CHOICE THEORY FOR CARDINAL SCALES I: RATIO-SCALES

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ABSTRACT

The classical choice theory considers three alternative approaches to the individual choice problem: a criterion language, a binary relation language and a language of choice functions. These approaches provide equivalent descriptions of choice function classes. The framework of rational choice is strongly based on the assumption that criteria are ordinal scales. For instance, two scalar criteria generate the same choice functions if and only if they are equivalent ordinal scales. Hence, the classical framework is not able to distinguish choices based on cardinal scales.

The aim of this paper is to present a theory of choice for ratio-scales. It turns out that fuzzy set theory provides a sufficient basis for costructing pair-dominant mechanisms of choice and a choice function language which give an adequate description of choice for ratio-scales. Scalar and vector criterion choice mechanisms are studied in detail in the paper.

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Introduction

The optimization problem is considered in this paper in a general framework of best variant choice. Many problems in decision theory, especially in economical, psychological and social applications, are reducible to making the best choice from a set of submitted variants with respect to some given optimality criterion. Mostly, this criterion is a real-valued function (scalar or vector) on the universum of variants. The main objectives of the theory in question are a study of behavior of best variants chosen under variations of sets submitted and establishing relations between different mechanisms of choice, rather than developing the technique of calculating the optimum.

The classical theory of choice considers the following framework (see, for example, [1]). Let A be a universe of variants which is supposed to be a finite set. A function $f : A \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, is said to be a scalar criterion (goal function, utility function, scale etc.). Values of this function are considered as measurements in an ordinal scale. It means that two such functions f_1 and f_2 are regarded as equivalent if (and only if) there is a monotone increasing transformation $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi o f_1 = f_2$. Any scalar criterion generates the following mechanism of choice

$$C_X^f = \{y \in X | f(y) \geq f(x) \text{ for all } x \in X\} \tag{1}$$

or, which equivalent,

$$C_X^f = \{y \in X | \text{there is no } x \in X \text{ such that } f(x) > f(y)\} \tag{2}$$

Elements of C_X^f are considered as "best variants" in a submitted set X. According to (1) an element $x \in X$ is regarded as the "best variant" in X if and only if the function f takes its maximum value on the set X in the point x. Hence, elements from C_X^f provide an "optimization" of a scalar criterion f on a given set X. Two choice mechanisms (1) and (2) have clearly a different meaning in spite of their mathematical equivalence: mechanism (1) defines C_X^f as a set of dominating variants as opposed to (2) which claims C_X^f to be a set of non-dominated variants.

The difference between (1) and (2) becomes more evident when we pass to vector criteria. Let $f = (f^1, f^2, \ldots, f^n)$ be a n-tuple of scalar functions on A. As usual, we define

$$f(x) \geq f(y) \text{ iff } f^i(x) \geq f^i(y) \text{ for all } i.$$

Generally speaking, (1) and (2) define different mechanisms of choice in this case. For example, C_X^f defined by (1) is very often an empty set but nonvoidness of C_X^f for mechanism (2) can be proved. Mechanism (1) defines the best variant $x \in X$ as an optimal one with respect to all components f^i of f simultaneously. On the other hand, (2) is a well-known mechanism of Pareto-optimal choice based on a vector criterion f.

It is established in classical choice theory that both scalar and vector criteria define mechanisms of choice which are equivalent to pair-dominant mechanisms based on binary relations on the universe A. The scalar criterion mechanism is the same as generated by ordering relations and different vector criterion mechanisms (1) and (2) coincide with mechanisms based on weak orderings and quasi-transitive relations, respectively. Moreover, it is possible to describe classes of choice functions based on pair-dominant mechanisms by some characteristic properties of choice functions. Indeed, let C_X be any choice function, i.e., a mapping which assigns a subset $C_X \subseteq X$ to each nonempty set $X \subseteq A$. The following properties are called characteristic properties

Heritage (H): $X' \subseteq X$ implies $C_{X'} \supseteq C_X \cap X'$.

Strict heritage (K): $X' \subseteq X$ and $X' \cap C_X \neq \phi$ imply $C_{X'} = C_X \cap X'$.

Concordance (C): $C_{X' \cup X''} \supseteq C_{X'} \cap C_{X''}$.

Independence of rejecting the outcaset variants (O):

$$C_X \subseteq X' \subseteq X \text{ imply } C_X = C_{X'}.$$

Remark. Properties (H) and (C) are Sen's conditions α and γ, respectively, [9]. Property (K) is Arrow's condition (C4) [2]. All these properties were also studied by Chernoff [3]. We use notations from [1] where these properties are studied in detail.

The class of choice functions which satisfy (K) is exactly the same as that of choice functions based on orderings and the class of choice functions fulfilling (H), (C) and (O) coincides with that based on quasi-transitive relations.

The main purpose of this paper is to present an extension of the framework described above on the case when criteria are measurements in ratio-scales. Let, for example, f_1 and f_2 be two scalar criteria which are equivalent ordinal scales, but $f_1 : f_2 \neq$ const. Then f_1 and f_2 are not equivalent ratio-scales although they yield the same choice mechanisms by (1). Therefore, the classical framework is not able to distinguish "optimizations" in scales which are stronger then ordinal ones and we "lose information" passing from criteria to choice mechanisms. It turns out that there is a model based on fuzzy set theory which provides the classical correspondence between "criterial language", "binary relation language" and the "language of choice functions" for the ratio-scale case. This model uses a notion of a maximizing set due to L. A. Zadeh [12]. Let X be a set and f- a positive real-valued function on X. Zadeh defines a maximizing set M^f by its membership function as follows

$$M^f(x) = \frac{f(x)}{\sup_X(f)}, \text{ for } x \in X. \quad (3)$$

"Intuitively, a maximizing set M^f... for a function f on X is a fuzzy subset of X such that the grade of membership of a point x in M^f represents the degree to which f(x) approximates to $\sup_X(f)$..." ([12]). Note, that it is clear from (3) that two functions f_1 and f_2 are equivalent ratio-scales if and only if they define the same maximizing sets. The notion of a maximizing set permits to give proper generalizations of choice mechanisms (1) and (2) which turn out to be equivalent to pair-dominant mechanisms based on fuzzy binary relations of certain types.

There are seven sections in the paper.

Abstract choice functions in fuzzy set theory are introduced in section 1.

Choice mechanisms based on criteria measured in ratio-scales and fuzzy binary relations are defined in sections 2 and 3. Some structural properties of fuzzy binary relations are established in section 3.

Characteristic properties of choice functions extending classical ones are introduced in section 4.

In section 5 main theorems are proved which establish equivalence of different approaches to choice theory in the framework developed.

A choice mechanism based on weak orderings is studied in section G.

We suppose that the reader is familiar, in general, with choice function theory (see, for example, [8] and, especially, [1]) and fuzzy set theory (see, for example, [4]).

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1. Choice functions

Let A be a set. A fuzzy set X with universe A is a mapping $X : A \to [0;1]$. X is completely defined by its membership function $X(x)$ with domain A and range $[0;1]$. $\underset{\sim}{P}(A)$ denotes the set of all fuzzy sets with universe A. We say that X is a subset of Y and write $X \subseteq Y$ if and only if $X(x) \leq Y(x)$ for all $x \in A$. Operations of union and intersection for fuzzy sets are defined point-wise by operations $\vee = \max$ and $\wedge = \min$ on $[0;1]$. carX denotes the carrier of fuzzy set X, i.e., a crisp subset $\{x \in A | X(x) > 0\}$. We write $x \in X$ for a fuzzy set X if $x \in \text{car}X$, i.e. if $X(x) > 0$.

Definition 1.1. A choice function C is a mapping $C : \underset{\sim}{P}(A) \to \underset{\sim}{P}(A)$ assigning a fuzzy subset $C_X \subseteq X$ to each fuzzy set X with universe A.

The subset C_X of X is considered as the subset of "best" elements in X. Then the value $C_X(x)$ may be regarded as a "degree of goodness" of the element $x \in X$.

Examples of choice functions can be found in sections 2 and 3 where some particular classes of choice functions are defined by means of criterial and pair-dominance choice mechanisms. These classes may be also described in an external way by some properties of abstract choice functions defined above. Most properties in question are defined in Section 4 and turn out to be quite similar to the classical ones. But there are two important properties which have no analogs in the classical theory. They are defined below.

Separation (S): $C_X = C_{\text{car}X} \cap X$ for all $X \in \underset{\sim}{P}(A)$.

This condition claims that the choice from a fuzzy set X is completely defined by the choice from a crisp set carX. Therefore, to define a choice function it is sufficient to define it for nonfuzzy sets. By (S), "fuzziness: of choice is separated into two parts: 1) fuzziness of choice "itself", and 2) fuzziness of a set submitted.

Positiveness (P): $C_X(x) > 0$ for all $x \in X$.

At the first glance, this condition looks very strangely, since it claims that every element of submitted set should be chosen (may be with a very small degree of belongness to the subset chosen). In other words, there are no rejected elements in any set X with respect to the choice function C fulfilling (P). But this circumstance is quite reasonable in the ratio-scale framework studied in the paper. In this framework, roughly speaking, the choice between two variants x and y is determinated by the intensity of preference over these variants. If, say, y is rejected, it means that x is infinitely times more preferable then y. Obviously, this possibility should be excluded in a reasonable mathematical model what is provided by (P).

2. Criterial choice

It is supposed further in the paper that a universe A is a finite set.

Let f be a positive real-valued function on A, which we shall call a scalar criterion. The following definitions are based on the idea of a maximizing set (see Introduction).

Definition 2.1 A dominating subset of a fuzzy set X with respect to a scalar criterion f is a fuzzy set D_X^f with a membership function

$$D_X^f(x) = \frac{f(x)}{\max_X(f)} \wedge X(x) \tag{2.1}$$

A dominating subset D_X^f can be also regarded as a fuzzy set of elements which are not dominated by some $y_{max} \in X$ such that

$f(y_{max}) = \max_X(f)$. Following this idea we consider $D_X^{f_y}$ as a fuzzy set of those elements which are not dominated by a given y, where f_y is a "cut off" function defined by

$$f_y(x) = \begin{cases} f(x), \text{ if } f(x) \leq f(y), \\ f(y), \text{ if } f(x) \geq f(y). \end{cases}$$

Then an undominated subset is defined as an intersection of the family of fuzzy subsets $\{D_X^{f_y}\}_{y \in X}$:

<u>Definition 2.2</u>. An <u>undominated subset</u> of X with respect to a scalar criterion f is a fuzzy set P_X^f with a membership function

$$P_X^f(x) = \bigwedge_{y \in X} \frac{fy(x)}{\max_X(fy)} \wedge X(x) \tag{2.2}$$

The following proposition shows that any dominating subset is an undominated subset and vise versa in the particular case of scalar criteria.

<u>Proposition 2.1</u>. $D_X^f = P_X^f$ for any scalar criterion f and for all $X \in \underset{\sim}{P}(A)$.

<u>Proof</u>. Let us choose $y_0 \in X$ such that $f(y_0) = \max_X(f)$. Then

$$D_X^f(x) = \frac{f(x)}{f(y_0)} \wedge X(x)$$

and, obviously,

$$P_X^f(x) \leq \frac{f_{y_0}(x)}{\max_X(f_{y_0})} \wedge X(x) = \frac{f_{y_0}(x)}{f(y_0)} \wedge X(x) = D_X^f(x).$$

On the other hand,

$$P_X^f(x) = \bigwedge_{y \in X} \frac{f_y(x)}{f(y)} \wedge X(x) \geq \bigwedge_{y \in X} \frac{f(x)}{\max_X(f)} \wedge X(x) = D_X^f(x),$$

since

$$\frac{f_y(x)}{f(y)} \geq \frac{f(x)}{\max_X(f)} \text{ for all } y \in X. \qquad \square$$

Let now $f = (f^1, f^2, \ldots, f^n)$ be a vector criterion on A with positive real-valued scalar components f^i. A Pareto-set for f is defined as a subset of all undominated elements with respect to a vector criterion f in the classical theory. Following the idea exposed above for scalar criteria we introduce a fuzzy set $\bigcup_i D_X^{f_y^i}$ which elements are regarded to be undominated by a given element $y \in X$ with respect to f. Then a Pareto-set can be defined as an intersection of these sets taken for all $y \in X$.

Definition 2.3. An undominated subset or a Pareto-set of a fuzzy set X with respect to a vector criterion f is a fuzzy set P_X^f with a membership function

$$P_X^f(x) = \bigwedge_{y \in X} \bigvee_i \frac{f_y^i(x)}{\max_X (f_y^i)} \wedge X(x) \tag{2.3}$$

Note, that (2.2) is a particular case of (2.3) for $n = 1$.

Dominating and undominated subsets defined above are considered in this paper as values of choice functions.

Definition 2.4. Let f be a (scalar or vector) criterion. Choice functions defined by

$$C_X = D_X^f \text{ and } C_X = P_X^f \tag{2.4}$$

are called choice functions based on a criterion f.

Ratio-scales are defined as scales which are unique up to dilations in the theory of measurement (see, for example, [7]). Let us suppose that a scalar criterion f is a ratio-scale. Then dominating subsets defined by (2.1) are invariant under dilations. Moreover, two scalar criteria f_1 and f_2 are equivalent ratio-scales if and only if they have the same dominating subsets for all $X \subseteq A$. In the classical theory the same statement is true for ordinal scales: two scalar criteria are equivalent ordinal scales if and only if they yield the same solutions of the optimization problem. Hence, the framework described above is an extension of the classical one on the case of ratio-scales. From this standpoint a (fuzzy) dominating set is a natural analog of the set of all optimal states with respect to a given scalar criterion.

In the same way, a vector criterion $f = (f^1, f^2, \dots, f^n)$ may be considered as a multidimensional ratio-scale if each component f^i is a ratio-scale. Then Pareto-sets defined by (2.3) are invariant under transformation defining ratio-scales. Moreover, ratio-scales f^i can be regarded as independent measurements in the sense that they admits distinct dilations leaving Pareto-sets invariant.

3. Pair-dominant choice

In this section a fuzzy pair-dominant mechanism of choice based on fuzzy binary relations is introduced and some structural properties of fuzzy binary relations are established.

Definition 3.1. A fuzzy binary relation R on A is a fuzzy set with the universe $A \times A$.

In the context of the problem studied a fuzzy binary relation R is regarded as a (weak) preference relation. It means that we consider the value $R(x,y)$ as a degree of certainty that an element x is preferred to an element y.

A preference relation R on A defines a choice function as follows:

Definition 3.2. A choice function C^R based on a fuzzy binary R is defined by its membership function

$$C_X^R(x) = \bigwedge_{y \in X} R(x,y) \wedge X(x) \tag{3.1}$$

Note, that the value C_X^R defined by (3.1) is a natural generalization of fuzzy upper bound defined in [11] for nonfuzzy sets. One can also consider (3.1) as a translation of the following definition into a fuzzy set theory language: "x is the best element in X with respect to R iff x is preferred to any element y in X."

A mechanism of choice given by (3.1) is said to be a <u>pair-dominant mechanism</u>, since it defines the best variants by means of "pair comparisons."

In order to separate important classes of pair-dominant mechanisms some properties of fuzzy binary relations are defined below.

<u>Definition 3.3</u>. A fuzzy binary relation R is said to be

i) <u>reflexive</u> if $R(x,x) = 1$ for all $x \in A$;

ii) <u>complete</u> if $R(x,y) > 0$ and $R(x,y) = 1$ or $R(y,x) = 1$ for all $x,y \in A$. (Note, that completeness implies reflexivity.)

The most important property of binary relations for choice theory is transivity. There are different ways to introduce this notion in fuzzy set theory (see, for example, [11]). For needs of our study some new concepts of transivity are introduced in this section.

<u>Definition 3.4</u>. A complete fuzzy binary relation R is said to be an <u>ordering</u> iff it satisfies the following transivity property:

$$\text{if } R(x,y) = 1 \text{ and } R(y,z) = 1, \text{ then } R(x,z) = 1 \text{ and}$$
$$R(z,x) = R(z,y) \cdot R(y,x). \tag{3.2}$$

Transivity property (3.2) is, in some sense, a "mixture" of usual crisp transivity and so-called max-product transivity introduced in fuzzy set theory [11]. Fuzzy orderings defined above are analogous to the classical ones and play an important role in the framework developed.

In choice theory the notion of a <u>strict</u> preference relation associated with a given weak preference relation is very useful. The following extension of this notion is suggested in this paper: a strict preference relation P_R associated with R is defined by its membership function as follows:

$$P_R(x,y) = \begin{cases} R(x,y), \text{ if } R(x,y) > R(y,x), \\ 0, \text{ otherwise.} \end{cases} \tag{3.3}$$

Note, that P_R thus defined is a crisp binary relation if R is a complete preference relation (for example, an ordering).

The following definition introduces an important notion of a quasi-transitive preference relation.

<u>Definition 3.5</u>. A complete fuzzy binary relation R is said to be a <u>quasi-transitive relation</u> iff it satisfies the following transivity property:

$$\text{if } P_R(x,y) = 1 \text{ and } P_R(y,z) = 1 \text{ then } P_R(x,z) = 1 \text{ and } R(z,x) \leq R(z,y) \cdot R(y,x) \tag{3.4}$$

<u>Proposition 3.1</u>. Any ordering is a quasi-transitive relation.

<u>Proof</u>. Let R be an ordering and $P_R(x,y) = 1$ and $P_R(y,z) = 1$. Then $R(x,y) = 1$ and $R(y,z) = 1$ which imply $R(x,z) = 1$, by (3.2). If $R(z,x) = 1$, then $R(z,y) = 1$, by (3.2), which contradicts $P_R(y,z) = 1$. Hence, $R(z,x) < 1$, i.e. $P_R(x,z) = 1$. Now (3.4) follows from (3.2). ¤

The following structural theorems are generalizations of the famous Szpilrajn's theorem [10] and will be used in section 5.

<u>Theorem 3.1</u>. Any quasi-transitive relation is a finite union of orderings.

<u>Proof</u>. It suffices to prove that for any pair (a,b) there is an ordering $\tilde{R}$ such that $\tilde{R} \subseteq R$ and $R(a,b) = R(a,b)$ for a given quasi-transitive relation R. Let us consider the following cases:

1) $R(a,b) = 1$

Since a crisp binary relation R^1 defined by: xR^1y iff $R(x,y) = 1$ is a crisp quasi-ordering, then, by Szpilrajn's theorem [10], there is a numeration $A = \{x_1,\ldots,x_n\}$ such that $i \leq j$ implies $x_jR^1x_i$ and $a = x_\ell$, $b = x_k$ for some $k < \ell$. Let $\varepsilon = \min\{R(x,y)\}$. Then $\varepsilon > 0$, since R is a complete relation. Let us define

$$\tilde{R}(x_i,x_j) = \begin{cases} 1, \text{ if } i \geq j, \\ \varepsilon^{j-i}, \text{ if } i < j \end{cases}$$

Then $\tilde{R}$ is complete, $\tilde{R} \subseteq R$ and $\tilde{R}(a,b) = R(a,b)$ (=1). To prove transivity, let us suppose that $\tilde{R}(x_i,x_j) = 1$ and $\tilde{R}(x_j,x_s) = 1$, i.e., $s \leq j \leq i$. Then, obviously, $R(x_i,x_s) = 1$ and $\tilde{R}(x_s,x_i) = \varepsilon^{i-s}$ $= \varepsilon^{j-s} \cdot \varepsilon^{i-j} = \tilde{R}(x_s,x_j) \cdot \tilde{R}(x_j,x_i)$. Hence, $\tilde{R}$ is an ordering.

2) $R(a,b) < 1$ and b is a unique maximal element with respect to P_R. Let us define

$$\tilde{R}(x,y) = \frac{R(x,b)}{R(y,b)} \wedge 1.$$

Obviously, $\tilde{R}(a,b) = R(a,b)$. Let us prove that $\tilde{R}(x,y) \leq R(x,y)$. It is true if $R(x,y) = 1$. If $R(x,y) < 1$, then yP_Rx. We also have bP_Ry, since b is a unique maximal element. Then, by (3.5), $R(x,b) \leq R(x,y)$ $\cdot R(y,b)$, which implies $\tilde{R}(x,y) \leq R(x,y)$.

$\tilde{R}$ is obviously a complete relation. To prove transitivity, let $\tilde{R}(x,y) = 1$ and $\tilde{R}(y,z) = 1$. Then $R(x,b) \geq R(y,b)$ and $R(y,b) \geq R(z,b)$ which immediately imply $\tilde{R}(x,z) = 1$. Further, we have

$$\tilde{R}(y,x) = \frac{R(y,b)}{R(x,b)}, \quad \tilde{R}(z,y) = \frac{R(z,b)}{R(y,b)} \text{ and } \tilde{R}(z,x) = \frac{R(z,b)}{R(x,b)},$$

which imply $\tilde{R}(z,x) = \tilde{R}(z,y) \cdot \tilde{R}(y,x)$. Hence, $\tilde{R}$ is an ordering.

3) $R(a,b) < 1$ and there is a maximal element $x_{max} \neq b$ with respect to P_R.

In this case the statement of the theorem is proved by induction. For n = 2 the statement is trivial. For n > 2 let us consider the set $A' = A \setminus \{x_{max}\}$. Then the restriction of R on the set A' is a quasi-transitive relation R'. By the induction hypothesis, there is an ordering $\tilde{R}'$ on A' such that $\tilde{R}' \subseteq R$ and $R'(a,b) = R'(a,b)$. Let us define $\tilde{R}$ on A by

$$\tilde{R}(x,y) = \tilde{R}'(x,y) \text{ if } x,y \in A',$$

$$\tilde{R}(x_{max},x) = 1 \text{ for all } x \in A, \text{ and}$$

$$\tilde{R}(x,x_{max}) = \frac{\varepsilon}{\tilde{R}'(x_{min},x)} \text{ for all } x \in A',$$

where $0 < \varepsilon < \min\{R(x,x_{max}) \cdot \tilde{R}'(x_{min},x)\}$ and x_{min} is some minimal element in A' with respect to $\tilde{P}_{R'}$. We have

$$\tilde{R}(a,b) = \tilde{R}'(a,b) = R'(a,b) = R(a,b),$$

since a, b ∈ A'. Further, $\tilde{R}(x,y) \leq R(x,y)$ for x,y ∈ A' and $\tilde{R}(x_{max},x) = R(x_{max},x) = 1$, since x_{max} is a maximal element with respect to P_R. We also have

$$\tilde{R}(x,x_{max}) = \frac{\varepsilon}{\tilde{R}'(x_{min},x)} < \frac{\tilde{R}(x,x_{max}) \cdot \tilde{R}'(x_{min},x)}{\tilde{R}'(x_{min},x)} = R(x,x_{max})$$

Hence, $\tilde{R} \subseteq R$. R is a complete relation, since so is $\tilde{R}'$.

Let us prove now that $\tilde{R}$ has transivity property (3.2). Suppose that $\tilde{R}(x,y) = 1$ and $\tilde{R}(y,z) = 1$ for distinct x,y,z ∈ A. If x,y,z ∈ A', then (3.2) is true by induction hypothesis. Since $\tilde{R}(x,x_{max}) < 1$ for all x ∈ A', then $\{x,y,z\} \not\subseteq A'$ only if $x = x_{max}$. Then $\tilde{R}(x_{max},z) = 1$ and

$$\tilde{R}(z,x_{max}) = \frac{\varepsilon}{\tilde{R}'(x_{min},z)} = \frac{\varepsilon \cdot \tilde{R}'(z,y)}{\tilde{R}'(x_{min},z) \cdot \tilde{R}'(z,y)}$$

$$= \tilde{R}'(z,y) \cdot \frac{\varepsilon}{\tilde{R}'(x_{min},y)} = \tilde{R}(z,y) \cdot \tilde{R}(y,x_{max}),$$

since $\tilde{R}'$ is an ordering and x_{min} is a minimal element with respect to $P_{\tilde{R}'}$. It is a trivial case when some of elements x,y,z are equal. ¤

It is much easier to prove the converse theorem.

Theorem 3.2. Any finite union of orderings is a quasi-transitive relation.

Proof. Let $R = \bigcup_i R_i$ where each R_i is an ordering. In order to prove (3.4), note that $aP_R b$ if and only if there is j such that $R_j(a,b) = 1$ and $R_i(b,a) < 1$ for all i, or, by completeness of R_i, if and only if $R_i(a,b) = 1$ and $R_i(b,a) < 1$ for all i. Then, obviously, $xP_R y$ and $yP_R z$ imply $xP_R z$ and

$$R(z,y) \cdot R(y,x) = [\bigvee_i R_i(z,y)] \cdot [\bigvee_i R_i(y,x)]$$

$$\geq \bigvee_i [R_i(z,y) \cdot (R_i(y,x)] = \bigvee_i R_i(z,x) = R(z,x),$$

since R_i are orderings. Hence, R is a quasi-transitive relation. ¤

Orderings and quasi-transitive relations play a distinguished role in the choice theory. It will be proved in section 5 that choice functions (3.1) based on these relations coincide with choice functions based on scalar and vector criteria, respectively. Hence, mechanisms of choice based on maximizing sets provide the same choice as fuzzy pair-dominant mechanisms - the result which is well-known in the classical choice theory.

4. Characteristic properties of choice functions

Two general properties of choice functions - separation and positiveness - were already introduced in section 1. In the rest of the paper only choice functions fulfilling these properties are considered.

In this section three additional characteristic properties are defined which are extensions of classical rationality properties.

These properties are called characteristic because their conjunctions separate important classes of choice functions (see section 5).

In order to introduce characteristic properties, we need the following notion. For any nonempty fuzzy set X a <u>normalizator</u> of X is a fuzzy set N(X) with a membership function

$$(N(X))(x) = \frac{X(x)}{h(X)},$$

where $h(X) = \max\{X(x)\}$ is a <u>height</u> of a set X.

Now, the characteristic properties are defined as follows:

<u>Heritage</u> (H): if $X' \subseteq X$ are crisp sets, then

$$C_{X'} \supseteq N(C_X \cap X').$$

<u>Strict heritage</u> (K): if $X' \subseteq X$ are crisp sets, then

$$C_{X'} = N(C_X \cap X').$$

<u>Concordance</u> (C): $C_{X' \cup X''} \supseteq C_{X'} \cap C_{X''}$.

Obviously, these conditions are of the same nature as classical ones described in the Introduction. Therefore, we assign them the same names.

Since the fulfillment of (P) is assumed in the paper, we have no rejected elements in any nonempty submitted set and a fuzzy form of the condition (O) (see Introduction) is not involved in the framework developed.

5. Main theorems

In this section two theorems are proved which establish equivalence of criterion language, binary relation language and choice function language in the framework of scalar and vector optimization.

<u>Theorem 5.1</u>. Let C be a choice function. The following statements are equivalent:

i) C is based on some scalar criterion f;

ii) C is based on some ordering R ;

iii) C fulfills properties (S), (P) and (K).

Proof. i) → ii).

Let f be a positive function and $C_X = D_X^f$. Let us define a fuzzy binary relation R by

$$R(x,y) = \frac{f(x)}{f(y)} \wedge 1. \tag{5.1}$$

Then

$$C_X^R(x) = \bigwedge_{y \in X} R(x,y) \wedge X(x) = \bigwedge_{y \in X} \frac{f(x)}{f(y)} \wedge X(x)$$

$$= \frac{f(x)}{\max\limits_X(f)} \wedge X(x) = D_X^f(x) = C_X(x),$$

i.e. a choice function based on R coincides with C. Let us prove that R is an ordering. It is clear from (5.1) that R is complete. Let us suppose that R(x,y) = 1 and R(y,z) = 1. Then, by (5.1), $f(x) \geq f(y) \geq f(z)$ which imply R(x,z) = 1 and

$$R(y,x) = \frac{f(y)}{f(x)}, \quad R(z,y) = \frac{f(z)}{f(y)}, \quad R(z,x) = \frac{f(z)}{f(x)}.$$

Hence, $R(z,x) = R(z,y) \cdot R(y,x)$ and R is an ordering.

ii) → iii).

Let R be an ordering and $C = C^R$, i.e.

$$C_X(x) = C_X^R(x) = \bigwedge_{y \in X} R(x,y) \wedge X(x).$$

First we prove that

$$R(x,y) = \frac{R(x,x_{max})}{\max\{R(x,x_{max}),R(y,x_{max})\}} \tag{5.2}$$

where x_{max} is a maximal element in A with respect to P_R, i.e. $R(x_{max},x) = 1$ for all $x \in A$. Two cases are considered:

1) $R(x,y) = 1$. Since $R(x_{max},x) = 1$, we have, by (3.2), $R(y,x_{max}) = R(y,x) \cdot R(x,x_{max})$. Hence, $R(y,x_{max}) \leq R(x,x_{max})$ and the right side of (5.2) is 1.

2) $R(x,y) < 1$. Then $R(y,x) = 1$, which together with $R(x_{max},y) = 1$ imply $R(x,x_{max}) = R(x,y) \cdot R(y,x_{max})$. Then $R(x,x_{max}) \leq R(y,x_{max})$ and (5.2) is true in this case too.

Let us prove now that C^R fulfills (K). We have, by (5.2)

$$C_X^R(x) = \bigwedge_{y \in X} \frac{R(x,x_{max})}{\max\{R(x,x_{max}),R(y,x_{max})\}} \wedge X(x)$$

$$= \frac{R(x,x_{max})}{\max_X\{R(x,x_{max})\}} \wedge X(x) \tag{5.3}$$

Further,

$$(N(C_X^R \cap X'))(x) = \frac{\dfrac{R(x,x_{max})}{\max_X\{R(x,x_{max})\}}}{\max_{X'} \dfrac{R(x,x_{max})}{\max_X\{R(x,x_{max})\}}} \wedge X'(x)$$

$$= \frac{R(x,x_{max})}{\max_{X'}\{R(x,x_{max})\}} \wedge X'(x) = C_{X'}^R(x),$$

by (5.3). Hence, (H) holds for C^R.

It is easy to see that C^R also fulfills (S) and (P).

iii) $\rightarrow$ i).

Let us define $f(x) = C_A(x)$. Then f is positive, by (P), and

$$D_X^f(x) = \frac{f(x)}{\max_X(f)} \wedge X(x) = \frac{C_A(x)}{\max_X\{C_A(x)\}} \wedge X(x)$$

$$= (N(C_A \cap \mathrm{car}X))(x) \wedge X(x) = C_{\mathrm{car}X}(x) \wedge X(x) = C_X(x),$$

by (K) and (S). The proof is over. ¤

Remark. The theorem proved is an extension of the famous Arrow's result [2, theorems 2 and 3].

Theorem 5.2. Let C be a choice function. The following statements are equivalent:

i) C is based on some vector criterion f;

ii) C is based on some quasi-transitive relation R;

iii) C fulfills properties (S), (P), (H) and (C).

Proof. i $\leftrightarrow$ ii).

Let C is based on a vector criterion f, i.e.

$$C_X(x) = P_X^f(x) = \bigwedge_{y \in X} \bigvee_i \frac{f_y^i(x)}{\max\limits_{x \in X}\{f_y^i(x)\}} \wedge X(x) \qquad (5.4)$$

We have, for $y \in X$,

$$\frac{f_y^i(x)}{\max\limits_{x \in X}\{f_y^i(x)} = \frac{f_y^i(x)}{f^i(y)} = \frac{f^i(x)}{f^i(y)} \wedge 1. \qquad (5.5)$$

Hence, by (5.1) and part i) $\to$ ii) of the proof of theorem 5.1,

$$R_i(x,y) = \frac{f_y^i(x)}{\max\limits_{x \in X}\{f_y^i(x)\}}$$

defines an ordering. By theorem 3.2, $R = \cup R_i$ is a quasi-transitive relation. It is obvious from (5.4) that $C = C^R$.

Conversely, let $C = C^R$ for some quasi-transitive relation R, i.e.

$$C_X(x) = C_X^R(x) = \bigwedge_{y \in X} R(x,y) \wedge X(x). \qquad (5.6)$$

By theorem 3.1,

$$R(x,y) = \bigvee_{i=1}^{m} R_i(x,y)$$

where R_i are orderings. Let x^i_{max} be a maximal element in A with respect to P_{R_i} and $f^i(x) = R_i(x,x^i_{max})$. It follows from (5.2), that $R_i(x,y) = \frac{f^i(x)}{f^i(y)} \wedge 1$. Hence, by (5.5) and (5.6),

$$C_X(x) = \bigwedge_{y\in X} \bigvee_{i=1}^{m} \frac{f^i_y(x)}{\max_{x\in X}\{f^i_y(x)\}} \wedge X(x) = P^f_X(x)$$

where $f = (f^1,f^2,\dots,f^m)$.

ii) $\leftrightarrow$ iii).

Let C is based on a quasi-transitive relation R (see 5.6)). Then (S) and (P) trivially hold. Let us prove (H). It is sufficient to prove that

$$\frac{\bigwedge_{y\in X} R(x,y)}{\max_{x\in X'}\{\bigwedge_{y\in X} R(x,y)\}} \leq \bigwedge_{y\in X'} R(x,y) \tag{5.7}$$

for all $x \in X'$, where $X' \subseteq X$ are crisp sets. Let $y_0 \in X$ be an element such that $R(x,y_0) \leq R(x,y)$ for all $y \in X$. Then (5.7) is equivalent to

$$\frac{R(x,y_0)}{\max_{x\in X'}\{R(x,y_0)\}} \leq R(x,z) \text{ for any } z \in X'. \tag{5.8}$$

Obviously, it suffices to prove that

$$R(x,y_0) \leq R(x,z) \cdot R(z,y_0) \text{ for any } z \in X'. \tag{5.9}$$

Note, that (5.8) is true if $R(x,z) = 1$ and we may suppose that $R(x,y) < 1$. If $R(z,y_0) = 1$, then (5.9) is true, by choice of y_0. Let $R(z,y_0) < 1$. Then $R(y_0,z) = 1$ and $R(z,x) = 1$ imply (5.9), by (3.4).

In order to prove (C), let $X = X' \cup X''$. Then

$$C^R_{X'}(x) \wedge C^R_{X''}(x) = [\bigwedge_{y\in X'} R(x,y) \wedge X'(x)] \wedge [\bigwedge_{y\in X''} R(x,y) \wedge X''(x)]:$$

$$= \bigwedge_{y\in X} R(x,y) \wedge X'(x) \wedge X''(x) \leq \bigwedge_{y\in X} R(x,y) \wedge X(x) = C^R_X(x),$$

i.e. $C^R_X \supseteq C^R_{X'} \cap C^R_{X''}$.

Conversely, let C fulfill (S), (P), (H) and (C). We define a fuzzy binary relation R by

$$R(x,y) = C_{\{x,y\}}(x).$$

Note, first of all, that (H) implies $C_X = N(C_X)$ if we take $X' = X$. Hence, $h(C_X) = 1$ for a nonempty X, i.e., there is $x \in X$ such that $C_X(x) = 1$. It is obvious now that R is a complete relation.

In order to prove transivity property (3.4), let us suppose that $P_R(x,y) = 1$ and $P_R(y,z) = 1$, i.e. that $R(y,x) < 1$ and $R(z,y) < 1$. Then $\{x,y\} \subset X_0 = \{x,y,z\}$ implies, by (H),

$$R(y,x) = C_{\{x,y\}}(y) \geq \frac{C_{X_0}(y)}{\max C_{X_0}(x), C_{X_0}(y)\}}$$

We have $C_{X_0}(y) < C_{X_0}(x)$, since $R(y,x) < 1$. In the same way, $R(z,y) < 1$ implies $C_{X_0}(z) \leq C_{X_0}(y)$. Further, by (H),

$$R(x,z) = C_{\{x,z\}}(x) \geq \frac{C_{X_0}(x)}{\max\{C_{X_0}(x), C_{X_0}(z)\}}$$

which implies $R(x,z) = 1$, since $C_{X_0}(x) > C_{X_0}(z)$.

Note, that $h(C_{X_0}) = 1$ implies $C_{X_0}(x) = 1$ in this case.

Let us consider now a representation $X_0 = \{x,z\} \cup \{y,z\}$. Then, by (C),

$$C_{X_0}(z) \geq C_{\{x,z\}}(z) \wedge C_{\{y,z\}}(z) = R(z,x) \wedge R(z,y)$$

and, by (H),

$$R(z,y) = C_{\{y,z\}}(z) \geq \frac{C_{X_0}(z)}{C_{X_0}(y)} \geq \frac{R(z,x) \wedge R(z,y)}{C_{X_0}(y)} \qquad (5.10)$$

But, in the same way,

$$R(y,x) = C_{\{x,y\}}(y) \geq \frac{C_{X_0}(y)}{C_{X_0}(x)} = C_{X_0}(y)$$

which together with (5.10) yield

$$R(z,y) \cdot R(y,x) \geq R(z,x) \wedge R(z,y).$$

Since $R(z,y) \geq R(z,y) \cdot R(y,x)$, we have

$$R(z,x) \leq R(z,y) \cdot R(y,x) < 1$$

which proves (3.5). Hence, R is a quasi-transitive relation.

Let us prove now that $C^R = C$. If $\{x,y\} \subseteq X$, where X is any crisp subset of A, then, by (H),

$$R(x,y) = C_{\{x,y\}}(x) \geq \frac{C_X(x)}{\max\{C_X(x), C_X(y)\}}$$

which implies, for $x \in X$,

$$C_X^R(x) = \bigwedge_{y \in X} R(x,y) \geq \bigwedge_{y \in X} \frac{C_X(x)}{\max\{C_X(x), C_X(y)\}} = \frac{C_X(x)}{h(C_X)} = C_X(x).$$

On the other hand, $X = \bigcup_{y \in X} \{x,y\}$ if $x \in X$, which implies, by (C),

$$C_X(x) \geq \bigwedge_{y \in X} C_{\{x,y\}}(x) = \bigwedge_{y \in X} R(x,y) = C_X^R(x).$$

Hence, $C_X^R = C_X$ for all crisp X and it is true for all fuzzy sets X, by (S). The proof is over. ¤

6. Consistent optimization and weak orderings

Let $f = (f^1, f^2, \ldots, f^n)$ be a vector criterion on A with positive real-valued scalar components f^i. We define a dominating subset of a given fuzzy set X with respect to f as a fuzzy set D_X^f with a membership function

$$D_X^f(x) = \bigwedge_i \frac{f^i(x)}{\max_X (f^i)} \wedge X(x).$$

The set D_X^f can be regarded as a fuzzy subset of elements of X which dominate all elements of X with respect to all functions f^i simultaneously. Obviously, $D_X^f = \bigcap_i D_X^{f^i}$ where $D_X^{f^i}$ are dominating subsets for scalar criteria f^i, i.e. D_X^f may be also considered as a "consistent" maximizing subset of X with respect to a vector criterion f. Note, that in the classical theory such a choice usually appears to be empty.

Generally speaking, the statement of Proposition 2.1 is false for vector criteria. It may be proved only that $D_X^f \subseteq P_X^f$ for any f and X.

A choice function defined by $C_X = D_X^f$ for a vector criterion is said to be a <u>choice function based on consistent optimization</u>. It turns out that such choice functions have a representation by a pair-dominant choice mechanism based on fuzzy weak orderings.

<u>Definition 6.1</u>. A reflexive fuzzy binary relation R is a <u>weak ordering</u> iff it has a max-product transivity property ([11]):

$$R(x,y) \cdot R(y,z) \leq R(x,z) \quad (6.1)$$

for all $x,y,z \in A$.

<u>Proposition 6.1</u>. Any ordering is a weak ordering.

<u>Proof</u>. Since a complete relation is reflexive, it is sufficient to prove that (3.2) implies (6.1). Let R be an ordering. If $R(x,z) = 1$, then (6.1) is trivial. Let $R(x,z) < 1$, i.e., $R(z,x) = 1$. If $R(x,y) = 1$, then $R(z,y) = 1$ and $R(y,x) \cdot R(x,z) = R(y,z)$, by (3.2), and

$$R(x,y) \cdot R(y,z) = R(y,z) \leq R(x,z).$$

Let $R(x,y) < 1$, i.e. $R(y,x) = 1$. If $R(y,z) = 1$, then $R(x,y) = R(x,z) \cdot R(z,y)$, by (3.2), which implies

$$R(x,y) \cdot R(y,z) = R(x,y) \leq R(x,z).$$

If $R(z,y) = 1$, then $R(x,y) \cdot R(y,z) = R(x,z)$. ¤

The following theorem shows that choice functions based on consistent optimization and on weak orderings yield the same choice.

Theorem 6.1. Let C be a choice function. The following statements are equivalent:

i) C is based on consistent optimization,

ii) C is based on some weak ordering R.

Proof. i) $\rightarrow$ ii).

Let $f = (f^1, f^2, \ldots, f^n)$ be a vector criterion and $C_X = D_X^f$. We define

$$R(x,y) = \bigwedge_i \frac{f^i(x)}{\max\{f^i(x), f^i(y)\}}$$

Then

$$D_X^f(x) = \bigwedge_i \frac{f^i(x)}{\max_X (f^i)} \wedge X(x) = \bigwedge_i \left[\bigwedge_{y \in X} \frac{f^i(x)}{\max\{f^i(x), f^i(x)\}}\right] \wedge X(x)$$

$$= \bigwedge_{y \in X} \left[\bigwedge_i \frac{f^i(x)}{\max\{f^i(x), f^i(y)\}}\right] \wedge X(x) = \bigwedge_{y \in X} R(x,y) \wedge X(x) = C_X^R(x),$$

i.e. $C = C^R$ for R defined above.

In order to prove that R is a weak ordering the following lemma is established.

Lemma 6.1. Let α, β, γ be positive numbers. Then

$$\frac{\alpha}{\max\{\alpha,\beta\}} \cdot \frac{\beta}{\max\{\beta,\gamma\}} \leq \frac{\alpha}{\max\{\alpha,\gamma\}} . \tag{6.2}$$

Proof. Let $\alpha < \beta$. Then (6.2) is true, since $\max\{\beta,\gamma\} \geq \max\{\alpha,\gamma\}$. Let $\alpha \geq \beta$. Then (6.2) is equivalent to

$$\frac{\beta}{\max\{\beta,\gamma\}} \leq \frac{\alpha}{\max\{\alpha,\gamma\}} . \tag{6.3}$$

If $\gamma \leq \alpha$, then (6.3) is true, since $\beta \leq \max\{\beta,\gamma\}$. If $\gamma > \alpha$ then (6.3) is true, since $\alpha \geq \beta$. ⌑

We return to the proof of the theorem. By the previous lemma we have

$$R(x,y) \cdot R(y,z) = \left[\bigwedge_{i=1}^{n} \frac{f^i(x)}{\max\{f^i(x),f^i(y)\}}\right] \cdot \left[\bigwedge_{i=1}^{n} \frac{f^i(y)}{\max\{f^i(y),f^i(z)\}}\right]$$

$$\leq \bigwedge_{i=1}^{n}\left[\frac{f^i(x)}{\max\{f^i(x),f^i(y)\}} \cdot \frac{f^i(y)}{\max\{f^i(y),f^i(z)\}}\right]$$

$$\leq \bigwedge_{i=1}^{n} \frac{f^i(x)}{\max\{f^i(x),f^i(y)\}} = R(x,z).$$

Hence, R has a max-product transivity property. R is a weak ordering, since it is obviously a reflexive relation.

Conversely, let R be a weak ordering and $A = \{x_1,x_2,\ldots,x_n\}$ any numeration of A. Let us define $f^i(x) = R(x,x_i)$. Then

$$R(x,y) \leq \frac{R(x,x_i)}{\max\{R(x,x_i),R(y,x_i)\}}$$

by max-product transivity of R. Hence,

$$C_X^R(x) = \bigwedge_{y\in X} R(x,y) \wedge X(x) \leq \bigwedge_{y\in X} \frac{f^i(x)}{\max\{f^i(x),f^i(y)\}} \wedge X(x)$$

$$= \frac{f^i(x)}{\max_X(f^i)} \wedge X(x) = D_X^{f^i}(x),$$

i.e., $C_X^R \subseteq D_X^{f^i}$ for all i, which implies $C_X^R \subseteq D_X^f$. On the other hand,

$$D_X^f(x) = \bigwedge_{i=1}^{n} \frac{f^i(x)}{\max_X(f^i)} \wedge X(x) = \bigwedge_{i=1}^{n} \frac{R(x,x_i)}{\max_X\{R(x,x_i)\}} \wedge X(x)$$

$$\leq \bigwedge_{i=1}^{n} R(x,x_i) \wedge X(x) = C_X^R(x).$$

Hence, $C_X^R = D_X^f$. ⌑

7. Discussion

In this section choice mechanisms introduced and some related topics are discussed.

1. Choice mechanisms based on dominating sets have the following interpretation which, possibly, makes the basic idea more clear. In some particular problems not optimal states are studied but so-called "quasi-optimal" or "ε-optimal" states. Let, for example, f be a scalar criterion, $\varepsilon > 0$ - some real number and X - a fixed set submitted. We denote $f_{max} = \max_X(f)$. A state $x \in X$ is said to be <u>ε-optimal</u> iff $|f(x)-f_{max}| \leq \varepsilon$. The set $C_X^{f,\varepsilon}$ of all ε-optimal states in X may be regarded as the set of "best states" in this case. Then $C_X^{f,\varepsilon}$ is a value of some choice function. If f is a ratio-scale, it is more natural to consider "relative quasi-optimal" states. Namely, let $\alpha \in (0;1]$ ("decisive level"). The state $x \in X$ is said to be <u>α-optimal</u> (in a relative sense) iff $f(x) \geq \alpha \cdot f_{max}$. Then

$$C_X^{f,\alpha} = \{x \in X | f(x) \geq \alpha \cdot f_{max}\}$$

again defines a choice function. (Note, that $C_X^{f,\alpha} = C_X^{f,\varepsilon}$, if $\alpha = 1 - \frac{\varepsilon}{f_{max}}$.) The choice of decisive level seems to be a subjective one: why should be chosen, for example, $\alpha = .9$ instead of $\alpha = .91$? In a general theory it is better to study the ensemble $\{C_X^{f,\alpha}\}$ taken for all $\alpha \in (0;1]$. But this idea leads exactly to the notion of a fuzzy set defined by means of level-sets in fuzzy set theory (see [4] or [11]). Hence, fuzzy set theory provides a natural basis for simultaneous study of quasi-optimal states taken for various levels.

2. Basic notions of dominating and undominated subsets introduced in section 2 may be considered as "translations" of classical definitions into the fuzzy set theory language. It is well-known (see, for example, [4]) that basic set-theoretic operations in fuzzy set theory are models of logical connectives and quantifiers:

corresponds to the connective "and",

———— // ———— "or",

$\bigwedge_x$ corresponds to the quantifier "for all x",

$\bigvee_x$ ———— // ———— "for some x"

If the notion of a maximizing set is chosen as a basic one, then all definitions in section 2 are translations of classical ones. Table 7.1 represents most important examples of these translations.

3. The interpretation of value R(x,y) for a given ordering R becomes more evident if we consider a standard decomposition of R into strict preference and indifference relations. This decomposition plays an important role in the classical theory. We have already defined a strict preference P_R by

$$P_R(x,y) = \begin{cases} R(x,y), \text{ if } R(x,y) > R(y,x), \\ 0, \text{ otherwise.} \end{cases}$$

An <u>indifference relation</u> I_R is defined by

$$I_R(x,y) = R(x,y) \wedge R(y,x).$$

It is easy to verify that P_R is a crisp quasi-series and $P_R \cup I_R = R$ for any ordering R. One may say that all fuzziness of an ordering R is concentrated in an indifference relation I_R. Indifference relations derived from orderings have many attractive properties which make it possible to consider them as equivalence relations and to define a proper notion of classes (see [6]). Then any ordering R may be regarded as a crisp linear ordering over the set of fuzzy classes of indifference relation I_R.

4. From the fuzzy set theory point of view all preference relations introduced in section 3 are not "very fuzzy" ones. Our definitions are more restrictive than usual definitions in fuzzy set

theory (cf. [4],[11]) (but provide a complete framework described in the paper). On the other hand, formula (3.1) defines a choice function for arbitrary fuzzy binary relation R. It turns out that certain pair-dominant choice mechanisms defined by (3.1) generate classes of choice functions which may be described by means of characteristic properties in the same way as it was done in sections 4 and 5. This approach, using standard fuzzy binary relations, is developed in [5]. Naturally, characteristic properties in [5] are different from those introduced in this paper.

5. The notion of a maximizing set may be also introduced in case of interval scales [4]. Then dominating and undominated subsets define choice functions in the same way as it was done for ratio-scales in this paper. It is easy to show that there is no pair-dominant mechanism which generates the same choice functions as based on maximizing sets for interval scales. But there is a mechanism of choice based on fuzzy ternary relations which provides a similar framework of choice in this case. The author intends to study this problem in a future publication.

	Classical notions	Translations
scalar criterion f	A set of maximal elements in X with respect to f.	$D_X^f(x) = \frac{f(x)}{\max_X(f)} \wedge X(x)$
	A subset of elements in X which are not dominated by a given $y \in X$, or, equivalently, a subset of maximal elements in X with respect to fy.	$D_X^{f_y}(x) = \frac{f_y(x)}{\max_X(fy)} \wedge X(x)$
	A subset of undominated elements in X, or, equivalently, a subset of elements in X which are not dominated by y for all $y \in X$.	$P_X^f(x) = \bigwedge_{y \in X} D_X^{fy}(x)$ $= \bigwedge_{y \in X} \frac{fy(x)}{\max_X(fy)} \wedge X(x)$
vector criterion f	A subset of elements in X which are not dominated by a given $y \in X$, or, equivalently, a subset of elements in X which are maximal with respect to f_y^i for some i.	$\bigvee_i D^{f_y^i}(x)$
	A Pareto set subset in X, i.e., a subset of elements which are not dominated by y for all $y \in X$.	$P_X^f(x) = \bigwedge_{y \in X} \bigvee_i D_X^{f_y^i}(x)$ $= \bigwedge_{y \in X} \bigvee_i \frac{f_y^i(x)}{\max_X(f_y^i)} \wedge X(x)$

Table 7.1

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