P_{L_2}} \\ \underset{\sim}{v}_{R_{T_2}} \\ \underset{\sim}{v}_{P_{T_2}} \end{bmatrix} = \begin{bmatrix} \cdot \\ -\underset{\sim}{B}_{PP}^{2} \\ \cdot \\ \underset{\sim}{1} \end{bmatrix} \underset{\sim}{v}_{P_{T_2}},\ \begin{bmatrix} \underset{\sim}{i}_{R_{L_1}} \\ \underset{\sim}{i}_{P_{L_1}} \\ \underset{\sim}{i}_{R_{T_1}} \\ \underset{\sim}{i}_{P_{T_1}} \end{bmatrix} = \begin{bmatrix} \cdot \\ \underset{\sim}{1} \\ \cdot \\ \underset{\sim}{B}_{PP}^{1T} \end{bmatrix} \underset{\sim}{i}_{P_{L_1}} \right\} \tag{36}$$

and hence

$$\dim \operatorname{Ker} \underset{\sim}{\chi}_R = n_{P_{T_2}} + n_{P_{L_1}} \tag{37}$$

Equations (32) and (37) imply (i). ¤

<u>Remark</u>. The above result says that rank $\underset{\sim}{\chi}_P$ and dim Ker $\underset{\sim}{\chi}_R$ are complementary with respect to $2n$ and $\rho + \mu$, i.e., if there are ρ independent port-only loops and μ independent port-only cut sets, then rank $\underset{\sim}{\chi}_P$ drops by $\rho + \mu$. This, in turn, forces dim Ker $\underset{\sim}{\chi}_R$ to increase by this same number $\rho + \mu$. This means that for any $(\underset{\sim}{v}_R,\underset{\sim}{i}_R) \in \underset{\sim}{\chi}_R(K)$, there are $\rho + \mu$ independent vectors $(\underset{\sim}{v}_{P_k},\underset{\sim}{i}_{P_k})$, $k = 1,\dots,\rho + \mu$, such that $(\underset{\sim}{v}_R,\underset{\sim}{v}_{P_k},\underset{\sim}{i}_R,\underset{\sim}{i}_{P_k}) \in K$ for all k.

Now observe that codim Im $\underset{\sim}{\chi}_P = 0$ (resp. dim Ker $\underset{\sim}{\chi}_R = 0$) means $\underset{\sim}{\chi}_P$ (resp. $\underset{\sim}{\chi}_R$) is surjective (resp. injective). This implies the next two facts.

<u>Corollary 1</u>. The following conditions are equivalent:

(i) <u>Condition P</u>.

(ii) $\underset{\sim}{\chi}_P$ is surjective.

(iii) $\underset{\sim}{\chi}_R$ is injective.

<u>Corollary 2</u>. The following conditions are equivalent:

(i) There are no resistor-only loops and no resistor-only cut sets.

(ii) $\underset{\sim}{\chi}_P$ is injective.

(iii) $\underset{\sim}{\chi}_R$ is surjective.

We are now ready to state the first of our main results.

<u>Theorem 1 (Characterization of Strong Structural Stability)</u>. Suppose that $\Lambda \cap K \neq \emptyset$.

(i) If $\Lambda \bar{\pitchfork}$ and if $\underset{\sim}{\pi}_P$ is a nice immersion, then the following hold:

(ia) For any small C^1 perturbation $\hat{\Lambda}$ of Λ, the map $\hat{\underset{\sim}{\pi}}_P$ persists to be a nice immersion. Furthermore, there is a C^1 diffeomorphism $\underset{\sim}{H} : \Sigma \to \hat{\Sigma}$ near the identity map and there is a homeomorphism $\underset{\sim}{H}_P : R \to \hat{R}$ such that the following diagram commutes:

$$\begin{tikzcd} \Sigma \arrow[r, "\underset{\sim}{H}"] \arrow[d, "\underset{\sim}{\pi}_P"'] & \hat{\Sigma} \arrow[d, "\hat{\underset{\sim}{\pi}}_P"] \\ R \arrow[r, "\underset{\sim}{H}_P"] & \hat{R} \end{tikzcd}$$

Therefore N is strongly structurally stable.

(ib) For any self intersection point $(\underset{\sim}{v}_P, \underset{\sim}{i}_P) \in R$, $\# \underset{\sim}{\pi}_P^{-1}(\underset{\sim}{v}_P, \underset{\sim}{i}_P) = 2$, where # denotes the cardinality of a set. Furthermore, self intersection points are isolated.

(ii) Assume Condition P and suppose that $\Lambda \not\pitchfork K$ or $\underset{\sim}{\pi}_P$ is not a nice immersion. Then there are arbitrarily small C^1 perturbations Λ' and Λ'' of Λ such that R' and R'' are not homeomorphic where Λ' and Λ'' are the constitutive relations of the perturbed n-ports. Therefore N is not strongly structurally stable.

Before we prove this theorem, let us state an important consequence.

Corollary 3. Suppose that $\Lambda \cap K \neq \emptyset$ and that Condition P is satisfied. Then N is strongly structurally stable if, and only if $\Lambda \bar{\pitchfork} K$ and $\underset{\sim}{\pi}_P$ is a nice immersion.

Remarks. 1) Notice the "if and only if" nature of Corollary 3. It completely characterizes strongly structurally stable n-ports. Recall that Proposition 1 says that an n-port is structurally stable if and only if $\Lambda \bar{\pitchfork} K$. Therefore, for strong structural stability, we need another condition; $\underset{\sim}{\pi}_P$ is a nice immersion and that is a necessary condition also.

2) Assume Condition P. If N is strongly structurally stable, then (ib) says that every self intersection point of R has exactly two points (no more and no less) in Σ which are mapped to this point by $\underset{\sim}{\pi}_P$. Therefore, one can immediately tell that the 1-port in Fig. 7(b) is not strongly structurally stable because there are three preimages of $\underset{\sim}{y}$ under $\underset{\sim}{\pi}_P$.

3) Let S (resp. $\hat{S}$) be the set of all self intersection points of R (resp $\hat{R}$). Then $R - S$ and $\hat{R} - \hat{S}$ are manifolds. If $H_p : R \to \hat{R}$ is the homeomorphism given in <u>Theorem 1</u>, then $H_p|(R - S) : (R - S) \to (\hat{R} - \hat{S})$ is a diffeomorphism.

<u>Proof of Theorem 1</u>. (ib) follows from the definition of a nice immersion. (ia) It follows from <u>Proposition 1</u> that there is a diffeomorphism $G : \Sigma \to \hat{\Sigma} = \hat{\Lambda} \cap K$ such that G is close to ι provided that $\hat{\Lambda}$ is close enough to Λ. Therefore $\chi_p \circ G : \Sigma \to \hat{R}$ is close to $\pi_p : \Sigma \to R$. It follows from <u>Lemma F</u> in Appendix II that $\chi_p \circ G$ persists to be a nice immersion. Since $\hat{\Sigma} = G(\Sigma)$ and since $\hat{\pi}_p = \chi_p|\hat{\Sigma}$, the map $\hat{\pi}_p$ is a nice immersion. By the definition of a nice immersion, for any self intersection point $y \triangleq (v_p, i_p) \in R$, <u>there are exactly two points</u> $x_1, x_2 \in \Sigma$ such that $\pi_p(x_1) = y = \pi_p(x_2)$. Moreover, there are disjoint neighborhoods U_k of x_k, $k = 1,2$, in Σ such that $\pi_p|U_k$ is a diffeomorphism, $\pi_p(U_1) \cap \pi_p(U_2) = \{y\}$ and $\pi_p(U_1) \pitchfork \pi_p(U_2)$. Let U_k^* be a neighborhood of x_k in U_k such that $\overline{U_k^*} \subset U_k$ and define $V_k \triangleq G(U_k) \subset \hat{\Sigma}$, $V_k^* \triangleq G(U_k^*) \subset \hat{\Sigma}$, $k = 1,2$, where a bar denotes the closure of a set. If $\chi_p \circ G$ is close to π_p, we may assume that $\chi_p(V_1) \cap \chi_p(V_2) = \{\hat{y}\}$, a singleton, and $\chi_p(V_1) \pitchfork \chi_p(V_2)$. Set

$$\hat{x}_k \triangleq \hat{\pi}_p^{-1}(\hat{y}) \cap V_k, \quad k = 1,2, \tag{38}$$

where $\hat{\pi}_p = \chi_p|\hat{\Sigma}$. Then there is a C^1 diffeomorphism $\phi_k : V_k \to V_k$ (see Fig. 8) such that (i) $\phi_k \circ G\,(x_k) = \hat{x}_k$ and (ii) ϕ_k is the identity map on a neighborhood of the boundary of V_k. Clearly, ϕ_k is close to the identity map if U_k^* is small enough. Define $H : \Sigma \to \hat{\Sigma}$ by

$$H\,(x) \triangleq \begin{cases} \phi_1 \circ G\,(x), & x \in U_1 \\ \phi_2 \circ G\,(x), & x \in U_2 \\ G\,(x) \quad , & x \notin U_1 \cup U_2 \quad . \end{cases} \tag{39}$$

Then H is a C^1 diffeomorphism close to the identity map. For any point $y \in R$, there is a point $x \in \Sigma$ such that $\pi_p(x) = y$. We define $H_p : R \to \hat{R}$ by

$$H_p(y) \triangleq \chi_p \circ H \circ \pi_p^{-1}(y) \tag{40}$$

where π_p^{-1} is the set theoretic inverse. We claim that this map is well-defined. To prove this (see Fig. 9) observe that $\underset{\sim}{y} = \underset{\sim}{\pi}_P(\underset{\sim}{x}_1) = \underset{\sim}{\pi}_P(\underset{\sim}{x}_2)$ implies $\underset{\sim}{H}(\underset{\sim}{x}_k) = \hat{\underset{\sim}{x}}_k$, $k = 1,2$, because of the property of $\underset{\sim}{\phi}_k$ and (39). It follows from (38) that $\underset{\sim}{\chi}_P(\hat{x}_1) = \underset{\sim}{\chi}_P(\hat{x}_2) = \hat{\underset{\sim}{y}}$. Therefore $\underset{\sim}{H}_P(\underset{\sim}{y}) = \underset{\sim}{\chi}_P \circ \underset{\sim}{H}(\underset{\sim}{x}_1) = \underset{\sim}{\chi}_P \circ \underset{\sim}{H}(x_2) = \hat{\underset{\sim}{y}}$ and $\underset{\sim}{H}_P$ is well-defined. Since $\underset{\sim}{\chi}_P \circ \underset{\sim}{H} = \hat{\underset{\sim}{\pi}}_P \circ \underset{\sim}{H}$, where $\hat{\underset{\sim}{\pi}}_P \triangleq \underset{\sim}{\chi}_P | \hat{\Sigma}$, we have from (40) that

$$\hat{\underset{\sim}{\pi}}_P \circ \underset{\sim}{H} = \underset{\sim}{H}_P \circ \underset{\sim}{\pi}_P \quad . \tag{41}$$

Also, the inverse

$$\underset{\sim}{H}_P^{-1} = \underset{\sim}{\chi}_P \circ \underset{\sim}{H}^{-1} \circ \hat{\pi}_P^{-1} \tag{42}$$

is continuous. Since $\underset{\sim}{H}_P$ and $\underset{\sim}{H}_P^{-1}$ are continuous, $\underset{\sim}{H}_P$ is a homeomorphism. This and (41) imply (ia).

(ii) There are four cases which can happen.

<u>Case 1</u>: $\Lambda \not\pitchfork K$. It follows from Theorem 2 of [1] that there is a C^1 perturbation Λ_1 of Λ such that $\Lambda_1 \cap K$ contains an open set U which is also an open set of an affine submanifold J of dimesnion $n + k$, $k > 0$. Consider the splitting $K = \text{Ker } \underset{\sim}{\chi}_P \oplus (\text{Ker } \underset{\sim}{\chi}_P)^{\perp}$. Since <u>Condition P</u> holds, <u>Corollary 1</u> implies that $\dim(\text{Ker } \underset{\sim}{\chi}_P)^{\perp} = 2n$. Let Λ' be a further perturbation of Λ_1 such that $\Lambda' \cap K$ contains an open subset U' of an affine submanifold J' such that J' has an $(n+k')$ - dimensional factor in $(\text{Ker } \underset{\sim}{\chi}_P)^{\perp}$ for some k', $1 \leq k' \leq k$. Since $\underset{\sim}{\chi}_P$ maps $(\text{Ker } \underset{\sim}{\chi}_P)^{\perp}$ onto $\underset{\sim}{\chi}_P(K)$ isomorphically, we see that $\underset{\sim}{\chi}_P(J')$ is an $(n+k')$ - dimensional affine submanifold. Hence $R' \triangleq \underset{\sim}{\chi}_P(\Lambda' \cap K)$ contains an $(n+k')$ - dimensional open set. On the other hand, it follows from <u>Theorem 2</u> which will be given shortly that there is another C^1 perturbation Λ'' of Λ such that $\underset{\sim}{\chi}_P | \Lambda'' \cap K$ is a nice immersion. Therefore $R'' \triangleq \underset{\sim}{\chi}_P(\Lambda'' \cap K)$ contains an n-dimensional open subset but it cannot contain an open subset whose dimension is greater than n. Therefore R' and R'' cannot be homeomorphic.

<u>Case 2</u>: $\Lambda \pitchfork K$, but $\underset{\sim}{\pi}_P$ is not an immersion. In this case Σ is an n-dimensional submanifold. Since $\underset{\sim}{\pi}_P$ is not an immersion, there is an $\underset{\sim}{x} \in \Sigma$ with

$$\dim \underset{\sim}{d\pi}_P(T_{\underset{\sim}{x}}\Sigma) < n \quad . \tag{43}$$

Let

$$J \triangleq T_{\underset{\sim}{x}}\Sigma \quad . \tag{44}$$

Then there is a C^1 perturbation Λ' of Λ such that $\Sigma' \cap J$ contains an open subset of Σ' and J. By a further perturbation, if necessary, we may assume that $\dim(d\underset{\sim}{\chi}_p)(J) = n - 1$. Then J is isomorphic to $\underset{\sim}{\chi}_p(J) \oplus J'$ for some J'. Now consider the following local coordinate system on a small neighborhood U of $\underset{\sim}{x}$ in Σ. (Fig. 10) Let D^{n-1} be an open disc of $\mathbb{R}^{n-1}$ centered at the origin with radius 1 and let $L \triangleq [-1,1] \subset \mathbb{R}$. Then there is a diffeomorphism $\underset{\sim}{\phi} : D^{n-1} \times L \to U$ such that $\underset{\sim}{\phi}(\underset{\sim}{0},0) = \underset{\sim}{x}$. Since $U \subset J$, we can choose $\underset{\sim}{\phi}$ in such a way that $\underset{\sim}{\phi}(D^{n-1} \times \{0\}) \subset \underset{\sim}{\chi}_p(J) \times \{0\}$. Since J is isomorphic to $\underset{\sim}{\chi}_p(J) \oplus J'$, there is a C^1 embedding $\underset{\sim}{F} : \Lambda' \to \mathbb{R}^{2b}$ close to $\underset{\sim}{\iota}_{\Lambda'}$ such that the following hold $(\Sigma'' \triangleq \underset{\sim}{F}(\Lambda') \cap K)$:

(i) $\underset{\sim}{F} \circ \underset{\sim}{\phi}(\underset{\sim}{u},0) \cong \underset{\sim}{\chi}_p \circ \underset{\sim}{F} \circ \underset{\sim}{\phi}(\underset{\sim}{u},0) \in \underset{\sim}{\chi}_p \circ \underset{\sim}{F} \circ \underset{\sim}{\phi}(D^{n-1} \times \{0\})$

(ii) For each $(\underset{\sim}{u},t)$, $\underset{\sim}{u} \in D^{n-1}$, $|t| \le \frac{1}{2}$,

$$\underset{\sim}{\chi}_p \circ \underset{\sim}{F} \circ \underset{\sim}{\phi}(\underset{\sim}{u},t) = \underset{\sim}{\chi}_p \circ \underset{\sim}{F} \circ \underset{\sim}{\phi}(\underset{\sim}{u},-t)$$

i.e. it is a self intersection point of $R'' \triangleq \underset{\sim}{\chi}_p(\Sigma'')$.

(iii) For any $(\underset{\sim}{u},t)$, $\underset{\sim}{u} \in D^{n-1}$, $|t| > \frac{1}{2}$, the point $\underset{\sim}{\chi}_p \circ \underset{\sim}{F} \circ \underset{\sim}{\phi}(\underset{\sim}{u},t)$ is not a self intersection point of R''.

(iv)

$$\dim(d\underset{\sim}{\chi}_p)\, T_{\underset{\sim}{F}\circ\underset{\sim}{\phi}(\underset{\sim}{u},t)}\, \Sigma'' = \begin{cases} n & , t \neq 0 \\ n-1 & , t = 0. \end{cases}$$

Then, for $\underset{\sim}{u} \in \operatorname{Int} D^{n-1}$, the point $\underset{\sim}{\chi}_p \circ \underset{\sim}{F} \circ \underset{\sim}{\phi}(\underset{\sim}{u},0)$ has a neighborhood in R'' of the form $\underset{\sim}{\chi}_p \circ \underset{\sim}{F} \circ \underset{\sim}{\phi}\,(D^{n-1} \times \{0\}) \times \{\prec\}$. On the other hand, for the perturbation $\hat{\Lambda}$ of Λ obtained by <u>Theorem 2</u> which will be given shortly, $\underset{\sim}{\chi}_p(\hat{\Sigma})$ has no points with such a neighborhood, because each self intersection point is isolated. Therefore, $\underset{\sim}{\chi}_p(\hat{\Sigma})$ and $\underset{\sim}{\chi}_p(\Sigma'')$ are not homeomorphic.

<u>Case 3</u>: $\Lambda \not\pitchfork K$ and $\underset{\sim}{\pi}_p$ is an immersion, but there are points $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ such that $\underset{\sim}{y} = \underset{\sim}{\pi}_p(\underset{\sim}{x}_1) = \underset{\sim}{\pi}_p(\underset{\sim}{x}_2)$, $\underset{\sim}{x}_1 \neq \underset{\sim}{x}_2$, and

$$(d\underset{\sim}{\pi}_p)_{\underset{\sim}{x}_1}\, T_{\underset{\sim}{x}_1}\Sigma + (d\underset{\sim}{\pi}_p)_{\underset{\sim}{x}_2}\, T_{\underset{\sim}{x}_2}\Sigma \neq T_{\underset{\sim}{y}}\mathbb{R}^{2n} \quad . \tag{45}$$

A simple example is the point $\underset{\sim}{y}$ of Fig. 7(a). The idea here is to obtain a C^1 perturbation Λ' of Λ such that $\underset{\sim}{\chi}_p(\Sigma')$ looks like R' of Fig. 11 and obtain another perturbation $\hat{\Lambda}$ of Λ such that $\underset{\sim}{\chi}_p(\hat{\Sigma})$ looks like $\hat{R}$ of Fig. 11. Certainly, they are not homeomorphic. First of all, observe that there is an affine submanifold $J \triangleq ((d\underset{\sim}{\pi}_p)_{\underset{\sim}{x}_1}\, T_{\underset{\sim}{x}_1}\Sigma) \cap ((d\underset{\sim}{\pi}_p)_{\underset{\sim}{x}_2}\, T_{\underset{\sim}{x}_2}\Sigma)$ in $T_{\underset{\sim}{y}}\mathbb{R}^{2n}$ such that $\dim J > 0$ because of (45). Without loss of generality, assume that J is a linear subspace of $\mathbb{R}^{2n}$. By perturbations on neighborhoods of x_1 and x_2 in Λ, we obtain a Λ' close to Λ such that there are neighborhoods U_k of x_k, $k = 1,2$, in $\Sigma' \triangleq \Lambda' \cap K$ and such that $V \triangleq \chi_p(U_1) \cap \chi_p(U_2)$ is a neighborhood of y in J. Let D^m be a disk centered at 0 in $\mathbb{R}^m$ and let $X^m \triangleq (D^m \times \{0\}) \cup (\{0\} \times D^m) \subset \mathbb{R}^m \times \mathbb{R}^m$. Then y has a neighborhood in $\chi_p(\Sigma')$ homeomorphic to $X^{n-k} \times D^k$, where $k = \dim J > 0$. (see Fig. 12) Now let $\hat{\Lambda}$ be the perturbation obtained in Theorem 2 which will be given shortly. Then any point $y \in \pi_p(\hat{\Sigma})$ has a neighborhood homeomorphic to D^n or X^n but it does not have a neighborhood homeomorphic to $X^{n-k} \times D^k$, $k > 0$. Therefore, $\chi_p(\Sigma')$ and $\chi_p(\hat{\Sigma})$ cannot be homeomorphic.

Case 4: $\Lambda \pitchfork K$ and conditions (i) and (ii) of Definition 3 hold, but (iii) fails to hold, i.e., there are three points x_1, x_2 and x_3 in Σ such that $\pi_p(x_1) = \pi_p(x_2) = \pi_p(x_3) \triangleq y \in R$. Let $\Delta \triangleq \{(u,u) \in \mathbb{R}^n \times \mathbb{R}^n\}$ which is called the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$. Then y has a neighborhood in R homeomorphic to $\Delta \cup (\mathbb{R}^n \times \{0\}) \cup (\{0\} \times \mathbb{R}^n) \subset \mathbb{R}^{2n}$. For example, point y of Fig. 7(b) has such a neighborhood. On the other hand, for the perturbation $\hat{\Lambda}$ of Λ obtained in Theorem 2 which will be given shortly, each point $y \in \hat{R} \triangleq \chi_p(\hat{\Sigma})$ has a neighborhood homeomorphic to D^n or X^n as in Case 3. (see Fig. 13) Therefore, R and $\hat{R}$ cannot be homeomorphic.

$\square$

Let us now give several examples to explain significance of the conditions in Theorem 1 and Corollary 1.

Example 4. Consider Example 1. Since $\Lambda \not\pitchfork K$, this 1-port is not strongly structurally stable as described in Example 1.

Example 5. Consider Example 2. At point y of R in Fig. 2(b), condition (ii) of Definition 3 is violated. Therefore this 1-port is not strongly structurally stable as was explained in Example 2. Notice that this 1-port is structurally stable as was explained after Example 3.

Example 6. Consider Example 3. Since all the conditions of Theorem 1 hold, this 1-port is strongly structurally stable as explained in Example 3.

Example 7. (A strongly structurally stable 2-port where π_p is not a nice immersion and Condition P is violated.) This example shows that Corollary 3 as well as (ii) of Theorem 1 is false without Condition P.

Consider the 2-port of Fig. 14(a) where the internal resistor constitutive relations Λ is given by the following parametric form:

$$v_{R_1} = \frac{1}{2}\rho_1^2 + \rho_1 \quad , \; v_{R_2} = \frac{1}{2}\rho_1^2 - \rho_1$$

$$i_{R_1} = \rho_1 i_{P_1} + i_{P_1}, \; i_{R_2} = \rho_2, \text{ where } \rho_1, \rho_2, i_{P_1} \in \mathbb{R} .$$

We will first show that Λ is a 6-dimensional submanifold. Taking the derivative of the parametrization, we have

$$\begin{array}{c} \\ v_{R_1} \\ v_{R_2} \\ i_{R_1} \\ i_{R_2} \\ v_{P_1} \\ v_{P_2} \\ i_{P_1} \\ i_{P_2} \end{array}
\begin{array}{c}
\begin{array}{cccccc} \rho_1 & \rho_2 & i_{P_1} & v_{P_1} & v_{P_2} & i_{P_2} \end{array} \\
\left[\begin{array}{ccc|ccc}
\rho_1+1 & \cdot & \cdot & & & \\
\rho_1-1 & \cdot & \cdot & & \cdot & \\
i_{P_1} & \cdot & \rho_1+1 & & & \\
\cdot & 1 & \cdot & & & \\
\hline
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right]
\end{array} \triangleq \underset{\sim}{E} . \qquad (46)$$

It is clear that this matrix has rank 6 and hence Λ is a 6-dimensional submanifold of $\mathbb{R}^8$. Let K be the Kirchhoff space [1]. Then K is parametrized by $(v_{R_1}, v_{R_2}, i_{P_1}, i_{P_2})$;

$$\begin{bmatrix} v_{R_1} \\ v_{R_2} \\ i_{R_1} \\ i_{R_2} \\ v_{P_1} \\ v_{P_2} \\ i_{P_1} \\ i_{P_2} \end{bmatrix} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & 1 \\ -1 & -1 & \cdot & \cdot \\ -1 & -1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} v_{R_1} \\ v_{R_2} \\ i_{P_1} \\ i_{P_2} \end{bmatrix} \triangleq \underset{\sim}{F} \begin{bmatrix} \underset{\sim}{v}_R \\ \underset{\sim}{i}_P \end{bmatrix}$$

Let $\underset{\sim}{\rho} = (\rho_1,\rho_2)$. Then, by the definition of transversality [1,2], $\Lambda \pitchfork K$ if and only if

$$\mathrm{Im}\ \underset{\sim}{E}_{(\underset{\sim}{\rho},\underset{\sim}{v}_P,\underset{\sim}{i}_P)} + \mathrm{Im}\ \underset{\sim}{F} = \mathbb{R}^8 \text{ for all } (\underset{\sim}{\rho},\underset{\sim}{v}_P,\underset{\sim}{i}_P)$$

which is equivalent to

$$\mathrm{rank}\ [\underset{\sim}{E}_{(\underset{\sim}{\rho},\underset{\sim}{v}_P,\underset{\sim}{i}_P)} \mid \underset{\sim}{F}] = 8 \text{ for all } (\underset{\sim}{\rho},\underset{\sim}{v}_P,\underset{\sim}{i}_P)\ . \tag{46}$$

It can be shown that (46) holds and hence $\Lambda \pitchfork K$. Therefore Σ is a 2-dimensional submanifold [1]. Since Kirchhoff laws are given by $v_{P_1} + v_{R_1} + v_{R_2} = 0$, $v_{P_2} + v_{R_1} + v_{R_2} = 0$, $i_{R_1} - i_{P_1} - i_{P_2} = 0$ and $i_{R_2} - i_{P_1} - i_{P_2} = 0$, we see that Σ is parametrized by (ρ_1, i_{P_1}) and hence $\mathcal{R}$ is described by

$$v_{P_1} = -\rho_1^2,\ v_{P_2} = -\rho_1^2,\ i_{P_1} \qquad ,\ i_{P_2} = \rho_1 i_{P_1}\ . \tag{47}$$

In Fig. 14(b) a picture of $\mathcal{R}$ embedded in $\mathbb{R}^3$ is given. Let $\underset{\sim}{\psi} : \Sigma \to \mathbb{R}^2$ be the above coordinate system. Then

$$(\underset{\sim}{d}\underset{\sim}{\pi}_P \circ \underset{\sim}{\psi}^{-1})_{(\rho_1, i_{P_1})} = \begin{bmatrix} -2\rho_1 & \cdot \\ -2\rho_1 & \cdot \\ \cdot & 1 \\ i_{P_1} & \rho_1 \end{bmatrix}_{(\rho_1, i_{P_1})} .$$

Since $\mathrm{rank}\ (\underset{\sim}{d}\underset{\sim}{\pi}_P \circ \underset{\sim}{\psi}^{-1})_{(0,0)} = 1$, $\underset{\sim}{\pi}_P$ is not an immersion at $\underset{\sim}{\psi}^{-1}(0,0) = \underset{\sim}{0} \in \Sigma$. (See Fig. 14(b)). One can see intuitively that $\mathcal{R}$ persists to be qualitatively the same object under small perturbations. (Proof of this fact is omitted since it is technically involved.) Therefore this 1-port is strongly structurally stable even though $\underset{\sim}{\pi}_P$ is not an immersion, where Condition P is violated. One can also see from Fig. 14 that for any small perturbation $\hat{\Lambda}$ of Λ, the map $\hat{\underset{\sim}{\pi}}_P$ will not be an immersion.

Remark. The proofs of our main results depend crucially on the C^1-ness of the perturbation $\hat{\Lambda}$ of Λ. If we consider C^2 perturbations, the proof for (ii) of Theorem 1 does not work. Roughly speaking, the reason is the following. Suppose that Λ is described by $y = x^2$. In Case 1 of the proof for (ii) of Theorem 1 and in the proof of Proposition 2, one has to take a procedure which is equivalent to flattening Λ onto the x-axis locally (see Fig. 15) in obtaining $\hat{\Lambda}$. This is impossible by a C^2 perturbation. To see this let us recall that the C^2 perturbation $\hat{\Lambda}$ of Λ must satisfy

$$|x^2 - \hat{f}(x)| + |2x - (D\hat{f})_x| + |2 - (D^2\hat{f})_x| < \varepsilon(x) \tag{48}$$

for small $\varepsilon(x)$, where $\hat{\Lambda}$ is described by $y = \hat{f}(x)$. It is impossible, however, to make the left hand side of (48) small, because at $x = 0$, we have $(D\hat{f})_0 = 0$, $(D^2\hat{f})_0 = 0, |2 - (D^2\hat{f})_0| = 2$. On the other hand, the C^1 distance $|x - \hat{f}(x)| + |2x - (D\hat{f})_x|$ can be made arbitrarily small. A similar procedure is taken in the proof for Case 3 of (ii) of Theorem 1. The above difficulty arises for all C^k perturbations, $k \geq 2$. On the other hand, if we consider C^0 perturbations of Λ, then all the transversality arguments do not make sense because of the very definition of transversality; $\Lambda \pitchfork K \Leftrightarrow T_{\underset{\sim}{x}}\Lambda + T_{\underset{\sim}{x}}K = \mathbb{R}^{2b}$. Namely tangent spaces cannot be defined on C^0 manifolds, in general. Also, the set of all C^1 functions $\underset{\sim}{F} : \Lambda \to \mathbb{R}^{2b}$ such that $\underset{\sim}{F}(\Lambda) \cap K$ is a submanifold, is not an open subset of $C^1(\Lambda;\mathbb{R}^{2b})$ with respect to the strong C^0 topology.

We will next give another important result on strong structural stability which says that there are "many" strongly structurally stable n-ports provided that Condition P holds.

Theorem 2 (Density of Strong Structural Stability)

Given an n-port N assume that $\Lambda \cap K \neq \emptyset$ holds and that Condition P is satisfied. Then there is an arbitrarily small C^1 perturbation $\hat{\Lambda}$ of Λ such that $\hat{\Lambda} \cap K \neq \emptyset$, $\hat{\Lambda} \pitchfork K$ and $\underset{\sim}{\pi}_P$ is a nice immersion. Therefore the perturbed n-port $\hat{N}$ is strongly structurally stable. Furthermore, if $\Lambda = \Lambda_R \times \mathbb{R}^{2n}$, then $\hat{\Lambda}$ can be obtained in the form $\hat{\Lambda} = \hat{\Lambda}_R \times \mathbb{R}^{2n}$, where $\hat{\Lambda}_R$ is a C^1 perturbation of Λ_R in $\mathbb{R}^{2n_R}$.

Proof. It follows from Proposition 3 that there is a C^1 perturbation $\hat{\Lambda}$ of Λ such that $\hat{\Lambda} \cap K \neq \emptyset$ and $\hat{\Lambda} \pitchfork K$. If $\Lambda = \Lambda_R \times \mathbb{R}^{2n}$, we have $\hat{\Lambda} = \hat{\Lambda}_R \times \mathbb{R}^{2n}$. Therefore, we may assume that $\Lambda \pitchfork K$ is satisfied already. By Condition P and Corollary 1, we know that $\underset{\sim}{\chi}_P$ defined by (25) is a surjection; $\underset{\sim}{\chi}_P(K) = \mathbb{R}^{2n}$. Since $\dim \Sigma = n$, it follows from Lemma F of Appendix II that there is an arbitrarily small C^1 perturbation $\underset{\sim}{F}_P : \Sigma \to \mathbb{R}^{2n}$ of $\underset{\sim}{\chi}_P|\Sigma = \underset{\sim}{\pi}_P$ such that $\underset{\sim}{F}_P$ is a nice immersion. It follows from $\underset{\sim}{\chi}_P(K) = \mathbb{R}^{2n}$ that there is an isomorphism $\underset{\sim}{A} : K \to \mathbb{R}^{2n} \oplus \mathrm{Ker}\ \underset{\sim}{\chi}_P$ such that $\underset{\sim}{A}(\underset{\sim}{x}) = (\underset{\sim}{\chi}_P(\underset{\sim}{x}), \underset{\sim}{\pi}_2 \circ \underset{\sim}{A}(\underset{\sim}{x}))$, where $\underset{\sim}{\pi}_2$ is the projection of $\mathbb{R}^{2n} \oplus \text{Ker } \underset{\sim}{\chi}_P$ onto the second factor. We define a C^1 map $\underset{\sim}{F}_0 : \Sigma \to K$ by

$$\underset{\sim}{F}_0(\underset{\sim}{x}) \triangleq \underset{\sim}{A}^{-1}(\underset{\sim}{F}_P(\underset{\sim}{x}), \underset{\sim}{\pi}_2 \circ \underset{\sim}{A}(\underset{\sim}{x})) \ . \tag{49}$$

Then $\underset{\sim}{F}_0$ is a C^1 perturbation of the inclusion map : $\Sigma \to K$. Therefore $\underset{\sim}{F}_0$ is an embedding. It follows from Lemma G of Appendix II that there is a C^1 extension $\underset{\sim}{F} : \Lambda \to \mathbb{R}^{2b}$ of $\underset{\sim}{F}_0$ which is close to the inclusion map $\underset{\sim}{\iota}_\Lambda : \Lambda \to \mathbb{R}^{2b}$. Observe that the extension $\underset{\sim}{F}$ obtained by Lemma G is so defined as $\underset{\sim}{F}(\underset{\sim}{x}) = \underset{\sim}{x} \in K$ for all $\underset{\sim}{x} \in K$. This implies that $\underset{\sim}{F}(\Lambda) \cap K = \underset{\sim}{F}_0(\Sigma)$. (See (A.43) of Appendix II in the proof of Lemma H.) Since $\Lambda \cap K \neq \emptyset$ and since $\Lambda \pitchfork K$, we have $\underset{\sim}{F}(\Lambda) \cap K \neq \emptyset$ and $\underset{\sim}{F}(\Lambda) \pitchfork K$, provided that $\underset{\sim}{F}$ is C^1 close enough to $\underset{\sim}{\iota}_\Lambda$. Since $\underset{\sim}{F}_P$ in (49) is a nice immersion and since $\underset{\sim}{A}$ is an isomorphism, we see that $\underset{\sim}{\chi}_P | \underset{\sim}{F}_0(\Sigma) = \underset{\sim}{\chi}_P | \underset{\sim}{F}(\Lambda) \cap K$ is a nice immersion. Therefore $\hat{\Lambda} \triangleq \underset{\sim}{F}(\Lambda)$ is the perturbation sought. Finally, consider the case $\Lambda = \Lambda_R \times \mathbb{R}^{2n}$. It follows from Lemma H of Appendix II that there is an $\underset{\sim}{F}_R : \Lambda_R \to \mathbb{R}^{2n_R}$ which is arbitrarily C^1 close to the inclusion : $\Lambda_R \to \mathbb{R}^{2n_R}$ such that for a given $\underset{\sim}{F}$, we have $\underset{\sim}{F}(\Sigma) = \hat{\Lambda} \cap K$, wehre $\hat{\Lambda} = \underset{\sim}{F}_R(\Lambda_R) \times \mathbb{R}^{2n}$. Since $\hat{\Lambda}$ is a small C^1 perturbation of Λ, we have $\hat{\Lambda} \cap K \neq \emptyset$, $\hat{\Lambda} \pitchfork K$. ¤

Remark. Observe that Whitney Immersion Theorem (Lemma F of Appendix II) is crucial in the proof of Theorem 1 as well as Theorem 2. The former, in turn, crucially depends on the fact that $\underset{\sim}{\chi}_P(K) = \mathbb{R}^{2n}$, i.e., Condition P. If this condition is not satisfied, the set of immersions is not dense in $C^1(\Lambda;\mathbb{R}^{2b})$. Example 7 is a case in point. Since there is a port-only loop, Proposition 6 tells us that dim Im $\underset{\sim}{\chi}_P = 1 < 2$, i.e., Condition P is violated. There is no way of making $\underset{\sim}{\pi}_P$ an immersion by perturbations. In order to further clarify the significance of Whitney Immersion Theorem, suppose that K is isomorphic to $\mathbb{R}^3$ and that $\underset{\sim}{\chi}_P$ is the projection map of $\mathbb{R}^3$ onto $\mathbb{R}^2$. Therefore dim Im $\underset{\sim}{\chi}_P = 2 < 4$. Suppose also that Σ is given as in Fig. 16. Then, it is clear that at $\underset{\sim}{x} \in \Sigma$, $(d\underset{\sim}{\pi}_P)_x = \underset{\sim}{0}$, where $\underset{\sim}{\pi}_P = \underset{\sim}{\chi}_P|\Sigma$. There is no way of making $\underset{\sim}{\pi}_P$ an immersion by perturbations.

We will next give a method of checking condition (ii) of Definition 3 concerning nice immersions. A method of checking condition (i) is given in [1]. Since Λ is a C^1 submanifold of dimension $2b-n_R$, for each point $(\underset{\sim}{v}_0, \underset{\sim}{i}_0) \in \Lambda$, there is a neighborhood $U \subset \mathbb{R}^{2b}$ of this point and there is a C^1 function $\underset{\sim}{f} : U \to \mathbb{R}^{n_R}$ such that

$$\Lambda \cap U = \mathbf{f}^{-1}(\mathbf{0}) \tag{50}$$

and

$$\operatorname{rank}(\mathbf{Df})_{(\mathbf{v},\mathbf{i})} = n_R \text{ for all } (\mathbf{v},\mathbf{i}) \in \Lambda \cap U\ . \tag{51}$$

Let T be a tree for N and let L be its associated cotree. Decompose $\mathbf{v}$ and $\mathbf{i}$ as $\mathbf{v} = (\mathbf{v}_L, \mathbf{v}_T)$ and $\mathbf{i} = (\mathbf{i}_L, \mathbf{i}_T)$. Let $\mathbf{B} = [\mathbf{1} \vdots \mathbf{B}_T]$ be the fundamental loop matrix and set

$$\mathbf{F}(\mathbf{v},\mathbf{i}) \triangleq [\mathbf{D}_{\mathbf{v}_T}\mathbf{f} - (\mathbf{D}_{\mathbf{v}_L}\mathbf{f})\mathbf{B}_T \vdots \mathbf{D}_{\mathbf{i}_L}\mathbf{f} + (\mathbf{D}_{\mathbf{i}_T}\mathbf{f})\mathbf{B}_T^T]_{(\mathbf{v},\mathbf{i})}\ . \tag{52}$$

This matrix plays an important role in checking transversality of Λ and K [1]. It turns out that this matrix is important for checking condition (ii) of nice immersion also.

<u>Proposition 7</u>. Let $\Lambda \cap K \neq \emptyset$, $\Lambda \pitchfork K$ and suppose that conditions (i) and (iii) of <u>Definition 3</u> are satisfied. Then $\boldsymbol{\pi}_P$ is a nice immersion if and only if $\boldsymbol{\pi}_P(\mathbf{v},\mathbf{i}) = \boldsymbol{\pi}_P(\tilde{\mathbf{v}},\tilde{\mathbf{i}})$ implies

$$\operatorname{rank} \begin{bmatrix} \mathbf{F}(\mathbf{v},\mathbf{i}) \\ \tilde{\mathbf{F}}(\tilde{\mathbf{v}},\tilde{\mathbf{i}}) \end{bmatrix} = b \tag{53}$$

where $\mathbf{F}$ is defined by (52) and $\tilde{\mathbf{F}}$ is defined similarly.

<u>Proof</u>. Let $(\boldsymbol{\psi}, \Sigma \cap U)$ be a local chart for Σ at $(\mathbf{v},\mathbf{i})$ and let

$$\mathbf{g}(\mathbf{v},\mathbf{i}) \triangleq \begin{bmatrix} \mathbf{Bv} \\ \mathbf{Qi} \\ \mathbf{f}(\mathbf{v},\mathbf{i}) \end{bmatrix} \tag{54}$$

where $\mathbf{f}$ is as in (50) and (51). Then $\Sigma \cap U = \mathbf{g}^{-1}(\mathbf{0})$. Similarly for $(\tilde{\mathbf{v}},\tilde{\mathbf{i}})$, we have $\Sigma \cap \tilde{U} = \tilde{\mathbf{g}}^{-1}(\mathbf{0})$, where $\tilde{U}$ and $\tilde{\mathbf{g}}$ are defined similarly. We first claim that

$$\operatorname{Im}(\mathbf{d}\boldsymbol{\pi}_P)_{(\mathbf{v},\mathbf{i})} + \operatorname{Im}(\mathbf{d}\boldsymbol{\pi}_P)_{(\tilde{\mathbf{v}},\tilde{\mathbf{i}})} = \mathbb{R}^{2n} \tag{55}$$

if and only if

$$\operatorname{rank} \begin{bmatrix} (\mathbf{Dg})_{(\mathbf{v},\mathbf{i})} \\ (\mathbf{D\tilde{g}})_{(\tilde{\mathbf{v}},\tilde{\mathbf{i}})} \end{bmatrix} = 2b\ . \tag{56}$$

To this end observe that

$$\mathrm{Im}(d\underset{\sim}{\pi}_P)_{(\underset{\sim}{v},\underset{\sim}{i})} + \mathrm{Im}(d\underset{\sim}{\pi}_P)_{(\tilde{\underset{\sim}{v}},\tilde{\underset{\sim}{i}})} = \underset{\sim}{\pi}'_P(\mathrm{Ker}(\underset{\sim}{Dg})_{(\underset{\sim}{v},\underset{\sim}{i})}+\mathrm{Ker}(\underset{\sim}{D\tilde{g}})_{(\tilde{\underset{\sim}{v}},\tilde{\underset{\sim}{i}})}) \quad (57)$$

where $\underset{\sim}{\pi}'_P$ is defined by (19). It follows from the proof of Proposition 3 in [1] that

$$\mathrm{rank}(d\underset{\sim}{\pi}_P)_{(\underset{\sim}{v},\underset{\sim}{i})} = \mathrm{rank}\begin{bmatrix} \underset{\sim}{\pi}'_P \\ (\underset{\sim}{Dg})_{(\underset{\sim}{v},\underset{\sim}{i})} \end{bmatrix} - \mathrm{rank}(\underset{\sim}{Dg})_{(\underset{\sim}{v},\underset{\sim}{i})} \ .$$

On the one hand, the fact that $\underset{\sim}{\pi}_P$ is an immersion implies $\mathrm{rank}(d\underset{\sim}{\pi}_P)_{(\underset{\sim}{v},\underset{\sim}{i})} = n$ and $\Lambda \,\bar{\pitchfork}\, K$ implies $\mathrm{rank}(\underset{\sim}{Dg})_{(\underset{\sim}{v},\underset{\sim}{i})} = b + n$. Therefore

$$\mathrm{rank}\begin{bmatrix} \underset{\sim}{\pi}'_P \\ (\underset{\sim}{Dg})_{(\underset{\sim}{v},\underset{\sim}{i})} \end{bmatrix} = 2b \ .$$

Similarly

$$\mathrm{rank}\begin{bmatrix} \underset{\sim}{\pi}'_P \\ (\underset{\sim}{D\tilde{g}})_{(\tilde{\underset{\sim}{v}},\tilde{\underset{\sim}{i}})} \end{bmatrix} = 2b \ .$$

Therefore

$$\begin{aligned} &\dim(\mathrm{Ker}\ \underset{\sim}{\pi}'_P \cap \mathrm{Ker}(\underset{\sim}{Dg})_{(\underset{\sim}{v},\underset{\sim}{i})}) \\ &\quad = \dim(\mathrm{Ker}\ \underset{\sim}{\pi}'_P \cap \mathrm{Ker}(\underset{\sim}{D\tilde{g}})_{(\tilde{\underset{\sim}{v}},\tilde{\underset{\sim}{i}})}) = 0 \ . \end{aligned} \quad (58)$$

Now suppose that (56) holds. Then

$$\dim(\mathrm{Ker}(\underset{\sim}{Dg})_{(\underset{\sim}{v},\underset{\sim}{i})} \cap \mathrm{Ker}(\underset{\sim}{D\tilde{g}})_{(\tilde{\underset{\sim}{v}},\tilde{\underset{\sim}{i}})}) = 0 \quad (59)$$

and hence

$$\begin{aligned} &\dim(\mathrm{Ker}(\underset{\sim}{Dg})_{(\underset{\sim}{v},\underset{\sim}{i})}+\mathrm{Ker}(\underset{\sim}{D\tilde{g}})_{(\tilde{\underset{\sim}{v}},\tilde{\underset{\sim}{i}})}) \\ &\quad = \dim(\mathrm{Ker}(\underset{\sim}{Dg})_{(\underset{\sim}{v},\underset{\sim}{i})} \oplus \mathrm{Ker}(\underset{\sim}{D\tilde{g}})_{(\tilde{\underset{\sim}{v}},\tilde{\underset{\sim}{i}})}) = 2n \ . \end{aligned} \quad (60)$$

Equations (57), (58) and (60) imply (55). Conversely, suppose that (55) holds. Since $\mathrm{rank}\ \underset{\sim}{\pi}'_P = 2n$, (57) implies that

$$\dim(\mathrm{Ker}(\underset{\sim}{Dg})_{(\underset{\sim}{v},\underset{\sim}{i})}+\mathrm{Ker}(\underset{\sim}{D\tilde{g}})_{(\tilde{\underset{\sim}{v}},\tilde{\underset{\sim}{i}})}) \geq 2n. \quad (61)$$

Since $\underset{\sim}{\pi}_P$ is an immersion, (58) implies that equality in (61) must hold. This implies (59) which, in turn, implies (56). Finally we will show that (56) is equivalent to (53). Since

$$(\mathbf{Dg})_{(\mathbf{v},\mathbf{i})} = \left[\begin{array}{cc|cc} \mathbf{1} & \mathbf{B}_T & \cdot & \cdot \\ \cdot & \cdot & -\mathbf{B}_T^T & \mathbf{1} \\ \mathbf{D}_{\mathbf{v}_L}\mathbf{f} & \mathbf{D}_{\mathbf{v}_T}\mathbf{f} & \mathbf{D}_{\mathbf{i}_L}\mathbf{f} & \mathbf{D}_{\mathbf{i}_T}\mathbf{f} \end{array}\right]_{(\mathbf{v},\mathbf{i})}$$

it is clear that (56) holds if and only if

$$\operatorname{rank} \left[\begin{array}{cc|cc} \mathbf{1} & \mathbf{B}_T & \cdot & \cdot \\ \cdot & \cdot & -\mathbf{B}_T^T & \mathbf{1} \\ (\mathbf{D}_{\mathbf{v}_L}\mathbf{f})(\mathbf{v},\mathbf{i}) & (\mathbf{D}_{\mathbf{v}_T}\mathbf{f})(\mathbf{v},\mathbf{i}) & (\mathbf{D}_{\mathbf{i}_L}\mathbf{f})(\mathbf{v},\mathbf{i}) & (\mathbf{D}_{\mathbf{i}_T}\mathbf{f})(\mathbf{v},\mathbf{i}) \\ (\mathbf{D}_{\mathbf{v}_L}\tilde{\mathbf{f}})(\tilde{\mathbf{v}},\tilde{\mathbf{i}}) & (\mathbf{D}_{\mathbf{v}_T}\tilde{\mathbf{f}})(\tilde{\mathbf{v}},\tilde{\mathbf{i}}) & (\mathbf{D}_{\mathbf{i}_L}\tilde{\mathbf{f}})(\tilde{\mathbf{v}},\tilde{\mathbf{i}}) & (\mathbf{D}_{\mathbf{i}_T}\tilde{\mathbf{f}})(\tilde{\mathbf{v}},\tilde{\mathbf{i}}) \end{array}\right] = 2b \, . \tag{62}$$

By elementary operations, one can show that (62) holds if and only if (53) holds. ¤

<u>Example 8</u>. Consider the 1-port of Fig. 1(a) where the internal resistor constitutive relations are given by Fig. 3(a). Choose $T \triangleq \{R_1, R_2\}$ to be our tree. Then $\mathbf{v}_T = (v_{R_1}, v_{R_2})$, $v_L = v_p$, $\mathbf{i}_T = (i_{R_1}, i_{R_2})$, $i_L = i_p$, $\mathbf{B}_T = [1 \;\; 1]$,

$$\mathbf{D}_{\mathbf{v}_T}\mathbf{f} = \begin{bmatrix} -Df_{R_1} & \cdot \\ \cdot & -Df_{R_1} \end{bmatrix}, \quad D_{v_L}\mathbf{f} = \begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$$

$$\mathbf{D}_{\mathbf{i}_T}\mathbf{f} = \begin{bmatrix} 1 & \cdot \\ \cdot & 1 \end{bmatrix}, \quad D_{i_L}\mathbf{f} = \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} .$$

Therefore the matrix of (53) is given by

$$\begin{bmatrix} (-Df_{R_1})_{v_{R_1}} & \cdot & 1 \\ \cdot & (-Df_{R_2})_{v_{R_2}} & 1 \\ (-Df_{R_1})_{\tilde{v}_{R_1}} & \cdot & 1 \\ \cdot & (-Df_{R_2})_{\tilde{v}_{R_2}} & 1 \end{bmatrix} . \tag{63}$$

At self intersection points of R, none of the derivatives $(Df_{R_k})_{v_{R_k}}$, $(Df_{R_k})_{\tilde{v}_{R_k}}$, k = 1, 2, vanishes. By checking Fig. 3(a) and Fig. 3(b) carefully, one sees that $(Df_{R_k})_{v_{R_k}} \neq (Df_{R_k})_{\tilde{v}_{R_k}}$, k = 1, 2, at self intersection points. Hence the matrix of (63) has rank 3. Since conditions (i) and (iii) of Definition 3 are satisfied, Proposition 7 tells us that $\underset{\sim}{\pi}_P$ is a nice immersion.

IV. Strong Structural Stabilization

Suppose that a given n-port N is not strongly structurally stable. Theorem 2 says that one can make N strongly structurally stable by a small C^1 perturbation of Λ, provided that Condition P holds. Such a perturbation is called element perturbation. Here we will give another strong structural stabilization procedure which is called network perturbation. It amounts to creating extra ports by "pliers-type entry" or "soldering-iron entry". Note that element perturbation gives rise to a new $\hat{\Lambda}$ but it keeps K unchanged, while network perturbation gives rise to a new ambient space $\mathbb{R}^{2(b+\hat{n})}$ where $\hat{n}$ is the number of ports created. Recall that norator imposes no constraints on the existing internal resistor constitutive relations. Therefore, network perturbation is equivalent to inserting norators by pliers-type entry or soldering-iron entry. Recall that N is the network obtained from N by terminating ports with norators. (See Section II).

Theorem 3 (Strong Structural Stabilization via Network Perturbation)

Given an n-port N let $\Lambda = \Lambda_R \times \mathbb{R}^{2n}$ and let $\Lambda \cap K \neq \emptyset$. Suppose that N is not strongly structurally stable. Let T be an arbitrary tree for N and let L be its associated cotree. Decompose T and L as $T = R_T \cup P_T$ and $L = R_L \cup P_L$, respectively, where R and P denote resistors and ports, respectively. Insert an extra port in parallel with each branch of R_T and insert an extra port in series with each branch of R_L. Then the resulting $(n+n_R)$-port $\hat{N}$ satisfies the following conditions: (i) $\hat{\Lambda} \cap \hat{K} \neq \emptyset$, (ii) $\hat{N}$ is strongly structurally stable.

For proof we will need two lemmas.

Lemma 2. Let T_1 and T_2 be arbitrary trees for N and let L_1 and L_2 be associated cotrees. Let $\underset{\sim}{B}_{T_1}$ (resp. $\underset{\sim}{B}_{T_2}$) be the main part of the fundamental loop matrix with respect to T_1(resp. T_2). Decompose v as $v = (v_{L_1}, v_{T_1})$ and decompose i as $i = (i_{L_2}, i_{T_2})$. Then $\Lambda \pitchfork K$ if and only if

$$\operatorname{rank} F^{12}(v,i) = n_R \text{ for all } (v,i) \in \Sigma \tag{64}$$

where

$$F^{12}(v,i) \triangleq [D_{v_{T_1}} f - (D_{v_{L_1}} f) B_{T_1} \,\vdots\, D_{i_{L_2}} f + (D_{i_{T_2}} f) B_{T_2}^T] \tag{65}$$

and f is as in (50) and (51).

Proof. Since the Kirchhoff space K is described by $B_1 v = 0$ and $Q_2 i = 0$, it follows from an argument in [1] that $\Lambda \pitchfork K$ if and only if for each $(v,i) \in \Sigma$

$$\operatorname{rank} \begin{bmatrix} B_1 & \cdot \\ \cdot & Q_2 \\ D_v f & D_i f \end{bmatrix}_{(v,i)} = b + n_R$$

where B_1 (resp. Q_2) is the fundamental loop (resp. cut set) matrix with respect to T_1 (resp. T_2). More explicitly, this matrix is given by

$$\left[\begin{array}{cc|cc} 1 & B_{T_1} & \cdot & \cdot \\ \cdot & \cdot & -B_{T_2}^T & 1 \\ D_{v_{L_1}} f & D_{v_{T_1}} f & D_{i_{L_2}} f & D_{i_{T_2}} f \end{array}\right]_{(v,i)}$$

By elementary operations, one can show that this matrix has rank $b + n_R$ if and only if (64) holds. ¤

Remark. Observe that we took full advantage of the fact that transversality is a coordinate-free property when we used two different trees simultaneously in (64) as well as (65). This enables us to prove Theorem 3.

Now let $\Lambda = \Lambda_R \times \mathbb{R}^{2n}$. Then f of (50) and (51) is independent of of (v_P, i_P). Let $U_R \triangleq U \cap \mathbb{R}^{2n_R}$ and define $f_R : U_R \to \mathbb{R}^{n_R}$ by

$$f_R(v_R, i_R) \triangleq f(v, i). \tag{66}$$

For the tree T_k, $k = 1,2$, decompose B_{T_k} as

$$\begin{array}{cc} & \begin{array}{cc} v_{R_{T_k}} & v_{P_{T_k}} \end{array} \\ \begin{array}{c} v_{R_{L_k}} \\ v_{P_{L_k}} \end{array} & \begin{bmatrix} B^k_{RR} & B^k_{RP} \\ B^K_{PR} & B^K_{PP} \end{bmatrix} \end{array} . \tag{67}$$

Recall π'_R of (7) and define $\pi_R : \Sigma \to \mathbb{R}^{2n_R}$ by

$$\pi_R \triangleq \pi'_R \circ \iota \tag{68}$$

where ι is defined by (21). Substituting (66) and (67) into (65), one obtains the following:

Corollary 4. Let $\Lambda = \Lambda_R \times \mathbb{R}^{2n}$, let T_1 and T_2 be arbitrary trees and let L_1 and L_2 be associated cotrees. Then $\Lambda \bar{\pitchfork} K$ if and only if

$$\text{rank } F^{12}_R(v_R, i_R) = n_R \text{ for all } (v_R, i_R) \in \pi_R(\Sigma)$$

where

$$F^{12}_R(v_R, i_R) \triangleq \left[\begin{array}{c|c} D_{v_{R_{T_1}}} f_R - (D_{v_{R_{L_1}}} f_R) B^1_{RR} & -(D_{v_{R_{L_1}}} f_R) B^1_{RP} \\ D_{i_{R_{L_2}}} f_R + (D_{i_{R_{T_2}}} f_R) B^{2T}_{RR} & (D_{i_{R_{T_2}}} f_R) B^{2T}_{PR} \end{array} \right]_{(v_R, i_R)} . \tag{69}$$

Proof of Theorem 3. The symbol ^ will denote a function or a set associated with perturbed $\hat{N}$. One can show that $\hat{\Lambda} \cap \hat{K} \neq \emptyset$ as in the proof of Theorem 4 of [1]. Let P_1 (resp. P_2) be the branches of the extra ports inserted in parallel (resp. series) with R_T (resp. R_L). Then $\hat{T}_1 \triangleq R_T \cup P_T \cup R_L$ is a tree and $\hat{L}_1 \triangleq P_L \cup P_1 \cup P_2$ is its associated co-tree. Therefore, $\hat{T}_1$ contains all the resistors and $v_{R_{\hat{T}_1}} = (v_{R_L}, v_{R_T}) = \hat{v}_R = v_R$, $\hat{B}^1_{RR} = \hat{B}^1_{RP} = \emptyset$, where $\hat{B}^1_{RR}$ and $\hat{B}^1_{RP}$ are as in (67) for $\hat{N}$ and $\emptyset$ denotes a 0 x 0 matrix. Set $\hat{T}_2 \triangleq P_T \cup P_1 \cup P_2$. Then $\hat{T}_2$ is another tree and $\hat{L}_2 \triangleq R_T \cup R_L \cup P_L$ is its associated cotree. Hence $\hat{L}_2$ contains all the resistors and $\underset{\sim}{i}_{R_{\hat{L}_2}} = (\underset{\sim}{i}_{R_L}, \underset{\sim}{i}_{R_T}) = \hat{\underset{\sim}{i}}_R = \underset{\sim}{i}_R$, $\hat{\underset{\sim}{B}}^2_{RR} = \emptyset$. Substituting these data into (69), we have

$$\hat{\underset{\sim}{F}}^{12}_R(\hat{\underset{\sim}{v}}_R, \hat{\underset{\sim}{i}}_R) = \left[D_{\underset{\sim}{v}_R} \underset{\sim}{f}_R \;\vdots\; D_{\underset{\sim}{i}_R} \underset{\sim}{f}_R \right]_{(\underset{\sim}{v}_R, \underset{\sim}{i}_R)} .$$

It follows from (51) that rank $\hat{\underset{\sim}{F}}^{12}_R (\hat{\underset{\sim}{v}}_R, \hat{\underset{\sim}{i}}_R) = n_R$ for all $(\hat{\underset{\sim}{v}}_R, \hat{\underset{\sim}{i}}_R) = (\underset{\sim}{v}_R, \underset{\sim}{i}_R) \in \Lambda_R \supset \hat{\underset{\sim}{\pi}}_R(\hat{\Sigma})$. By <u>Corollary 4</u>, we have $\hat{\Lambda} \pitchfork \hat{K}$. Now, since $\hat{T}_2$ consists only of ports and since $\hat{L}_2$ contains all the resistors, it follows from an argument similar to <u>Proposition 5</u> that there are no resistor-only loops and no resistor-only cut sets in $\hat{N}$. It follows from <u>Corollary 2</u> that $\hat{\underset{\sim}{\chi}}_P$ is injective and hence $\hat{\underset{\sim}{\pi}}_P \triangleq \hat{\underset{\sim}{\chi}}_P|\hat{\Sigma}$ is an embedding. Afortiori $\hat{\underset{\sim}{\pi}}_P$ is a nice immersion. It follows from <u>Theorem 1</u> that $\hat{N}$ is strongly structurally stable. ¤

We can reduce the number of extra ports by choosing appropriate trees. Let T_1 be a tree containing maximum number of ports and let L_1 be its associated cotree. Let T_2 be a tree containing maximum number of resistors and let L_2 be its associated cotree such that $R_{T_1} \subset T_2$ and $P_{L_1} \subset L_2$, where R_{T_1} (resp. P_{L_1}) denotes resistors (resp. ports) in T_1 (resp. L_1). It is not difficult to show that such a pair of trees exists. We will use the following notation in decomposing the branches of N:

	resistors	ports
$T_1 \cap T_2$	$R_{T_{12}} (= R_{T_1})$	$P_{T_{12}}$
$L_1 \cap L_2$	$R_{L_{12}}$	$P_{L_{12}} (= P_{L_1})$
$T_1 \cap L_2$	ϕ	P_{12}
$L_1 \cap T_2$	R_{12}	ϕ

<u>Proposition 8</u>. Given an n-port N let $\Lambda = \Lambda_R \times \mathbb{R}^{2n}$ and let $\Lambda \cap K \neq \emptyset$. Suppose that N is not strongly structurally stable. Let T_1 (resp. T_2) be a tree containing maximum number of ports (resp. resistors) and decompose the branches of N as above. Insert an extra port in series

with each branch of $R_{T_{12}}$ and insert an extra port in parallel with each branch of $R_{L_{12}}$. Then the perturbed $\hat{N}$ is an $(n+n_{R_{T_{12}}}+n_{R_{L_{12}}})$-port satisfying the following conditions: (i) $\hat{\Lambda} \cap \hat{K} \neq \emptyset$, (ii) $\hat{N}$ is strongly structurally stable, where $n_{R_{T_{12}}}$ (resp. $n_{R_{L_{12}}}$) is the number of branches in $R_{T_{12}}$ (resp. $R_{L_{12}}$).

Proof. The fact $\hat{\Lambda} \cap \hat{R} \neq \emptyset$ can be proved in the same manner as the proof of Theorem 4 in [1]. Let $\hat{T}_1 \triangleq R_{T_{12}} \cup R_{L_{12}} \cup R_{12} \cup P_{T_{12}}$. Then $\hat{T}_1$ is a tree and $\hat{L}_1 \triangleq P_{L_{12}} \cup P_{12} \cup P_1 \cup P_2$ is its associated cotree. Therefore, $\hat{T}_1$ contains all the resistors and $\underset{\sim}{v}_{R\hat{T}_1} = \hat{\underset{\sim}{v}}_R = \underset{\sim}{v}_R$, $\hat{\underset{\sim}{B}}^1_{RR} = \emptyset$. Let $\hat{T}_2 \triangleq P_{T_{12}} \cup P_{12} \cup P_1 \cup P_2$. Then $\hat{T}_2$ is a tree and $\hat{L}_2 \triangleq P_{L_{12}} \cup R_{T_{12}} \cup R_{L_{12}} \cup R_{12}$ is its associated cotree. Therefore $\hat{L}_2$ contains all the resistors and $\underset{\sim}{i}_{R\hat{L}_2} = \hat{\underset{\sim}{i}}_R = \underset{\sim}{i}_R$, $\hat{\underset{\sim}{B}}^2_{RR} = \emptyset$. It follows from the same argument as that of Theorem 3 that rank $\hat{\underset{\sim}{F}}^{12}_R(\hat{\underset{\sim}{v}}_R, \hat{\underset{\sim}{i}}_R) = n_R$ and that $\hat{\underset{\sim}{\pi}}_P$ is a nice immersion. Therefore $\hat{N}$ is strongly structurally stable. ¤

Remark. The number $n_{R_{T_{12}}}$ (resp. $n_{R_{L_{12}}}$) is the number of independent resistor-only cut sets (resp. loops). Therefore, the network perturbation used in Proposition 8 as well as in Theorem 3 eliminates resistor-only loops and resistor-only cut sets. It, then, transversalizes $\hat{\Lambda}$ and $\hat{K}$, and makes $\hat{\underset{\sim}{\chi}}_P$ an injection. The fact that $\hat{\underset{\sim}{\chi}}_P$ is injective forces $\hat{R}$ to have no self intersection points. In fact, $\hat{R} = \hat{\underset{\sim}{\chi}}_P(\hat{\Sigma}) = \hat{\underset{\sim}{\pi}}_P(\hat{\Sigma})$ is an $n_R(=\hat{n}_R)$-dimensional submanifold, because $\hat{\underset{\sim}{\pi}}_P = \hat{\underset{\sim}{\chi}}_P|\hat{\Sigma} : \hat{\Sigma} \to \hat{R}$ is now an embedding. This means that if resistor-only cutsets and resistor-only loops are eliminated, then nice properties of $\Lambda_R(\hat{\Lambda}_R)$ are inherited to $\hat{R}$. Observe that nice properties associated with Λ_R, a submanifold, may be destroyed by resistor - only cut sets and resistor-only loops if it is mapped into the port space by $\underset{\sim}{\chi}_P$. The network perturbation, therefore, is a kind of "blowing up" procedure for eliminating self intersection points. Notice the distinction between this condition and Condition P; the former excludes resistor-only cut sets and resistor-only loops while the latter excludes port-only cut sets and port-only loops.

Example 9. Consider the 1-port N of Fig. 1(a) where internal resistor constitutive relations are given by Fig. 2(a). As was explained in Example 2 this 1-port is not strongly structurally stable. Let $T_1 \triangleq \{P,R_1\}$ and $T_2 \triangleq \{R_1,R_2\}$ be the trees discussed in Proposition 8. Then the procedure of Proposition 8 tells us to create an extra port in parallel with R_1 as in Fig. 17(a) and $\hat{N}$ is strongly structurally stable. Roughly speaking, this procedure provides more free space for $\hat{R}$ so that $\hat{R}$ would look like the configuration space Σ of Fig. 4, where there are no self intersection points. Observe that if one follows the procedure of Theorem 3 then one has to add one more port as in Fig. 17(b).

V. Strong Structural R-stability

Recall that in Theorem 1 and Theorem 2, Condition P was crucial. In this section, we will relax Condition P but restrict ourselves to those n-ports whose internal resistor constitutive relations are of the form $\Lambda = \Lambda_R \times \mathbb{R}^{2n}$. In this section, therefore, a perturbation $\hat{\Lambda}$ of Λ is of the form $\hat{\Lambda} = \hat{\Lambda}_R \times \mathbb{R}^{2n}$, where $\hat{\Lambda}_R$ is a perturbation of Λ_R in $\mathbb{R}^{2n_R}$. We will give an "if and only if" condition for a special case of strong structural stability without Condition P.

In order to simplify notation, we will identify two linear subspaces if they have the same dimension, i.e., if they are isomorphic. We will write $\doteq$ to denote that two objects are isomorphic.

Recall $\underset{\sim}{\chi}_R$ of (8) and recall that dim Ker $\underset{\sim}{\chi}_R = \mu + \rho$, where ρ (resp. μ) is the number of independent port-only loops (resp. port-only cut sets) (see Proposition 6). Also recall $\underset{\sim}{\chi}_P$ of (25). In order to state the results of this section, we will need several lemmas and definitions.

Lemma 3. Ker $\underset{\sim}{\chi}_R \cap$ Ker $\underset{\sim}{\chi}_P \doteq \{\underset{\sim}{0}\}$.

Proof. Recall the coordinate system $\underset{\sim}{\psi}_1$ used in the proof of Proposition 6. Let $(\underset{\sim}{v},\underset{\sim}{i}) \in$ Ker $\underset{\sim}{\chi}_R$. Then, in terms of $\underset{\sim}{\psi}_1$, we have (see (29) and (30))

$$\underset{\sim}{v} = \begin{bmatrix} \underset{\sim}{v}_{R_{L_1}} \\ \underset{\sim}{v}_{P_{L_1}} \\ \underset{\sim}{v}_{R_{T_1}} \\ \underset{\sim}{v}_{P_{T_1}} \end{bmatrix} = \begin{bmatrix} \cdot \\ -\underset{\sim}{B}^1_{PP} \\ \cdot \\ \underset{\sim}{1} \end{bmatrix} \underset{\sim}{v}_{P_{T_1}} , \quad \underset{\sim}{i} = \begin{bmatrix} \underset{\sim}{i}_{R_{L_2}} \\ \underset{\sim}{i}_{P_{L_2}} \\ \underset{\sim}{i}_{R_{T_2}} \\ \underset{\sim}{i}_{P_{T_2}} \end{bmatrix} = \begin{bmatrix} \cdot \\ \underset{\sim}{1} \\ \cdot \\ \underset{\sim}{B}^{2T}_{PP} \end{bmatrix} \underset{\sim}{i}_{P_{L_2}} .$$

It follows from this that $(\underset{\sim}{v},\underset{\sim}{i}) \in \operatorname{Ker} \underset{\sim}{\chi}_P$ only if $(\underset{\sim}{v},\underset{\sim}{i}) = \underset{\sim}{0}$. ¤

Now, Lemma 3 implies that Ker $\underset{\sim}{\chi}_R$ is contained in a complement of Ker $\underset{\sim}{\chi}_P$. Since $\underset{\sim}{\chi}_P$ maps any complement of Ker $\underset{\sim}{\chi}_P$ onto its image, it follows from (i) of Proposition 6 that there is a $2n-(\rho+\mu)-(\rho+\mu)$ $= 2(n-\rho-\mu)$ - dimensional linear subspace H of $\mathbb{R}^{2n}$ such that

$$\operatorname{Im} \underset{\sim}{\chi}_P = H \oplus \underset{\sim}{\chi}_P(L) \cong H \oplus L \tag{72}$$

where $L = \operatorname{Ker} \underset{\sim}{\chi}_R$. Define

$$K_0 \triangleq \underset{\sim}{\chi}_P^{-1}(H) \ . \tag{73}$$

Lemma 4. $K_0 \oplus L = K$. (74)

Proof. We first claim that

$$K_0 \cap L = \{\underset{\sim}{0}\} \ . \tag{75}$$

To prove this let $\underset{\sim}{x} \in K_0 \cap L$. Then $\underset{\sim}{\chi}_P(\underset{\sim}{x}) \in \underset{\sim}{\chi}_P(K_0) \cap \underset{\sim}{\chi}_P(L) = H \cap \underset{\sim}{\chi}_P(L)$ $\cong H \cap L$. This and (72) imply $\underset{\sim}{\chi}_P(\underset{\sim}{x}) = \underset{\sim}{0}$. Hence $\underset{\sim}{x} \in \operatorname{Ker} \underset{\sim}{\chi}_P \cap \operatorname{Ker} \underset{\sim}{\chi}_R$. This and Lemma 3 imply (75).

By (i) of Proposition 6, we have

$$\begin{aligned}
\dim K_0 = \dim \underset{\sim}{\chi}_P^{-1}(H) &= \dim H + \dim \operatorname{Ker} \underset{\sim}{\chi}_P \\
&= \dim H + \dim K - \dim \operatorname{Im} \underset{\sim}{\chi}_P \\
&= 2(n-\rho-\mu) + (n+n_R) - (2n-\rho-\mu) \\
&= n + n_R - \rho - \mu \\
&= \dim K - \dim L
\end{aligned}$$

which implies $\dim K_0 + \dim L = \dim K$. This and (75) imply (74). ¤

Since $L = \operatorname{Ker} \underset{\sim}{\chi}_R$, it follows from Lemma 4 that

$$\underset{\sim}{\chi}_R | K_0 : K_0 \to \underset{\sim}{\chi}_R(K) \tag{76}$$

is an isomorphism. Assume that Λ is of the form $\Lambda_R \times \mathbb{R}^{2n}$ and recall $\underset{\sim}{\pi}_P'$ of (19). Then, we have

$$\begin{aligned}
\Lambda \cap \underset{\sim}{\pi}_P'^{-1}(\underset{\sim}{\chi}_P(K)) &\cong (\Lambda_R \times \mathbb{R}^{2n}) \cap (\mathbb{R}^{2n_R} \times \underset{\sim}{\chi}_P(K)) \\
&= \Lambda_R \times \underset{\sim}{\chi}_P(K) \\
&\cong \Lambda_R \times (H \oplus L) \\
&= (\Lambda_R \times H) \times L \ .
\end{aligned}$$

It follows from this, Lemma 4 and $K \subset \underset{\sim}{\pi}_p'^{-1}(\underset{\sim}{\chi}_p(K))$ that the following holds:

$$
\begin{aligned}
\Sigma = \Lambda \cap K &= (\Lambda \cap \underset{\sim}{\pi}_p'^{-1}(\underset{\sim}{\chi}_p(K))) \cap K \\
&\cong ((\Lambda_R \times H) \times L) \cap (K_0 \oplus L) \\
&= ((\Lambda_R \times H) \cap K_0) \times L \,.
\end{aligned}
$$

Define

$$
\Lambda_0 \triangleq \Lambda_R \times H, \; \Sigma_0 \triangleq \Lambda_0 \cap K_0 \,. \tag{77}
$$

Lemma 5.

(i) $\underset{\sim}{\chi}_p | K_0 : K_0 \to H$ is surjective.

(ii) $\underset{\sim}{\chi}_R | K_0 : K_0 \to \mathbb{R}^{2n_R}$ is injective.

Proof. (i) follows from definition (73). (ii) follows from the fact that the map defined by (76) is an isomorphism. ¤

Letting

$$
\mathbb{R}_0 \triangleq \mathbb{R}^{2n_R} \oplus H \tag{78}
$$

we have from (72) that $\underset{\sim}{\pi}_p'^{-1}(\underset{\sim}{\chi}_p(K)) \cong \mathbb{R}^{2n_R} \times \underset{\sim}{\chi}_p(K) \cong \mathbb{R}^{2n_R} \times H \oplus L = \mathbb{R}_0 \oplus L$. Summarizing the preceding arguments, we see that for a given N, there is a unique N_0 given by the following:

$$
\begin{tikzcd}
 & \mathbb{R}^{2n_R} \\
\mathbb{R}_0 \supset K_0 \arrow[ur, "\underset{\sim}{\chi}_R | K_0"] \arrow[dr, "\underset{\sim}{\chi}_p | K_0"'] & \\
 & H
\end{tikzcd}
\qquad
\begin{array}{l}
\mathbb{R}_0 \supset \Lambda_0 \triangleq \Lambda_R \times H \\[3em]
\Sigma_0 \triangleq \Lambda_0 \cap K_0
\end{array}
$$

In this section, we will always assume that Λ is of the form $\Lambda_R \times \mathbb{R}^{2n}$ and by a perturbation $\hat{\Lambda}$ (resp. $\hat{\Lambda}_0$) of Λ (resp. Λ_0), we mean $\hat{\Lambda}_R \times \mathbb{R}^{2n}$ (resp. $\hat{\Lambda}_R \times H$), where $\hat{\Lambda}_R$ is a perturbation of Λ_R in $\mathbb{R}^{2n_R}$. Also, by the constitutive relation of N_0, we mean the set $\mathcal{R}_0 \triangleq \underset{\sim}{\chi}_p(\Sigma_0)$.

Lemma 6. Assume that $\Lambda = \Lambda_R \times \mathbb{R}^{2n}$. Then for N_0, statements of Theorem 1 and Theorem 2 hold without Condition P.

Proof. There is a C^1 perturbation $\hat{\Lambda}_R$ of Λ_R such that $\hat{\Lambda}_R \pitchfork \underset{\sim}{\chi}_R(K_0)$. Since $\hat{\Lambda}_0 = \hat{\Lambda}_R \times H$, it follows from an argument similar to the proof of Lemma 1 that $\hat{\Lambda}_0 \pitchfork K_0$. Therefore $\hat{\Sigma}_0 = \hat{\Lambda}_0 \cap K_0$ is a C^1 submanifold. It follows from (76) - (78) that

$$\begin{aligned}\dim \hat{\Sigma}_0 &= \dim \hat{\Lambda}_0 + \dim K_0 - \dim \mathbb{R}_0 \\ &= (n_R+2(n-\rho-\mu)) + (n+n_R-\rho-\mu) \\ &\quad - 2(n+n_R-\rho-\mu) = n - \rho - \mu \quad .\end{aligned}$$

It follows from Lemma 5 that Condition P for N_0 is unnecessary to prove Theorem 1. Since $\dim H = 2(n-\rho-\mu) = 2 \dim \hat{\Sigma}_0$, one can show a result similar to Lemma F of Appendix II. Therefore Theorem 2 holds without Condition P. ¤

Definition 5. Let Λ be of the form $\Lambda_R \times \mathbb{R}^{2n}$ and let N_0 be as above. Then $\underset{\sim}{\pi}_P = \underset{\sim}{\chi}_P|\Sigma : \Sigma \to \mathbb{R}^{2n}$ is said to be an admissible immersion if $\underset{\sim}{\chi}_P|\Sigma_0 : \Sigma_0 \to H$ is a nice immersion.

Remark. It can be shown that the definition of admissible immersion does not depend on the particular choice of H satisfying (72).

The following fact follows from the proof of Theorem 1.

Proposition 9. Assume $\Lambda = \Lambda_R \times \mathbb{R}^{2n}$ and $\Lambda \cap K \neq \emptyset$, $\Lambda \pitchfork K$. If $\underset{\sim}{\pi}_P$ is an admissible immersion, then the following hold:

(i) $R = \underset{\sim}{\pi}_P(\Sigma) \approx \underset{\sim}{\chi}_P(\Sigma_0) \times L$ and for any self intersection point $(\underset{\sim}{v}_P, \underset{\sim}{i}_P) \in R$,

$$\# \underset{\sim}{\pi}_P^{-1}(\underset{\sim}{v}_P, \underset{\sim}{i}_P) = 2,$$

where $\approx$ denotes that two objects are homeomorphic. Furthermore, every connected component σ of the self intersection set of R is of the form

$$\sigma \approx \{\underset{\sim}{x}\} \times L \subset H \oplus L$$

where $\underset{\sim}{x} \in H$ is a self intersection point of R_0.

(iia) For a small C^1 perturbation $\hat{\Lambda}_R$ of Λ_R, the map $\underset{\sim}{\chi}_P|\hat{\Sigma}_0$ is also a nice immersion. Furthermore, there is a C^1 diffeomorphism $\underset{\sim}{H} : \Sigma \to \hat{\Sigma}$ near the identity map and there is a homeomorphism $\underset{\sim}{H}_P : R \to \hat{R}$ such that the following diagram commutes:

$$\begin{tikzcd}
\Sigma \arrow[r, "\underset{\sim}{H}"] \arrow[d, "\underset{\sim}{\chi}_P"'] & \hat{\Sigma} \arrow[d, "\underset{\sim}{\chi}_P"] \\
R \arrow[r, "\underset{\sim}{H}_P"] & \hat{R}
\end{tikzcd}$$

(iib) $\underset{\sim}{H}$ is of the form $\underset{\sim}{H}_0 \times \text{id.}$ for some C^1 diffeomorphism $\underset{\sim}{H}_0 : \Sigma_0 \to \hat{\Sigma}_0$, where $\text{id.} : L \to L$ is the identity map. $\underset{\sim}{H}_P$ is of the form $\underset{\sim}{H}_{P0} \times \text{id.}$ for some homeomorphism $\underset{\sim}{H}_{P0} : R_0 \to \hat{R}_0$. Furthermore, the following diagram commutes:

$$\begin{tikzcd}
\Sigma \approx \Sigma_0 \times L \arrow[r, "\underset{\sim}{H}_0 \times \text{id.}"] \arrow[d, "(\underset{\sim}{\chi}_P | \Sigma_0) \times \text{id.}"] & \hat{\Sigma}_0 \times L \approx \hat{\Sigma} \arrow[d, "(\underset{\sim}{\chi}_P | \hat{\Sigma}_0) \times \text{id.}"] \\
R \approx R_0 \times L \arrow[r, "\underset{\sim}{H}_{P0} \times \text{id.}"'] & \hat{R}_0 \times L \approx \hat{R}
\end{tikzcd}$$

We are now ready to define strong structural R-stability and state its characterization result.

<u>Definition 6</u>. Let Λ be of the form $\Lambda_R \times \mathbb{R}^{2n}$. Then N is said to be <u>strongly structurally R-stable</u> if for any small C^1 perturbation $\hat{\Lambda}_R$ of Λ_R, R and $\hat{R}$ are homeomorphic.

<u>Theorem 4</u>. (<u>Characterization of Strong Structural R-Stability</u>)

Let Λ be of the form $\Lambda_R \times \mathbb{R}^{2n}$ and let $\Lambda \cap K \neq \emptyset$. Then N is strongly structurally R-stable <u>if and only if</u>

(i) $\Lambda \pitchfork K$.

(ii) $\underset{\sim}{\pi}_P$ is an admissible immersion.

<u>Proof</u>. Sufficiency follows from <u>Proposition 9</u>. In order to prove necessity we consider the following four cases:

<u>Case 1</u>. $\Lambda \not\pitchfork K$. It is clear that $\Lambda_0 \not\pitchfork K_0$. It follows from an argument similar to the proof of <u>Lemma 1</u> that $\Lambda_R \not\pitchfork \underset{\sim}{\chi}_R(K_0)$. Since $\underset{\sim}{\chi}_R | K_0$ is

injective, dim $\chi_R(K_0)$ = dim $K_0 = n + n_R - \rho - \mu$. Therefore, it follows from an argument similar to the proof of Theorem 2 in [1], we have a small C^1 perturbation Λ_R' of Λ_R such that $\Lambda_R' \cap \chi_R(K_0)$ contains an open subset of an $(n-\rho-\mu+k)$ - dimensional affine submanifold for some $k \geq 1$. Since $\chi_P|K_0$ is surjective, it follows from an argument similar to the proof of Theorem 1 that $R_0' \triangleq \chi_P(\Sigma_0')$ contains an open subset of an $(n-\rho-\mu+k')$ - dimensional affine submanifold, $1 \leq k' \leq k$. It also follows from a similar argument to the proof of Theorem 1 that there is another C^1 perturbation Λ_R'' of Λ_R such that $\chi_P|\Sigma_0''$ is a nice immersion. Therefore R' and R'' cannot be homeomorphic.

Case 2. $\Lambda \pitchfork K$ but $\chi_P|\Sigma_0$ is not an immersion. By assumption, Σ_0 is a C^1 submanifold of K_0. Since $\chi_R|K_0$ is injective, $\chi_R|\Sigma_0 : \Sigma_0 \to \chi_R(\Sigma_0)$ is a diffeomorphism. Since $\chi_P|\Sigma_0$ is not an immersion and since $\chi_R|\Sigma_0$ is a diffeomorphism one sees that $\chi_P \circ (\chi_R|\Sigma_0)^{-1} : \chi_R(\Sigma_0) \to H \subset \mathbb{R}^{2n}$ is not an immersion. Since $\chi_R(\Sigma_0) = \Lambda_R \cap \chi_R(K)$, using an argument similar to the proof of (ii) of Theorem 1, one can show that there are small C^1 perturbations Λ_R' and Λ_R'' of Λ_R such that $\chi_P(\Sigma_0')$ and $\chi_P(\Sigma_0'')$ are not homeomorphic, where $\Sigma_0' \triangleq \Lambda_0' \cap K_0$, $\Lambda_0' \triangleq \Lambda_R' \times H$, $\Sigma_0'' \triangleq \Lambda_0'' \cap K_0$, $\Lambda_0'' \triangleq \Lambda_R'' \times H$.

The remaining two cases can be proved in a manner similar to that of Case 3 and Case 4 of (ii) of Theorem 1. ¤

Remark. In Case 2 of the proof of Theorem 4, we must perturb Λ_R but not Λ_0. This is the reason why we consider the map $\chi_P \circ (\chi_R|\Sigma_0)^{-1}$ instead of $\chi_P|\Sigma_0$.

The following is a density result for strong structural R-stability. The proof is similar to that of Theorem 2.

Theorem 5. (Density of Strong Structural R-Stability)

Let Λ be of the form $\Lambda_R \times \mathbb{R}^{2n}$ and let $\Lambda \cap K \neq \emptyset$. Then there is an arbitrarily small C^1 perturbation $\hat{\Lambda}_R$ of Λ_R such that

(i) $\hat{\Lambda} \cap K \neq \emptyset$, where $\hat{\Lambda} = \hat{\Lambda}_R \times \mathbb{R}^{2n}$

(ii) $\hat{\Lambda} \pitchfork K$.

(iii) $\hat{\pi}_P$ is an admissible immersion.

Therefore the perturbed $\hat{N}$ is strongly structurally R-stable.

So far, we have only shown the existence of K_0 and Λ_0. The following result describes a simple way of obtaining Λ_0 and K_0. Recall that

ρ (resp. μ) is the number of independent port-only loops (resp. cut sets).

<u>Proposition 10</u>. Let N be an n-port with $\Lambda = \Lambda_R \times \mathbb{R}^{2n}$. Let T_1 (resp. T_2) be a tree containing maximum number of ports (resp. resistors) and let L_1 (resp. L_2) be its associated cotree such that $R_{T_1} \subset T_2$ and $P_{L_1} \subset L_2$. Open branches of P_{L_1} (port branches belonging to L_1) and short P_{T_2} (port branches belonging to T_2) and call the resulting $(n\text{-}\rho\text{-}\mu)$-port N_0. Then

(i) K_0 is isomorphic to the Kirchhoff space of N_0.

(ii) Λ_0 is diffeomorphic to the internal resistor constitutive relations of N_0.

<u>Proof</u>. We decompose the port branches of N in the following manner:

<table>
<tr><td>L_1</td><td colspan="2">T_1</td></tr>
<tr><td>P_{L_1}</td><td>$P^1_{T_1}$</td><td>$P^2_{T_1}$</td></tr>
<tr><td>$P^1_{L_2}$</td><td>$P^2_{L_2}$</td><td>P_{T_2}</td></tr>
<tr><td colspan="2">L_2</td><td>T_2</td></tr>
</table>

Fundamental loop and cut set matrices with respect to T_1 and T_2 are given, respectively, by

$$\begin{array}{ccccc}
\underset{\sim}{v}_{R_{L_1}} & \underset{\sim}{v}_{P_{L_1}} & \underset{\sim}{v}_{R_{T_1}} & \underset{\sim}{v}_{P^1_{T_1}} & \underset{\sim}{v}_{P^2_{T_1}} \\
\end{array}$$

$$\left[\begin{array}{cc|ccc}
\underset{\sim}{1} & \cdot & \underset{\sim}{B}_{RR} & \underset{\sim}{B}^1_{RP} & \underset{\sim}{B}^2_{RP} \\
\cdot & \underset{\sim}{1} & \cdot & \underset{\sim}{B}^1_{PP} & \underset{\sim}{B}^2_{PP}
\end{array}\right]$$

$$\begin{array}{ccccc}
\underset{\sim}{i}_{R_{L_1}} & \underset{\sim}{i}_{P_{L_1}} & \underset{\sim}{i}_{R_{T_1}} & \underset{\sim}{i}_{P^1_{T_1}} & \underset{\sim}{i}_{P^2_{T_1}} \\
\end{array}$$

$$\left[\begin{array}{cc|ccc}
-\underset{\sim}{B}^T_{RR} & \cdot & \underset{\sim}{1} & \cdot & \cdot \\
-\underset{\sim}{B}^{1T}_{RP} & -\underset{\sim}{B}^{1T}_{PP} & \cdot & \underset{\sim}{1} & \cdot \\
-\underset{\sim}{B}^{2T}_{RP} & -\underset{\sim}{B}^{2T}_{PP} & \cdot & \cdot & \underset{\sim}{1}
\end{array}\right]$$

$$
\begin{array}{ccccc}
v_{R_{L_2}} & v_{P^1_{L_2}} & v_{P^2_{L_2}} & v_{R_{T_2}} & v_{P_{T_2}} \\
\end{array}
$$

$$
\left[\begin{array}{ccc|cc}
1 & \cdot & \cdot & \bar{B}_{RR} & \cdot \\
\cdot & 1 & \cdot & \bar{B}^{1}_{PR} & \bar{B}^{1}_{PP} \\
\cdot & \cdot & 1 & \bar{B}^{2}_{PR} & \bar{B}^{2}_{PP}
\end{array}\right]
$$

$$
\begin{array}{ccccc}
i_{R_{L_2}} & i_{P^1_{L_2}} & i_{P^2_{L_2}} & i_{R_{T_2}} & i_{P_{T_2}} \\
\end{array}
$$

$$
\left[\begin{array}{ccc|cc}
-\bar{B}^{T}_{RR} & -\bar{B}^{1T}_{PR} & -\bar{B}^{2T}_{PR} & 1 & \cdot \\
\cdot & -\bar{B}^{1T}_{PP} & -\bar{B}^{2T}_{PP} & \cdot & 1
\end{array}\right]
$$

where subscript R denotes variables associated with resistors and a bar denotes that matrix is with respect to T_2 and L_2. Let $P_{T_1} \triangleq P^1_{T_1} \cup P^2_{T_1}$ and $P_{L_2} \triangleq P^1_{L_2} \cup P^2_{L_2}$. Then

$$
\operatorname{Ker} \chi_R = \left\{ (v, i) \;\middle|\;
\begin{array}{l}
(v_R, i_R) = (0, 0) \\
\left[\begin{array}{c|c} 1 & \begin{array}{c} \bar{B}^{1}_{PP} \\ \bar{B}^{2}_{PP} \end{array} \end{array}\right]
\begin{bmatrix} v_{P_{L_2}} \\ v_{P_{T_2}} \end{bmatrix} = 0 \\
\left[\begin{array}{c|c} \begin{array}{c} -B^{1T}_{PP} \\ -B^{2T}_{PP} \end{array} & 1 \end{array}\right]
\begin{bmatrix} i_{P_{L_1}} \\ i_{P_{T_1}} \end{bmatrix} = 0
\end{array}
\right\}
$$

$$
\cong \left\{ (v_P, i_P) \;\middle|\;
\begin{array}{l}
\left[\begin{array}{c|c} 1 & \begin{array}{c} -\bar{B}^{1}_{PP} \\ \bar{B}^{2}_{PP} \end{array} \end{array}\right]
\begin{bmatrix} v_{P_{L_2}} \\ v_{P_{T_2}} \end{bmatrix} = 0 \\
\left[\begin{array}{c|c} \begin{array}{c} -B^{1T}_{PP} \\ -B^{2T}_{PP} \end{array} & 1 \end{array}\right]
\begin{bmatrix} i_{P_{L_1}} \\ i_{P_{T_1}} \end{bmatrix} = 0
\end{array}
\right\}
$$

$$\cong \{(\underset{\sim}{v}_P, \underset{\sim}{i}_P) \,|\, \underset{\sim}{v}_{P_{L_2}} = \underset{\sim}{0},\ \underset{\sim}{i}_{P_{T_1}} = \underset{\sim}{0}\} \tag{79}$$

$$\cong \{(\underset{\sim}{v}_{P_{T_2}}, \underset{\sim}{i}_{P_{L_1}}) \in \mathbb{R}^{\rho+\mu}\} = L\ . \tag{80}$$

We next compute H of (72). To this end observe that

$$\underset{\sim}{\chi}_P(K) = \left\{ (\underset{\sim}{v},\underset{\sim}{i}) \;\middle|\; \begin{array}{c} (\underset{\sim}{v}_R,\underset{\sim}{i}_R) = (\underset{\sim}{0},\underset{\sim}{0}) \\ [\,\underset{\sim}{1} \mid \underset{\sim}{B}^1_{PP} \quad \underset{\sim}{B}^2_{PP}\,] \begin{bmatrix} \underset{\sim}{v}_{P_{L_1}} \\ \underset{\sim}{v}_{p^1 T_1} \\ \underset{\sim}{v}_{p^2 T_1} \end{bmatrix} = \underset{\sim}{0} \\ [-\underset{\sim}{\bar{B}}^{1T}_{PP} \quad -\underset{\sim}{\bar{B}}^{2T}_{PP} \mid \underset{\sim}{1}\,] \begin{bmatrix} \underset{\sim}{i}_{p^1 L_2} \\ \underset{\sim}{i}_{p^2 L_2} \\ \underset{\sim}{i}_{P_{T_2}} \end{bmatrix} = \underset{\sim}{0} \end{array} \right\}$$

$$\cong \left\{ (\underset{\sim}{v}_P,\underset{\sim}{i}_P) \;\middle|\; \begin{array}{c} [\,\underset{\sim}{1} \mid \underset{\sim}{B}^1_{PP} \quad \underset{\sim}{B}^2_{PP}\,] \begin{bmatrix} \underset{\sim}{v}_{P_{L_1}} \\ \underset{\sim}{v}_{p^1 T_1} \\ \underset{\sim}{v}_{p^2 T_1} \end{bmatrix} = \underset{\sim}{0} \\ [-\underset{\sim}{\bar{B}}^{1T}_{PP} \quad -\underset{\sim}{\bar{B}}^{2T}_{PP} \mid \underset{\sim}{1}\,] \begin{bmatrix} \underset{\sim}{i}_{p^1 L_2} \\ \underset{\sim}{i}_{p^2 L_2} \\ \underset{\sim}{i}_{P_{T_2}} \end{bmatrix} = \underset{\sim}{0} \end{array} \right\} \tag{81}$$

Since $P_{T_2} = P^2_{T_1}$ and since $P_{L_1} = P^1_{L_2}$, we have from (72), (79), (80) and (81) that[2]

$$H \cong \left\{ (\underset{\sim}{v}_P, \underset{\sim}{i}_P) \;\middle|\; \begin{array}{l} [\, \underset{\sim}{1} \;\vdots\; \underset{\sim}{B}^1_{PP} \,] \begin{bmatrix} \underset{\sim}{v}_{P_{L_1}} \\ \underset{\sim}{v}_{P^1_{T_1}} \end{bmatrix} = \underset{\sim}{0} \\ [-\overline{\underset{\sim}{B}}^{2T}_{PP} \;\vdots\; \underset{\sim}{1} \,] \begin{bmatrix} \underset{\sim}{i}_{P^2_{L_2}} \\ \underset{\sim}{i}_{P_{T_2}} \end{bmatrix} = \underset{\sim}{0} \\ \underset{\sim}{v}_{P^2_{T_1}} = \underset{\sim}{0},\; \underset{\sim}{i}_{P^1_{L_2}} = \underset{\sim}{0} \end{array} \right\}$$

$$\cong \left\{ (\underset{\sim}{v}_{P_{L_1}}, \underset{\sim}{v}_{P^1_{T_1}}, \underset{\sim}{i}_{P^2_{L_2}}, \underset{\sim}{i}_{P_{T_2}}) \;\middle|\; \begin{array}{l} [\, \underset{\sim}{1} \;\vdots\; \underset{\sim}{B}^1_{PP} \,] \begin{bmatrix} \underset{\sim}{v}_{P_{L_1}} \\ \underset{\sim}{v}_{P^1_{T_1}} \end{bmatrix} = \underset{\sim}{0} \\ [-\overline{\underset{\sim}{B}}^{2T}_{PP} \;\vdots\; \underset{\sim}{1} \,] \begin{bmatrix} \underset{\sim}{i}_{P^2_{L_2}} \\ \underset{\sim}{i}_{P_{T_2}} \end{bmatrix} = \underset{\sim}{0} \end{array} \right\}$$

$$\cong \{ (\underset{\sim}{v}_{P^1_{T_1}}, \underset{\sim}{i}_{P^2_{L_2}}) \in \mathbb{R}^{2(n-\rho-\mu)} \}. \tag{82}$$

It follows from Lemma 4 and (80) that K_0 of (73) is given by

$$K_0 \cong \{ (\underset{\sim}{v}, \underset{\sim}{i}) \in K \,|\, (\underset{\sim}{v}, \underset{\sim}{i}) \perp L \}$$

$$= \{ (\underset{\sim}{v}, \underset{\sim}{i}) \in K \,|\, \underset{\sim}{v}_{P_{T_2}} = \underset{\sim}{0},\; \underset{\sim}{i}_{P_{L_1}} = \underset{\sim}{0} \}.$$

But since $P_{T_2} = P^2_{T_1}$ and $P_{L_1} = P^1_{L_2}$, we have

$$K_0 \cong \{ (\underset{\sim}{v}, \underset{\sim}{i}) \in K \,|\, \underset{\sim}{v}_{P^2_{T_1}} = \underset{\sim}{0},\; \underset{\sim}{i}_{P^1_{L_2}} = \underset{\sim}{0} \}$$

$$
= \left\{ (\mathbf{v},\mathbf{i}) \;\middle|\; \begin{array}{l}
\left[\begin{array}{cc|cc} \mathbf{1} & \cdot & \mathbf{B}_{RR} & \mathbf{B}^1_{RP} \\ \cdot & \mathbf{1} & \cdot & \mathbf{B}^1_{PP} \end{array}\right]
\begin{bmatrix} \mathbf{v}_{R_{L_1}} \\ \mathbf{v}_{P_{L_1}} \\ \mathbf{v}_{R_{T_1}} \\ \mathbf{v}_{P^1_{T_1}} \end{bmatrix} = \mathbf{0}, \; \mathbf{v}_{P^2_{T_1}} = \mathbf{0} \\[2em]
\left[\begin{array}{cc|cc} -\bar{\mathbf{B}}^T_{RR} & -\bar{\mathbf{B}}^{2T}_{PR} & \mathbf{1} & \cdot \\ \cdot & -\bar{\mathbf{B}}^{2T}_{PP} & \cdot & \mathbf{1} \end{array}\right]
\begin{bmatrix} \mathbf{i}_{R_{L_2}} \\ \mathbf{i}_{P^2_{L_2}} \\ \mathbf{i}_{R_{T_2}} \\ \mathbf{i}_{P_{T_2}} \end{bmatrix} = \mathbf{0}, \; \mathbf{i}_{P^1_{L_2}} = \mathbf{0}
\end{array} \right\}
$$

$$
\cong \left\{ \left(\mathbf{v}_{R_{L_1}}, \mathbf{v}_{R_{T_1}}, \mathbf{v}_{P^1_{T_1}} \;\vdots\; \mathbf{i}_{R_{L_2}}, \mathbf{i}_{P^2_{L_2}}, \mathbf{i}_{R_{T_2}}\right) \;\middle|\; \begin{array}{l}
\left[\mathbf{1} \,\vdots\, \mathbf{B}_{RR} \;\; \mathbf{B}^1_{RP}\right]
\begin{bmatrix} \mathbf{v}_{R_{L_1}} \\ \mathbf{v}_{R_{T_1}} \\ \mathbf{v}_{P^1_{T_1}} \end{bmatrix} = \mathbf{0} \\[2em]
\left[-\bar{\mathbf{B}}^T_{RR} \;\; -\bar{\mathbf{B}}^{2T}_{PR} \,\vdots\, \mathbf{1}\right]
\begin{bmatrix} \mathbf{i}_{R_{L_2}} \\ \mathbf{i}_{P^2_{L_2}} \\ \mathbf{i}_{R_{T_2}} \end{bmatrix} = \mathbf{0}
\end{array} \right\}.
$$

Since $P^1_{T_1} = P^2_{L_2}$, we see that K_0 is the Kirchhoff space of N_0. This proves (i). Finally, it follows from (82) that Λ_0 of (77) is given by the following:

$$\Lambda_0 = \Lambda_R \times H \approx \{(\underset{\sim}{v}_R, \underset{\sim}{v}_{P^1_{T_1}} \mid \underset{\sim}{i}_R, \underset{\sim}{i}_{P^2_{L_2}}) \mid (\underset{\sim}{v}_R, \underset{\sim}{i}_R) \in \Lambda_R\}$$

where $\approx$ denotes that the two objects are diffeomorphic. Since $P^1_{T_1} = P^2_{L_2}$, this set is the internal resistor constitutive relations of N_0. This proves (ii). ¤

Proposition 10 gives an easy way of checking conditions (i) and (ii) of Theorem 4. Namely, pick T_1 and T_2, open P_{L_1}, short P_{T_2} and obtain an $(n-\rho-\mu)$-port N_0. Then check transversality of Λ_0 and K_0 and check if $\underset{\sim}{\chi}_P|\Sigma_0$ is a nice immersion.

Example 10. Consider the 2-port of Fig. 14(a) where Λ_R is described by the following parametrized form:

$$v_{R_1} = \frac{\rho_1}{2(1+\rho_1^3)} \quad , \quad v_{R_2} = \frac{\rho_1}{2(1+\rho_1^3)}$$

$$i_{R_1} = \frac{\rho_1^2}{1+\rho_1^3} \quad , \quad i_{R_2} = \rho_2 \quad , \quad \rho_1, \rho_2 \in \mathbb{R} .$$

Taking the derivative of these functions with respect to (ρ_1, ρ_2), one can show that Λ_R is a 2-dimensional submanifold. In order to check strong structural R-stability, we choose $T_1 = \{P_1, R_1\}$ and $T_2 = \{R_1, R_2\}$. Then $P_{L_1} = \{P_2\}$ and we open P_2. We then apply Theorem 4 to this 1-port N_0. In order to check transversality we apply Proposition 2 of [1]. Choose $T_0 \triangleq \{R_1, R_2\}$ to be a tree for N_0 and let L_0 be its associated cotree. Then $R_{T_0} = \{R_1, R_2\}$, $R_{L_0} = \emptyset$, $\underset{\sim}{B}_{RP} = \underset{\sim}{B}_{RR} = \emptyset$, $\underset{\sim}{B}^T_{PR} = [1 \quad 1]^T$ so that

$$\underset{\sim}{F}^*(\rho_1, \rho_2) = \left[\begin{array}{c|c} \underset{\sim}{D}\,\underset{\sim}{i}_R & \begin{matrix} 1 \\ 1 \end{matrix} \end{array}\right]_{(\rho_1,\rho_2)} = \left[\begin{array}{cc|c} \dfrac{\rho_1(2-\rho_1^3)}{(1+\rho_1^3)^2} & \cdot & 1 \\ \cdot & 1 & 1 \end{array}\right]_{\rho_1} .$$

Since this matrix has rank 2, it follows from Proposition 2 of [1] that $\Lambda_0 \pitchfork K_0$. Observe that any $(\underset{\sim}{v}_R, v_{P_1} \mid \underset{\sim}{i}_R, i_{P_1}) \in \Sigma_0$ is given by

$$(v_{R_1}, v_{R_2}, v_{P_1} \mid i_{R_1}, i_{R_2}, i_{P_1}) = \left(\frac{\rho_1}{2(1+\rho_1^3)}, \frac{\rho_1}{2(1+\rho_1^3)}, \frac{-\rho_1}{1+\rho_1^3} \,\middle|\, \frac{\rho_1^2}{1+\rho_1^3}, \frac{\rho_1^2}{1+\rho_1^3}, \frac{\rho_1^2}{1+\rho_1^3}\right).$$

Therefore

$$\underset{\sim}{\chi}_P(\Sigma_0) = \left\{ \left(\frac{-\rho_1}{1+\rho_1^3}, \frac{\rho_1^2}{1+\rho_1^3}\right) \middle| \rho_1 \in \mathbb{R} \right\} .$$

Picture of this set is given by Fig. 18(a). By inspection $\underset{\sim}{\chi}_P|\Sigma_0$ is a nice immersion. Therefore <u>Theorem 4</u> tells us that N is strongly structurally R-stable. Notice that $(\underset{\sim}{v}_P, \underset{\sim}{i}_P) \in R = \underset{\sim}{\chi}_P(\Sigma)$ is given by

$$(v_{P_1}, v_{P_2}, i_{P_1}, i_{P_2}) = \left(\frac{-\rho_1}{1+\rho_1^3}, \frac{-\rho_1}{1+\rho_1^3}, i_{P_1}, \frac{\rho_1^2}{1+\rho_1^3} - i_{P_1}\right), \ \rho_1, i_{P_1} \in \mathbb{R} .$$

Therefore R looks like Fig. 18(b). As was described by <u>Proposition 9</u>, R is of the form $\underset{\sim}{\chi}_P(\Sigma_0) \times L$.

<u>Remark</u>. If N is strongly structurally R-stable, then one can show that N is strongly structurally stable, i.e., R persists under small perturbations $\hat{\Lambda}$ of Λ, where $\hat{\Lambda}$ is not necessarily of the form $\hat{\Lambda}_R \times \mathbb{R}^{2n}$. The proof is very involved, however.

Appendix I

Proof of Proposition 1. We will first explain the idea of the proof. Given $\Sigma = \Lambda \cap K$, let $\hat{\Lambda}$ be a C^1 perturbation of Λ and let $\hat{\Sigma} \triangleq \hat{\Lambda} \cap K$. (Fig. 19). We would like to define a diffeomorphism between Σ and $\hat{\Sigma}$. This is not as easy as it looks. Here, we first consider the tangent space $T_{\underset{\sim}{x}}\Lambda$ and let $N_{\underset{\sim}{x}}\Sigma$ be the orthogonal complement of $T_{\underset{\sim}{x}}\Sigma$ in $T_{\underset{\sim}{x}}\Lambda$. (Fig. 20). Recall that $\hat{\Lambda} = \underset{\sim}{F}(\Lambda)$ for some C^1 embedding $\underset{\sim}{F}$. (See Section II) Observe that for $\underset{\sim}{x} \in \Lambda \cap K$, the point $\underset{\sim}{F}(\underset{\sim}{x})$ may not be in K. Speaking very roughly, we map $N_{\underset{\sim}{x}}\Sigma$ by $\underset{\sim}{\mathcal{F}}$, which is a certain modification of $\underset{\sim}{F}$, and set $\underset{\sim}{y} \triangleq \underset{\sim}{\mathcal{F}}(N_{\underset{\sim}{x}}\Sigma) \cap K$. (Fig. 20) The map : $\Sigma \to \hat{\Sigma}$, $\underset{\sim}{x} \to \underset{\sim}{y}$, is essentially the one which we look for. There is a problem, however, because $N_{\underset{\sim}{x}}\Sigma$ may not be C^1 in $\underset{\sim}{x}$ when Σ is only C^1. Therefore the above map may not be C^1. This stems from the fact that $N_{\underset{\sim}{x}}\Sigma$ is defined in terms of $T_{\underset{\sim}{x}}\Sigma$ and $T_{\underset{\sim}{x}}\Lambda$, and the fact that $T_{\underset{\sim}{x}}\Sigma$ and $T_{\underset{\sim}{x}}\Lambda$ are defined in terms of derivatives of functions. A similar difficulty arises in the proof of Theorem 2 of [1]. That is the very reason we had to assume that Λ is C^2 in defining a diffeomorphism : $\Sigma \to \hat{\Sigma}$. Here we overcome this difficulty by approximating the C^1 submanifold by a C^∞ submanifold and then extend $\underset{\sim}{F}$ of (6) to $\underset{\sim}{\mathcal{F}}$ using a special map called exponential map. The extension is necessary in order to map $N_{\underset{\sim}{x}}\Sigma$ since $\underset{\sim}{F}$ is defined only on Λ.

For proof, we will need several terminologies and lemmas. For the convenience of the reader, we will give simple pictures explaining the ideas involved.

If Λ is a C^r manifold, the set $T\Lambda \triangleq \bigcup_{\underset{\sim}{x}\in\Lambda} T_{\underset{\sim}{x}}\Lambda$ is called the tangent bundle. The zero section of $T\Lambda$ is the set $\bigcup_{\underset{\sim}{x}\in\Lambda} \underset{\sim}{0}_{\underset{\sim}{x}}$, where $\underset{\sim}{0}_{\underset{\sim}{x}}$ is the zero vector of $T_{\underset{\sim}{x}}\Lambda$. If $r \geq 3$, then there are a neighborhood $W \subset T\Lambda$ of the zero section and a C^{r-2} map, called an exponential map

$$\exp : W \to \Lambda \tag{A.1}$$

such that ([4,p.72])

(i) $\exp(\underset{\sim}{0}_{\underset{\sim}{x}}) = \underset{\sim}{x}$

(ii) $\exp_{\underset{\sim}{x}} \triangleq \exp | T_{\underset{\sim}{x}}\Lambda : (T_{\underset{\sim}{x}}\Lambda) \cap W \to \Lambda$ is a diffeomorphism

(iii) $(\underset{\sim}{d} \exp_{\underset{\sim}{x}})_{\underset{\sim}{0}_{\underset{\sim}{x}}} : T_{\underset{\sim}{x}}\Lambda \to T_{\underset{\sim}{x}}\Lambda$ is the identity.

Intuitively, exponential map pushes $T\Lambda$ onto Λ as in Fig. 21. It provides us with a convenient means for extending a map on Λ.

If Σ is a C^r submanifold of Λ, let $N_{\underset{\sim}{x}}\Sigma$ be the orthogonal complement of $T_{\underset{\sim}{x}}\Sigma$ in $T_{\underset{\sim}{x}}\Lambda$. (Fig. 22) The set

$$N\Sigma \triangleq \bigcup_{\underset{\sim}{x}\in\Sigma} N_{\underset{\sim}{x}}\Sigma \tag{A.2}$$

is called the normal bundle of Σ in Λ. Then we have a bundle splitting [2]

$$T\Lambda|\Sigma = T\Sigma \oplus N\Sigma \tag{A.3}$$

where $T\Lambda|\Sigma \triangleq \bigcup_{\underset{\sim}{x}\in\Sigma} T_{\underset{\sim}{x}}\Lambda$. A C^r tubular neighborhood of Σ in Λ consists of $N\Sigma$, an open neighborhood V of the zero section $\underset{\sim}{\zeta} : \Sigma \to N\Sigma$ and a C^r diffeomorphism $\underset{\sim}{\phi}$ of V onto an open neighborhood U of Σ in Λ which commutes with $\underset{\sim}{\zeta}$. (Fig. 23) The map $\underset{\sim}{\phi}$ is called the tubular map and $U = \underset{\sim}{\phi}(V)$ is called the tube. Using [4,p.96, Theorem 9] and its proof, one has the following lemma:

<u>Lemma A</u>. Let Σ be a C^∞ submanifold of Λ. Then there is a C^∞ tubular neighborhood of Σ in Λ with a C^∞ tubular map $\underset{\sim}{\phi} : U \to V$ such that

$$\underset{\sim}{\phi}|N_{\underset{\sim}{x}}\Sigma \cap V = \exp_{\underset{\sim}{x}}|N_{\underset{\sim}{x}}\Sigma \cap V. \tag{A.4}$$

The following lemma can be proved in a similar manner to that of [5,p.41, Theorem 4.8].

<u>Lemma B</u>. Given a C^1 submanifold Λ of $\mathbb{R}^{2b}$ such that $\Lambda \pitchfork K$ for an affine submanifold K of $\mathbb{R}^{2b}$, there is a C^1 embedding $\underset{\sim}{H} : \Lambda \to \mathbb{R}^{2b}$ with the following properties:

(i) $\underset{\sim}{H}$ is arbitrarily close to $\underset{\sim}{\imath}_\Lambda$ in the strong C^1 topology.

(ii) $\underset{\sim}{H}(\Lambda)$ is a C^∞ submanifold of $\mathbb{R}^{2b}$.

(iii) $\underset{\sim}{H}(\Sigma)$ is a C^∞ submanifold of K where $\Sigma = \Lambda \cap K$.

(iv) $\underset{\sim}{H}(\Lambda) \pitchfork K$, $\underset{\sim}{H}(\Lambda) \cap K = \underset{\sim}{H}(\Sigma)$.

Let G_b be the set of all b-dimensional linear subspaces of $\mathbb{R}^{2b}$ and let M_b be the set of all b x 2b matrices having rank b. For any $\underset{\sim}{A} \in M_b$, the rows of $\underset{\sim}{A}$ are linearly independent in $\mathbb{R}^{2b}$. Therefore, the rows determine an element of G_b which is denoted by $\underset{\sim}{\lambda}(\underset{\sim}{A})$. The set M_b is an open subset of $\mathbb{R}^{2b^2}$ so that it has a natural C^∞ differentiable structure. If we define $V \subset G_b$ to be open iff $\underset{\sim}{\lambda}^{-1}(V)$ is open in M_b, then G_b is a

C^∞ submanifold and the map $\underset{\sim}{\lambda} : M_b \to G_b$ is C^∞. [5,p.43, Theorem 5.2].

<u>Lemma C</u>. [5,p.45, Lemma 5.3]. Let N be a C^r manifold and let $\underset{\sim}{f} : N \to G_b$ be a C^r map. Given $\underset{\sim}{x} \in N$, there are a neighborhood U of $\underset{\sim}{x}$ and a C^r map $\underset{\sim}{f}_* : U \to M_b$ such that $\underset{\sim}{\lambda} \circ \underset{\sim}{f}_* = \underset{\sim}{f}$.

Let J and K be two affine submanifolds of $\mathbb{R}^{2b}$ such that $J \cap K \neq \emptyset$, $J \pitchfork K$ and $\dim J + \dim K = 2b$. Then $J \cap K$ is a single point $\underset{\sim}{P}_{JK}$. We define the <u>angle</u> between J and K to be the following quantity:

$$\theta(J,K) \triangleq \sup\left\{ \langle \underset{\sim}{u}, \underset{\sim}{v} \rangle \;\middle|\; \begin{array}{l} \underset{\sim}{u} = \underset{\sim}{P}_J - \underset{\sim}{P}_{JK},\ \underset{\sim}{v} = \underset{\sim}{P}_K - \underset{\sim}{P}_{JK} \\ \\ \underset{\sim}{P}_J \in J,\ \underset{\sim}{P}_K \in K,\ \|\underset{\sim}{u}\| = \|\underset{\sim}{v}\| = 1 \end{array} \right\} \tag{A.5}$$

where $\langle\ ,\ \rangle$ denotes the inner product.

<u>Lemma D</u>. Let N be a manifold and let K be a b-dimensional affine submanifold of $\mathbb{R}^{2b}$. If $\underset{\sim}{J} : N \to G_b$ is continuous, $\underset{\sim}{J}(\underset{\sim}{x}) \cap K \neq \emptyset$ and $\underset{\sim}{J}(\underset{\sim}{x}) \pitchfork K$ for $\underset{\sim}{x} \in N$, then the map : $N \to \mathbb{R}$ defined by

$$\underset{\sim}{x} \to \theta(\underset{\sim}{J}(\underset{\sim}{x}),K) \tag{A.6}$$

is continuous.

<u>Proof</u>. It follows from <u>Lemma C</u> that there is a set of b row vectors $\underset{\sim}{J}^1(\underset{\sim}{x}),\dots,\underset{\sim}{J}^b(\underset{\sim}{x})$ continuous in $\underset{\sim}{x}$ such that $\underset{\sim}{J}(\underset{\sim}{x}) = \operatorname{span}\{\underset{\sim}{J}^1(\underset{\sim}{x}),\dots,\underset{\sim}{J}^b(\underset{\sim}{x})\}$. Hence the result follows from definition (A.5). ¤

Let K be a linear subspace of $\mathbb{R}^{2b}$ and consider the splitting $\mathbb{R}^{2b} = K \oplus K^\perp$. We will write

$$\mathbb{R}^{2b} \ni \underset{\sim}{x} = (\underset{\sim}{x}_1,\underset{\sim}{x}_2),\ \underset{\sim}{x}_1 \in K,\ \underset{\sim}{x}_2 \in K^\perp.$$

<u>Lemma E</u>. Let Γ be a submanifold of $\mathbb{R}^{2b}$ such that $\dim K + \dim \Gamma = 2b$ and $K \pitchfork \Gamma$. Let $\underset{\sim}{P} \in K \cap \Gamma$ and let $\underset{\sim}{\gamma} : [0,1] \to \Gamma$ be a C^1 map satisfying

(i) $\underset{\sim}{\gamma}(0) = \underset{\sim}{P}$

(ii) $T_{\underset{\sim}{\gamma}(t)}\Gamma \pitchfork K$.

Let $L(t)$ (resp. $L_2(t)$) be the length of the path $\underset{\sim}{\gamma}([0,t])$ (resp. $\underset{\sim}{\gamma}_2([0,t])$), where $\underset{\sim}{\gamma}(t) = (\underset{\sim}{\gamma}_1(t),\underset{\sim}{\gamma}_2(t)) \in K \oplus K^\perp)$. Then

$$L(1) \le \int_0^1 \frac{1}{\sqrt{1-\theta(T_{\underset{\sim}{\gamma}(t)}\Gamma,K)^2}} \; \frac{dL_2(t)}{dt}\, dt \tag{A.7}$$

Remark. Since $T_{\gamma(t)}\Gamma \pitchfork K$ and since $\dim T_{\gamma(t)}\Gamma + \dim K = 2b$, we have $0 < \theta(T_{\gamma(t)}\Gamma, K) < 1$ so that (A.7) makes sense.

Proof. Let us first consider an affine submanifold J of $\mathbb{R}^{2b}$ with $J \pitchfork K$. Let P, $\Delta P \in \mathbb{R}^{2b}$ satisfy P, $P + \Delta P \in J$ and write $P = (P_1, P_2) \in K \oplus K^{\perp}$, $\Delta P = ((\Delta P)_1, (\Delta P)_2) \in K \oplus K^{\perp}$ (Fig. 24). Then

$$\frac{\|(\Delta P)_1\|}{\|\Delta P\|} = \left\langle \frac{(\Delta P)_1}{\|\Delta P\|}, \frac{(\Delta P)_1}{\|(\Delta P)_1\|} \right\rangle \le \left\langle \frac{\Delta P}{\|\Delta P\|}, \frac{(\Delta P)_1}{\|(\Delta P)_1\|} \right\rangle \le \theta(J,K)$$

implies $\|(\Delta P)_1\| \le \|\Delta P\|\ \theta(J,K)$. Since $\|\Delta P\|^2 = \|(\Delta P)_1\|^2 + \|(\Delta P)_2\|^2 \le \|\Delta P\|^2\ \theta(J,K)^2 + \|(\Delta P)_2\|^2$, we have

$$\|\Delta P\| \le \frac{\|(\Delta P)_2\|}{\sqrt{1-\theta(J,K)^2}} . \tag{A.8}$$

Next, consider the submanifold Γ and γ in the hypothesis and put $\Delta\gamma \triangleq \gamma(t+\Delta t) - \gamma(t)$. If $\Delta\gamma$ is small enough one can think of $\Delta\gamma$ belonging to $T_{\gamma(t)}\Gamma$. Then, it follows from (A.8) that

$$\|\Delta\gamma\| \le \frac{\|(\Delta\gamma)_2\|}{\sqrt{1-\theta(T_{\gamma(t)}\Gamma, K)^2}} . \tag{A.9}$$

Integrating (A.9) over $[0,1]$, we have (A.7). ¤

Proof of Proposition 1

Step 1: Approximation of Λ by a C^{∞} manifold.

It follows from Lemma B that there is a C^1 embedding $H : \Lambda \to \mathbb{R}^{2b}$ satisfying (i) - (iv) of Lemma B. We fix one such H and set $\tilde{\Lambda} \triangleq H(\Lambda)$, $\tilde{\Sigma} \triangleq H(\Lambda) \cap K$. Since $\tilde{\Lambda}$ and $\tilde{\Sigma}$ are C^{∞} manifolds, it follows from Lemma A that there is a tubular neighborhood of $\tilde{\Sigma}$ in $\tilde{\Lambda}$. Let $N\tilde{\Sigma}$ be the normal bundle on $\tilde{\Sigma}$ and let $N_y\tilde{\Sigma}$ be the orthogonal complement of $T_y\tilde{\Sigma}$ in $T_y\tilde{\Lambda}$. There are a neighborhood $\tilde{V}$ of the zero section in $N\tilde{\Sigma}$, a neighborhood $\tilde{U}$ of $\tilde{\Sigma}$ in $\tilde{\Lambda}$ and the tubular map $\phi : \tilde{V} \to \tilde{U}$ such that

$$\phi | N_y\tilde{\Sigma} \cap \tilde{V} = \exp_y | N_y\tilde{\Sigma} \cap \tilde{V} .$$

Then $\phi(0_y) = y$ and

$$(d(\phi | N_y\tilde{\Sigma} \cap \tilde{V}))_{0_y} : N_y\tilde{\Sigma} \to T_y\tilde{\Lambda} \tag{A.10}$$

is a linear injection. Therefore setting $\tilde{V}_y \triangleq N_y\tilde{\Sigma} \cap \tilde{V}$ one sees that

$$\tilde{U}_{\underset{\sim}{y}} \triangleq \underset{\sim}{\phi}(\tilde{V}_{\underset{\sim}{y}}) \tag{A.11}$$

is an open ball of dimension $b = n + n_R$ which is transversal to $\tilde{\Sigma}$ at $\underset{\sim}{y}$. Set

$$U \triangleq \underset{\sim}{H}^{-1}(\tilde{U}),\ U_{\underset{\sim}{x}} \triangleq \underset{\sim}{H}^{-1}(\tilde{U}_{\underset{\sim}{H}(\underset{\sim}{x})}) \ . \tag{A.12}$$

Since $\underset{\sim}{H}^{-1} : \tilde{\Lambda} \to \Lambda$ is a C^1 diffeomorphism, U is a tubular neighborhood of Σ in Λ and $U_{\underset{\sim}{x}}$ is an open ball of dimension $b = n + n_R$ which is transversal to Σ at $\underset{\sim}{x}$. (Fig. 25)

<u>Step 2</u>: Definition of $\underset{\sim}{G} : \Sigma \to \hat{\Sigma} \triangleq \underset{\sim}{F}(\Lambda) \cap K$.

Let $\underset{\sim}{H}$, U, V, $\tilde{U}$, $\tilde{V}$ etc. be as in <u>Step 1</u>. We claim that if $\underset{\sim}{F}$ is close to $\underset{\sim}{\imath}_\Lambda$ in the strong C^1 topology, then

$$\underset{\sim}{F}(U_{\underset{\sim}{x}}) \bar{\pitchfork} K \tag{A.13}$$

and

$$\underset{\sim}{F}(U_{\underset{\sim}{x}}) \cap K \text{ is a single point .} \tag{A.14}$$

In order to prove this observe

$$T_{\underset{\sim}{y}}\tilde{\Lambda} = T_{\underset{\sim}{y}}\tilde{\Sigma} \oplus \tilde{V}_{\underset{\sim}{y}} \ . \tag{A.15}$$

Since the map defined by (A.10) is a linear injection, (A.11) and (A.15) imply

$$T_{\underset{\sim}{y}}\tilde{\Lambda} = T_{\underset{\sim}{y}}\tilde{\Sigma} \oplus T_{\underset{\sim}{y}}\tilde{U}_{\underset{\sim}{y}} \ . \tag{A.16}$$

Since $\underset{\sim}{H}$ is an embedding, if $\underset{\sim}{y} = \underset{\sim}{H}(\underset{\sim}{x})$, then (A.16) implies

$$T_{\underset{\sim}{x}}\Lambda = T_{\underset{\sim}{x}}\Sigma \oplus T_{\underset{\sim}{x}}U_{\underset{\sim}{x}} \ . \tag{A.17}$$

Since $T_{\underset{\sim}{x}}\Sigma \subset T_{\underset{\sim}{x}}K$ and since $\Lambda \bar{\pitchfork} K$, (A.17) implies

$$\mathbb{R}^{2b} = T_{\underset{\sim}{x}}\Lambda + T_{\underset{\sim}{x}}K = T_{\underset{\sim}{x}}U_{\underset{\sim}{x}} + T_{\underset{\sim}{x}}\Sigma + T_{\underset{\sim}{x}}K = T_{\underset{\sim}{x}}U_{\underset{\sim}{x}} + T_{\underset{\sim}{x}}K. \tag{A.18}$$

Since $\dim T_{\underset{\sim}{x}}U_{\underset{\sim}{x}} + \dim T_{\underset{\sim}{x}}K = 2b$, (A.18) implies

$$T_{\underset{\sim}{x}}U_{\underset{\sim}{x}} \oplus T_{\underset{\sim}{x}}K = \mathbb{R}^{2b} \ . \tag{A.19}$$

Now, since $\exp_{\underset{\sim}{y}}(\tilde{V}_{\underset{\sim}{y}}) = \tilde{U}_{\underset{\sim}{y}}$, one has

$$\tilde{U}_{\underset{\sim}{y}} \cap K = \{\underset{\sim}{y}\} \ . \tag{A.20}$$

We claim that

$$U_{\underset{\sim}{x}} \cap K = \{\underset{\sim}{x}\} \ . \tag{A.21}$$

If not, there is an $\underset{\sim}{x}' \neq \underset{\sim}{x}$ in $U_{\underset{\sim}{x}} \cap K$. Then

$$x' \in U_x \cap K \subset \Lambda \cap K = \Sigma \;. \tag{A.22}$$

Since $H(\Sigma) = H(\Lambda) \cap K$, (A.22) implies

$$H(x') \in \tilde{\Lambda} \cap K \subset K$$

and hence

$$H(x') \in \tilde{U}_y \;. \tag{A.23}$$

But since H is an embedding $H(x') \neq H(x)$. Therefore, (A.23) contradicts (A.20). Hence if F is close enough to ι_Λ, then (A.19) and (A.21) imply (A.13) and (A.14).

For $x \in \Sigma$ define (Fig. 26)

$$G(x) \triangleq F(U_x) \cap K \;.$$

It follows from (A.14) that G is a well-defined function. G is a surjection onto $F(\Lambda) \cap K$ because $F(\Lambda) \cap K = F(U) \cap K$ if F is close enough to ι_Λ. By definition, G is an injection. We next claim that G is a local diffeomorphism at each point. To prove this, let $\nu : N\tilde{\Sigma} \to \tilde{\Sigma}$ be the normal bundle map defined by $\nu(\eta) = y$, where $\eta \in N_y\tilde{\Sigma}$. Let $\tilde{\psi} \triangleq \nu \circ \phi^{-1}$ and (Fig. 27)

$$\psi \triangleq H^{-1} \circ \tilde{\psi} \circ H \;.$$

Generally, the degree of differentiability of ν is lower than that of $\tilde{\Sigma}$. But since $\tilde{\Sigma}$ is C^∞, we have that ν is C^∞. Since ϕ^{-1} is also C^∞, we know that $\tilde{\psi}$ is C^∞ and ψ is C^1. Since $\hat{\Sigma} = F(\Lambda) \cap K$ is a C^1 submanifold, we see that

$$\psi \circ F^{-1}|\hat{\Sigma} : \hat{\Sigma} \to \Sigma$$

is C^1. Observe that for any $y \in \hat{\Sigma}$, there is an $x \in \Sigma$ such that $y \in F(U_x)$. Since $F(U_x) \pitchfork K$, we have

$$(F(U_x) \cap F(\Lambda)) \pitchfork (K \cap F(\Lambda)) \text{ in } F(\Lambda) \;. \tag{A.24}$$

Since $F(U_x) \cap F(\Lambda) = F(U_x)$ and since $K \cap F(\Lambda) = \hat{\Sigma}$, (A.24) implies $F(U_x) \pitchfork \hat{\Sigma}$, i.e.,

$$T_P F(U) = T_P F(U_x) \oplus T_P \hat{\Sigma} \tag{A.25}$$

for all $P \in F(U_x) \cap \hat{\Sigma}$. By the definition of ψ, we know that $F \circ \psi^{-1}(x) = F(U_x)$. Set $P = y$ in (A.25) and map both sides of (A.25) by $(d(\psi \circ F^{-1}))_y$.

By the definition of ψ, the map $(\underset{\sim}{d}(\underset{\sim}{\psi}\circ\underset{\sim}{F}^{-1}))_{\underset{\sim}{y}}$ annihilates the direction perpendicular to $T_{\underset{\sim}{y}}\hat{\Sigma}$. Therefore, we have $(\underset{\sim}{d}(\underset{\sim}{\psi}\circ\underset{\sim}{F}^{-1}))_{\underset{\sim}{y}} T_{\underset{\sim}{y}} \underset{\sim}{F}(U_{\underset{\sim}{x}}) = \underset{\sim}{0}$ and $(\underset{\sim}{d}(\underset{\sim}{\psi}\circ\underset{\sim}{F}^{-1}))_{\underset{\sim}{y}} T_{\underset{\sim}{y}} \underset{\sim}{F}(U) = T_{\underset{\sim}{x}}\Sigma$. Hence $T_{\underset{\sim}{x}}\Sigma = \{\underset{\sim}{0}\} \oplus (\underset{\sim}{d}(\underset{\sim}{\psi}\circ\underset{\sim}{F}^{-1}))_{\underset{\sim}{y}} T_{\underset{\sim}{y}}\hat{\Sigma}$. Since $T_{\underset{\sim}{x}}\Sigma$ and $T_{\underset{\sim}{y}}\hat{\Sigma}$ have the same dimension, the map $\underset{\sim}{d}(\underset{\sim}{\psi}\circ\underset{\sim}{F}^{-1})_{\underset{\sim}{y}}$ is an isomorphism. By definition, we know that $\underset{\sim}{\psi} \circ \underset{\sim}{F}^{-1}|\hat{\Sigma} = \underset{\sim}{G}^{-1}$. Since we already showed that $\underset{\sim}{G}$ is bijective, we conclude that $\underset{\sim}{G}$ is a C^1 diffeomorphism.

<u>Step 3</u>: Estimate of $\|\underset{\sim}{G}(\underset{\sim}{x}) - \underset{\sim}{x}\|$.

Let Λ, Σ, $\underset{\sim}{F}$, $\underset{\sim}{G}$ etc. be the same as in <u>Step 1</u> and <u>Step 2</u>. There is a sequence $\phi = \Lambda_0 \subset \Lambda_1 \subset \Lambda_2 \subset \ldots.$ of compact submanifolds with boundary such that $\bigcup_{k=0}^{\infty} \Lambda_k = \Lambda$ and $\Lambda_k \subset \text{Int}\, \Lambda_{k+1}$, where Int denotes the interior of a set. Let U be the tube of Σ in Λ obtained in <u>Step 1</u> and set

$$U_k \triangleq (\Lambda_{k+1} - \Lambda_k) \cap U \ . \tag{A.26}$$

Then $\overline{U}_k$ is a compact manifold with boundary where the bar denotes closure of a set. Since each $\underset{\sim}{y} \in U$ belongs to a unique $U_{\underset{\sim}{x}}$, we denote it by $U_{\underset{\sim}{x}(\underset{\sim}{y})}$. By the definition of $U_{\underset{\sim}{x}}$, the map : $U \to G_b$, $\underset{\sim}{y} \mapsto T_{\underset{\sim}{y}} U_{\underset{\sim}{x}(\underset{\sim}{y})}$, is continuous, where G_b is the set of all b-dimensional linear subspaces. It follows from (A.5) and $\Lambda \pitchfork K$ that there is an A_k', $0 \leq A_k' < 1$ such that

$$\theta(T_{\underset{\sim}{y}} U_{\underset{\sim}{x}(\underset{\sim}{y})}, K) \leq A_k' \text{ for all } \underset{\sim}{y} \in U_k \ .$$

Define $A_k \triangleq A_k' + \dfrac{1-A_k'}{2} < 1$. Then, there is a continuous function $\delta_1 : \Lambda \to \mathbb{R}^+$ such that $d_1(\underset{\sim}{F}, \underset{\sim}{\iota}_\Lambda)(\underset{\sim}{y}) < \delta_1(\underset{\sim}{y})$ implies

$$\theta(T_{\underset{\sim}{F}(\underset{\sim}{y})}(\underset{\sim}{F}(U_{\underset{\sim}{x}(\underset{\sim}{y})})), K) \leq A_k \text{ for all } \underset{\sim}{y} \in U_k \ . \tag{A.27}$$

Next, we will estimate the distance between $\underset{\sim}{x} \in \Sigma$ and $\underset{\sim}{G}(\underset{\sim}{x}) = \underset{\sim}{F}(U_{\underset{\sim}{x}}) \cap K$. Let us write $\underset{\sim}{F}(\underset{\sim}{x}) - \underset{\sim}{G}(\underset{\sim}{x}) = (\underset{\sim}{u}_1, \underset{\sim}{u}_2)$, where $\underset{\sim}{u}_1 \in K$, $\underset{\sim}{u}_2 \in K^\perp$. (Fig. 26) Let U_2 be the linear subspace spanned by $\underset{\sim}{u}_2$. Then $\underset{\sim}{F}(U_{\underset{\sim}{x}}) \pitchfork (K \oplus U_2)$ because $T_{\underset{\sim}{y}} \underset{\sim}{F}(U_{\underset{\sim}{x}}) \pitchfork K$ for all $\underset{\sim}{y} \in \underset{\sim}{F}(U_{\underset{\sim}{x}})$. Therefore $M \triangleq \underset{\sim}{F}(U_{\underset{\sim}{x}}) \cap (K \oplus U_2)$ is a 1-dimensional submanifold. Let $\underset{\sim}{\pi}_2 : K \oplus U_2 \to U_2$ be the natural projection map. We claim that $\underset{\sim}{\pi}_2 | M : M \to \underset{\sim}{\pi}_2(M) \subset U_2$ is a diffeomorphism. To prove this let $\underset{\sim}{\pi}_\perp : K \oplus K^\perp \to K^\perp$ be the natural projection map. Then $\underset{\sim}{d\pi}_\perp$ maps $T_{\underset{\sim}{y}} \underset{\sim}{F}(U_{\underset{\sim}{x}})$ isomorphically onto $K^\perp$ because $T_{\underset{\sim}{y}} \underset{\sim}{F}(U_{\underset{\sim}{x}}) \pitchfork K$ and

$\dim K^{\perp} = \dim T_{\underset{\sim}{y}}\underset{\sim}{F}(U_{\underset{\sim}{x}})$. Therefore, $\underset{\sim}{\pi}_{\perp}|\underset{\sim}{F}(U_{\underset{\sim}{x}}) : \underset{\sim}{F}(U_{\underset{\sim}{x}}) \to K^{\perp}$ is a local diffeomorphism. Since $U_{\underset{\sim}{x}}$ is close to a part of an affine submanifold and since $\underset{\sim}{F}$ is close to the identity map, $\underset{\sim}{\pi}_{\perp}|\underset{\sim}{F}(U_{\underset{\sim}{x}})$ is injective. Finally, since $\underset{\sim}{\pi}_2|M = \underset{\sim}{\pi}_{\perp}|M$, we conclude that $\underset{\sim}{\pi}_2|M : M \to \underset{\sim}{\pi}_2(M)$ is a diffeomorphism. Hence $\underset{\sim}{\pi}_2(M)$ is a C^1 curve. Observe that $\underset{\sim}{F}(\underset{\sim}{x}), \underset{\sim}{G}(\underset{\sim}{x}) \in M$ and let $\underset{\sim}{\gamma} : [0,1] \to M$ be a C^1 arc from $\underset{\sim}{G}(\underset{\sim}{x})$ to $\underset{\sim}{F}(\underset{\sim}{x})$. It follows from the above argument that $\underset{\sim}{\pi}_2 \circ \underset{\sim}{\gamma}$ is also an arc. It follows from Lemma D that $\theta(T_{\underset{\sim}{F}(\underset{\sim}{y})}(\underset{\sim}{F}(U_{\underset{\sim}{x}(\underset{\sim}{y})})),K)$ of (A.27) is continuous in $\underset{\sim}{y}$. Let $L(t)$ and $L_2(t)$ be as in Lemma E. Then Lemma E and (A.27) imply that if $d_1(\underset{\sim}{F},\underset{\sim}{\iota}_{\Lambda})(\underset{\sim}{y}) < \delta_1(\underset{\sim}{y})$ then for $\underset{\sim}{x} \in \Sigma \cap U_k$, the following is true:

$$\begin{aligned}
\|\underset{\sim}{G}(\underset{\sim}{x}) - \underset{\sim}{F}(\underset{\sim}{x})\| &\le L(1) \\
&\le \int_0^1 \frac{1}{\sqrt{1-\theta(T_{\underset{\sim}{\gamma}(t)}\underset{\sim}{F}(U_{\underset{\sim}{x}}),K)^2}} \frac{dL_2(t)}{dt}\, dt \\
&\le \int_0^1 \frac{1}{\sqrt{1-A_k^2}} \frac{dL_2(t)}{dt}\, dt \\
&= \frac{1}{\sqrt{1-A_k^2}} \|\underset{\sim}{u}_2\| \qquad \text{(A.28)}
\end{aligned}$$

where $0 \le A_k < 1$. Let us denote the $K^{\perp}$ - component of a vector by $(\ )_2$. Since $\underset{\sim}{x}$, $\underset{\sim}{G}(\underset{\sim}{x}) \in K$ for $\underset{\sim}{x} \in \Sigma$ we have

$$\begin{aligned}
\underset{\sim}{u}_2 = (\underset{\sim}{F}(\underset{\sim}{x})-\underset{\sim}{G}(\underset{\sim}{x}))_2 &= (\underset{\sim}{F}(\underset{\sim}{x})-\underset{\sim}{G}(\underset{\sim}{x}))_2 + (\underset{\sim}{G}(\underset{\sim}{x})-\underset{\sim}{x})_2 \\
&= (\underset{\sim}{F}(\underset{\sim}{x})-\underset{\sim}{x})_2 \ . \qquad \text{(A.29)}
\end{aligned}$$

Since $\|(\underset{\sim}{F}(\underset{\sim}{x})-\underset{\sim}{x})_2\| \le \|\underset{\sim}{F}(\underset{\sim}{x})-\underset{\sim}{x}\|$, (A.28) and (A.29) imply

$$\|\underset{\sim}{G}(\underset{\sim}{x}) - \underset{\sim}{F}(\underset{\sim}{x})\| \le \frac{1}{\sqrt{1-A_k^2}} \|\underset{\sim}{F}(\underset{\sim}{x}) - \underset{\sim}{x}\|$$

for $\underset{\sim}{x} \in \Sigma \cap U_k$. Hence, for $\underset{\sim}{x} \in \Sigma \cap U_k$,

$$\|G(x) - x\| \le \left(\frac{1}{\sqrt{1-A_k^2}} + 1\right) \|F(x) - x\| \tag{A.30}$$

if $d_1(F,\iota_\Lambda)(y) < \delta_1(y)$ for all $y \in \Lambda$.

Step 4: Estimate of $\|(dG)_x - 1\|$.

For a point $x \in \Sigma$, let N be a neighborhood of x in $\mathbb{R}^{2b}$. There is a local chart $(\bar{\psi},N)$ of x such that $\bar{\psi} : N \to V \times W \times Z$ is a diffeomorphism where Z is a neighborhood of the origin of $\mathbb{R}^b$, W is a neighborhood of the origin of $\mathbb{R}^{n_R}$ and V is a neighborhood of the origin of $\mathbb{R}^n$, $\bar{\psi}(N\cap\Sigma) = V$, $\bar{\psi}(N\cap K) = V \times W$ and $\bar{\psi}(N\cap U_x) = \{\bar{\psi}(x)\} \times \{0\} \times Z$. In order to simplify the arguments, we will identify N with $V \times W \times Z$. (Fig. 28). If $x \in \Lambda\cap N$, then x is of the form $x = (v,0,z) \in V \times W \times Z$. Let $F(v,0,z) = (F_1(v,0,z), F_2(v,0,z), F_3(v,0,z)) \in V \times W \times Z$ where F is as before. If F is C^1 close to ι_Λ, then $F_3(v,0,z) \doteq z$ where $\doteq$ means that the left hand side is approximately equal to the right hand side. Although we could give a precise estimate we do not need it and this approximation simplifies the argument significantly. It follows from the implicit function theorem that there is a function $z = Q(v)$ satisfying $F_3(v,0,Q(v)) = 0$. We can also show that $(dQ) = -(D_zF_3)^{-1}(D_vF_3)$ where (D_zF_3) (resp. (D_vF_3)) is the partial derivative of F_3 with respect to z (resp. v). Therefore, we can make $\|(dQ)_v\|$ arbitrarily small by making F sufficiently C^1 close to ι_Λ. If $x \in \Sigma \cap N$, then x is of the form $x = (v,0,0)$ and $F^{-1} \circ G(v,0,0) = (v,0,Q(v))$. (Fig. 28) Hence for any $\alpha > 0$ there is a $\beta > 0$ such that $d_1(F,\iota_\Lambda)(x) < \beta$, $x \in \Sigma \cap N$ implies $\|(d(F^{-1}\circ G))_x - 1\| < \alpha$, $x \in \Sigma \cap N$. Since $G = F \circ (F^{-1}\circ G)$, we have $(dG)_x = (dF)_{F^{-1}\circ G(x)}(d(F^{-1}\circ G))_x$. Since $\|(dF)\|$ is bounded in $\Lambda \cap N$, there is an $a > 0$ such that $\|(dF)_x\| < a$ for $x \in \Lambda \cap N$. Without loss of generality assume $\beta < \alpha$. Then

$$\begin{aligned}
\|(dG)_x - 1\| &= \|(dF)_{F^{-1}\circ G(x)}(d(F^{-1}\circ G))_x - 1\| \\
&= \|(dF)_{F^{-1}\circ G(x)}((d(F^{-1}\circ G))_x - 1) + ((dF)_{F^{-1}\circ G(x)} - 1)\| \\
&\le \|(dF)_{F^{-1}\circ G(x)}\| \, \|(d(F^{-1}\circ G))_x - 1\| + \|(dF)_{F^{-1}\circ G(x)} - 1\| \\
&< a\,\alpha + \beta < (a+1)\beta, \qquad x \in \Sigma \cap N \ .
\end{aligned} \tag{A.31}$$

Now recall U_k defined by (A.26). Since $\overline{U}_k$ is compact, it is covered by a finite number of neighborhoods of the form N given in the above argument. Therefore, (A.31) implies that for any $\varepsilon_k > 0$, there is a $\overline{\delta}_k > 0$ such that

$$d_1(\underset{\sim}{F},\underset{\sim}{\iota}_\Lambda)(\underset{\sim}{x}) < \overline{\delta}_k \text{ implies } \|(\underset{\sim}{dG})_{\underset{\sim}{x}} - \underset{\sim}{1}\| < \frac{\varepsilon_k}{2} \tag{A.32}$$

where $\underset{\sim}{x} \in \Sigma \cap U_k$.

Step 5: Completion of the Proof.

Let $\varepsilon : \Sigma \to \mathbb{R}^+$ be the continuous function given in the statement of Proposition 1. Recall Λ_k and U_k defined in Step 1. Since $(\Lambda_{k+1}-\Lambda_k) \subset \text{Int } \Lambda$ and since $\Sigma \cap \overline{U}_k = \overline{((\Lambda_{k+1}-\Lambda_k) \cap U)} \cap K$, the set $\Sigma \cap \overline{U}_k$ is compact. Since ε is continuous $\varepsilon_k \triangleq \inf\{\varepsilon(\underset{\sim}{x}) | \underset{\sim}{x} \in \Sigma \cap \overline{U}_k\}$ is attained on $\Sigma \cap \overline{U}_k$ and $\varepsilon_k > 0$. Choose a continuous function $\delta_2 : \Sigma \to \mathbb{R}^+$ in such a way that

$$\delta_2(\underset{\sim}{x}) < \frac{\varepsilon_k}{2\left(\frac{1}{\sqrt{1-A_k^2}} + 1\right)}$$

holds. It follows from (A.30) that if $\|\underset{\sim}{F}(\underset{\sim}{x}) - \underset{\sim}{x}\| < \delta_2(\underset{\sim}{x})$ and if $d_1(\underset{\sim}{F},\underset{\sim}{\iota}_\Lambda)(\underset{\sim}{y}) < \delta_1(\underset{\sim}{y})$ for all $\underset{\sim}{y} \in \Lambda$, then

$$\|\underset{\sim}{G}(\underset{\sim}{x}) - \underset{\sim}{x}\| \leq \left(\frac{1}{\sqrt{1-A_k^2}} + 1\right) \|\underset{\sim}{F}(\underset{\sim}{x}) - \underset{\sim}{x}\| < \frac{\varepsilon_k}{2} \leq \frac{\varepsilon(\underset{\sim}{x})}{2}$$

where $\underset{\sim}{x} \in \Sigma \cap U_k$. Hence, for any $\underset{\sim}{x} \in \Sigma$, if $\|F(\underset{\sim}{x}) - \underset{\sim}{x}\| < \delta_2(\underset{\sim}{x})$ and if $d_1(\underset{\sim}{F},\underset{\sim}{\iota}_\Lambda)(\underset{\sim}{y}) < \delta_1(\underset{\sim}{y})$ for all $\underset{\sim}{y} \in \Lambda$, then

$$\|\underset{\sim}{G}(\underset{\sim}{x}) - \underset{\sim}{x}\| < \frac{\varepsilon(\underset{\sim}{x})}{2}. \tag{A.33}$$

Next define a continuous function $\delta_3 : \Sigma \to \mathbb{R}^+$ in such a way that

$$\delta_3(\underset{\sim}{x}) \leq \overline{\delta}_k, \; \underset{\sim}{x} \in \Sigma \cap U_k$$

where $\overline{\delta}_k$ is as in (A.32). Then, for $\underset{\sim}{x} \in \Sigma \cap U_k$, $d_1(\underset{\sim}{F},\underset{\sim}{\iota}_\Lambda)(\underset{\sim}{x}) < \delta_3(\underset{\sim}{x})$ implies

$$\|(\underset{\sim}{dG})_{\underset{\sim}{x}} - \underset{\sim}{1}\| < \frac{\varepsilon_k}{2} \leq \frac{\varepsilon(\underset{\sim}{x})}{2} \quad . \tag{A.34}$$

Finally define $\delta : \Sigma \to \mathbb{R}^+$ by

$$\delta(\underset{\sim}{x}) \triangleq \min\{\delta_1(\underset{\sim}{x}), \delta_2(\underset{\sim}{x}), \delta_3(\underset{\sim}{x})\} .$$

Then, it follows from (A.33) and (A.34) that $d_1(\underset{\sim}{F},\underset{\sim}{\iota}_\Lambda)(\underset{\sim}{x}) < \delta(\underset{\sim}{x})$, $\underset{\sim}{x} \in \Sigma$, implies

$$d_1(\underset{\sim}{G},\underset{\sim}{\iota}_\Lambda)(\underset{\sim}{x}) < \varepsilon(\underset{\sim}{x}) \ . \qquad \square$$

Appendix II

The following is a version of Whiteny Immersion Theorem [6]. Although the fact is not stated in [6] as it is stated here, it is not difficult to prove it.

Lemma F. Let M be an m-dimensional C^1 manifold and let $\underset{\sim}{F} : M \to \mathbb{R}^{2m}$ be an arbitrary C^1 map. Then the following hold:

(i) There is a C^1 nice immersion $\underset{\sim}{G} : M \to \mathbb{R}^{2m}$ which is arbitrarily C^1 close to $\underset{\sim}{F}$.

(ii) If $\underset{\sim}{F}$ is a nice immersion, then every C^1 map $\underset{\sim}{G}$ which is C^1 close to $\underset{\sim}{F}$ is a nice immersion.

In other words, the set of all nice immersions is a dense and open subset of $C^1(M;\mathbb{R}^{2m})$ in the strong C^1 topology.

Lemma G. Let Λ be a C^1 submanifold of $\mathbb{R}^{2b}$ and let Σ be a C^1 submanifold of Λ. Then, for any continuous function $\varepsilon : \Lambda \to \mathbb{R}^+$, there is a continuous function $\delta : \Sigma \to \mathbb{R}^+$ with the following property : for an arbitrary embedding $\underset{\sim}{F}_0 : \Sigma \to \mathbb{R}^{2b}$ with $d_1(\underset{\sim}{F}_0,\underset{\sim}{\iota})(\underset{\sim}{x}) < \delta(\underset{\sim}{x})$, there is a C^1 embedding $\underset{\sim}{F} : \Lambda \to \mathbb{R}^{2b}$ such that $\underset{\sim}{F}|\Sigma = \underset{\sim}{F}_0$ and $d_1(\underset{\sim}{F}(\underset{\sim}{x}),\underset{\sim}{\iota}_\Lambda)(\underset{\sim}{x}) < \varepsilon(\underset{\sim}{x})$, where $\underset{\sim}{\iota}$ is defined by (21).

Proof. Let $\dim \Sigma = n$. It follows from [5, Theorem 5.5] and its proof that there are a neighborhood W of Σ in $\mathbb{R}^{2b}$ and a family of (2b-n)-dimensional affine submanifolds $\{M_{\underset{\sim}{x}} | \underset{\sim}{x} \in \Sigma\}$ such that

(i) $M_{\underset{\sim}{x}}$ is C^1 in $\underset{\sim}{x} \in \Sigma$,

(ii) $M_{\underset{\sim}{x}} \pitchfork \Sigma$, $\underset{\sim}{x} \in \Sigma$,

(iii) $(M_{\underset{\sim}{x}} \cap W) \cap (M_{\underset{\sim}{y}} \cap W) = \emptyset$ if $\underset{\sim}{x} \neq \underset{\sim}{y}$.

Itfollows from these properties that there is a C^1 function $\alpha : \Sigma \to \mathbb{R}^+$ such that $0 < \alpha(\underset{\sim}{x}) < 1$ and the ball $B_{\underset{\sim}{x}} \triangleq \{\underset{\sim}{y} \in M_{\underset{\sim}{x}} | \|\underset{\sim}{x} - \underset{\sim}{y}\| \leq \alpha(\underset{\sim}{x})\}$ contained in W. (Fig. 29) It is clear that there is a C^1 function $\beta : [0,1] \to [0,1]$ such that $\beta(0) = 1$, $\beta(1) = 0$, $(D\beta)_t \leq 0$, $t \in [0,1]$, and $(D\beta)_0 = (D\beta)_1 = 0$. For example,

$$\beta(t) \triangleq \int_0^t e^{-\frac{1}{s}+\frac{1}{s-1}} ds \Big/ \int_0^1 e^{-\frac{1}{s}+\frac{1}{s-1}} ds$$

will do. Now, for the given $\varepsilon : \Sigma \to \mathbb{R}^+$ in the statement let $\varepsilon' : \Sigma \to \mathbb{R}^+$ satisfy $\varepsilon'(\underset{\sim}{x}) < \varepsilon(\underset{\sim}{y})$ for any $\underset{\sim}{y} \in B_{\underset{\sim}{x}}$ and define γ and $\delta : \Sigma \to \mathbb{R}^+$ by

$$\gamma(\underset{\sim}{x}) \triangleq \max\left\{\left(\left\|\frac{(D\alpha)_{\underset{\sim}{x}}}{\alpha(\underset{\sim}{x})}\right\| + \frac{1}{\alpha(\underset{\sim}{x})}\right)\left(\max_{t\in[0,1]}(D\beta)_t\right), 1\right\} \tag{A.35}$$

and

$$\delta(\underset{\sim}{x}) \triangleq \frac{\alpha(\underset{\sim}{x})\varepsilon'(\underset{\sim}{x})}{4\gamma(\underset{\sim}{x})} \tag{A.36}$$

respectively. Without loss of generality one may assume that $W = \bigcup_{\underset{\sim}{x}\in\Sigma} B_{\underset{\sim}{x}}$. Then, for any $\underset{\sim}{y} \in W$ there is a unique $\underset{\sim}{x} \in \Sigma$ such that $\underset{\sim}{y} \in B_{\underset{\sim}{x}}$. Define $\underset{\sim}{F} : \mathbb{R}^{2b} \to \mathbb{R}^{2b}$ by

$$\underset{\sim}{F}(\underset{\sim}{y}) \triangleq \begin{cases} \underset{\sim}{y} + \beta\left(\frac{\|\underset{\sim}{y}-\underset{\sim}{x}\|}{\alpha(\underset{\sim}{x})}\right)(\underset{\sim}{F}_0(\underset{\sim}{x})-\underset{\sim}{x}) & \text{if } \underset{\sim}{y} \in W \\ \underset{\sim}{y} & \text{if } \underset{\sim}{y} \notin W . \end{cases} \tag{A.37}$$

It is clear that $\underset{\sim}{F}$ is well-defined and is C^1. We will show that if $d_1(\underset{\sim}{F}_0,\underset{\sim}{\iota})(\underset{\sim}{x}) < \delta(\underset{\sim}{x})$, then

$$d_1(\underset{\sim}{F},\underset{\sim}{\iota}_\Lambda)(\underset{\sim}{y}) < \varepsilon(\underset{\sim}{y}), \quad \underset{\sim}{y} \in \Lambda . \tag{A.38}$$

In order to show this observe that at each $\underset{\sim}{x}_0 \in \Sigma$, there is a chart $(\underset{\sim}{\phi},U)$, where U is a neighborhood of $\underset{\sim}{x}_0$ in Σ such that $\underset{\sim}{\phi}(U) \subset \mathbb{R}^n \times \{\underset{\sim}{0}\}$ and $\underset{\sim}{\phi}(B_{\underset{\sim}{x}}) \subset \{\underset{\sim}{z}\} \times \mathbb{R}^{2b-n}$ for some point $\underset{\sim}{z} \in \mathbb{R}^n$. Let $\underset{\sim}{\phi}^{-1}(\underset{\sim}{v},\underset{\sim}{w}) \triangleq \underset{\sim}{y}$ and denote the derivative of $\underset{\sim}{F}$ in terms of $\underset{\sim}{\phi}$ by

$$\begin{aligned}(\underset{\sim}{D}(\underset{\sim}{F}\circ\underset{\sim}{\phi}^{-1}))_{(\underset{\sim}{v},\underset{\sim}{w})} &= ((\underset{\sim}{D}_{\underset{\sim}{v}}\underset{\sim}{F}\circ\underset{\sim}{\phi}^{-1})_{(\underset{\sim}{v},\underset{\sim}{w})}, (\underset{\sim}{D}_{\underset{\sim}{w}}\underset{\sim}{F}\circ\underset{\sim}{\phi}^{-1})_{(\underset{\sim}{v},\underset{\sim}{w})}) \\ &\triangleq ((\underset{\sim}{D}_\Sigma\underset{\sim}{F})_{\underset{\sim}{y}}, (\underset{\sim}{D}_B\underset{\sim}{F})_{\underset{\sim}{y}}) .\end{aligned}$$

Then, it follows from (A.35) and (A.36) $\left(t \triangleq \frac{\|\underset{\sim}{y}-\underset{\sim}{x}\|}{\alpha(\underset{\sim}{x})}\right)$ that

$$\begin{aligned}\|(\underset{\sim}{D}_\Sigma(\underset{\sim}{F}-\underset{\sim}{\iota}_\Lambda))_{\underset{\sim}{y}}\| &= \left\|\left(\underset{\sim}{D}_{\underset{\sim}{x}}\left(\beta\left(\frac{\|\underset{\sim}{y}-\underset{\sim}{x}\|}{\alpha(\underset{\sim}{x})}\right)(\underset{\sim}{F}_0(\underset{\sim}{x})-\underset{\sim}{x})\right)\right)_{(\underset{\sim}{x},\underset{\sim}{y})}\right\| \\ &= \left\|(D\beta)_t \frac{-(D\alpha)_{\underset{\sim}{x}}\|\underset{\sim}{y}-\underset{\sim}{x}\| - \frac{\underset{\sim}{y}-\underset{\sim}{x}}{\|\underset{\sim}{y}-\underset{\sim}{x}\|}\alpha(\underset{\sim}{x})}{(\alpha(\underset{\sim}{x}))^2}(\underset{\sim}{F}_0(\underset{\sim}{x})-\underset{\sim}{x}) + \beta(t)((\underset{\sim}{D}\underset{\sim}{F}_0)_{\underset{\sim}{x}}-\underset{\sim}{1})\right\| \\ &\leq (D\beta)_t\left(\left\|\frac{(D\alpha)_{\underset{\sim}{x}}}{\alpha(\underset{\sim}{x})}\right\| + \frac{1}{\alpha(\underset{\sim}{x})}\right)\|\underset{\sim}{F}_0(\underset{\sim}{x}) - \underset{\sim}{x}\| + \|(\underset{\sim}{D}\underset{\sim}{F}_0)_{\underset{\sim}{x}} - \underset{\sim}{1}\|\end{aligned}$$

$$\leq \gamma(\underset{\sim}{x})(\|\underset{\sim}{F}_0(\underset{\sim}{x})-\underset{\sim}{x}\| + \|(\underset{\sim}{DF}_0)_{\underset{\sim}{x}}-\underset{\sim}{1}\|)$$

$$= \gamma(\underset{\sim}{x})\ d_1(\underset{\sim}{F}_0,\underset{\sim}{\iota})(\underset{\sim}{x}) < \gamma(\underset{\sim}{x})\delta(\underset{\sim}{x}) \leq \frac{1}{4}\varepsilon'(\underset{\sim}{x}) < \frac{1}{4}\varepsilon(\underset{\sim}{y}). \tag{A.39}$$

Also

$$\|(\underset{\sim}{D}_B(\underset{\sim}{F}-\underset{\sim}{\iota}_\Lambda))_{\underset{\sim}{y}}\| = \left\| \left(\underset{\sim}{D}_{\underset{\sim}{y}}\left(\beta\left(\frac{\|\underset{\sim}{y}-\underset{\sim}{x}\|}{\alpha(\underset{\sim}{x})}\right)(\underset{\sim}{F}_0(\underset{\sim}{x})-\underset{\sim}{x})\right)\right)_{(\underset{\sim}{x},\underset{\sim}{y})}\right\|$$

$$\leq \left\| (D\beta)_t \frac{\frac{\underset{\sim}{y}-\underset{\sim}{x}}{\|\underset{\sim}{y}-\underset{\sim}{x}\|}}{\alpha(\underset{\sim}{x})} (\underset{\sim}{F}_0(\underset{\sim}{x})-\underset{\sim}{x}) \right\|$$

$$\leq (D\beta)_t \frac{\|\underset{\sim}{F}_0(\underset{\sim}{x})-\underset{\sim}{x}\|}{\alpha(\underset{\sim}{x})} \leq (D\beta)_t \frac{\delta(\underset{\sim}{x})}{\alpha(\underset{\sim}{x})}$$

$$< \frac{1}{4}\varepsilon'(\underset{\sim}{x}) < \frac{1}{4}\varepsilon(\underset{\sim}{y}) . \tag{A.40}$$

It follows from (A.39) and (A.40) that

$$d_1(\underset{\sim}{F},\underset{\sim}{\iota}_\Lambda)(\underset{\sim}{y}) = \|\underset{\sim}{F}(\underset{\sim}{y})-\underset{\sim}{y}\| + \|(\underset{\sim}{DF})_{\underset{\sim}{y}} - \underset{\sim}{1}\|$$

$$\leq \beta(t)\|\underset{\sim}{F}_0(\underset{\sim}{x}) - \underset{\sim}{x}\| + \|(\underset{\sim}{DF})_{\underset{\sim}{y}} - \underset{\sim}{1}\|$$

$$\leq d_1(\underset{\sim}{F}_0,\underset{\sim}{\iota})(\underset{\sim}{x}) + \frac{1}{4}\varepsilon(\underset{\sim}{y}) + \frac{1}{4}\varepsilon(\underset{\sim}{y})$$

$$\leq \frac{1}{4}\varepsilon(\underset{\sim}{y}) + \frac{1}{4}\varepsilon(\underset{\sim}{y}) + \frac{1}{4}\varepsilon(\underset{\sim}{y}) = \frac{3}{4}\varepsilon(\underset{\sim}{y}) .$$

This implies (A.38). Since $\underset{\sim}{\iota}_\Lambda$ is an embedding, $\underset{\sim}{F}$ is an embedding also. Therefore this is the desired map. ¤

<u>Lemma H</u>. Let $\Lambda = \Lambda_R \times \mathbb{R}^{2n}$, $\Lambda \cap K \neq \emptyset$ and $\Lambda \pitchfork K$. If <u>Condition P</u> holds, then for a sufficiently small C^1 perturbation $\underset{\sim}{F} : \Sigma \to K$ of the inclusion $\underset{\sim}{\iota}_\Sigma : \Sigma \to K$, there is an arbitrarily small C^1 perturbation $\underset{\sim}{F}_R : \Lambda_R \to \mathbb{R}^{2n_R}$ of the inclusion $\underset{\sim}{\iota}_{\Lambda_R} : \Lambda_R \to \mathbb{R}^{2n_R}$ such that

$$\hat{\Sigma} \triangleq \underset{\sim}{F}(\Sigma) = \underset{\sim}{F}_R(\Lambda_R) \times \mathbb{R}^{2n} \cap K .$$

<u>Proof</u>. Since <u>Condition P</u> holds, <u>Corollary 1</u> implies that $\underset{\sim}{\chi}_R$ defined by (8) is injective. Therefore $\underset{\sim}{\pi}_R \triangleq \underset{\sim}{\chi}_R|\Sigma$ and $\hat{\underset{\sim}{\pi}}_R \triangleq \underset{\sim}{\chi}_R|\hat{\Sigma}$ are diffeomorphisms.

Define $\underset{\sim}{F}_{R0} : \underset{\sim}{\chi}_R(\Sigma) \to \underset{\sim}{\chi}_R(\hat{\Sigma})$ by

$$\underset{\sim}{F}_{R0} \triangleq \hat{\underset{\sim}{\pi}}_R \circ \underset{\sim}{F} \circ \underset{\sim}{\pi}_R^{-1} \quad . \tag{A.41}$$

If $\underset{\sim}{F} : \Sigma \to K$ is C^1 close to the inclusion : $\Sigma \to K$, then $\underset{\sim}{F}_{R0}$ is also C^1 close to the inclusion : $\underset{\sim}{\chi}_R(\Sigma) \to \mathbb{R}^{2n_R}$. It follows from an argument similar to that of the proof of Lemma G that there is an embedding $\underset{\sim}{F}_R : \Lambda_R \to \mathbb{R}^{2n_R}$ which is C^1 close to the inclusion : $\Lambda_R \to \mathbb{R}^{2n_R}$ such that

$$\underset{\sim}{F}_R | \underset{\sim}{\pi}_R(\Sigma) = \underset{\sim}{F}_{R0} \; . \tag{A.42}$$

We first claim that

$$\underset{\sim}{F}_R(\Lambda_R) \cap \underset{\sim}{\chi}_R(K) = \underset{\sim}{F}_R(\Lambda_R \cap \underset{\sim}{\chi}_R(K)) \; . \tag{A.43}$$

In order to prove this recall (A.37) and observe that the extension $\underset{\sim}{F}_R$ of $\underset{\sim}{F}_{R0}$ is defined in such a way that $\underset{\sim}{F}_R(\underset{\sim}{y}) - \underset{\sim}{y} = \beta\left(\frac{\|\underset{\sim}{y}-\underset{\sim}{x}\|}{\alpha(\underset{\sim}{x})}\right)(\underset{\sim}{F}_{R0}(\underset{\sim}{x})-\underset{\sim}{x})$. Since $\underset{\sim}{F}_{R0} : \underset{\sim}{\pi}_R(\Sigma) \to \hat{\underset{\sim}{\pi}}_R(\hat{\Sigma}) \subset \underset{\sim}{\chi}_R(K)$, we have $\underset{\sim}{F}_{R0}(\underset{\sim}{x}) - \underset{\sim}{x} \in \underset{\sim}{\chi}_R(K)$. Therefore $\underset{\sim}{F}_R(\underset{\sim}{y}) - \underset{\sim}{y} \in \underset{\sim}{\chi}_R(K)$. This implies that $\underset{\sim}{F}_R(\underset{\sim}{y}) \in \underset{\sim}{\chi}_R(K)$ if and only if $\underset{\sim}{y} \in \underset{\sim}{\chi}_R(K)$ which, in turn, implies (A.43). Since $\Lambda_R \cap \underset{\sim}{\chi}_R(K) = \underset{\sim}{\pi}_R(\Sigma)$, we have, from (A.42) and (A.43) that

$$\begin{aligned} \underset{\sim}{F}_R(\Lambda_R) \cap \underset{\sim}{\chi}_R(K) = \underset{\sim}{F}_R(\Lambda_R \cap \underset{\sim}{\chi}_R(K)) &= \underset{\sim}{F}_R \circ \underset{\sim}{\pi}_R(\Sigma) \\ &= \underset{\sim}{F}_{R0} \circ \underset{\sim}{\pi}_R(\Sigma) \quad . \end{aligned} \tag{A.44}$$

Since $\hat{\underset{\sim}{\pi}}_R^{-1}(\underset{\sim}{F}_R(\Lambda_R) \cap \underset{\sim}{\chi}_R(K)) = \underset{\sim}{F}_R(\Lambda_R) \times \mathbb{R}^{2n} \cap K$, we have from (A.41) and (A.44) that

$$\underset{\sim}{F}_R(\Lambda_R) \times \mathbb{R}^{2n} \cap K = \hat{\underset{\sim}{\pi}}_R^{-1} \circ \underset{\sim}{F}_R \circ \underset{\sim}{\pi}_R(\Sigma) = \underset{\sim}{F}(\Sigma)$$

which is the desired equality. ¤

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FIGURE CAPTIONS

Fig. 1. A 1-port which is not strongly structurally stable. (a) The circuit diagram. (b) Constitutive relations of R_1 and R_2. (c) Constitutive relations of the composite 1-port before and after perturbations.

Fig. 2. A 1-port which is not strongly structurally stable. (a) Constitutive relations of R_1 and R_2. (b) Constitutive relations of the composite 1-port before and after perturbations.

Fig. 3. A 1-port which is strongly structurally stable. (a) Constitutive relations of R_1 and R_2. (b) Constitutive relations of the composite 1-port before and after perturbations.

Fig. 4. A 1-port which is structurally stable but not strongly structurally stable; configuration spaces and constitutive relations of the composite 1-port before and after perturbations.

Fig. 5. A 1-port which is neither structurally stable nor strongly structurally stable; configuration spaces and constitutive relations of the composite 1-port before and after perturbations.

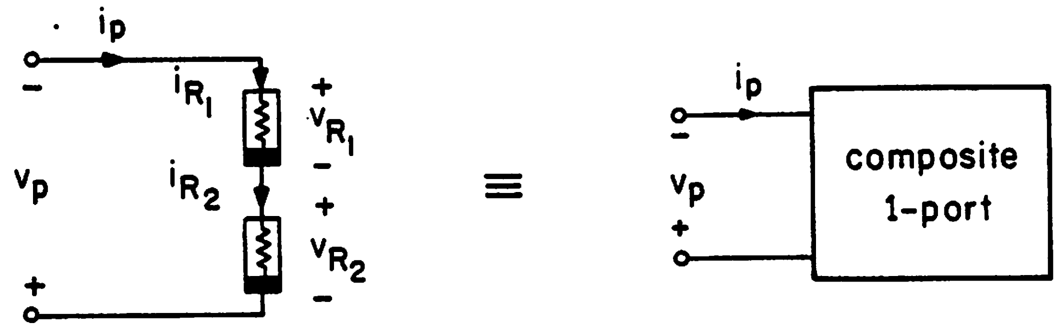
Fig. 6. A 1-port which is structurally stable and strongly structurally stable; configuration spaces and constitutive relations of the composite 1-port before and after perturbations.

Fig. 7. Illustration of nice immersion. (a) $\underset{\sim}{\pi}_p$ violates condition (ii) of nice immersion. (b) $\underset{\sim}{\pi}_p$ violates condition (iii) of nice immersion. (c) $\underset{\sim}{\pi}_p$ is a nice immersion.

Fig. 8. The diffeomorsism $\underset{\sim}{\phi}_k : V_k \to V_k$.

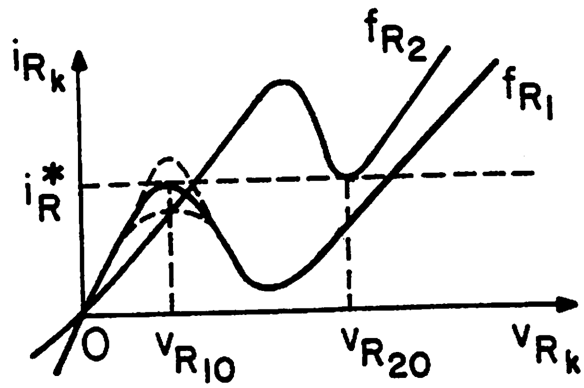
Fig. 9. A diagram illusteating the relationships among $\underset{\sim}{H}$, $\underset{\sim}{H}_p$, $\underset{\sim}{\chi}_p$ and $\underset{\sim}{\pi}_p^{-1}$.

Fig. 10. The sets Σ'', R'' and $D^{n-1} \times L$ in <u>Case 2</u> of the proof of <u>Theorem 1</u>.



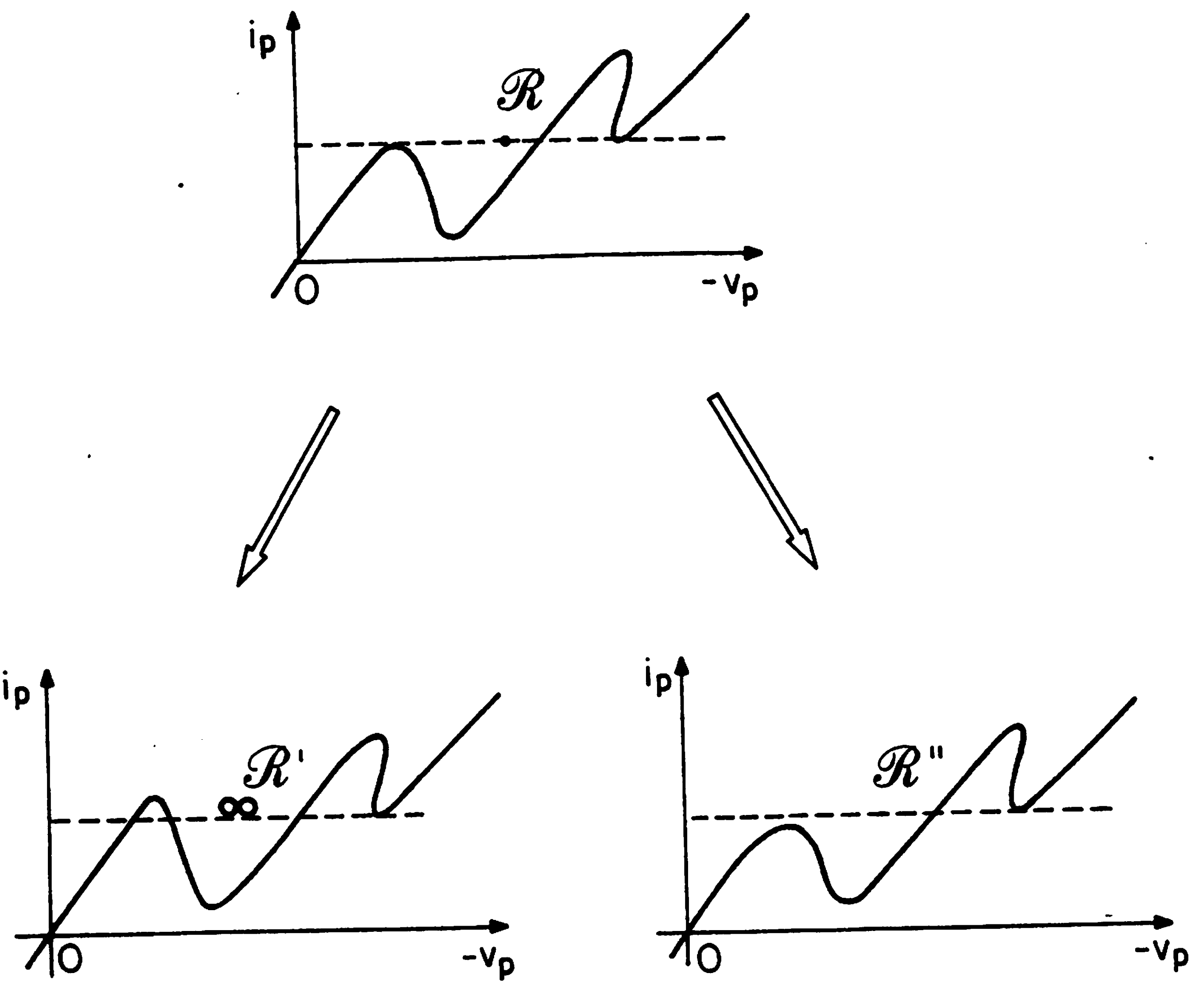
(a)

Fig. 1



i_{R_k}
f_{R_2}
f_{R_1}
i_R^*
O
v_{R10}
v_{R20}
v_{R_k}

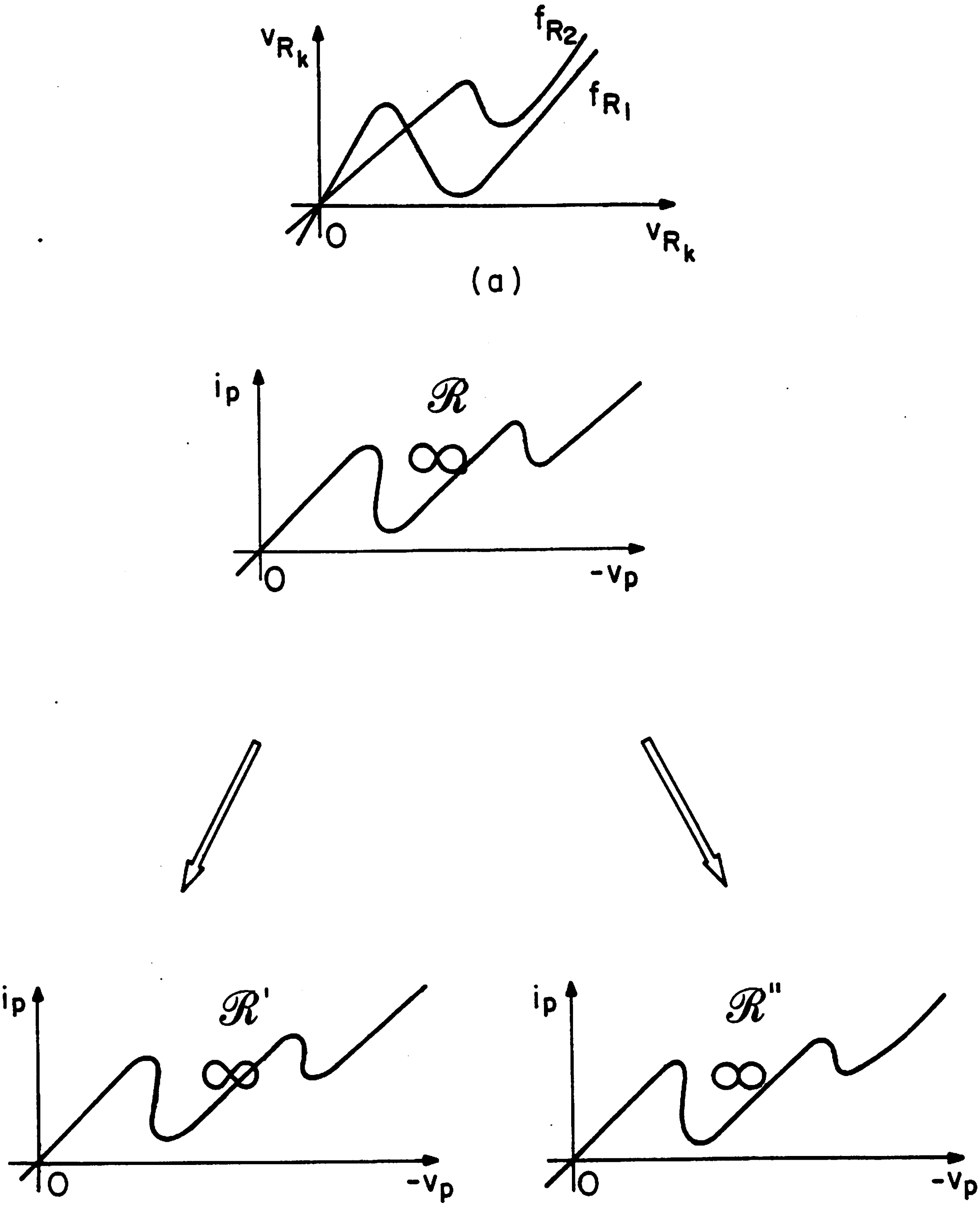
(b)



i_p
$\mathscr{R}$
O
$-v_p$
i_p
$\mathscr{R}'$
O
$-v_p$
i_p
$\mathscr{R}''$
O
$-v_p$

(c)

Fig. 1


v_{R_k}
f_{R_2}
f_{R_1}
O
v_{R_k}
i_p
$\mathscr{R}$
O
$-v_p$
i_p
$\mathscr{R}'$
O
$-v_p$
i_p
$\mathscr{R}''$
O
$-v_p$

(a)

(b)

Fig. 2

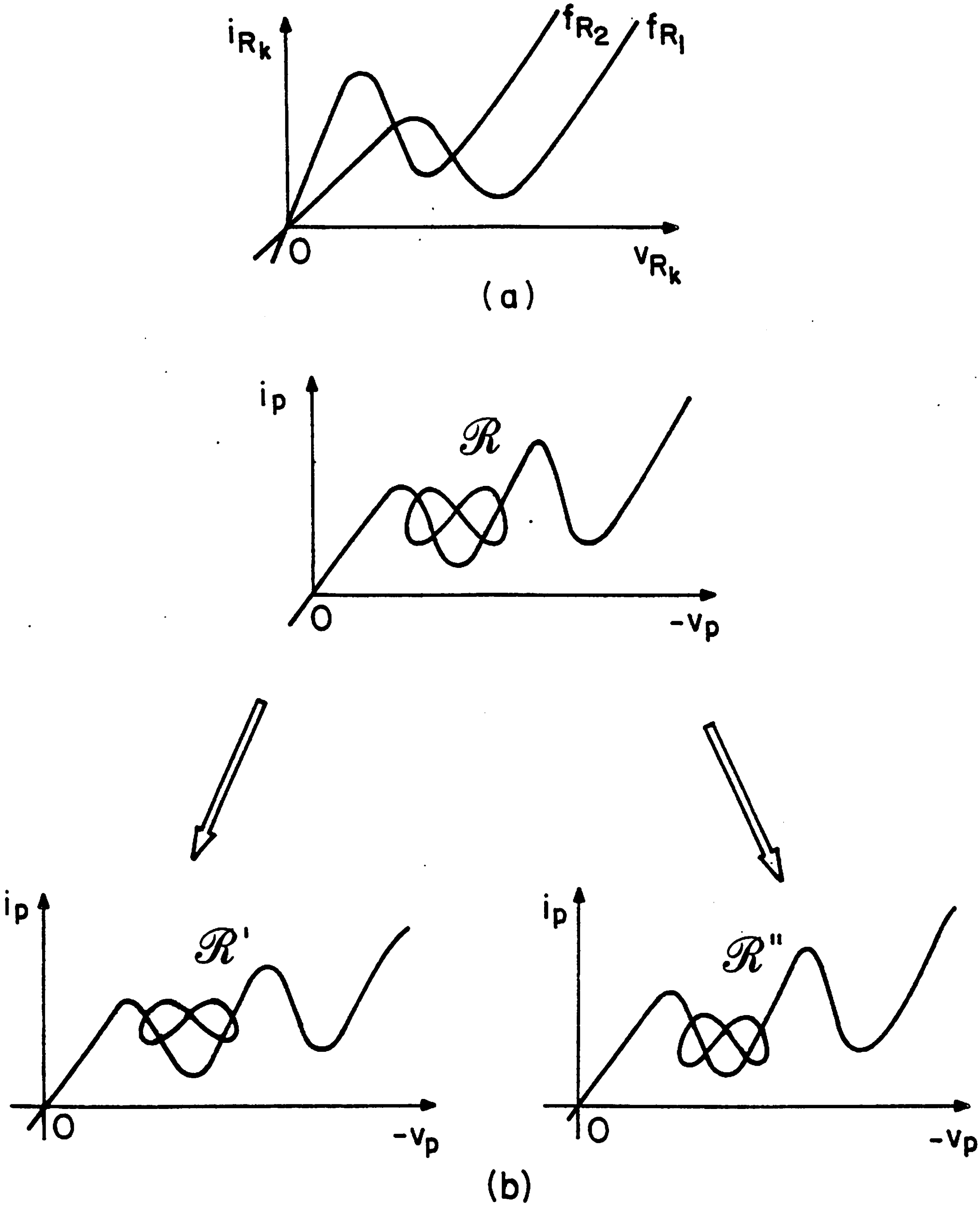
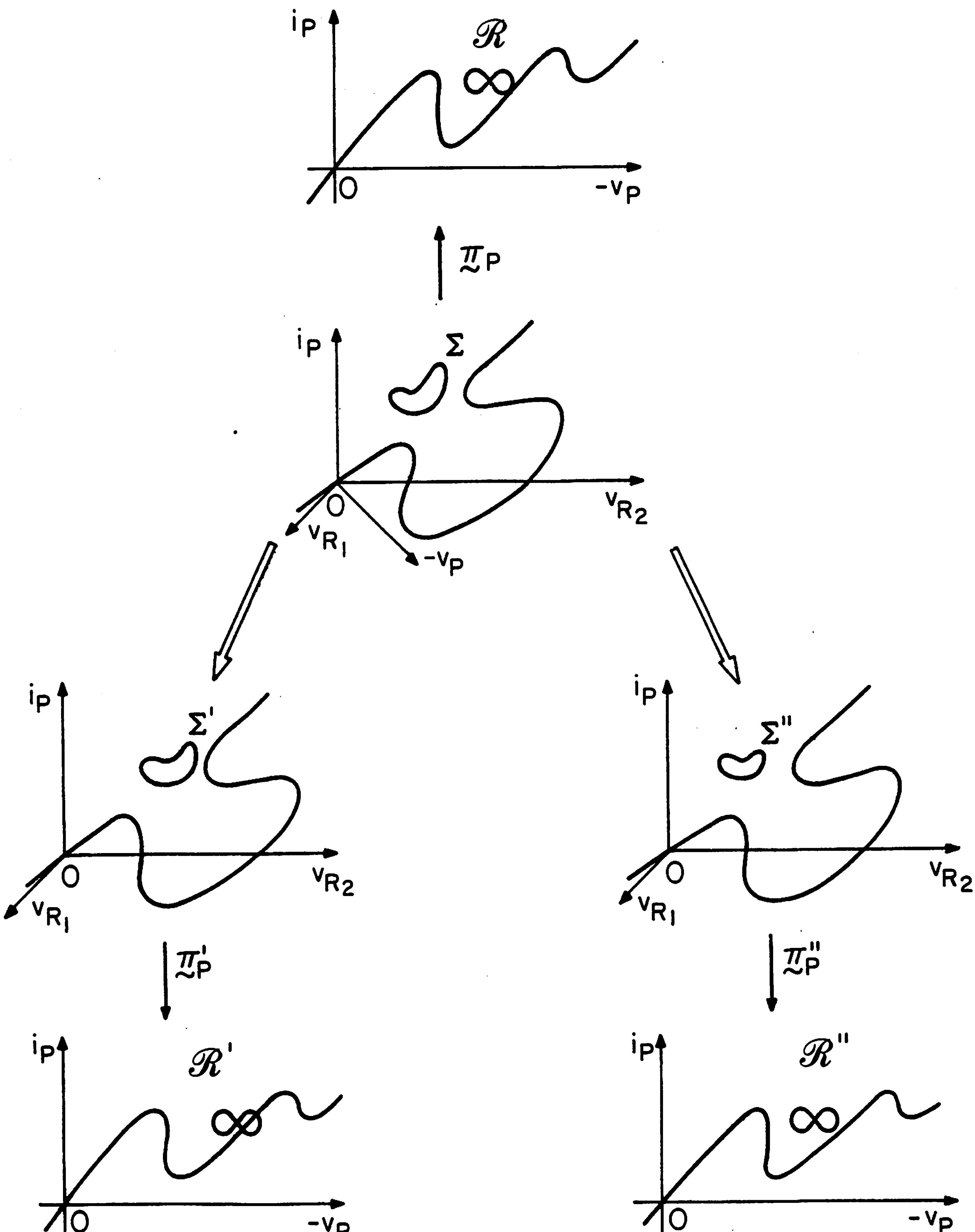

i_{R_k}
f_{R_2}
f_{R_1}
O
v_{R_k}
(a)
i_p
$\mathscr{R}$
O
$-v_p$
i_p
$\mathscr{R}'$
O
$-v_p$
i_p
$\mathscr{R}''$
O
$-v_p$
(b)

Fig. 3



i_P
$\mathscr{R}$
O
$-v_P$
$\underset{\sim}{\pi}_P$
i_P
Σ
O
v_{R_2}
v_{R_1}
$-v_P$
i_P
Σ'
O
v_{R_2}
v_{R_1}
i_P
Σ''
O
v_{R_2}
v_{R_1}
$\underset{\sim}{\pi}_P'$
$\underset{\sim}{\pi}_P''$
i_P
$\mathscr{R}'$
O
$-v_P$
i_P
$\mathscr{R}''$
O
$-v_P$

Fig. 4

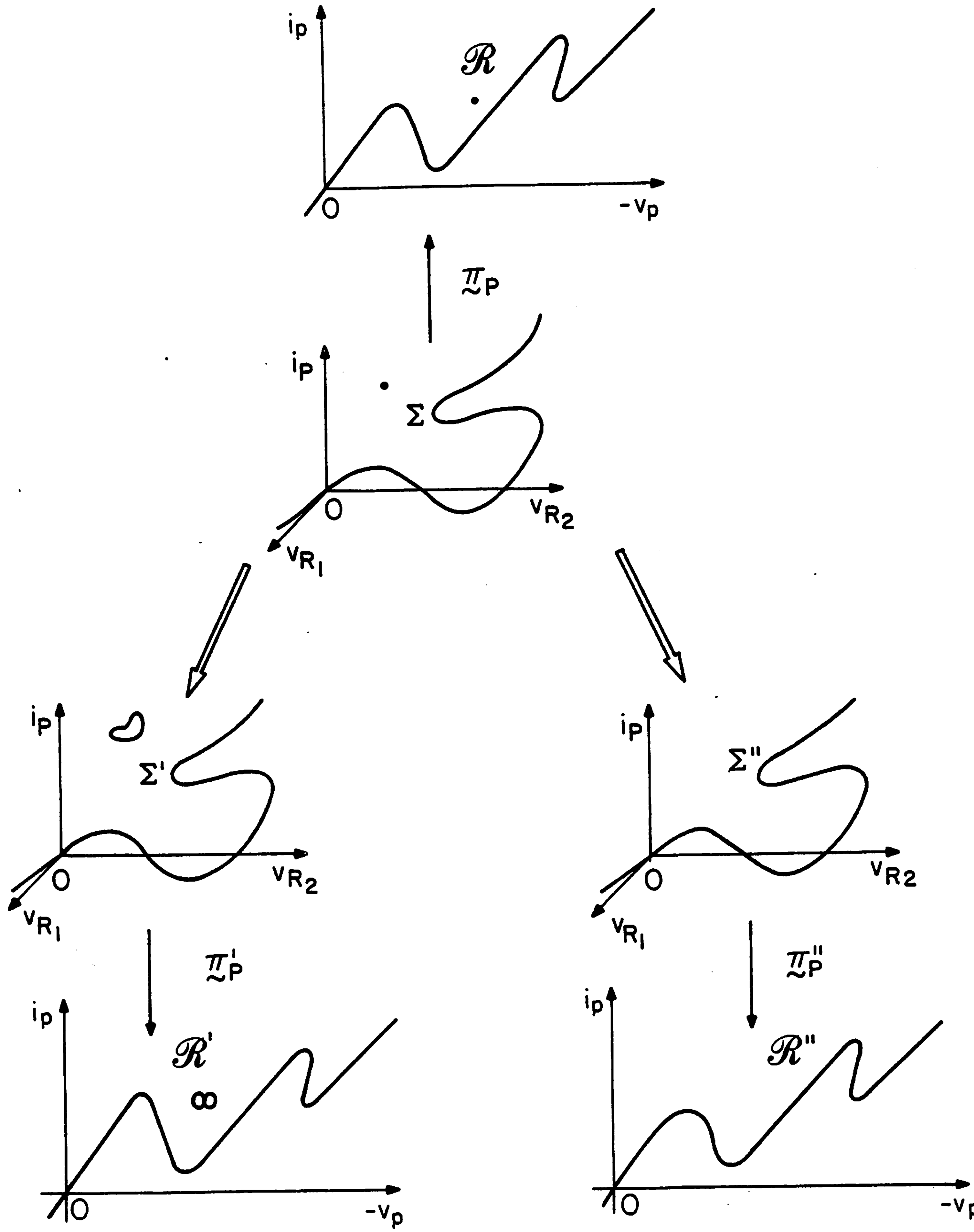
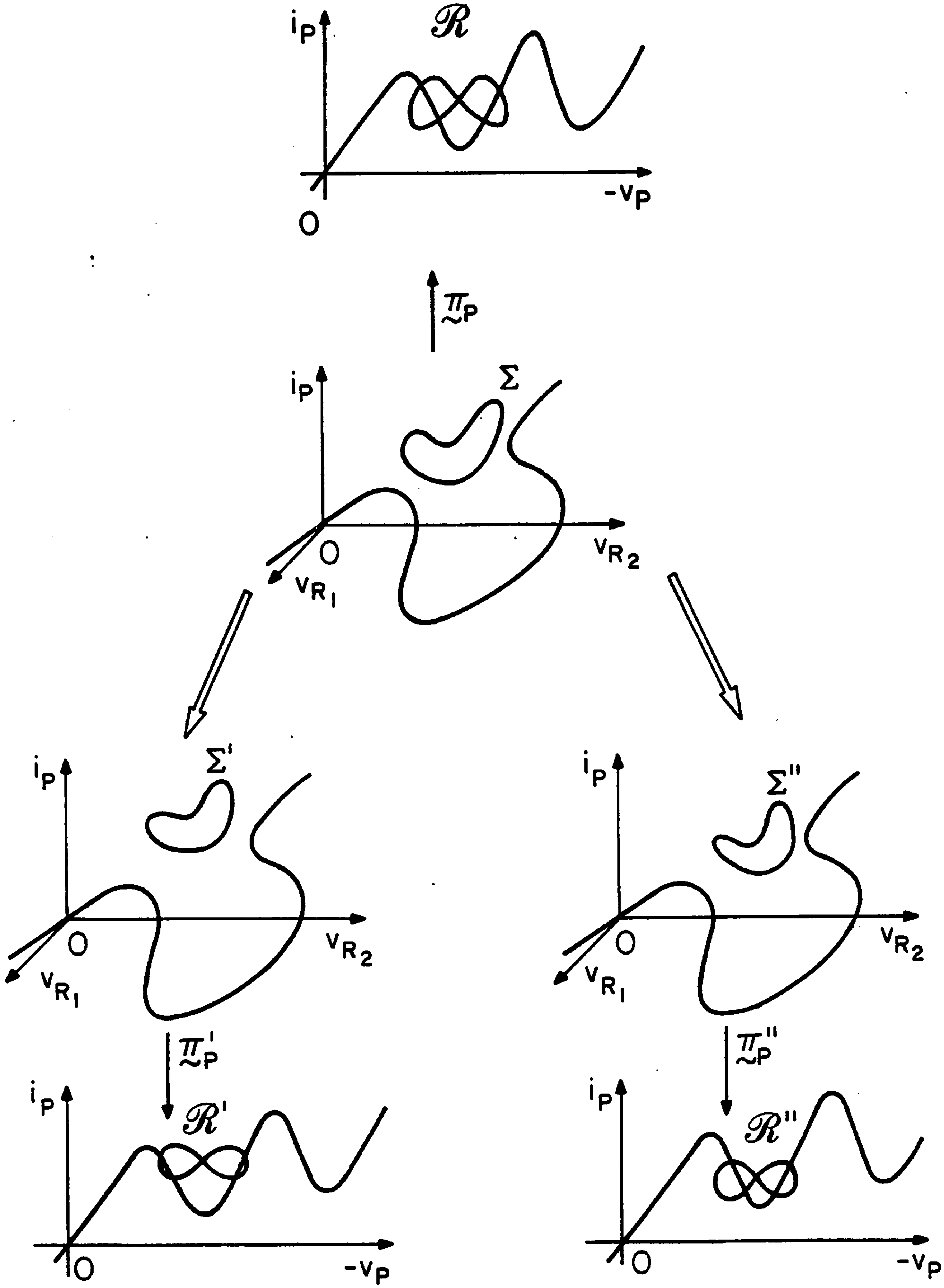
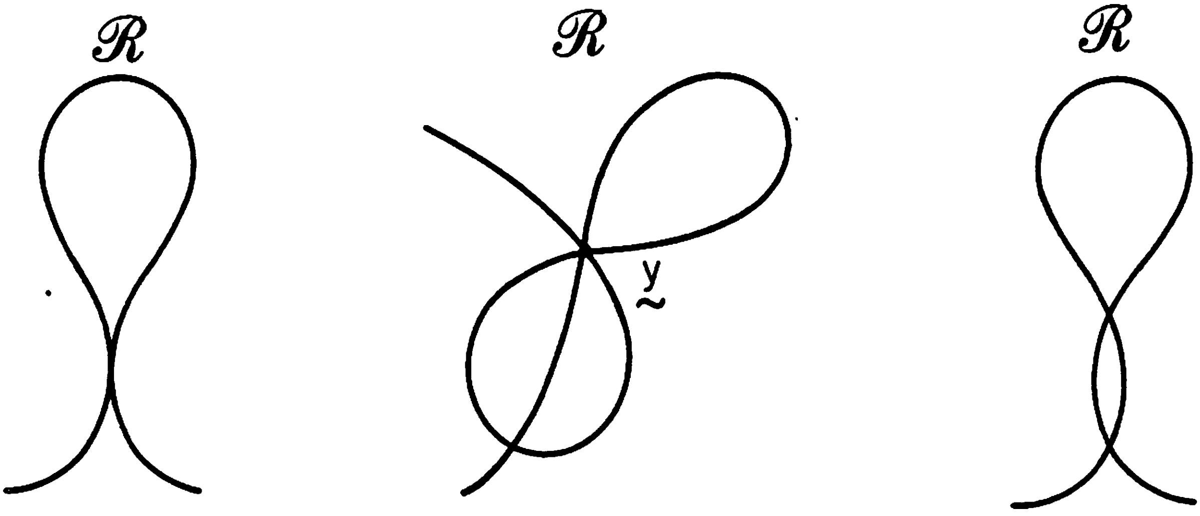

ip
𝓡
O
-vp
πP
iP
Σ
O
vR2
vR1
iP
Σ'
O
vR2
vR1
iP
Σ''
O
vR2
vR1
π'P
π''P
ip
𝓡'
O
-vp
ip
𝓡''
O
-vp

Fig. 5



i_P
$\mathscr{R}$
O
$-v_P$
$\underset{\sim}{\pi}_P$
i_P
Σ
O
v_{R_2}
v_{R_1}
i_P
Σ'
O
v_{R_2}
v_{R_1}
i_P
Σ''
O
v_{R_2}
v_{R_1}
$\underset{\sim}{\pi}_P'$
$\underset{\sim}{\pi}_P''$
i_P
$\mathscr{R}'$
O
$-v_P$
i_P
$\mathscr{R}''$
O
$-v_P$

Fig. 6



(a) (b) (c)

Fig. 7

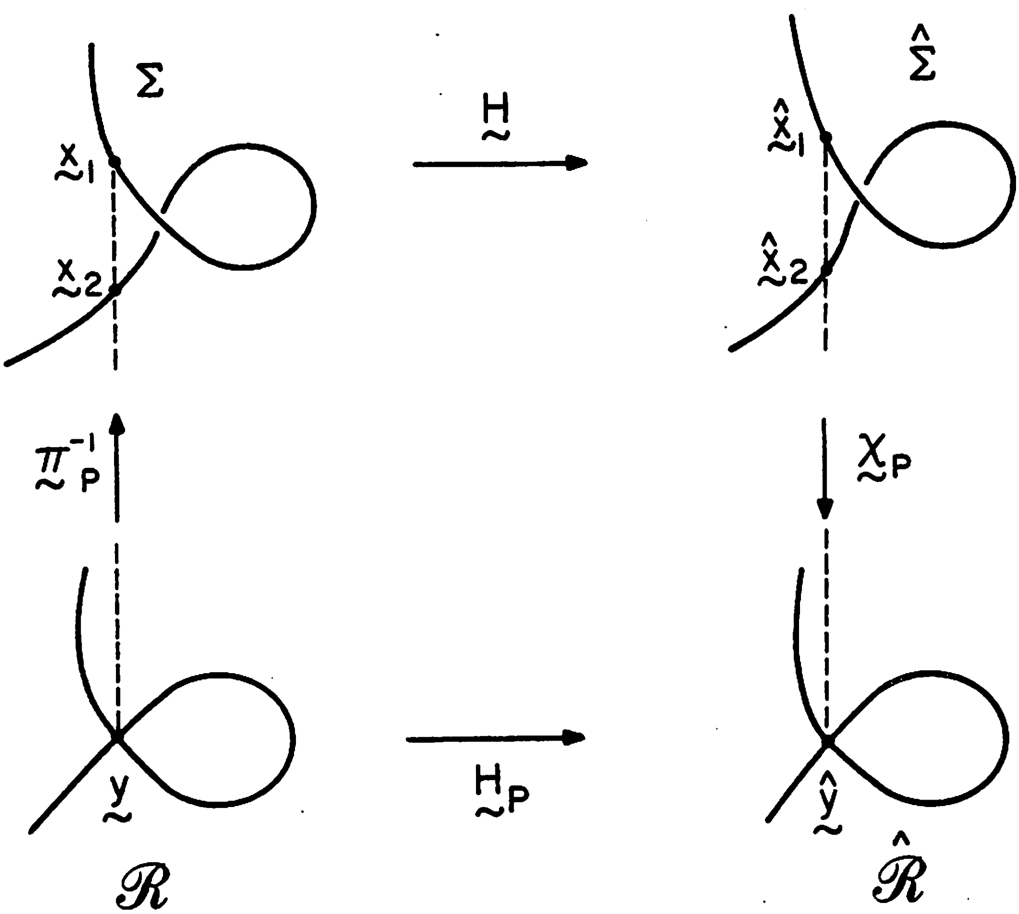
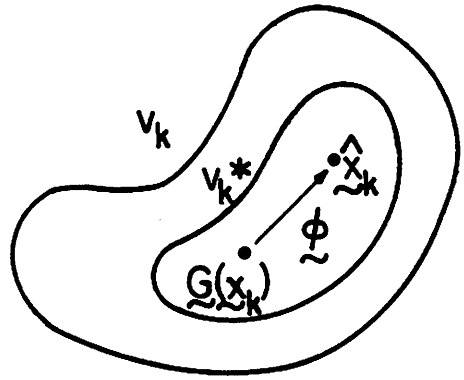
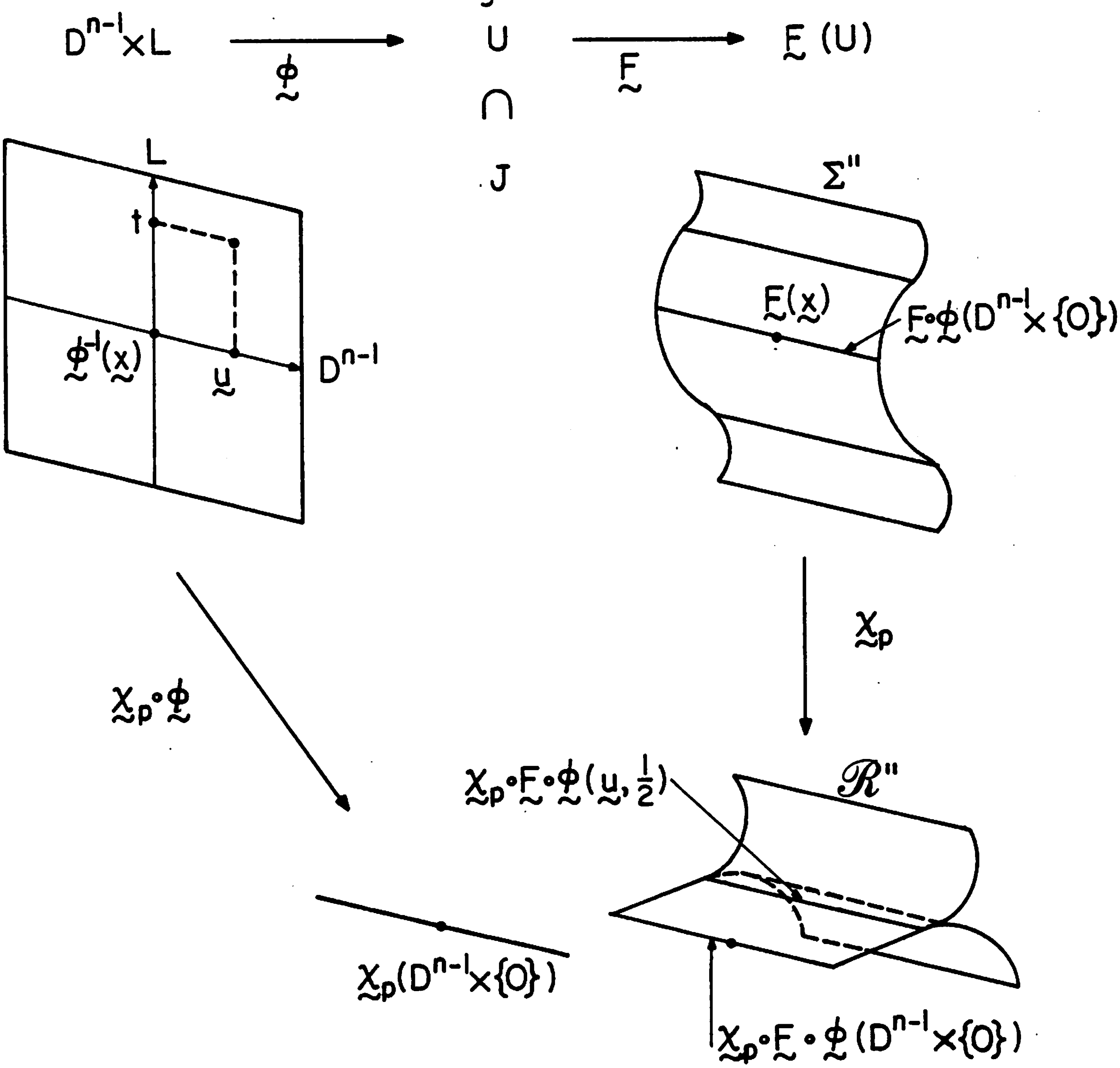


Fig. 8



V_k
V_k^*
$\hat{x}_k$
ϕ
$G(x_k)$

Fig. 9



$D^{n-1} \times L$
ϕ
U
F
F (U)
J
L
t
$\phi^{-1}(x)$
u
D^{n-1}
Σ''
F(x)
$F \circ \phi(D^{n-1} \times \{0\})$
χ_p
$\chi_p \circ \phi$
$\chi_p \circ F \circ \phi(u, \frac{1}{2})$
$\mathscr{R}''$
$\chi_p(D^{n-1} \times \{0\})$
$\chi_p \circ F \circ \phi(D^{n-1} \times \{0\})$

Fig. 10

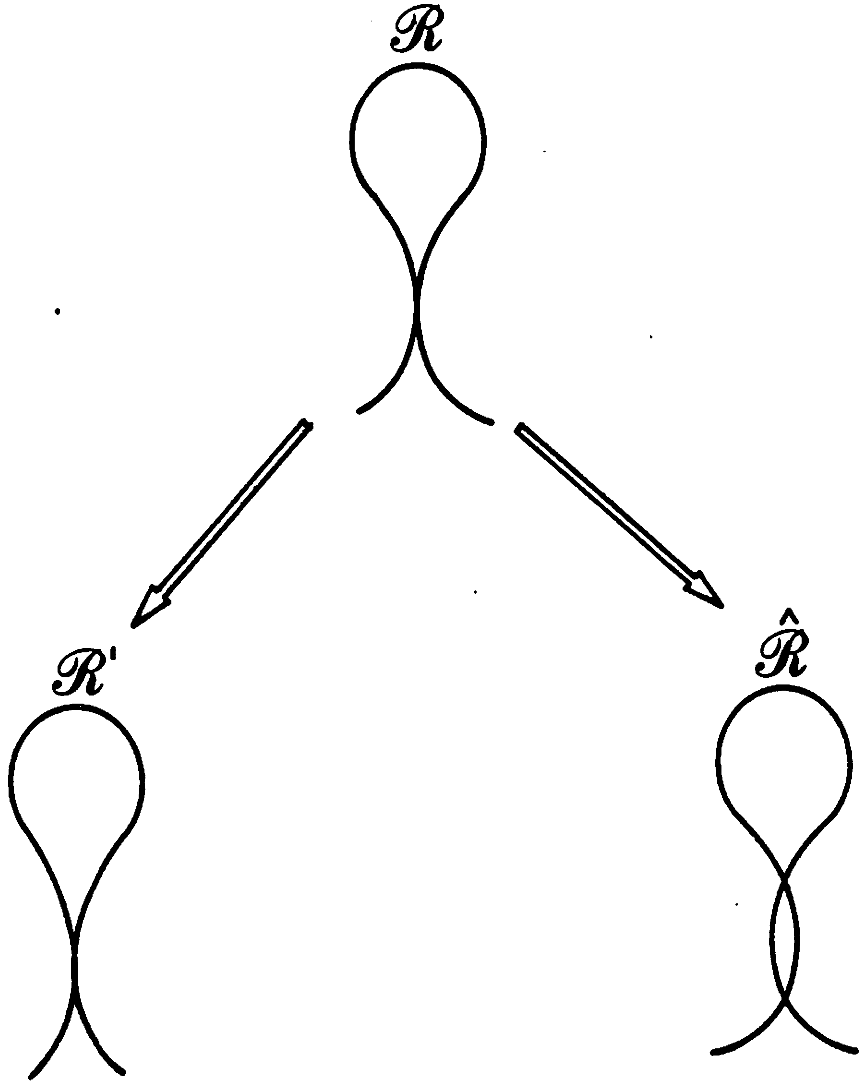

$\mathscr{R}$
$\mathscr{R}'$
$\hat{\mathscr{R}}$

Fig. 11

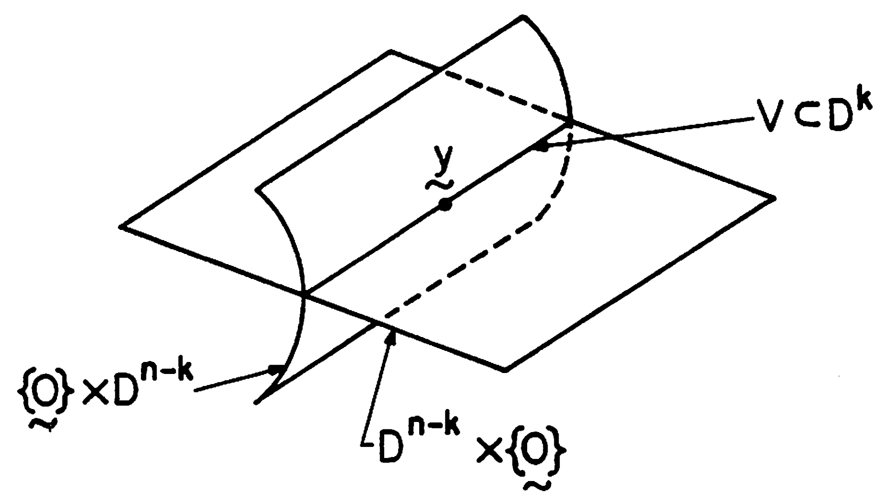

$V \subset D^k$
$\underset{\sim}{y}$
$\{\underset{\sim}{0}\} \times D^{n-k}$
$D^{n-k} \times \{\underset{\sim}{0}\}$

Fig. 12

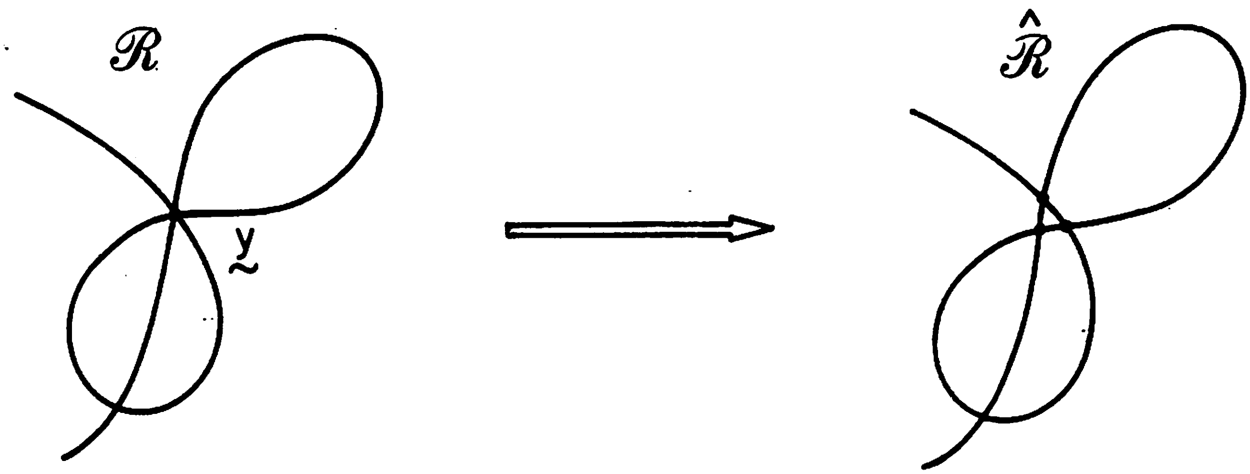

$\mathscr{R}$
$\underset{\sim}{y}$
$\hat{\mathscr{R}}$

Fig. 13

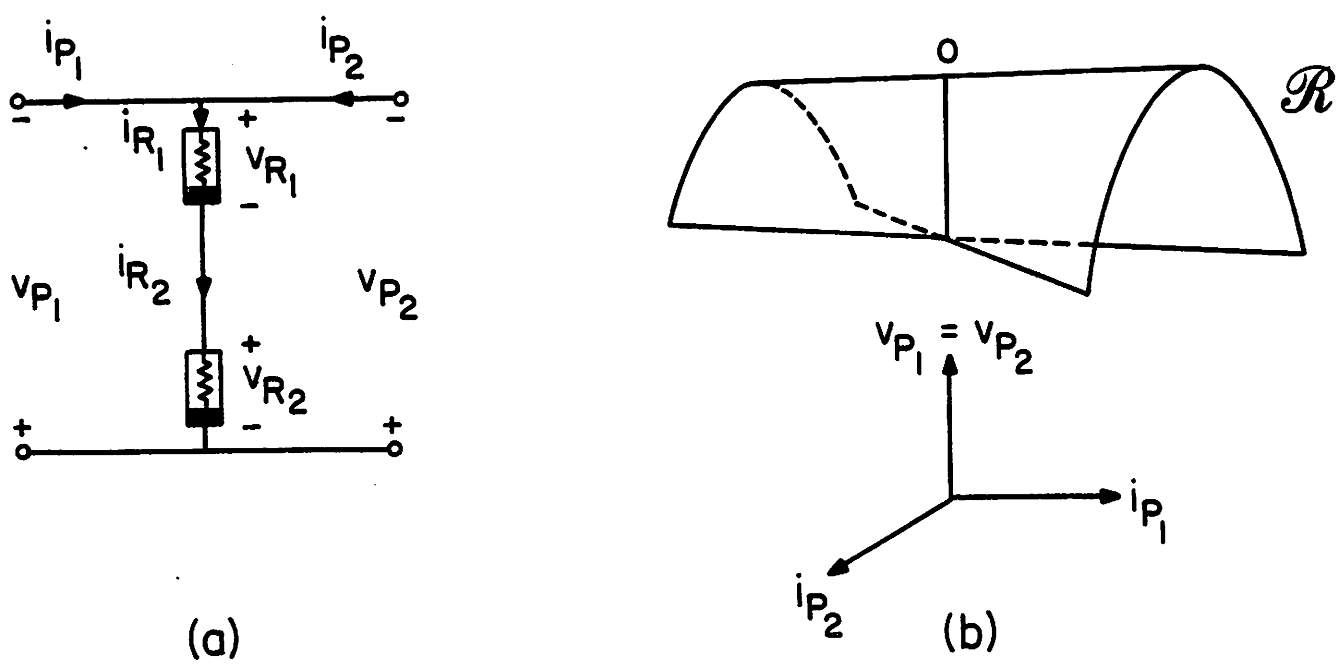

i_{P_1}
i_{P_2}
i_{R_1}
v_{R_1}
i_{R_2}
v_{P_1}
v_{P_2}
v_{R_2}
(a)
0
$\mathscr{R}$
$v_{P_1} = v_{P_2}$
i_{P_1}
i_{P_2}
(b)

Fig. 14

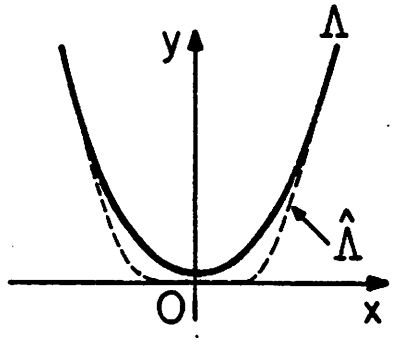


Fig. 15

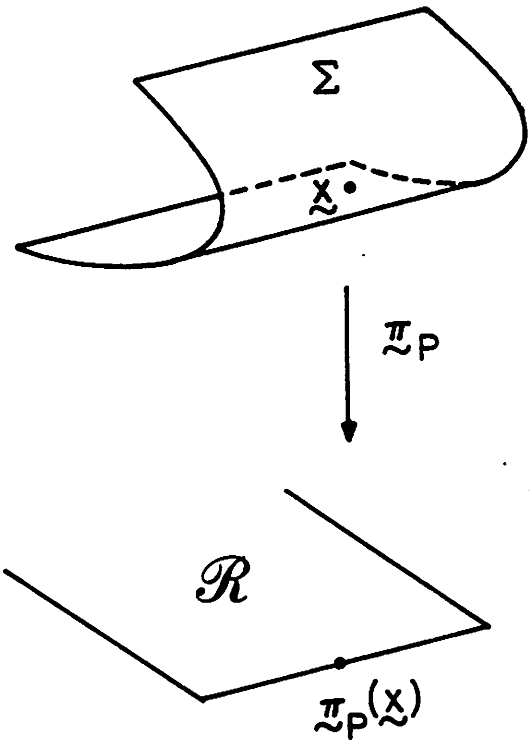
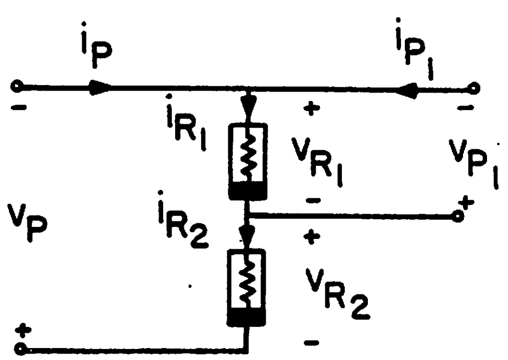
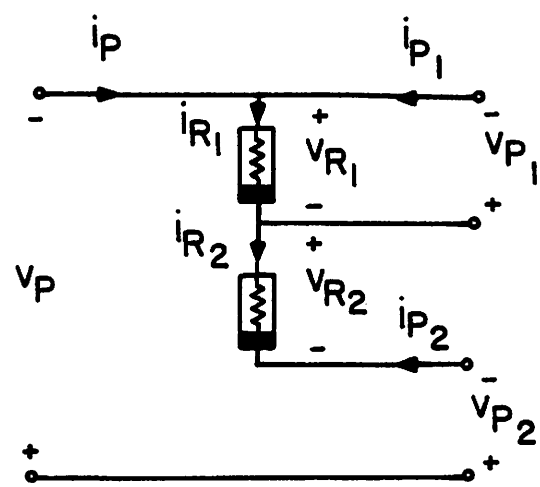


Fig. 16

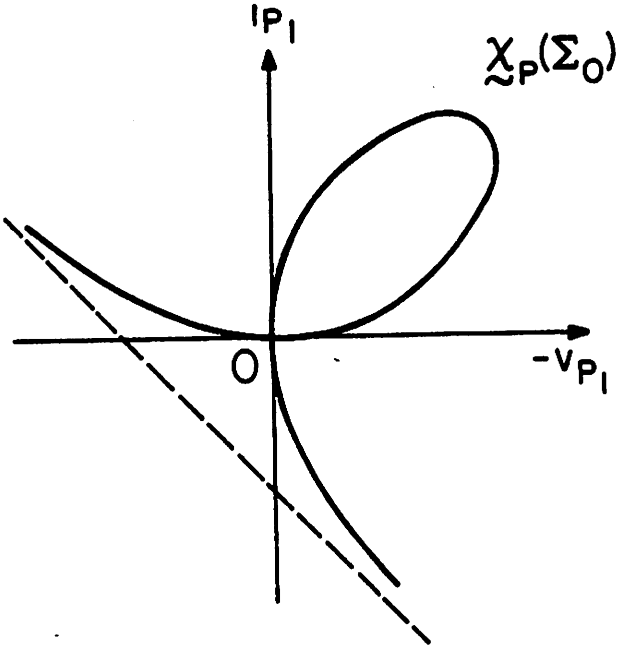


(a)

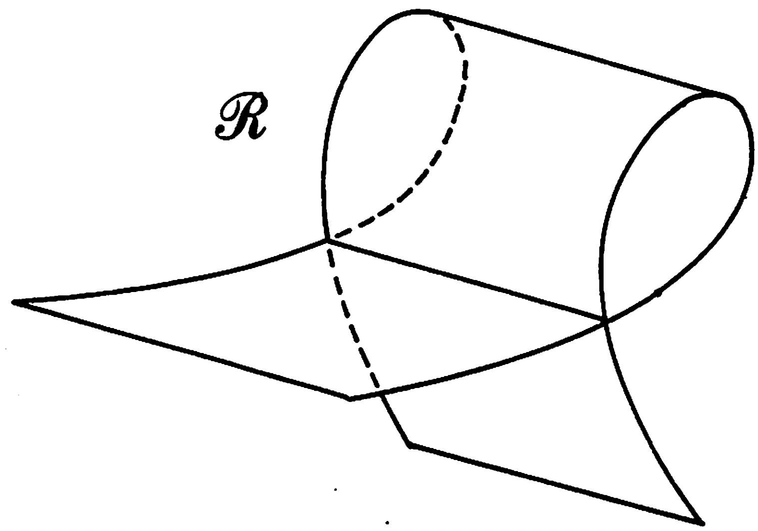


(b)

Fig. 17



(a)



(b)

Fig. 18

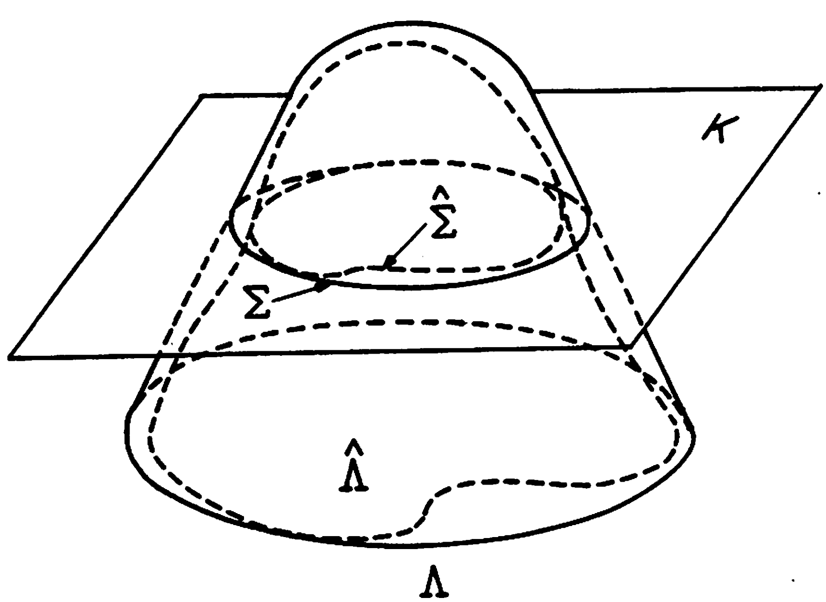


Fig. 19

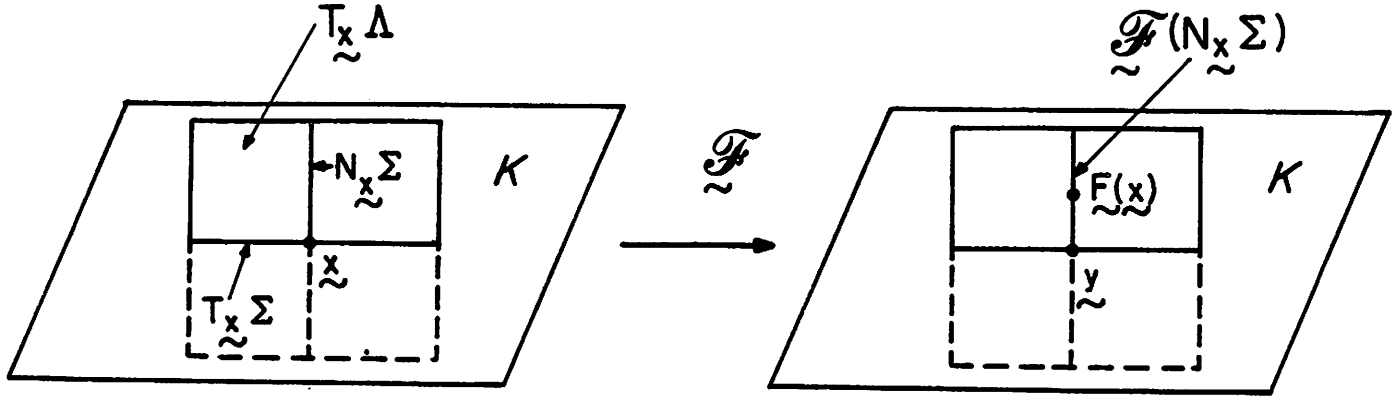


Fig. 20

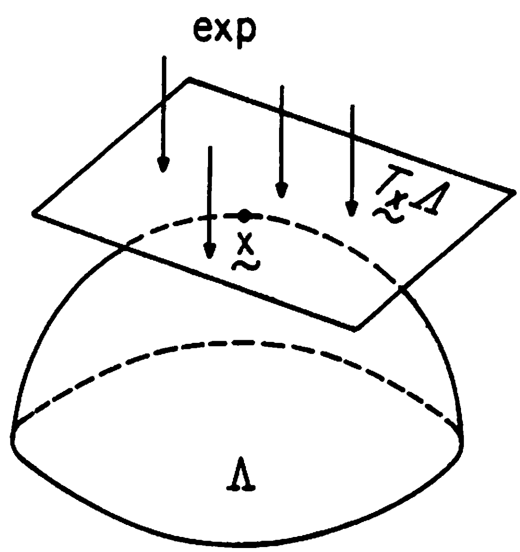


Fig. 21

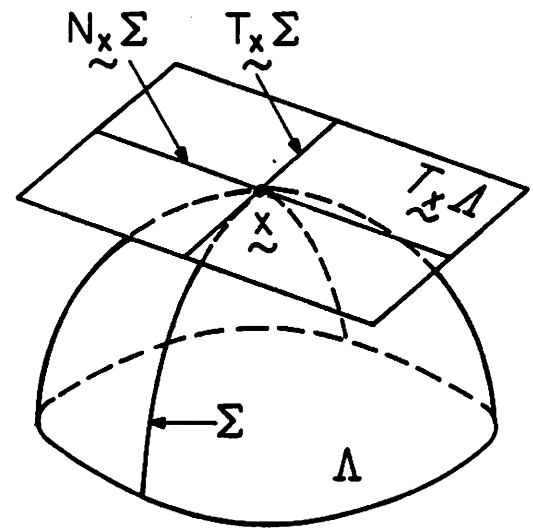
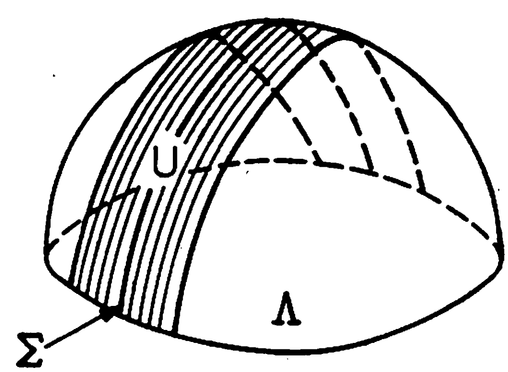


Fig. 22

$$
\begin{tikzcd}
V \arrow[dr, "\phi"] & \\
\Sigma \arrow[u, "\zeta"] \arrow[r, "\text{inclusion}"'] & \Lambda
\end{tikzcd}
$$

(a)



(b)

Fig. 23

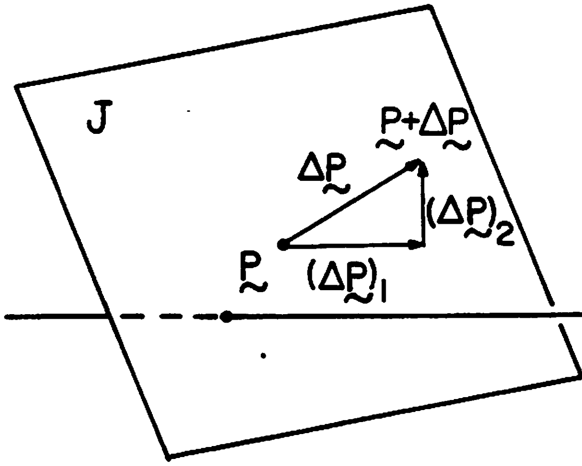


Fig. 24

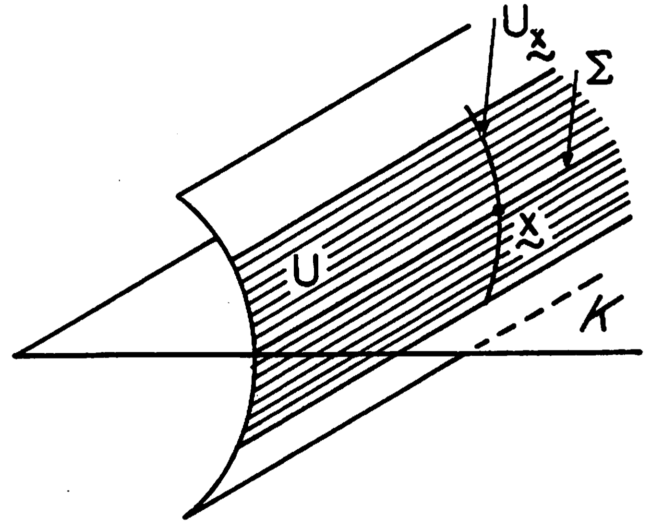


Fig. 25

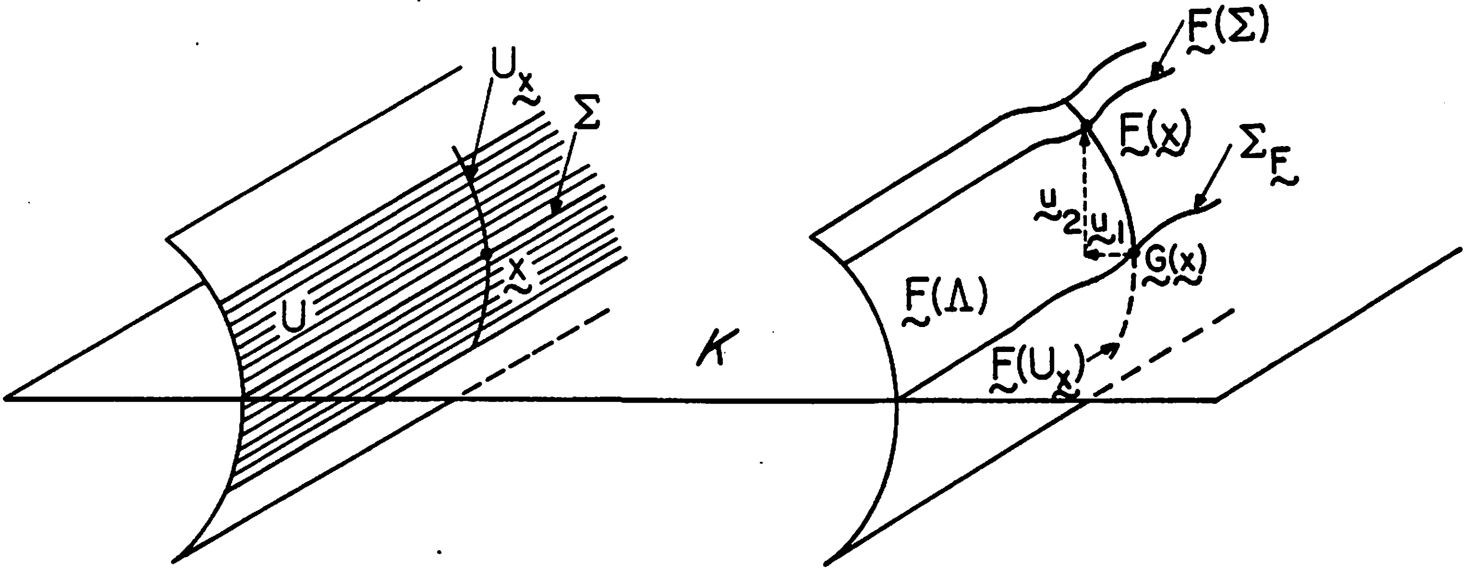


Fig. 26

$$\begin{tikzcd} \tilde{U} \arrow[r, "\Phi^{-1}"] & \tilde{V} \arrow[r, "\nu"] & \tilde{\Sigma} \\ & \cap & \\ & N\tilde{\Sigma} & \end{tikzcd}$$

(a)

$$\begin{tikzcd} \hat{\Sigma} \subset F(U) \arrow[drr, "\psi\circ F^{-1}\mid\hat{\Sigma}"'] & & U \arrow[ll, "F"'] \arrow[r, "H"] \arrow[d, "\psi"] & \tilde{U} \arrow[d, "\tilde{\psi}"] \\ & & \Sigma \arrow[r, "H"'] & \tilde{\Sigma} \end{tikzcd}$$

(b)

Fig. 27

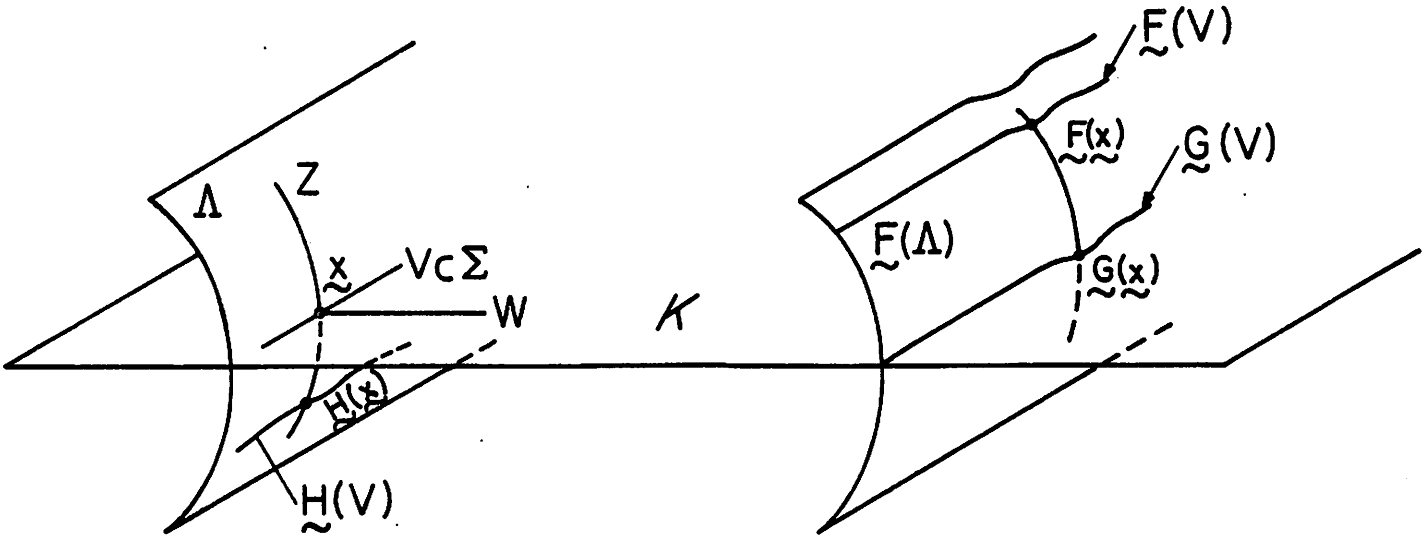


Fig. 28

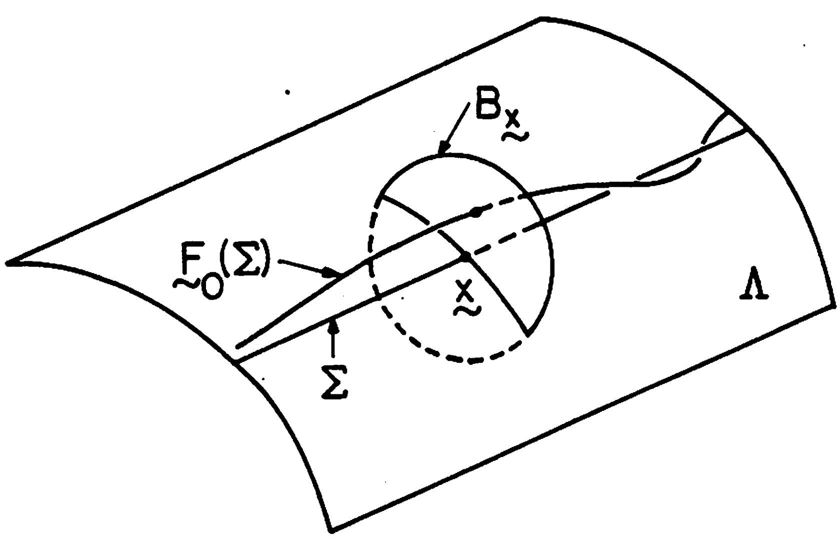


Fig. 29