RELIABILITY SYSTEMS MODELED AS CONTINUOUS-TIME MARKOV CHAINS WITH PERIODIC TRANSITION RATES

by

Yu-Kun Tsai and Felix F. Wu

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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Yu-Kun Tsai and Felix F. Wu

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

ABSTRACT

Systems modeled as continuous-time finite-state Markov chains with periodic transition rates are considered. The system states are partitioned into the set of up-states and the set of down-states. The probability distributions of remaining in up-states and in down-states are derived. The limiting distributions of up-time, down-time and cycle-time, and the limiting expected up-time, down-time, and cycle-time, as well as the limiting average availability and the limiting expected frequency of entering down-states are derived. Some analogous relations are shown to be true as in Renewal Theory.

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1. INTRODUCTION

Consider a complex system consists of r repairable components that are either in operation or in repair, there are altogether $n = 2^r$ system states. The state transition in the system is modeled as a continuous-time finite-state Markov chain. We consider the case where the state transition rates are periodic. The model arises in our study of interconnected electric power network security assessment and reliability analysis, where the transition rates involve load demands that are periodic functions of time [1].

For some of the system states, the system is considered functioning or operational, we call them the set of up-states and denoted by U. For the remaining of the system states, the system is considered failure, we call them the set of down-states and denoted by D. For reliability analysis we are interested in the time during which the system remains in up-states if presently it is in U or the time during which the system remains in down-states if presently it is in D. In this paper we first derive the probability distributions of these up-time and down-time for general nonstationary continuous-time finite-state Markov chains. We then derive the limiting distributions of up-time, down-time, and cycle-time, and the limiting expected up-time, down-time, and cycle-time for continuous-time finite-state Markov chains with periodic transition rates. Furthermore we derive the limiting average probability that the system is in an up-state, and the limiting expected frequency of entering down-states. We show that some analogous relations hold as in Renewal Theory.

Some reliability models in the form of continuous-time finite-state Markov chains with constant transition rates are studied in [2, Ch. 7]. The time to failure distributions for such systems starting from all the components up are derived in [3]. A simple derivation is presented in [4].

2. UP-TIME AND DOWN-TIME DISTRIBUTIONS OF NONSTATIONARY CONTINUOUS-TIME MARKOV CHAINS

Consider a continuous-time finite-state Markov chain $\underset{\sim}{x}(t)$ with n states denoted by 1,2,...n. We make the following assumptions†

Assumption 1. For all $i \neq j$, and $\Delta t > 0$,

$$\Pr\{\underset{\sim}{x}(t+\Delta t)=j \mid \underset{\sim}{x}(t)=i\} = \lambda_{ij}(t)\Delta t + o(\Delta t) \tag{1}$$

Assumption 2.

$$\Pr\{\text{The number of transitions in } [t,t+\Delta t] \geq 2\} = o(\Delta t) \tag{2}$$

We are concerned with reliability analysis of complex systems modeled as continuous-time finite-state Markov chains. For example, for a system of r repairable components that are either in operation or in repair, there are altogether $n = 2^r$ states of the system. The system is functional for some of these states, called up-states, and is considered failure for the remaining states, called down-states. Let $U = \{1,2,\ldots m\}$ denote the set of up-states and $D = \{m+1,\ldots n\}$ the set of down-states.

Initially at time t_0 we have the probability distribution of the states that the system resides on,

$$p_i(t_0) \triangleq \Pr\{x(t_0) = i\} \qquad i = 1,2,\ldots n \tag{3}$$

Suppose that given $x(t_0) \in U$ the probability distribution of states at t_0 is denoted by

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†The little-oh notation represents a term such that $\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$.

$$u_i(t_o) \overset{\Delta}{=} \Pr\{\underset{\sim}{x}(t_o) = i \,|\, \underset{\sim}{x}(t_o) \in U\} \qquad i = 1,2,\ldots m \tag{4}$$

Similarly given $\underset{\sim}{x}(t_o) \in D$ we define

$$d_i(t_o) \overset{\Delta}{=} \Pr\{\underset{\sim}{x}(t_o) = i \,|\, \underset{\sim}{x}(t_o) \in D\} \qquad i = m+1,\ldots n \tag{5}$$

We will use $\underline{p}(t_o)$, $\underline{u}(t_o)$ and $\underline{d}(t_o)$ to denote the vectors whose components are $p_i(t_o)$, $u_i(t_o)$ and $d_i(t_o)$, respectively.

Suppose that at time t_o the system is in an up-state $\underset{\sim}{x}(t_o) \in U$, we are interested in the duration that the system remains in up-states, i.e., the time to failure. More precisely we define, given that at t_o $\underset{\sim}{x}(t_o) \in U$, the <u>up-time</u> to be

$$T_U \overset{\Delta}{=} T_U \text{ such that } \underset{\sim}{x}(t) \in U \text{ for } t \in [t_o, t_o + T_U)$$
$$\text{and } \underset{\sim}{x}(t_o + T_U) \in D$$

To simplify notation we shall write $\underset{\sim}{x}[t_o, t_o + T_U) \subset U$ to represent $\underset{\sim}{x}(t) \in U$ for $t \in [t_o, t_o + T_U)$. Similarly if at t_o $\underset{\sim}{x}(t_o) \in D$ we define the <u>down-time</u>, or the time to repair, to be

$$T_D \overset{\Delta}{=} T_D \text{ such that } \underset{\sim}{x}[t_o, t_o + T_D) \subset D,\ \underset{\sim}{x}(t_o + T_D) \in U$$

The up-time T_U and the down-time T_D are random variables. We are interested in finding the <u>probability distributions of up-time</u> $F_U(t,t_o)$ and <u>down-time</u> $F_D(t,t_o)$, viz.

$$F_U(t,t_o) \overset{\Delta}{=} \Pr\{T_U > t \,|\, \underset{\sim}{x}(t_o) \in U,\ \underline{u}(t_o)\} \tag{6}$$

$$F_D(t,t_o) \overset{\Delta}{=} \Pr\{T_D > t \,|\, \underset{\sim}{x}(t_o) \in D,\ \underline{d}(t_o)\} \tag{7}$$

Clearly we have

$$F_U(t,t_o) = \Pr\{\underset{\sim}{x}[t_o, t_o+t] \subset U \,|\, \underset{\sim}{x}(t_o) \in U,\ \underline{u}(t_o)\}$$
$$= \sum_{i=1}^{m} \Pr\{\underset{\sim}{x}(t_o+t)=i, \underset{\sim}{x}[t_o, t_o+t] \subset U \,|\, \underset{\sim}{x}(t_o) \in U, \underline{u}(t_o)\} \tag{8}$$

$$F_D(t,t_0) = \Pr\{\underset{\sim}{x}[t_0,t_0+t] \subset D \,|\, \underset{\sim}{x}(t_0) \in D,\ \underline{d}(t_0)\}$$

$$= \sum_{i=m+1}^{n} \Pr\{\underset{\sim}{x}(t_0+t)=i,\ \underset{\sim}{x}[t_0,t_0+t] \subset D \,|\, \underset{\sim}{x}(t_0) \in D,\ \underline{d}(t_0)\} \qquad (9)$$

Let us define

$$u_i(t) \overset{\Delta}{=} \Pr\{\underset{\sim}{x}(t)=i,\ \underset{\sim}{x}[t_0,t] \subset U \,|\, \underset{\sim}{x}(t_0) \in U,\ \underline{u}(t_0)\} \qquad (10)$$

$$d_i(t) \overset{\Delta}{=} \Pr\{\underset{\sim}{x}(t)=i,\ \underset{\sim}{x}[t_0,t] \subset D \,|\, \underset{\sim}{x}(t_0) \in D,\ \underline{d}(t_0)\} \qquad (11)$$

Hence

$$F_U(t,t_0) = \underline{1}^T \underline{u}(t+t_0) \qquad (12)$$

$$F_D(t,t_0) = \underline{1}^T \underline{d}(t+t_0) \qquad (13)$$

where $\underline{1}^T$ is a row vector whose elements are all 1. The dimension of $\underline{1}$ is clear from the context.

We further define

$$\lambda_{ii}(t) \overset{\Delta}{=} - \sum_{\substack{j=1 \\ j \neq i}}^{n} \lambda_{ij}(t) \qquad (14)$$

Let $R(t)$ be the $n \times n$ matrix whose ij-th element is $\lambda_{ji}(t)$. We partition the matrix $R(t)$ into submatrices corresponding to the sets U and D,

$$R(t) = \begin{bmatrix} R_{UU}(t) & R_{UD}(t) \\ R_{DU}(t) & R_{DD}(t) \end{bmatrix} \qquad (15)$$

where $R_{UU}(t)$ is $m \times m$, and $R_{DD}(t)$ is $(n-m) \times (n-m)$.

Theorem 1 below states that the probability distributions of up-time $F_U(t,t_0)$ and down-time $F_D(t,t_0)$ can be obtained from the solution of a differential equation.

Theorem 1. Under assumptions 1 and 2,

(i) for a given initial distribution $\underline{p}(t_0)$, we have

$$\frac{d\underline{p}(t)}{dt} = R(t)\ \underline{p}(t) \tag{16}$$

(ii) given that $\underset{\sim}{x}(t_0) \in U$ and $\underline{u}(t_0)$, we have

$$\frac{d\underline{u}(t)}{dt} = R_{UU}(t)\ \underline{u}(t) \tag{17}$$

$$\text{and } F_U(t,t_0) = \underline{1}^T\underline{u}(t+t_0) \tag{18}$$

(iii) given that $\underset{\sim}{x}(t_0) \in D$ and $\underline{d}(t_0)$, we have

$$\frac{d\underline{d}(t)}{dt} = R_{DD}(t)\underline{d}(t) \tag{19}$$

$$\text{and } F_D(t,t_0) = \underline{1}^T\underline{d}(t+t_0) \tag{20}$$

Proof. The derivation of (i) is standard for Markov chains [5, p. 842]. (iii) is similar to (ii). Thus we prove (ii) only.

$$u_i(t+\Delta t) = Pr\{\underset{\sim}{x}(t+\Delta t)=i, \underset{\sim}{x}[t_0,t+\Delta t] \subset U | \underset{\sim}{x}(t_0) \in U;\ \underline{u}(t_0)\} \tag{21}$$

$$= \sum_{j=1}^{m} Pr\{\underset{\sim}{x}(t+\Delta t)=i, \underline{x}[t,t+\Delta t] \subset U | \underset{\sim}{x}(t)=j, \underset{\sim}{x}[t_0,t] \subset U;\ \underset{\sim}{x}(t_0) \in U, \underline{u}(t_0)\}\ Pr\{\underset{\sim}{x}(t)=j, \underset{\sim}{x}[t_0,t] \subset U | \underset{\sim}{x}(t_0) \in U,\ \underline{u}(t_0)\} \tag{22}$$

$$= \sum_{j=1}^{m} Pr\{\underset{\sim}{x}(t+\Delta t)=i,\ \underset{\sim}{x}[t,t+\Delta t] \subset U | \underset{\sim}{x}(t)=j\} u_j(t) \tag{23}$$

$$= \sum_{j=1}^{m} \sum_{\ell=1}^{\infty} Pr\{\underset{\sim}{x}(t+\Delta t)=i,\ \underset{\sim}{x}[t,t+\Delta t] \subset U, \text{ \# of transition in } [t,t+\Delta t] = \ell | \underset{\sim}{x}(t)=j\} u_j(t) \tag{24}$$

$$= \sum_{j \neq i}^{m} Pr\{x(t+\Delta t)=i,\ x[t,t+\Delta t] \subset U, \text{ \# of transition in } [t,t+\Delta t] = 1 | \underset{\sim}{x}(t)=j\} u_j(t)$$
$$+ Pr\{\underset{\sim}{x}(t+\Delta t)=i,\ \underset{\sim}{x}[t,t+\Delta t] \subset U, \text{ \# of transition in } [t,t+\Delta t] = 0 | \underset{\sim}{x}(t)=i\} u_i(t)$$
$$+ o(\Delta t) \quad \text{(by Ass. 2)} \tag{25}$$

Since "$\underset{\sim}{x}(t+\Delta t)=i$, $\underset{\sim}{x}(t)=j$, $1 \leq i,j \leq m$, and # of transitions in $[t,t+\Delta t] = 1$" $\Rightarrow$ "$\underset{\sim}{x}[t,t+\Delta t] \subset U$" and "$\underset{\sim}{x}(t+\Delta t)=i$, $\underset{\sim}{x}(t)=i$, $1 \leq i \leq m$, # of transitions in $[t,t+\Delta t] = 0$" $\Rightarrow$ "$\underset{\sim}{x}[t,t+\Delta t] \subset U$." So,

$$u_i(t+\Delta t) = \sum_{j \neq i}^{m} [\lambda_{ji}(t)\Delta t + o(\Delta t)]u_j(t) + (1 + \lambda_{ii}(t)\Delta t + o(\Delta t))u_i(t) + o(\Delta t) \tag{26}$$

$$\Rightarrow \frac{d}{dt}\underline{u}(t) = R_{UU}(t)\underline{u}(t) \tag{27}$$

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3. CONTINUOUS-TIME MARKOV CHAINS WITH PERIODIC TRANSITION RATES

We now consider continuous-time finite-state Markov chains with periodic transition rates. We make the following assumption

Assumption 3. The elements of $R(t)$ are periodic with period T_0. The limiting solution of

$$\frac{d\underline{p}(t)}{dt} = R(t)\underline{p}(t) \tag{28}$$

exists and is unique. This limiting distribution is denoted by $\underline{\pi}(t)$.

Lemma 1 Under assumptions 1-3, $\underline{\pi}(t)$ is also periodic with the same period T_0.

Proof. Since $R(t)$ is periodic with period T_0, the state transition matrix of (28) may be written as [6, pp. 126]

$$\Phi(t,t_0) = Q(t,t_0)\exp[(t-t_0)B] \tag{29}$$

where B is a constant matrix $\overset{\Delta}{=} \frac{1}{T_0} \ln \Phi(t_0 + T_0, t_0)$, and the matrix valued function $t \to Q(t,t_0)$ is periodic with period T_0, i.e. $Q(t+T_0,t_0) = Q(t,t_0)$

We first show that 1 is a left eigenvalue which implies that 1 is also a right eigenvalue of $\Phi(t_0+T_0,t_0)$. Note that

$$\underline{1}^T R(t) = \underline{0} \quad \forall t \tag{30}$$

$$\Rightarrow \underline{1}^T \frac{\partial}{\partial t} \Phi(t,t_0) = \underline{1}^T R(t)\Phi(t,t_0) = \underline{0} \tag{31}$$

$$\Rightarrow \underline{1}^T \Phi(t,t_0) = \underline{1}^T \Phi(t_0,t_0) = \underline{1}^T \cdot \underline{I} = \underline{1}^T \tag{32}$$

$$\Rightarrow 1 \text{ is an eigenvalue of } \Phi(t_0+T_0,t_0) \tag{33}$$

$$\Rightarrow 0 \text{ is an eigenvalue of } B \tag{34}$$

Let $\underline{v}$ be the right eigenvector of B corresponding to the eigenvalue 0. By Ass. 3, the limiting distribution $\underline{\pi}(t)$ is unique.

$$\underline{\pi}(t) \overset{\Delta}{=} \Phi(t,t_0)\underline{v} = Q(t,t_0)\, e^{(t-t_0)}\, \underline{v} \tag{35}$$

$$= Q(t,t_0)\underline{v} \tag{36}$$

$$Q(t+T_0,t_0) = Q(t,t_0) \tag{37}$$

$$\Rightarrow \underline{\pi}(t+T_0) = \underline{\pi}(t) \tag{38}$$

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The system state resides alternatively in U and D. However the resulting process is not an alternating renewal process because the up-time (down-time) distributions are not identical. In the following sections, we derive the limiting distributions and expected values of up-time, down-time and cycle-time, and compare the results with Renewal Theory.

3.1 Limiting Distributions of Up-Time, Down-Time and Cycle-Time

Suppose that we look at time t_1 ahead. Let $T_U(t_1)$ be the first full up-time duration after t_1. To be more precise, if t' is the first transition from D to U after t_1, $T_U(t_1)$ is defined to be the time such that

$$\underset{\sim}{x}(t'-)\in D, \underset{\sim}{x}[t',t'+T_U(t_1))\subset U, \text{ and } \underset{\sim}{x}(t'+T_U(t_1))\in D$$

The limiting distribution of up-time $F_U(t)$ is defined to be

$$F_U(t) \overset{\Delta}{=} \lim_{t_1\to\infty} \Pr\{T_U(t_1) > t\} \tag{39}$$

$$= \lim_{t_1\to\infty} \Pr\left\{\underset{\sim}{x}[t',t'+t]\subset U \,\middle|\, \begin{matrix}t' \text{ is the first transition}\\ \text{from D to U after } t_1\end{matrix}\right\} \tag{40}$$

Similarly, we define the limiting distribution of down-time $F_D(t)$ to be

$$F_D(t) \overset{\Delta}{=} \lim_{t_1\to\infty} \Pr\{T_D(t_1) > t\} \tag{41}$$

$$= \lim_{t_1\to\infty} \Pr\left\{\underset{\sim}{x}[t',t'+t]\subset D \,\middle|\, \begin{matrix}t' \text{ is the first transition}\\ \text{from U to D after } t_1\end{matrix}\right\} \tag{42}$$

Let $T_C(t_1)$ denote the first complete cycle after t_1 that starts from U, i.e., if t' is the first transition from D to U after t_1, then

$$\underset{\sim}{x}(t'-)\in D,\ \underset{\sim}{x}[t',t'+t_2)\subset U,\ \underset{\sim}{x}[t'+t_2,t'+T_C(t_1))\subset D$$
$$\text{and} \quad \underset{\sim}{x}(t'+T_C(t_1))\in U$$

The limiting distribution of cycle-time $F_C(t)$ is defined to be

$$F_C(t) = \lim_{t_1\to\infty} \Pr\{T_C(t_1) > t\} \tag{43}$$

Theorem 2 below gives explicit expressions for the limiting distributions of up-time, down-time, and cycle-time. We first introduce some notations. The vector $\underline{\pi}(t)$ is partioned into $\underline{\pi}(t) = (\underline{\pi}_U(t), \underline{\pi}_D(t))$. The state transition matrix of $R_{UU}(t)$ from t' to t is denoted by $\Phi_U(t,t')$, i.e., $\Phi_U(t,t')$ satisfies

$$\frac{\partial}{\partial t} \Phi_U(t,t') = R_{UU}(t)\Phi_U(t,t'), \Phi_U(t',t') = \underset{\sim}{I} \tag{44}$$

The state transition matrix of $R_{DD}(t)$ from t' to t is denoted by $\Phi_D(t,t')$.

Theorem 2. Under assumptions 1-3,

(i) $$F_U(t) = \frac{1}{K}\int_0^{T_o} \underline{1}^T \Phi_U(t+t',t')R_{UD}(t')\underline{\pi}_D(t')dt' \tag{45}$$

where $$K = \int_0^{T_o} \underline{1}^T R_{UD}(t)\underline{\pi}_D(t)dt \tag{46}$$

(ii) $$F_D(t) = \frac{1}{K^*}\int_0^{T_o} \underline{1}^T \Phi_D(t+t',t')R_{DU}(t')\underline{\pi}_U(t')dt' \tag{47}$$

where $$K^* = \int_0^{T_o} \underline{1}^T R_{DU}(t)\underline{\pi}_U(t)dt \tag{48}$$

(iii) $$K^* = K \tag{49}$$

(iv) $$F_C(t) = F_U(t) + \frac{1}{K}\int_0^{T_o}\int_0^{t} \underline{1}^T \Phi_D(t+t',t'+x)R_{DU}(t'+x)\Phi_U(t'+x,t') R_{UD}(t')\underline{\pi}_D(t')dxdt' \tag{50}$$

Proof.

(i) Let $t_1 \to \infty$ be represented by a sequence $t_1 = \ell T_o$, $\ell = 1,2,\ldots$. Hence $t_1 \to \infty \Rightarrow \ell \to \infty$. Let $\underset{\sim}{N}_\ell$ be the random variable such that the interval $[\underset{\sim}{N}_\ell T_o,(\underset{\sim}{N}_\ell+1)T_o)$ contains the first transition from D to U after t_1. Note that $t_1 < (\underset{\sim}{N}_\ell+1)T_o$ almost surely, i.e., $\underset{\sim}{N}_\ell \geq \ell$.

It follows from Eq. (40) that

$$F_U(t) = \lim_{\ell\to\infty} \sum_{j\geq\ell} \int_{jT_o}^{(j+1)T_o} \sum_{i=1}^{m} \Pr\{T_U > t \mid \underset{\sim}{x}(t')=i, \underset{\sim}{x}(t')\in U, x(t'-)\in D, \underset{\sim}{N}_\ell=j\}$$
$$\cdot \Pr\{\underset{\sim}{x}(t')=i \mid \underset{\sim}{x}(t'-)\in D, \underset{\sim}{x}(t')\in U, \underset{\sim}{N}_\ell=j\}$$
$$\cdot \Pr\{\underset{\sim}{x}(t'-)\in D, \underset{\sim}{x}(t')\in U \mid \underset{\sim}{N}_\ell=j\}dt' \cdot \Pr\{\underset{\sim}{N}_\ell=j\} \tag{51}$$

Note that the conditional probability density to make a first transition from D to U at t is proportional to $\underline{1}^T R_{UD}(t)\underline{p}_D(t)$ where $\underline{p}_U(t) =$ $(\underline{p}_U(t),\underline{p}_D(t))$, i.e.,

$$\Pr\{\underset{\sim}{x}(t-)\in D,\underset{\sim}{x}(t)\in U|\underset{\sim}{N}_\ell=j\} = \frac{1}{K(j)}[\underline{1}^T R_{UD}(t)\underline{p}_D(t)],$$

$$jT_o \leq t \leq (j+1)T_o \tag{52}$$

We use the following fact to find the proportion factor K(j).

$$\Pr\left\{\begin{matrix}\text{a transition from D to U}\\ \text{occurs during } [jT_o,(j+1)T_o)\end{matrix} \middle| \underset{\sim}{N}_\ell=j\right\} = 1$$

$$= \int_{jT_o}^{(j+1)T_o} \Pr\{\underset{\sim}{x}(t-)\in D,\underset{\sim}{x}(t)\in U|\underset{\sim}{N}_\ell=j\}dt \tag{53}$$

Hence

$$K(j) = \int_{jT_o}^{(j+1)T_o} \underline{1}^T R_{UD}(t)\underline{p}_D(t)dt \tag{54}$$

Note that

$$\Pr\{T_U>t|\underset{\sim}{x}(t')=i,\ \underset{\sim}{x}(t')\in U,\underset{\sim}{x}(t'-)\in D,\underset{\sim}{N}_\ell=j\}$$

$$= \Pr\{\underset{\sim}{x}(t',t'+t]\subset U|\underset{\sim}{x}(t')=i,\underset{\sim}{x}(t')\in U\} \tag{55}$$

$$= \underline{1}^T\Phi_U(t+t',t')\underline{e}_i \qquad (\underline{e}_i \text{ standard vector})$$

The last equality follows from Theorem 1. The other term in (51) may be expressed as,

$$\Pr\{\underset{\sim}{x}(t')=i|\underset{\sim}{x}(t'-)\in D,\underset{\sim}{x}(t')\in U,N_\ell=j\}$$

$$= \Pr\{\underset{\sim}{x}(t')=i|\underset{\sim}{x}(t'-)\in D,\underset{\sim}{x}(t')\in U\}$$

$$= \frac{[R_{UD}(t')\underline{p}_D(t')]_i}{\underline{1}^T R_{UD}(t')\underline{p}_D(t')} \tag{56}$$

Substituting (52) (55) (56) into (51), we obtain

$$F_U(t) = \lim_{\ell\to\infty} \sum_{j\geq\ell} \frac{1}{K(j)} \int_{jT_o}^{(j+1)T_o} \underline{1}^T \Phi_U(t+t',t') R_{UD}(t') \underline{p}_D(t')dt' \cdot Pr\{\underset{\sim}{N}_\ell = j\} \tag{57}$$

where K(j) is defined on (54).

Let us consider K(j). We are going to show that $K(j)\to K$. First we claim that

$$\Phi_U(t+T_o, t_o+T_o) = \Phi_U(t,t_o) \quad \text{for } \forall\ t_o, t \tag{58}$$

$$\because\ \Phi_U(t+T_o,t_o+T_o)|_{t=t_o} = I = \Phi_U(t_o,t_o)$$

$$\frac{\partial}{\partial t}\Phi_U(t+T_o,t_o+T_o) = R_{UU}(t+T_o)\Phi_U(t+T_o,t_o+T)$$

$$= R_{UU}(t)\Phi_U(t+T_o,t_o+T)$$

The claim (58) follows from the uniqueness of solution to the differential equation.

On the other hand, by definition of limiting distribution, we have

$$\underline{p}(t) = \underline{\pi}(t) + \bar{0}(t), \quad \text{where } \bar{0}(t) \xrightarrow[t\to\infty]{} 0 \tag{59}$$

Thus,

$$K(j) = \int_{jT_o}^{(j+1)T_o} \underline{1}^T R_{UD}(t)\underline{p}_D(t)dt$$

$$= \int_{jT_o}^{(j+1)T_o} \underline{1}^T R_{UD}(t)(\underline{\pi}_D(t)+\bar{0}(t))dt$$

$$= \int_0^{T_o} \underline{1}^T R_{UD}(x+jT_o)\underline{\pi}_D(x+jT_o)dx + \bar{0}(jT_o)$$

$$= \int_0^{T_o} \underline{1}^T R_{UD}(x)\underline{\pi}_D(x)dx + \bar{0}(jT_o)$$

$$\lim_{j\to\infty} K(j) = K \tag{60}$$

Next we consider the numerator of (57),

$$\int_{jT_0}^{(j+1)T_0} \underline{1}^T\Phi_U(t+t',t')R_{UD}(t')\underline{p}_D(t')dt' \tag{61}$$

$$= \int_0^{T_0} \underline{1}^T\Phi_U(t+x+jT_0,x+jT_0)R_{UD}(x+jT_0)\underline{p}_D(x+jT_0)dx \tag{62}$$

Note that since $\lambda_{ij}(t) \geq 0$, $i \neq j$. $\int_0^{T_0} \underline{1}^T R_{UD}(x)\underline{\pi}_D(x)dx > 0$, implies

$$\frac{(61)}{K(j)} = \frac{(61)}{K+\bar{0}(jT_0)} = \frac{(61)}{K} + \bar{0}(jT_0) \tag{63}$$

Substituting (60), (62) and (63) into (57), it becomes

$$F_U(t) = \lim_{\ell\to\infty} \sum_{j\geq\ell}\left[\frac{\int_0^{T_0} \underline{1}^T\Phi_U(t+x,x)R_{UD}(x)\underline{\pi}_D(x)dx}{K} + \bar{0}(jT_0)\right] Pr\{N_\ell=j\}$$

$$= \lim_{\ell\to\infty}\left\{\sum_{j\geq\ell}\left[\frac{\int_0^{T_0} \underline{1}^T\Phi_U(t+x,x)R_{UD}(x)\underline{\pi}_D(x)dx}{K}\cdot Pr\{\underset{\sim}{N}_\ell=j\}\right] + \bar{0}(\ell T_0)\right\}$$

$$= \lim_{\ell\to\infty}\left\{\frac{\int_0^{T_0} \underline{1}^T\Phi_U(t+x,x)R_{UD}(x)\underline{\pi}_D(x)dx}{K}\cdot\sum_{j\geq\ell} Pr\{\underset{\sim}{N}_\ell=j\} + \bar{0}(\ell T_0)\right\}$$

$$= \frac{1}{K}\int_0^{T_0} \underline{1}^T\Phi_U(t+x,x)R_{UD}(x)\underline{\pi}_D(x)dx \tag{64}$$

where we use the fact that $\sum_{j\geq\ell} Pr\{\underset{\sim}{N}_\ell=j\} = 1$.

(ii) Similar to (i).

(iii) By definition, we have $\underline{1}^T R(t) = \underline{0}$, i.e.,

$$\underline{1}^T R_{UU}(t) + \underline{1}^T R_{DU}(t) = \underline{0} \quad \forall t \tag{65}$$

Now consider

$$K-K^* = \int_0^{T_o} \underline{1}^T(R_{UD}(t)\underline{\pi}_D(t) - R_{DU}(t)\underline{\pi}_U(t))dt$$

$$= \int_0^{T_o} \underline{1}^T(R_{UD}(t)\underline{\pi}_D(t) + R_{UU}(t)\underline{\pi}_U(t))dt \quad \text{(by substituting (65))}$$

$$= \int_0^{T_o} \underline{1}^T \frac{d}{dt} \underline{\pi}_U(t)\cdot dt \qquad \text{(by definition)}$$

$$= \underline{1}^T(\underline{\pi}_U(T_o) - \underline{\pi}_U(0))$$

which is 0 as a consequence of lemma 1.

(iv) Let us use the same notations as in the proof of (i)

$$F_C(t) = \lim_{\ell\to\infty} \sum_{j\geq\ell} \Pr\{T_C > t|\underset{\sim}{N}_\ell=j\} \Pr\{\underset{\sim}{N}_\ell=j\}$$

$$= \lim_{\ell\to\infty} \sum_{j\geq\ell}\int_{jT_o}^{(j+1)T_o} \sum_{i=1}^{m} \Pr\{T_C > t|\underset{\sim}{x}(t')=i,\underset{\sim}{x}(t'-)\in D,\underset{\sim}{x}(t')\in U\}$$

$$\cdot \Pr\{\underset{\sim}{x}(t')=i|\underset{\sim}{x}(t'-)\in D,\underset{\sim}{x}(t')\in U\}$$

$$\cdot \Pr\{\underset{\sim}{x}(t'-)\in D,\underset{\sim}{x}(t')\in U|\underset{\sim}{N}_\ell=j\}dt' \cdot \Pr\{\underset{\sim}{N}_\ell=j\} \tag{66}$$

$$\Pr\{T_C > t|\underset{\sim}{x}(t')=i,\ \underset{\sim}{x}(t'-)\in D,\underset{\sim}{x}(t')\in U\}$$

$$= \int_0^t \Pr\{\underset{\sim}{x}[t',t'+x)\subset U,\underset{\sim}{x}[t'+x,t'+t]\subset D|\underset{\sim}{x}(t')=i,\underset{\sim}{x}(t'-)\in D,\underset{\sim}{x}(t')\in U\}dx$$

$$+ \Pr\{T_U > t|\underset{\sim}{x}(t')=i,\underset{\sim}{x}(t'-)\in D,\underset{\sim}{x}(t')\in U\} \tag{67}$$

Substitute (67) into (66), manipulate term $\Pr\{T_U > t|\ldots\}$ as in (i), we get

$$F_C(t) = F_U(t) + \lim_{\ell\to\infty} \sum_{j\geq\ell} \int_{jT_o}^{(j+1)T_o} \sum_{i=1}^{m} \int_0^t$$

$$\Pr\{\underset{\sim}{x}[t',t'+x)\subset U,\underset{\sim}{x}[t'+x,t'+t]\subset D|$$

$$\underset{\sim}{x}(t')=i,\underset{\sim}{x}(t'-)\in D,\underset{\sim}{x}(t')\in U\} \frac{(R_{UD}(t')\underline{p}_D(t'))_i}{K(j)} dxdt'$$

$$\Pr\{N_\ell=j\} \tag{68}$$

Now

$$\Pr\{\underset{\sim}{x}[t',t'+x)\subset U,\underset{\sim}{x}[t'+x,t'+t]\subset D|\underset{\sim}{x}(t')=i,\underset{\sim}{x}(t'-)\in D,\underset{\sim}{x}(t')\in U\}$$

$$= \sum_{i_1=m+1}^{n} \Pr\{\underset{\sim}{x}[t'+x,t'+t]\subset D|\underset{\sim}{x}[t',t'+x)\subset U,\underset{\sim}{x}(t'+x)=i_1,\underset{\sim}{x}(t'+x)\in D, \underset{\sim}{x}(t'-)\in D,\underset{\sim}{x}(t')\in U, x(t')=i\}$$

$$\Pr\{\underset{\sim}{x}(t'+x)=i_1|\underset{\sim}{x}((t'+x)-)\in U, x(t'+x)\in D, \underset{\sim}{x}[t',t'+x)\subset U,\underset{\sim}{x}(t')=i\}$$

$$\Pr\{\underset{\sim}{x}(t'+x)\in D,\underset{\sim}{x}[t',t'+x)\subset U, |\underset{\sim}{x}(t')=i,\underset{\sim}{x}(t'-)\in D,\underset{\sim}{x}(t')\in U\}$$

$$= \sum_{i_1=m+1}^{n} \underline{1}^T\Phi_D(t'+t,t'+x)\underline{e}_{i_1}\cdot\frac{(R_{DU}(t'+x)\Phi_U(t'+x,t')\underline{e}_i)_{i_1}}{\underline{1}^T R_{DU}(t'+x)\Phi_U(t'+x,t')\underline{e}_i} \cdot -\frac{d}{dx}\underline{1}^T\Phi_U(t'+x,t')\underline{e}_i \tag{69}$$

where we have used Therem 1. But

$$\frac{d}{dx}\underline{1}^T\Phi_U(t'+x,t') = \underline{1}^T R_{UU}(t'+x)\Phi_U(t'+x,t')$$

$$= -\underline{1}^T R_{DU}(t'+x)\Phi_U(t'+x,t')$$

Hence (69) becomes

$$\underline{1}^T\Phi_D(t'+t;t'+x)R_{DU}(t'+x)\Phi_U(t'+x,t')\underline{e}_i \tag{70}$$

Substituting (69) (70) into (68), we get

$$F_C(t) = F_U(t) + \lim_{\ell\to\infty}\sum_{j\geq\ell}\frac{1}{K(j)}\int_{jT_o}^{(j+1)T_o}\int_0^t \underline{1}^T\Phi_D(t'+t,t'+x) R_{DU}(t'+x)\Phi_U(t'+x,t')R_{UD}(t')\underline{p}_D(t')\, dxdt'\cdot\Pr\{\underset{\sim}{N}_\ell=j\} \tag{71}$$

$$= F_U(t) + \lim_{\ell\to\infty}\sum_{j\geq\ell}\frac{1}{K(j)}\int_0^{T_o}\int_0^t \underline{1}^T\Phi_D(t+t'+jT_o,t'+jT_o+x)R_{DU}(t'+jT_o+x)$$

$$\Phi_U(t'+jT_o+x,t'+jT_o)R_{UD}(t'+jT_o)[\underline{\pi}_D(t'+jT_o)+\bar{0}(t'+jT_o)]$$

$$dxdt' \cdot \Pr\{N_\ell=j\}$$

$$= F_U(t) + \frac{1}{K}\int_0^{T_o}\int_0^t \underline{1}^T\Phi_D(t+t',t'+x)R_{DU}(t'+x)\Phi_U(t'+x,t')R_{UD}(t')\underline{\pi}_D(t')\,dxdt'$$ ¤

3.2 Limiting Expected Up-Time, Down-Time, and Cycle-Time

The limiting expected up-time $E(T_U)$ is defined to be

$$E(T_U) \triangleq \lim_{t_1 \to \infty} E(T_U(t_1)) \tag{72}$$

The limiting expected down-time $E(T_D)$ and the limiting expected cycle-time $E(T_C)$ are defined to be

$$E(T_D) \triangleq \lim_{t_1 \to \infty} E(T_D(t_1)) \tag{73}$$

$$E(T_C) \triangleq \lim_{t_1 \to \infty} E(T_C(t_1)) \tag{74}$$

Theorem 3 below gives explicit expressions for the limiting expected up-time, down-time, and cycle-time.

Theorem 3. Under assumptions 1-3, we have

$$E(T_U) = \frac{\int_0^{T_o} \underline{1}^T\underline{\pi}_U(t)dt}{\int_0^{T_o} \underline{1}^T R_{UD}(t)\underline{\pi}_D(t)dt} \tag{75}$$

$$E(T_D) = \frac{\int_0^{T_o} \underline{1}^T\underline{\pi}_D(t)dt}{\int_0^{T_o} \underline{1}^T R_{DU}(t)\underline{\pi}_U(t)dt} \tag{76}$$

$$E(T_C) = E(T_U) + E(T_D) \tag{77}$$

Proof.

(i) We are going to show first that

$$E(T_U) = \int_0^\infty F_U(t)dt \tag{78}$$

Let $F_U^i(t) \overset{\Delta}{=} \Pr\{T_U(iT_o) > t\}$

then $E(T_U(iT_o)) = \int_0^\infty F_U^i(t)dt$

and $E(T_U) = \lim_{i\to\infty} \int_0^\infty F_U^i(t)dt$

Since $F_U^i(t)$ is nonnegative measureable function, if we allow ∞ as a limit then, by Fubini's Theorem [7, p. 355], $\{F_U^i\}$ is a sequence of integrable functions, and by (64) $F_U^i(t)$ also converges pointwise to $F_U(t)$. Furthermore $|F_U^i| \leq 1$ $\forall i$, hence by Dominated Convergence Theorem [7, p. 331], $F_U(t)$ is integrable. And

$$\int_0^\infty F_U(t)dt = \lim_{i\to\infty} \int_0^\infty F_U^i(t)dt \overset{\Delta}{=} E(T_U)$$

Next we are going to show that

$$\int_0^\infty F_U(t)dt = \frac{\int_0^{T_o} \underline{1}^T \underline{\pi}_U(t)dt}{\int_0^{T_o} \underline{1}^T R_{UD}(t)\underline{\pi}_D(t)dt}$$

$$E(T_U) = \int_0^\infty F_U(t)dt = \frac{1}{K}\int_0^\infty \int_0^{T_o} \underline{1}^T \Phi_U(t+t',t')R_{UD}(t')\underline{\pi}_D(t')dt'dt$$

With the change of variables $z = t+t'$, $t' = t'$, the corresponding change in the domain of integration can be easily seen as in Fig. 1.

$$E(T_U) = \frac{1}{k}\int_{z=0}^{T_o} \int_{t'=0}^{z} \underline{1}^T \Phi_U(z,t')R_{UD}(t')\underline{\pi}_D(t')dt'dz$$

$$+ \frac{1}{K}\int_{z=T_o}^{\infty} \int_{t'=0}^{T_o} \underline{1}^T \Phi_U(z,t')R_{UD}(t')\underline{\pi}_D(t')dt'dz \tag{79}$$

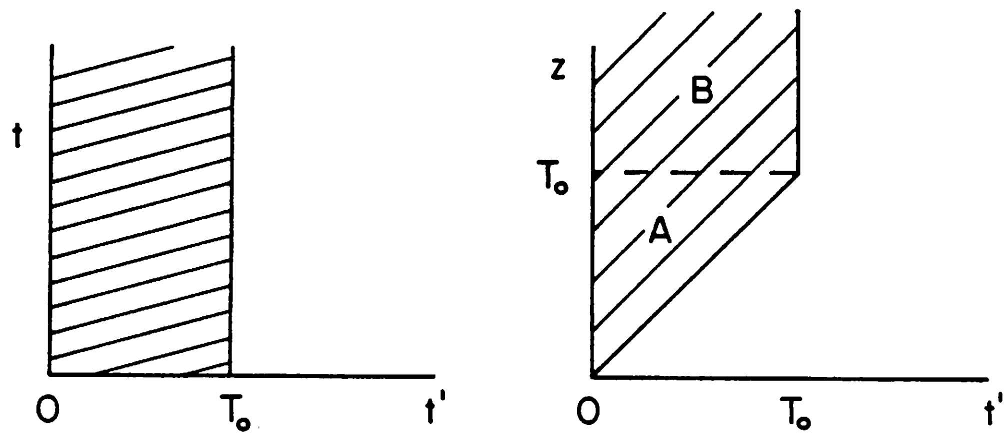

Fig. 1. The domain of integration after change of variables $z = t + t'$, $t' = t'$.

On the other hand,

$$\dot{\underline{\pi}}_U(t') = R_{UU}(t')\underline{\pi}_U(t') + R_{UD}(t')\underline{\pi}_D(t')$$

$$\Rightarrow R_{UD}(t')\underline{\pi}_D(t') = \dot{\underline{\pi}}_U(t') - R_{UU}(t')\underline{\pi}_U(t') \tag{80}$$

Furthermore

$$\frac{\partial}{\partial t'} \Phi_U(z,t')\underline{\pi}_U(t')$$

$$= [\frac{\partial}{\partial t'} \Phi_U(z,t')]\,\underline{\pi}_U(t') + \Phi_U(z,t')\dot{\underline{\pi}}_U(t')$$

$$= -\Phi_U(z,t')R_{UU}(t')\underline{\pi}_U(t') + \Phi_U(z,t')\dot{\underline{\pi}}_U(t') \tag{81}$$

(80) and (81) give

$$\Phi_U(z,t')R_{UD}(t')\underline{\pi}_D(t') = \frac{\partial}{\partial t'} \Phi_U(z,t')\underline{\pi}_U(t') \tag{82}$$

Substituting (82) into (79) and integrating over t', we have

$$E(T_U) = \frac{1}{K}\int_{z=0}^{T_0} \underline{1}^T\underline{\pi}_U(z)dz - \frac{1}{K}\int_{z=0}^{T_0} \underline{1}^T\Phi_U(z,0)\underline{\pi}_U(0)dz$$

$$+ \frac{1}{K}\int_{z=T_0}^{\infty} \underline{1}^T\Phi_U(z,T_0)\underline{\pi}_U(T_0)dz - \frac{1}{K}\int_{z=T_0}^{\infty} \underline{1}^T\Phi_U(z,0)\underline{\pi}_U(0)dz$$

$$= \frac{1}{K}\int_0^{T_0} \underline{1}^T\underline{\pi}_U(z)dz + \frac{1}{K}\int_{T_0}^{\infty} \underline{1}^T\Phi_U(z,T_0)\underline{\pi}_U(T_0)dz$$

$$- \frac{1}{K}\int_0^{\infty} \underline{1}^T\Phi_U(z,0)\underline{\pi}_U(0)dz$$

But $$\int_{T_0}^{\infty} \underline{1}^T \Phi_U(z,T_0)\underline{\pi}_U(T_0)dz$$

$$= \int_0^{\infty} \underline{1}^T \Phi_U(z'+T_0,T_0)\underline{\pi}_U(T_0)dz'$$

$$= \int_0^{\infty} \underline{1}^T \Phi_U(z',0)\underline{\pi}_U(0)dz'$$

Hence we arrive at the result

$$E(T_U) = \frac{1}{K}\int_0^{T_0} \underline{1}^T \underline{\pi}_U(z)dz$$

Similarly for $E(T_D)$.

(ii) The proof that $E(T_C) = \int_0^{\infty} F_C(t)dt$ is the same as in (i).

Applying the result of Theorem 2, we have

$$\int_0^{\infty} F_C(t)dt = \int_0^{\infty} F_U(t)dt + \frac{1}{K}\int_0^{\infty}\int_{t'=0}^{T_0}\int_{x=0}^{t} \underline{1}^T \Phi_D(t+t',t'+x)$$

$$R_{DU}(t'+x)\Phi_U(t'+x,t')R_{UD}(t')\underline{\pi}_D(t')dxdt'dt$$

With the change of variables $\tau = t-x$, $x = x$. (Since the integrand is a nonnegative measurable function the interchange of order of integration is justified by the Fubini Theorem.)

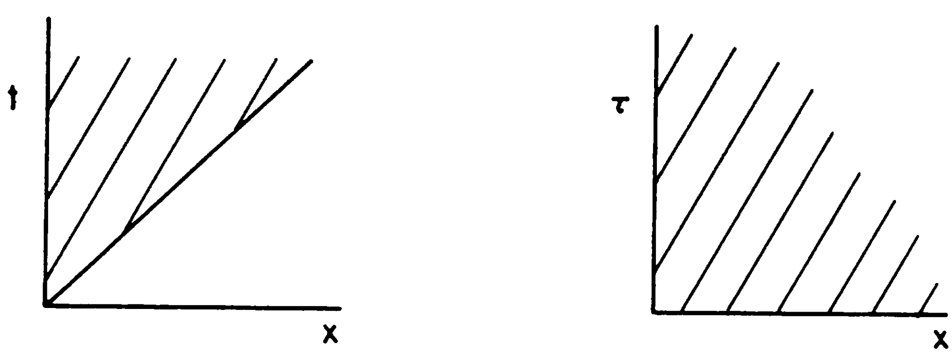

Fig. 2. The change in the domain of integration as a result of the change of variables $\tau = t-x$, $x = \chi$.

$$E(T_C) = E(T_U) + \frac{1}{K}\int_0^\infty \int_0^{T_o} \int_0^\infty \underline{1}^T \Phi_D(\tau+t'+x,t'+x)R_{DU}(t'+x)\Phi_U(t'+x,t')$$
$$R_{UD}(t')\underline{\pi}_D(t')dx\,dt'\,d\tau$$

Interchange the order of integration $dxdt' \to dt'dx$, and with another change of variables $z = x+t'$, $t' = t'$.

$$E(T_C) = E(T_U) + \frac{1}{K}\int_0^\infty \left[\int_{z=0}^{T_o} \int_{t'=0}^{z} \underline{1}^T \Phi_D(t+z,z)R_{DU}(z)\Phi_U(z,t')R_{UD}(t')\underline{\pi}_D(t')\,dt'dz\right]dt$$
$$+\frac{1}{K}\int_0^\infty \left[\int_{z=T_o}^{\infty} \int_{t'=0}^{T_o} \underline{1}^T \Phi_D(t+z,z)R_{DU}(z)\Phi_U(z,t')R_{UD}(t')\underline{\pi}_D(t')\,dt'dz\right]dt$$

Similarly as in the proof of (i)

$$E(T_C) = E(T_U) + \frac{1}{K}\int_0^\infty \left\{ \int_{z=0}^{T_o} \underline{1}^T \Phi_D(t+z,z)R_{DU}(z)[\Phi_U(z,z)\underline{\pi}_U(z)-\Phi_U(z,0)\underline{\pi}_U(0)]dz \right.$$
$$\left. + \int_{z=T_o}^{\infty} \underline{1}^T \Phi_D(t+z,z)R_{DU}(z)[\Phi_U(z,T_o)\underline{\pi}_U(T_o)-\Phi_U(z,0)\underline{\pi}_U(0)]dz \right\} dt$$
$$= E(T_U) + \frac{1}{K}\int_0^\infty \left\{ \int_{z=0}^{T_o} \underline{1}^T \Phi_D(t+z,z)R_{DU}(z)\underline{\pi}_U(z)dz \right.$$
$$+ \int_{z=T_o}^{\infty} \underline{1}^T \Phi_D(t+z,z)R_{DU}(z)\Phi_U(z,T_o)\underline{\pi}_U(T_o)dz$$
$$\left. - \int_{z=0}^{\infty} \underline{1}^T \Phi_D(t+z,z)R_{DU}(z)\Phi_U(z,0)\underline{\pi}_U(0)dz \right\} dt$$

But $\int_{z=T_o}^{\infty} \underline{1}^T \Phi_D(t+z,z)R_{DU}(z)\Phi_U(z,T_o)\underline{\pi}_U(T_o)dz$

$$= \int_{z=0}^{\infty} \underline{1}^T \Phi_D(t+z+T_o,z+T_o)R_{DU}(z+T_o)\Phi_U(z+T_o,T_o)\underline{\pi}_U(T_o)dz$$
$$= \int_{z=0}^{\infty} \underline{1}^T \Phi_D(t+z,z)R_{DU}(z)\Phi_U(z,0)\underline{\pi}_U(0)dz$$

$$\Rightarrow E(T_C) = E(T_U) + \frac{1}{K}\int_0^{\infty}\int_{z=0}^{T_0} \underline{1}^T \Phi_D(t+z,z)R_{DU}(z)\underline{\pi}_U(z)dz$$

$$= E(T_U) + \frac{1}{K^*}\int_0^{\infty}\int_{z=0}^{T_0} \underline{1}^T \Phi_D(t+z,z)R_{DU}(z)\underline{\pi}_U(z)dz$$

$$= E(T_U) + \int_0^{\infty} F_D(t)dt \quad \text{(by Theorem 2)}$$

$$= E(T_U) + E(T_D)$$

3.3 Limiting Average Availability, Limiting Expected Frequency and Duration

The availability at time t, A(t), is defined to be the probability that the system is in an up-state at time t, i.e.,

$$A(t) \triangleq \Pr\{x(t) \in U\} \tag{83}$$

The limiting average availability A_{av} is defined to be

$$A_{av} \triangleq \lim_{T\to\infty} \frac{1}{T}\int_0^T A(t)dt \tag{84}$$

The unavailability at t, $\overline{A}(t)$, and the limiting average unavailability $\overline{A}_{av}$ are similarly defined.

The system resides alternatively in U and D. Let $N_U(t)$ denote the number of transitions into U in the time period [0,t]. We define the limiting expected frequency of entering up-states U, f_U, to be

$$f_U \triangleq \lim_{t\to\infty} E\,\frac{N_U(t)}{t} \tag{85}$$

Similarly $N_D(t)$ denotes the number of transitions into down-states D in [0,t] and the limiting expected frequency of entering down-states D is

$$f_D \triangleq \lim_{t\to\infty} E\,\frac{N_D(t)}{t} \tag{86}$$

Assumption 4.

$$E[N_U(t)] < \infty \quad \text{and} \quad E[N_D(t)] < \infty \text{ for any } t < \infty.$$

Theorem 4 below gives explicit expressions for the limiting average availability, and limiting expected frequency. Theorem 5 below presents some relationships among various limiting expected quantities, similar to the ones in Renewal Theory.

Theorem 4. Under Assumptions 1-4,

$$\text{(i)} \quad A_{av} = \frac{1}{T_o} \int_0^{T_o} \underline{1}^T \underline{\pi}_U(t)dt \tag{87}$$

$$\overline{A}_{av} = \frac{1}{T_o} \int_0^{T_o} \underline{1}^T \underline{\pi}_D(t)dt \tag{88}$$

$$\text{(ii)} \quad f_U = \frac{1}{T_o} \int_0^{T_o} \underline{1}^T R_{UD}(t) \underline{\pi}_D(t)dt \tag{89}$$

$$f_D = \frac{1}{T_o} \int_0^{T_o} \underline{1}^T R_{DU}(t) \underline{\pi}_U(t)dt \tag{90}$$

Proof.

$$\text{(i)} \quad A_{av} = \lim_{T\to\infty} \frac{1}{T} \int_0^T A(t)dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_0^T \underline{1}^T \underline{p}_U(t)dt \qquad (\Pr\{x(t)\in U\} = \underline{1}^T \underline{p}_U(t))$$

$$= \frac{1}{T_o} \lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} \int_{iT_o}^{(i+1)T_o} \underline{1}^T \underline{p}_U(t)dt$$

Since $\int_{iT_o}^{(i+1)T_o} \underline{1}^T \underline{p}_U(t)dt \to \int_0^{T_o} \underline{1}^T \underline{\pi}_U(t)dt$ as $i\to\infty$,

$$A_{av} = \frac{1}{T_o} \int_0^{T_o} \underline{1}^T \underline{\pi}_U(t)dt$$

Similarly for $\bar{A}_{av}$.

(ii) Let us divide t into k subintervals, $\Delta t = \frac{t}{k}$. Then,

$$EN_U(t) = E[N_U(t)-N_U(t-\Delta t) + N_U(t-\Delta t) - N_U(t-2\Delta t) + \dots - N_U(0)]$$

$$= \sum_{i=0}^{k-1} E[N_U((i+1)\Delta t) - N_U(i\Delta t)]$$

By assumptions 2 and 4,

$$E[N_U((i+1)\Delta t) - N_U(i\Delta t)]$$

$$= Pr\{\text{there is a transition into up-state in } [i\Delta t, (i+1)\Delta t]\} + o(\Delta t)$$

$$= \underline{1}^T R_{UD}(i\Delta t)\, \underline{P}_D(i\Delta t)\Delta t + o(\Delta t)$$

Therefore

$$EN_U(t) = \sum_{i=0}^{k-1} \underline{1}^T R_{UD}(i\Delta t)\, \underline{P}_D(i\Delta t)\, t + k\, o(\Delta t)$$

Let $k \to \infty$ (i.e., $\Delta t \to 0$), we have

$$EN_U(t) = \int_0^t \underline{1}^T R_{UD}(t')\, \underline{P}_D(t')dt'$$

So,

$$f_U \triangleq \lim_{t\to\infty} \frac{EN_U(t)}{t}$$

$$= \lim_{t\to\infty} \frac{1}{t} \int_0^t \underline{1}^T R_{UD}(t')\, \underline{P}_D(t')dt'$$

$$= \frac{1}{T_o} \int_o^{T_o} \underline{1}^T R_{UD}(t')\, \underline{\pi}_D(t')dt'$$

Similarly for f_D ¤

Theorem 5. Under assumptions 1-4,

(i) $$A_{av} = \frac{E(T_U)}{E(T_U)+E(T_D)}$$

$$\bar{A}_{av} = \frac{E(T_D)}{E(T_U)+E(T_D)}$$

(ii) $$f_u \cdot E(T_U) = A_{av}$$

$$f_D \cdot E(T_D) = \bar{A}_{av}$$

Proof (i) From Theorem 3, we have

$$\frac{E(T_U)}{E(T_C)} = \frac{\int_0^{T_o} \underline{1}^T \underline{\pi}_U(t)dt}{\int_0^{T_o} \underline{1}^T \underline{\pi}_U(t) + \underline{1}^T \underline{\pi}_D(t)dt}$$

$$= \frac{\int_0^{T_o} \underline{1}^T \underline{\pi}_U(t)dt}{\int_0^{T_o} \underline{1}^T \underline{\pi}(t)dt}$$

$$= \frac{\int_0^{T_o} \underline{1}^T \underline{\pi}_U(t)dt}{T_o}$$

$$= A_{av}$$

(ii) The results are immediate by comparing eqs. (87), (89), and (75). ¤

It is reasonable to define the limiting average failure rate λ_{UD} to be

$$\lambda_{UD} \triangleq \lim_{T\to\infty} \frac{1}{T} \int_0^T \left[\lim_{\Delta t \to o^+} \frac{\Pr\{\underset{\sim}{x}(t+\Delta t)\in D \mid \underset{\sim}{x}(t)\in U\}}{\Delta t} \right] dt$$

It can be shown that

$$\lambda_{UD} = \frac{1}{T_o} \int_o^{T_o} \frac{\underline{1}^T R_{DU}(t)\, \underline{\pi}_U(t)}{\underline{1}^T \underline{\pi}_U(t)}\, dt$$

On the other hand,

$$E(T_U) = \frac{\int_o^{T_o} \underline{1}^T \underline{\pi}_U(t)dt}{\int_o^{T_o} \underline{1}^T R_{DU}(t)\underline{\pi}_U(t)dt}$$

Hence in general $\lambda_{UD} \neq \frac{1}{E(Tu)}$, i.e., the limiting average failure rate is not equal to the reciprocal of the limiting expected up-time as in the case with constant transition rates.

4. CONCLUSIONS

In this paper we have defined several limiting reliability indices of a continuous-time Markov Chain with periodic transition rates. Explicit expressions for the limiting distributions of up-time, down-time and cycle-time, and the limiting expected up-time, down-time, and cycle-time, as well as the limiting average availability and the limiting expected frequency of entering up-states, down-states are derived.

The results are generalizations to stationary Markov Chains [4]. Analogous relations among availability, expected up-time, down-time, frequency and duration as in the renewal theory are shown to be true. However, in this more general case the limiting average failure rate is no longer equal to the reciprocal of the limiting expected up-time.

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