NONLINEAR OSCILLATION VIA VOLTERRA SERIES

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Abstract

Using a novel approach, the amplitude and frequency of nearly sinusoidal nonlinear oscillators can be calculated by solving two algebraic nonlinear equations. These determining equations can be generated to within any desired accuracy using a recursive algorithm based on Volterra series.

Our method inherits many desirable features of the harmonic balance method, the describing function method, and the averaging method. Our technique is analogous to, but is much simpler than, the classic approach due to Krylov, Bogoliubov and Mitropolsky. Unlike conventional techniques, however, our approach imposes no severe restriction on either the degree of nonlinearity, or the amplitude of oscillation. Moreover, the accuracy of the solution can be determined by a constructive algorithm.

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1. INTRODUCTION

The problem of determining the amplitude A and frequency ω of weakly nonlinear (nearly sinusoidal) oscillators dates back at least to van der Pol [1]. Since then, several methods have been developed: they include the describing function method [2], the harmonic balance method [3], and the averaging method due to Krylov, Bogoliubov, and Mitropolsky [4-5].

The describing function and harmonic balance methods are widely used in design problems when the oscillator can be modeled by a single-loop feedback system as shown in Fig. 1.[†] Here, G(s) denotes the transfer function of a single-input single-output linear system made of linear time-invariant elements (e.g., resistors, inductors, capacitors, transmission lines, etc.), and f(•) denotes a memoryless (possibly hysteretic) scalar nonlinear function. Since these methods neglect all harmonics of the fundamental frequency ω, they are valid only when G(s) behaves essentially like a "low-pass" filter. Although rigorous mathematical theorems are available for checking the validity of these methods, they are often impractical to apply. Since these methods are known to predict, incorrectly, oscillations in systems where there are none [6], the answers should be carefully checked in doubtful situations.

The averaging method is applicable for systems described by

$$\dot{\underset{\sim}{x}} = \varepsilon \underset{\sim}{f}(\underset{\sim}{x},t), \quad \underset{\sim}{x} \in \mathbb{R}^n$$

where ε is a small parameter. In the case of oscillators, this method can, in principle, allow one to calculate A and ω to have any desired accuracy by choosing a suitable "order" of determining equations [4]. However, these equations become extremely complex beyond the second order. In practice, this method is usually chosen only when $n = 2$.

More recently, the Hopf bifuraction theorem [7] offers yet another tool for predicting the frequency "ω" of oscillation, provided the amplitude "A" is sufficiently small. Unfortunately, no simple guideline is available for determining how small is "small".

Our objective in this paper is to develop an entirely new approach which inherits many desirable features of the preceding methods. Some interesting properties of this new approach are:

[†]Although these methods can be generalized for multi-loop feedback systems, they are often impractical.

(1) Like the harmonic balance and describing function methods, our approach is formulated in terms of a single-loop nonlinear feedback system (Fig. 2(a)).[†] Our only assumption is that the associated open-loop system F (Fig. 2(b)) has a convergent Volterra series representation [8]:[§]

$$y(t) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1,\tau_2,\cdots,\tau_n) \prod_{i=1}^{n} u(t-\tau_i)d\tau_i \qquad (1.1)$$

Unlike the harmonic balance and describing function methods, ours includes significant effects contributed by the higher harmonics.

(2) Like the averaging method, our approach reduces to solving a pair of "algebraic" determining equations. In our case, the equations assume the form

$$\boxed{\begin{aligned} \text{Re } d_N(A,\omega) &= 0 \\ \text{Im } d_N(A,\omega) &= 0 \end{aligned}} \qquad \begin{aligned}(1.2a)\\(1.2b)\end{aligned}$$

where $d_N(A,\omega)$ is an algebraic function of A and ω involving complex numbers, and where Re(•) and Im(•) denote the real and imaginary part respectively. Like the averaging method, our approach is capable, in principle, of finding A and ω to any desired accuracy.

Unlike the averaging method, our method is applicable to nth-order differential equations with n > 2 and does not require the presence of a (often artificial) "small parmeter" ε.

(3) Unlike the Hopf bifurcation theorem, which is a "local" result, ours is global in the sense that "A" need not be small.

In order to make this paper accessible to the non-specialist, we present first the determining equations (for calculating amplitude and frequency) in Section 2 using a "handbook" style. For oscillations which can be modeled by the special feedback structure in Fig. 1, the first-order determining equations are extremely simple. In fact, the reader need not even have to be familiar with Volterra series. We then illustrate several practical examples in Section 3.

[†]From the circuit design point of view, a feedback system formulation is highly desirable because most electronic oscillators are in fact designed as feedback systems having a unity closed loop gain [9].

[§]Readers unfamiliar with Volterra series needs only to assume (1.1) as a definition and refer to Section 4 for a straightforward description of the few additional details needed to apply our new approach.

For more general systems, only the rudiments of Volterra series are needed in deriving the determining equations. Whatever background that is needed is given in Section 4 and Appendix A.

Finally, the mathematical justification of our approach is given in Section 5 along with the complete proof of all theorems.

2. THE AMPLITUDE-FREQUENCY DETERMINING EQUATION

The main result of this paper is to develop a systematic method for generating the Nth-order algebraic determining equation (1.2) so that its solution gives the amplitude A and frequency ω of the sinusoidal oscillation to any desired accuracy.

In this section, we will present the determining equation for various cases without proof so that the user can apply it directly without being distracted by the rather involved mathematical justification to be given in Section 5.

A. First-Order Determining Equation

The first-order determining equation is given by:

$$\boxed{d_1(A,\omega) \triangleq H_1(j\omega) + \Omega_1(j\omega)A^2 - 1 = 0} \tag{2.1}$$

where $H_1(j\omega)$ and $\Omega_1(j\omega)$ are functions of ω only whose explicit form will be given below.

Equating the real and imaginary parts of both sides of (2.1) to zero, we obtain the following two equivalent equations:

$$\text{Re } d_1(A,\omega) \triangleq \text{Re}\{H_1(j\omega) + \Omega_1(j\omega)A^2 - 1\} = 0 \tag{2.1a}$$

$$\text{Im } d_1(A,\omega) \triangleq \text{Im}\{H_1(j\omega) + \Omega_1(j\omega)A^2 - 1\} = 0 \tag{2.1b}$$

Solving (2.1b) for A^2 and substituting the result into (2.1a), we obtain the following explicit "frequency" determining equation:

$$\boxed{d_0(\omega) \triangleq \text{Re } H_1(j\omega) - \left[\frac{\text{Im } H_1(j\omega)}{\text{Im } \Omega_1(j\omega)}\right] \text{Re } \Omega_1(j\omega) - 1 = 0} \tag{2.2}$$

Since (2.2) is a scalar equation in ω, it can easily be solved either graphically, or by standard numerical techniques [11]. For each solution $\omega = \omega_i$ of (2.2), we can calculate the corresponding amplitude A_i by direct substitution into (2.1a) or (2.1b).

Let us now define $H_1(j\omega)$ and $\Omega_1(j\omega)$:

A.1. Special Case: Feedback Loop in Fig. 1 $\{f(0) = 0\}$

Let us assume that the nonlinear function in the single-loop feedback system of Fig. 1 is represented by a polynomial†

$$f(u) = a_1 u + a_2 u^2 + a_3 u^3 + \cdots \tag{2.3}$$

In this case, we have simply:

$$H_1(j\omega) = a_1 \ G(j\omega) \tag{2.4}$$

$$\Omega_1(j\omega) = \frac{1}{4}\left\{\frac{2a_2^2 G(j\omega)G(j2\omega)}{1-a_1 G(j2\omega)} + \frac{4a_2^2 G(0)G(j\omega)}{1-a_1 G(0)} + 3a_3 \ G(j\omega)\right\} \tag{2.5}$$

A.2. General Case: Feedback Loop in Fig. 2

In this case, we assume the open-loop system F in Fig. 2(b) is described by a convergent Volterra series (1.1). If we apply an input $u(t)$ consisting of a sum of exponentials, then it is shown in Section 4 that the response $y(t)$ in Fig. 2(b) can be calculated in the frequency domain in terms of higher-order transfer functions $H_1(s_1)$, $H_2(s_1,s_2)$, $H_3(s_1,s_2,s_3)$, $\cdots$, etc. These transfer functions are completely analogous to the familiar transfer functions from Linear System Theory. They can be generated using a recursive algorithm described in Appendix A which consists of solving a succession of Linear Systems. Here, we will assume that these higher-order transfer functions have been found and simply present the determining equations in terms of $H_1(s_1)$, $H_2(s_1,s_2)$, $H_3(s_1,s_2,s_3)$, $\cdots$, etc. In particular, we have:

$$H_1(j\omega) \triangleq H_1(s_1)\Big|_{s_1=j\omega} \tag{2.6}$$

$$\Omega_1(j\omega) \triangleq \frac{1}{4}\{\mathcal{H}_3(j\omega,j\omega,-j\omega) + \mathcal{H}_3(j\omega,-j\omega,j\omega) + \mathcal{H}_3(-j\omega,j\omega,j\omega)\} \tag{2.7}$$

where

$$\mathcal{H}_3(j\omega,j\omega,-j\omega) \triangleq H_2(j2\omega,-j\omega)\,\frac{H_2(j\omega,j\omega)}{1-H_1(j2\omega)} + H_2(j\omega,0)\,\frac{H_2(j\omega,-j\omega)}{1-H_1(0)} + H_3(j\omega,j\omega,-j\omega) \tag{2.8a}$$

$$\mathcal{H}_3(-j\omega,j\omega,j\omega) \triangleq H_2(0,j\omega)\,\frac{H_2(-j\omega,j\omega)}{1-H_1(0)} + H_2(-j\omega,j2\omega)\,\frac{H_2(j\omega,j\omega)}{1-H_1(j2\omega)} + H_3(-j\omega,j\omega,j\omega) \tag{2.8b}$$

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†If $f(\cdot)$ is not a polynomial, replace it by a Taylor series expansion about its dc operating point, and retain only the first 3 terms.

$$H_3(j\omega,-j\omega,j\omega) \triangleq H_2(0,j\omega)\,\frac{H_2(j\omega,-j\omega)}{1-H_1(0)} + H_2(j\omega,0)\,\frac{H_2(-j\omega,j\omega)}{1-H_1(0)} + H_3(j\omega,-j\omega,j\omega) \tag{2.8c}$$

Observe that $H_1(j\omega)$ and $\Omega_1(j\omega)$ can be written down as soon as $H_1(s_1)$, $H_2(s_1,s_2)$, and $H_3(s_1,s_2,s_3)$ of F are known.

Straightforward methods for deriving these higher-order transfer functions are given in [10], and in Appendix A. For example, applying [10] to the system in Fig. 1, we obtain

$$H_1(s_1) = a_1\ G(s_1) \tag{2.9a}$$

$$H_2(s_1,s_2) = a_2\ G(s_1+s_2) \tag{2.9b}$$

$$H_3(s_1,s_2,s_3) = a_3\ G(s_1+s_2+s_3) \tag{2.9c}$$

Substituting (2.9) into (2.6)-(2.8) and simplifying, we obtain (2.4) and (2.5).

B. Second-Order Determining Equation

The second-order determining equation is given by:

$$d_2(A,\omega) \triangleq H_1(j\omega) + \Omega_1(j\omega)A^2 + \Omega_2(j\omega)A^4 - 1 = 0 \tag{2.10}$$

or equivalently:

$$\text{Re } d_2(A,\omega) \triangleq \text{Re}\{H_1(j\omega) + \Omega_1(j\omega)A^2 + \Omega_2(j\omega)A^4 - 1\} = 0 \tag{2.10a}$$

$$\text{Im } d_2(A,\omega) \triangleq \text{Im}\{H_1(j\omega) + \Omega_1(j\omega)A^2 + \Omega_2(j\omega)A^4 - 1\} = 0 \tag{2.10b}$$

Either (2.10a) or (2.10b) can be solved for A^2 and substituted into the other to obtain a single equation in terms of only ω. For example, if $\text{Im }\Omega_2(j\omega) \neq 0$, then the result is:

$$d_0(\omega) \triangleq \text{Re } H_1(j\omega) + \text{Re }\Omega_1(j\omega)A^2(\omega) + \text{Re }\Omega_2(j\omega)[A^2(\omega)]^2 - 1 = 0 \tag{2.11a}$$

where

$$A^2(\omega) \triangleq \frac{-\text{Im }\Omega_1(j\omega) \pm \sqrt{[\text{Im }\Omega_1(j\omega)]^2 - 4[\text{Im }\Omega_2(j\omega)][\text{Im } H_1(j\omega)]}}{2\text{Im }\Omega_2(j\omega)} \tag{2.11b}$$

Let us now define $H_1(j\omega)$, $\Omega_1(j\omega)$, and $\Omega_2(j\omega)$.

B.1. Special Case: Feedback Loop in Fig. 1 $(f(u) = -f(-u))$

Let us assume that the nonlinear function is represented by an "odd" polynomial

$$f(u) = a_1u + a_3u^3 + a_5u^5 + \cdots \tag{2.12}$$

In this case, we have simply:

$$H_1(j\omega) = a_1\ G(j\omega) \tag{2.13}$$

$$\Omega_1(j\omega) = \frac{3}{4}a_3\ G(j\omega) \tag{2.14}$$

$$\Omega_2(j\omega) = \frac{1}{16}\left\{\frac{3a_3^2G(j\omega)G(j3\omega)}{1-a_1G(j3\omega)} + 10a_5G(j\omega)\right\} \tag{2.15}$$

B.2. <u>General Case: Feedback Loop in Fig. 2</u>

In this case, $H_1(j\omega)$ and $\Omega_1(j\omega)$ are given by (2.6) and (2.7), respectively, whereas

$$\begin{aligned}\Omega_2(j\omega) \triangleq \frac{1}{16}\{&\mathcal{H}_5(j\omega,j\omega,j\omega,-j\omega,-j\omega) + \mathcal{H}_5(j\omega,j\omega,-j\omega,j\omega,-j\omega)\\ &+ \mathcal{H}_5(j\omega,j\omega,-j\omega,-j\omega,j\omega) + \mathcal{H}_5(j\omega,-j\omega,j\omega,j\omega,-j\omega)\\ &+ \mathcal{H}_5(j\omega,-j\omega,j\omega,-j\omega,j\omega) + \mathcal{H}_5(j\omega,-j\omega,-j\omega,j\omega,j\omega)\\ &+ \mathcal{H}_5(-j\omega,j\omega,j\omega,j\omega,-j\omega) + \mathcal{H}_5(-j\omega,j\omega,j\omega,-j\omega,j\omega)\\ &+ \mathcal{H}_5(-j\omega,j\omega,-j\omega,j\omega,j\omega) + \mathcal{H}_5(-j\omega,-j\omega,j\omega,j\omega,j\omega)\}\end{aligned} \tag{2.16}$$

The expression defining $\mathcal{H}_5(s_1,s_2,s_3,s_4,s_5)$ is quite involved and is best generated using the recursive algorithm described in <u>Appendix A</u> with the help of a <u>symbolis software</u> system [12]. However, in the special case where F is <u>odd</u> symmetric (i.e., $H_{2n}(s_1,s_2,\cdots,s_{2n})=0$ for all n), we have:

$$\mathcal{H}_5(j\omega,j\omega,j\omega,-j\omega,-j\omega) = \frac{H_3(j3\omega,-j\omega,-j\omega)H_3(j\omega,j\omega,j\omega)}{1-H_1(j3\omega)} + H_5(j\omega,j\omega,j\omega,-j\omega,-j\omega) \tag{2.17a}$$

$$\mathcal{H}_5(j\omega,j\omega,-j\omega,j\omega,-j\omega) = H_5(j\omega,j\omega,-j\omega,j\omega,-j\omega) \tag{2.17b}$$

$$\mathcal{H}_5(j\omega,j\omega,-j\omega,-j\omega,j\omega) = H_5(j\omega,j\omega,-j\omega,-j\omega,j\omega) \tag{2.17c}$$

$$\mathcal{H}_5(j\omega,-j\omega,j\omega,j\omega,-j\omega) = H_5(j\omega,-j\omega,j\omega,j\omega,-j\omega) \tag{2.17d}$$

$$\mathcal{H}_5(j\omega,-j\omega,j\omega,-j\omega,j\omega) = H_5(j\omega,-j\omega,j\omega,-j\omega,j\omega) \tag{2.17e}$$

$$\mathcal{H}_5(j\omega,-j\omega,-j\omega,j\omega,j\omega) = H_5(j\omega,-j\omega,-j\omega,j\omega,j\omega) \tag{2.17f}$$

$$\mathcal{H}_5(-j\omega,j\omega,j\omega,j\omega,-j\omega) = \frac{H_3(-j\omega,j3\omega,-j\omega)H_3(j\omega,j\omega,j\omega)}{1-H_1(j3\omega)} + H_5(-j\omega,j\omega,j\omega,j\omega,-j\omega) \tag{2.17g}$$

$$\mathcal{H}_5(-j\omega,j\omega,j\omega,-j\omega,j\omega) = H_5(-j\omega,j\omega,j\omega,-j\omega,j\omega) \tag{2.17h}$$

$$\mathcal{H}_5(-j\omega,j\omega,-j\omega,j\omega,j\omega) = H_5(-j\omega,j\omega,-j\omega,j\omega,j\omega) \quad (2.17i)$$

$$\mathcal{H}_5(-j\omega,-j\omega,j\omega,j\omega,j\omega) = \frac{H_3(-j\omega,-j\omega,j3\omega)H_3(j\omega,j\omega,j\omega)}{1-H_1(j3\omega)} + H_5(-j\omega,-j\omega,j\omega,j\omega,j\omega) \quad (2.17j)$$

C. Nth-order Determining Equation

The Nth-order determining equation is given by

$$d_N(A,\omega) \triangleq H_1(j\omega) + \Omega_1(j\omega)A^2 + \Omega_2(j\omega)A^4 + \cdots\cdots + \Omega_N(j\omega)A^{2N} - 1 = 0 \quad (2.18)$$

or equivalently:

$$\mathrm{Re}\, d_N(A,\omega) \triangleq \mathrm{Re}\{H_1(j\omega) + \Omega_1(j\omega)A^2 + \Omega_2(j\omega)A^4 + \cdots\cdots + \Omega_N(j\omega)A^{2N} - 1\} = 0 \quad (2.18a)$$

$$\mathrm{Im}\, d_N(A,\omega) \triangleq \mathrm{Im}\{H_1(j\omega) + \Omega_1(j\omega)A^2 + \Omega_2(j\omega)A^4 + \cdots\cdots + \Omega_N(j\omega)A^{2N} - 1\} = 0 \quad (2.18b)$$

Here, $H_1(j\omega)$ and $\Omega_1(j\omega)$ are given by (2.6) and (2.7) as before, whereas $\Omega_2(j\omega)$, $\Omega_3(j\omega)$, $\cdots$, $\Omega_N(j\omega)$ can be generated as described in Section 4 and Appendix A.

The expressions $H_1(j\omega)$, $\Omega_1(j\omega)$, $\cdots$, $\Omega_{N-1}(j\omega)$ are identical to the corresponding expressions in the (N-1)th order determining equation. Since the "magnitude" of each additional term $\Omega_N(j\omega)A^{2N}$ will usually be at least an order of magnitude smaller than the preceding term, we can interpret this additional term as a "higher-order" correction analogous to that characterizing the averaging method [4].

3. ILLUSTRATIVE EXAMPLES

In many applications, the designer is interested in knowing only whether a circuit or system will oscillate, and if so, its "approximate" frequency ω and amplitude A. On such situations, it is usually quite satisfactory to choose the first-order determining equation in view of its simplicity. If $f(\cdot)$ is an odd function, then increased accuracy could be obtained with the second-order determining equation with very little additional work.

If one is interested in a "nearly" exact value of ω and A, one could always resort to a more efficient computer simulation algorithm [11] using the above "approximate" ω and A as initial condition. In fact, one important application of the extremely simple first-order determining equation is precisely to calculate a good "initial condition" which is essential in the rapid convergence of the subsequent exact computer simulation.

Our objective in this section is to illustrate the application of these determining equations to some typical nonlinear circuits.

Example 1. Linear one-port terminated in a nonlinear resistor

Conisder the circuit shown in Fig. 3(a) where N denotes an arbitrary linear time-invariant one-port described by an impedance Z(s) or admittance Y(s).

If the nonlinear resistor is voltage-controlled $(i=f(v))$, then the equivalent feedback representation is shown in Fig. 3(b), where $G(s) = -Z(s)$.

If the nonlinear resistor is current-controlled $(v=f(i))$, then the equivalent feedback representation is shown in Fig. 3(c), where $G(s) = -Y(s)$.

In either case, we can apply the explicit formulas in the preceding section to calculate the amplitude A and frequency ω of the oscillation, assuming the circuit oscillates.

Example 2. Linear one-port terminated in a nonlinear inductor

Consider the circuit shown in Fig. 4(a) where $\mathcal{L}$ denotes either a nonlinear flux-controlled $(i=f(\phi))$ or current-controlled $(\phi=f(i))$ inductor. The corresponding equivalent feedback system is shown respectively in Fig. 4(b), with $G(s) = -Z(s)/s$, and in Fig. 4(c), with $G(s) = -sY(s)$.

Example 3. Linear one-port terminated in a nonlinear capacitor

Consider the circuit shown in Fig. 5(a) where $\mathcal{C}$ denotes either a nonlinear charge-controlled $(v=f(q))$ or voltage-controlled $(q=f(v))$ capacitor. The corresponding equivalent feedback system is shown respectively in Fig. 5(b), with $G(s) = -Y(s)/s$, and in Fig. 5(c), with $G(s) = -sZ(s)$.

Example 4. van der Pol Oscillator

The circuit shown in Fig. 6 is described by:

$$\ddot{v} + \frac{1}{C}(1-v^2)\dot{v} + \frac{1}{LC}v = 0 \tag{3.1}$$

If we assume R = 1 and $\frac{1}{C} = L \triangleq -\varepsilon$, then (3.1) reduces to the well-known van der Pol equation [3-5]:

$$\boxed{\ddot{v} - \varepsilon(1-v^2)\dot{v} + 1 = 0} \tag{3.2}$$

This celebrated equation has been extensively studied and many properties of its solution are now well known. In particular, we have:

1. For small positive ε, (3.2) has a stable sinusoidal solution of frequency $\omega \approx 1$ and amplitude $A \approx 2$. This corresponds to a stable "circular" limit cycle of radius 2 in the phase plane.
2. For small negative ε, (3.2) has an unstable sinusoidal solution of frequency $\omega \approx 1$ and amplitude $A \approx 2$. This corresponds to an unstable "circular" limit cycle of radius 2 in the phase plane.

Let us analyze the van der Pol equation (3.2) using the first-order determining equation (2.1). Comparing Fig. 6 with Fig. 3(a), we identify:

$$G(s) = \frac{\varepsilon}{-\varepsilon + (s + \frac{1}{s})} \tag{3.3}$$

Since $f(v) = -\frac{1}{3}v^3$, we have $a_i = 0$ for all $i \neq 3$ and $a_3 = -\frac{1}{3}$. Consequently, (2.4) and (2.5) give:

$$H_1(j\omega) = 0, \qquad \Omega_1(j\omega) = \frac{-\varepsilon}{4}\left[\frac{1}{-\varepsilon + j(\omega - \frac{1}{\omega})}\right] \tag{3.4}$$

Substituting (3.4) into (2.1), we obtain the following first-order determining equation:

$$d_1(A,\omega) = \frac{-\varepsilon}{4}\left[\frac{1}{-\varepsilon + j(\omega - \frac{1}{\omega})}\right] A^2 - 1 = 0 \tag{3.5}$$

Simplifying (3.5), we obtain

$$-\frac{\varepsilon}{4}A^2 + \varepsilon - j(\omega - \frac{1}{\omega}) = 0 \tag{3.6}$$

Solving (3.6) we obtain the first-order solution:

$$\boxed{\omega = 1, \quad A = 2} \tag{3.7}$$

Consequently, our first-order determining equation gives exactly the same answer as that obtained from solving an analogous first-order equation derived from the method of averaging [4]. Since neither A nor ω in this case depends on ε, it is clear that (3.7) is only an approximate solution.

To determine the effect of the parameter ε on A and ω, let us write the following second-order determining equation. Since f(v) is an odd function of v, we can use (2.15) to obtain:

$$\Omega_2(j\omega) = \frac{\varepsilon^2}{48}\left\{\frac{1}{[-\varepsilon + j(\omega - \frac{1}{\omega})][-\varepsilon + j(3\omega - \frac{1}{3\omega})]}\right\} \tag{3.7}$$

Substituting (3.4) and (3.7) into (2.10), we obtain:

$$d_2(A,\omega) = -\frac{\varepsilon}{4}\left[\frac{A^2}{-\varepsilon + j(\omega - \frac{1}{\omega})}\right] + \frac{\varepsilon^2}{48}\left\{\frac{A^4}{[-\varepsilon + j(\omega - \frac{1}{\omega})][-\varepsilon + j(3\omega - \frac{1}{3\omega})]}\right\} - 1 = 0 \tag{3.8}$$

Simplifying (3.8) and equating the respective real and imaginary parts to zero, we obtain:

$$\mathrm{Re}\ \hat{d}_2(A,\omega) = \varepsilon^2 A^4 + 12\varepsilon^2 A^2 - 48\varepsilon^2 + 144\omega^2 + \frac{16}{\omega^2} - 160 = 0 \tag{3.8a}$$

$$\text{Im } \hat{d}_2(A,\omega) = -36\varepsilon\omega A^2 + 4\varepsilon A^2/\omega + 192\varepsilon\omega - 64\ \varepsilon/\omega = 0^{\dagger} \tag{3.8b}$$

Solving (3.8) numerically with $\varepsilon = 0.2$, we obtain:

$$\text{Second-order Solution: } \boxed{\omega = 0.9975 \approx 1, \quad A = 1.998 \approx 2} \tag{3.9}$$

The error resulting from a first-order analysis can be analyzed by substituting (3.7) into (3.8a) to obtain the "slack" equation

$$\begin{aligned} \frac{\text{Re } \hat{d}_2(2,\omega)}{16} &= \varepsilon^2 + 9\omega^2 + \frac{1}{\omega^2} - 10 = 9(\omega^2-1) + (\frac{1}{\omega^2}-1) + \varepsilon^2 \\ &= 9(\omega+1)(\omega-1) + (\frac{1}{\omega}+1)(\frac{1}{\omega}-1) + \varepsilon^2 = 0 \end{aligned} \tag{3.10}$$

Now if we let $\omega = 1 + \delta\omega$ and make use of the approximations:

$$\left.\begin{aligned} &\omega + 1 \approx 2, \quad \frac{1}{\omega} + 1 \approx 2 \\ &\frac{1}{\omega} - 1 = \frac{1}{1+\delta\omega} - 1 \approx (1-\delta\omega) - 1 = -\delta\omega \end{aligned}\right\} \tag{3.11}$$

we would obtain

$$18\delta\omega - 2\delta\omega + \varepsilon^2 = 0 \tag{3.12}$$

Hence $\delta\omega = -\frac{1}{16}\varepsilon^2$ and we can write the second-order solution as

$$\boxed{\omega = 1 - \frac{1}{16}\varepsilon^2} \tag{3.13}$$

This answer is identical to that obtained from solving an analogous second-order equation derived from the method of averaging [4]. In other words, the solution derived from our <u>second-order determining equation</u> has the same degree of accuracy as that obtained from applying the <u>averaging method</u> of the <u>same order</u>.

If we repeat our analysis using a <u>3rd-order determining equation</u>, we will see that a correction term proportional to ε^4 will have to be subtracted from (3.13). Repeating this analysis using a higher-order determining equation of a sufficiently high order we can in principle generate an <u>analytical</u> expression giving ω as a function of ε, which is correct to any desired accuracy.

<u>Example 5</u>. <u>Unforced Duffing's Equation</u>

The circuit shown in Fig. 7 is described by:

$$\ddot{\phi} + \frac{1}{RC}\dot{\phi} + \frac{1}{LC}\phi + \frac{1}{C}\phi^3 = 0 \tag{3.14}$$

$^{\dagger}\hat{d}_2(A,\omega)$ represents the left-hand-side of the equation obtained by simplifying (3.8).

If we assume R = 1 and $\frac{1}{C} = L \triangleq \varepsilon$, then (3.14) reduces to the well-known "unforced" Duffing's Equation:

$$\boxed{\ddot{\phi} + \varepsilon\dot{\phi} + \phi + \varepsilon\phi^3 = 0} \tag{3.15}$$

It is well known that this equation is globally asymptotically stable [5,13] and hence there is no oscillation. This implies that our determining equation can not have a solution. Let us confirm this conclusion.

Comparing Fig. 7 with Fig. 4(b), we identify

$$G(s) = \frac{-Z(s)}{s} = -\left(\frac{\varepsilon}{s}\right)\left[\frac{1}{\varepsilon+(s+\frac{1}{s})}\right] \tag{3.16}$$

Since $f(\phi) = \phi^3$, we have $a_i = 0$ for all $i \neq 3$, and $a_3 = 1$. Consequently, (2.4) and (2.5) give:

$$H_1(j\omega) = 0, \quad \Omega_1(j\omega) = -\left(\frac{3\varepsilon}{j4\omega}\right)\left[\frac{1}{\varepsilon+j(\omega-\frac{1}{\omega})}\right] \tag{3.17}$$

Substituting (3.17) into (2.1), we obtain the following first-order determining equation:

$$d_1(A,\omega) = -\left(\frac{3\varepsilon}{j4\omega}\right)\left[\frac{A^2}{\varepsilon+j(\omega-\frac{1}{\omega})}\right] - 1 = 0 \tag{3.18}$$

In order for (3.18) to have a real solution A and ω, it is necessary that $A \neq 0$ and $\varepsilon + j(\omega - \frac{1}{\omega})$ be purely imaginary. But this is possible only if $\varepsilon = 0$. Hence, the first-order determining equation (3.18) does not have a solution, as expected.

Since $f(\phi)$ is odd symmetric, we can use (2.15) to obtain

$$\Omega_2(j\omega) = \left(\frac{3}{16}\right)\left(-\frac{\varepsilon}{j\omega}\right)\left(-\frac{\varepsilon}{j3\omega}\right)\left[\frac{1}{\varepsilon+j(\omega-\frac{1}{\omega})}\right]\left[\frac{1}{\varepsilon+j(3\omega-\frac{1}{3\omega})}\right] \tag{3.19}$$

substituting (3.17) and (3.19) into (2.10), we obtain:

$$d_2(A,\omega) = -\left(\frac{3\varepsilon}{j4\omega}\right)\left[\frac{A^2}{\varepsilon+j(\omega-\frac{1}{\omega})}\right] - \left(\frac{\varepsilon^2}{16\omega^2}\right)\left[\frac{1}{\varepsilon+j(\omega-\frac{1}{\omega})}\right]\left[\frac{A^4}{\varepsilon+j(3\omega-\frac{1}{3\omega})}\right] - 1 = 0 \tag{3.20}$$

Simplifying this equation and equating the respective real and imaginary parts to zero, we obtain:

$$\text{Re}\ \hat{d}_2(A,\omega) = -\frac{3}{4}\varepsilon^2A^2 + \varepsilon\omega\left(4\omega - \frac{1}{\omega} - \frac{1}{3\omega}\right) = 0 \tag{3.20a}$$

$$\mathrm{Im}\,\hat{d}_2(A,\omega) = -\frac{3}{4}\,\varepsilon(3\omega-\frac{1}{3\omega})A^2 - \frac{1}{16}\frac{\varepsilon^2}{\omega}A^4 - \omega\varepsilon^2 + \omega(\omega-\frac{1}{\omega})(3\omega-\frac{1}{3\omega}) = 0 \qquad (3.20b)$$

We can recast (3.20a) as follow:

$$-\frac{3}{4}\,\varepsilon A^2 = -\omega[(\omega-\frac{1}{\omega}) + (3\omega-\frac{1}{3\omega})] \qquad (3.21)$$

Substituting (3.21) into (3.20b) and simplifying, we obtain

$$\omega(3\omega-\frac{1}{3\omega})^2 + \frac{1}{16}\frac{\varepsilon^2}{\omega}A^4 + \omega\varepsilon^2 = 0 \qquad (3.22)$$

Since the first term is non-negative and the last two terms in (3.22) are positive, it follows that (3.22), and hence the second-order determining equation, can not have a solution, as expected.

Example 6. Tunnel Diode Oscillator

Consider the circuit shown in Fig. 6 again but with a new set of parameters:

$R = 250\ \Omega$, $L = 200$ nH, $C = 500$ pf.

Let the nonlinear resistor be described by a tunnel-diode like characteristic:

$$i = f(v) = -0.0108v - 0.003v^2 + 0.1v^3 \qquad (3.23)$$

The impedance in this case is given by

$$Z(s) = \frac{1}{(1/250)+(5\text{x}10^6/s)+5\text{x}10^{-10}s} \qquad (3.24a)$$

and the coefficients a_i are:

$$a_1 = -0.0108,\quad a_2 = -0.003,\quad a_3 = 0.1 \qquad (3.24b)$$

Substituting (3.24) into (2.4) and (2.5), we obtain

$$H_1(j\omega) = 0.0108\left[\frac{1}{5\text{x}10^{-10}j\omega-5\text{x}10^6 j/\omega+1/250}\right] \qquad (3.25)$$

$$\Omega_1(j\omega) = \frac{1}{4}\left\{\frac{1.8\text{x}10^{-5}}{(5\text{x}10^{-10}j\omega-5\text{x}10^6 j/\omega+0.004)(10^{-9}j\omega-2.5\text{x}10^6 j/\omega+0.004)(1-\frac{0.0108}{10^{-9}j\omega-2.5\text{x}10^6 j/\omega+0.004})} - \frac{}{5\text{x}10^{-10}j\omega-5\text{x}10^6 j/\omega+0.004}\right\} \qquad (3.26)$$

Substituting (3.25) and (3.26) into (2.1) we obtain

$$d_1(A,\omega) = \frac{1}{4}\left\{\frac{1.8\times10^{-5}}{(5\times10^{-10}j\omega-5\times10^{6}j/\omega+0.004)(10^{-9}j\omega-2.5\times10^{6}j/\omega+0.004)\left(1-\frac{0.0108}{10^{-9}j\omega-2.5\times10^{6}j/\omega+0.004}\right)} - \frac{0.3}{5\times10^{-10}j\omega-5\times10^{6}j/\omega+0.004}\right\}A^2 + \frac{0.0108}{5\times10^{-10}j\omega-5\times10^{6}j/\omega+0.004} - 1 = 0 \quad (3.27) \tag{3.27}$$

Solving (3.27) numerically, we find:

$$\boxed{A = 0.301, \quad \omega = 99.99\times10^{6}} \tag{3.28}$$

<u>Example 7</u>. <u>Wien-Bridge Oscillator</u>

As our final example, consider the Wien-Bridge oscillator circuit shown in Fig. 8(a), where $R_1 = R_2 = 1$, and $C_1 = C_2 = 1$. The op amp is modeled by a nonlinear voltage-controlled voltage source and the resulting circuit is shown in Fig. 8(b), where

$$f(v) = 3.234v - 2.195v^3 + 0.666v^5 \tag{3.29}$$

This circuit can in turn be described by the equivalent single-loop feedback system shown in Fig. 8(c), where

$$G(s) = \frac{1}{3+(s+\frac{1}{s})} \tag{3.30a}$$

and the coefficient a_i are

$$a_1 = 3.234, \quad a_3 = -2.195, \quad a_3 = 0.666 \tag{3.30b}$$

Substituting (3.30) into (2.13), (2.14) and (2.15), we obtain

$$H_1(j\omega) = \frac{3.234}{j\omega-j/\omega+3} \tag{3.31}$$

$$\Omega_1(j\omega) = \frac{-1.646}{j\omega-j/\omega+3} \tag{3.32}$$

$$\Omega_2(j\omega) = 0.0625\left\{\frac{14.45}{(j\omega - j/\omega+3)(3j\omega - j/3\omega+3)(1 - \frac{3.234}{3j\omega - j/3\omega+3})} + \frac{6.66}{j\omega - j/\omega+3}\right\} \tag{3.33}$$

Substituting (3.31), (3.32) and (3.33) into (2.10) we obtain

$$\boxed{\begin{aligned} d_2(A,\omega) = 0.0625\left\{\frac{14.45}{(j\omega - j/\omega+3)(3j\omega - j/3\omega+3)(1 - \frac{3.234}{3j\omega - j/3\omega+3})} + \frac{6.66}{j\omega - j/\omega+3}\right\} A^4 \\ - \frac{1.646}{j\omega - j/\omega+3} A^2 + \frac{3.234}{j\omega - j/\omega+3} - 1 = 0 \end{aligned}} \tag{3.34}$$

Solving (3.34) numerically, we obtain

$$\boxed{A = 0.384, \quad \omega = 0.996} \tag{3.35}$$

4. DERIVING THE DETERMINING EQUATIONS: INTUITIVE APPROACH

Our objective in this section is to derive the formulas given in Section 2 using an intuitive "frequency-domain" approach familiar to engineers. The mathematical justification of the validity of this approach will be given in Section 5.

In the frequency-domain approach, we assume the system in Fig. 2(a) is in "steady state" in the sense that all waveforms can be expressed as a sum of sinusoidal signals of various component frequencies. In particular, let the input to the system F be

$$u(t) = \sum_{i=1}^{M} A_i e^{p_i t} \tag{4.1}$$

Substituting (4.1) into the Volterra series (1.1), we find the output of F is given by:

$$y(t) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1,\tau_2,\cdots,\tau_n)\left(\sum_{i=1}^{M} A_i e^{p_i(t-\tau_1)}\right)\left(\sum_{i=1}^{M} A_i e^{p_i(t-\tau_2)}\right)$$

$$\cdots\left(\sum_{i=1}^{M} A_i e^{p_i(t-\tau_n)}\right) d\tau_1 d\tau_2 \cdots d\tau_n$$

$$= \sum_{n=1}^{\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1,\tau_2,\cdots,\tau_n) \sum_{i_1=1}^{M} \sum_{i_2=1}^{M} \cdots \sum_{i_n=1}^{M}$$

$$\left(A_{i_1}A_{i_2}\cdots A_{i_n}\right)e^{p_{i_1}(t-\tau_1)+\cdots p_{i_n}(t-\tau_n)}\,d\tau_1\cdots d\tau_n$$

$$= \sum_{n=1}^{\infty}\left\{\sum_{i_1=1}^{M}\sum_{i_2=1}^{M}\cdots\sum_{i_n=1}^{M}\left[\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\tau_2,\cdots\tau_n)e^{-p_{i_1}\tau_1-p_{i_2}\tau_2\cdots-p_{i_n}\tau_n}\right.\right.$$

$$\left.\left.d\tau_1 d\tau_2\cdots d\tau_n\right]A_{i_1}A_{i_2}\cdots A_{i_n}e^{(p_{i_1}+p_{i_2}+\ldots+p_{i_n})t}\right\} \qquad (4.2)$$

We can simplify the expression within the bracket in (4.2) by introducing the notation:†

$$H_n(s_1,s_2,\cdots,s_n) \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\tau_2,\cdots,\tau_n)e^{-s_1\tau_1-s_2\tau_2\cdots-s_n\tau_n}d\tau_1 d\tau_2\cdots d\tau_n \qquad (4.3)$$

Since $H_n(s_1,s_2,\cdots,s_n)$ is of fundamental importance in this paper and plays the same role as that of a "transfer function" in linear system theory, we will henceforth call it an nth order transfer function of F. Using this notation, (4.2) becomes

$$y(t) = \sum_{n=1}^{\infty}\left\{\sum_{i_1,i_2,\cdots,i_n=1}^{M}H_n(p_{i_1},p_{i_2},\cdots,p_{i_n})A_{i_1}A_{i_2}\cdots A_{i_n}\,e^{(p_{i_1}+p_{i_2}+\cdots+p_{i_n})t}\right\} \qquad (4.4)$$

where

$$H_n(p_{i_1},p_{i_2},\cdots,p_{i_n}) \triangleq H_n(s_1,s_2,\cdots,s_n)\Big|_{s_1=p_{i_1},s_2=p_{i_2},\cdots,s_n=p_{i_n}} \qquad (4.5)$$

denotes the nth-order transfer function evaluated at $s_i = p_{i_1}$, $s_2 = p_{i_2},\cdots,s_n=p_{i_n}$, and where the second simulation index in (4.4) covers all possible combinations of $(s_1,s_2,\cdots,s_n)$ as each argument s_i goes over $p_1,p_2,\cdots,p_n$, respectively.

Equation (4.4) shows that if the input $u(t)$ is a sum of exponentials with exponents $p_1,p_2,\cdots,p_n$, then the output $y(t)$ of F is also a sum of exponentials

†By analogy to Laplace transform in the single variable case (n=1), $H_n(s_1,s_2,\cdots,s_n)$ is also called the n-dimensional Laplace transform of $h_n(t_1,t_2,\cdots,t_n)$ [8].

with exponents $p_{i_1}+\cdots+p_{i_n}$, each weighted by the nth-order transfer function $H_n(p_{i_1},p_{i_2},\cdots,p_{i_n})$. Hence, once the transfer functions $H_1(s_1)$, $H_2(s_1,s_2)$, $\cdots,H_n(s_1,s_2,\cdots,s_n),\cdots$ of F is known, the response of F to $u(t)$ can be written down explicitly using (4.4). Fortunately, these higher-order transfer functions can be evaluated by a recursive algorithm given in [10] for nonlinear circuits, or by the analogous algorithm given in Appendix A for nonlinear systems.

Now if the system in Fig. 2 has a periodic solution of frequency ω, then in general, the Fourier spectrum of $u(t)$ and $y(t)$ will contain all harmonics $k\omega$ of the fundamental frequency ω. Now let us extract the fundamental frequency component from $y(t)$ using an ideal filter P and let us extract the remaining components by another ideal filter $I-P$. It is convenient to think of P as an "operator"[†] and I as an "identity" operator, so that $I-P$ means whatever remains after the fundamental signal component has been extracted. Using these two operators, we can transform Fig. 2(a) into the equivalent system shown in Fig. 9(a).

By definition of P, we can write:

$$u(t) = P(y(t)) = |A|\cos(\omega t + \measuredangle A) = \frac{A}{2}e^{j\omega t} + \frac{\bar{A}}{2}e^{-j\omega t} \qquad (4.6)$$

where $A \triangleq |A|e^{j\measuredangle A}$ is a complex phasor and $\bar{A}$ denotes the complex conjugate of A.

Now cut the loop in Fig. 9(a) and redraw the resulting system in Fig. 9(b). If we apply $u(t)$ given by (4.6), then, because of the "ideal" filter P, the output is:

$$z(t) = |A_z|\cos(\omega t + \measuredangle A_z) = \frac{A_z}{2}e^{j\omega t} + \frac{\bar{A}_z}{2}e^{-j\omega t} \qquad (4.7)$$

It follows from Figs. 9(a) and 9(b) that a necessary and sufficient condition for the system in Fig. 2(a) to have a periodic solution of frequency ω is that $A_z = A$. Since A_z depends in general on both A and ω, let us determine next the function $A_z = A_z(A,\omega)$.

The system S in Fig. 9(b) consists of a cascade of two subsystems S_1 and S_2. In Appendix A, we show that given the transfer functions $H_1(s_1)$, $H_2(s_1,s_2)$, $\cdots,H_n(s_1,s_2,\cdots s_n),\cdots$, of F, we can generate a "formal" Volterra series[§] between $z(t)$ and $u(t)$ for S. In particular, we give a recursive algorithm for generating

[†]Mathematically, P is called a projection operator.

[§]To be rigorous, we must first prove that $z(t)$ can be expressed by a "convergent" Volterra series in terms of $u(t)$. Here, we are interested only in generating the series "formally" in the same spirit as that of a "formal" power series expansion. In Section 5, it will be clear that this "formal" series -- which is always well defined in view of our recursive algorithm -- is all that is needed to prove the main result.

$\mathcal{H}_1(s_1), \mathcal{H}_2(s_1,s_2), \cdots, \mathcal{H}_n(s_1,s_2,\cdots,s_n), \cdots$, in terms of $H_1(s_1)$, $H_2(s_1,s_2)$, $\cdots, H_n(s_1,s_2,\cdots,s_n), \cdots$, etc. Moreover, we show how the higher-order transfer functions can be generated in a symbolic form -- e.g., Eq. (2.17) -- using the Macsyma software system [12]. Alternatively, given any $(s_1,s_2,\cdots,s_n) = (jk_1\omega, jk_2\omega, \cdots, jk_n\omega)$, our recursive algorithm allows us to calculate the numerical value of $\mathcal{H}_n(jk_1\omega, jk_2\omega, \cdots, jk_n\omega)$.

To derive A_z as a function of A and ω, compare (4.6) and (4.1) and identify $M = 2$, $A_1 = A/2$, $A_2 = \bar{A}/2$, $p_1 = j\omega$, and $p_2 = -j\omega$. It follows from (4.4) (with y replaced by z, and H_n by $\mathcal{H}_n$) that

$$z(t) = \sum_{n=1}^{\infty} \left\{ \sum_{i_1,i_2,\cdots,i_n=1}^{2} \mathcal{H}_n(p_{i_1},p_{i_2},\cdots p_{i_n}) A_{i_1} A_{i_2} \cdots A_{i_n} \, e^{j\overbrace{(\omega \pm \omega \pm \cdots \pm \omega)}^{n \text{ terms}} t} \right\} \tag{4.8}$$

where $A_{i_k} = \frac{A}{2}$ or $\frac{\bar{A}}{2}$, and the "$\pm$" signs denote all possible combinations satisfying (recall z(t) is given by (4.7)):

$$\underbrace{\omega \pm \omega \pm \cdots \pm \omega}_{n \text{ terms}} = \pm \omega \tag{4.9}$$

Since (4.9) can not be satisfied if n is even, it follows that

$$\mathcal{H}_n(s_1,s_2,\cdots,s_n) \equiv 0 \qquad \text{for } n = \text{even integer} \tag{4.10}$$

Moreover, since

$$\mathcal{H}_n(jk_1\omega, jk_2\omega, \cdots, jk_n\omega) = \bar{\mathcal{H}}_n(-jk_1\omega, -jk_2\omega, \cdots, -jk_n\omega) \tag{4.11}$$

where $k_i = \pm 1$, it suffices to sum only those terms in (4.8) which contribute to $\frac{A_z}{2} e^{j\omega t}$; namely,†

$$\begin{aligned} \frac{A_z}{2} e^{j\omega t} = {} & \mathcal{H}_1(j\omega) \frac{A}{2} e^{j\omega t} \qquad \text{(1st-order term)} \\ & \left.\begin{aligned} &+ \mathcal{H}_3(j\omega, j\omega, -j\omega) \frac{A}{2}\frac{A}{2}\frac{\bar{A}}{2} e^{j(\omega+\omega-\omega)t} \\ &+ \mathcal{H}_3(-j\omega, j\omega, j\omega) \frac{\bar{A}}{2}\frac{A}{2}\frac{A}{2} e^{j(-\omega+\omega+\omega)t} \\ &+ \mathcal{H}_3(j\omega, -j\omega, j\omega) \frac{A}{2}\frac{A}{2}\frac{A}{2} e^{j(\omega-\omega+\omega)t} \end{aligned}\right\} \quad \text{3 rd-order terms} \end{aligned}$$

†The sum of the remaining terms is just the complex conjugate of (4.9) and contributes therefore to $\frac{\bar{A}_z}{2} e^{-j\omega t}$.

$$
\left.
\begin{array}{l}
+ H_5(j\omega,j\omega,j\omega,-j\omega,-j\omega)\,\frac{A}{2}\frac{A}{2}\frac{A}{2}\frac{\bar{A}}{2}\frac{\bar{A}}{2}\,e^{j(\omega+\omega+\omega-\omega-\omega)t} \\
+ H_5(j\omega,j\omega,-j\omega,j\omega,-j\omega)\,\frac{A}{2}\frac{A}{2}\frac{\bar{A}}{2}\frac{A}{2}\frac{\bar{A}}{2}\,e^{j(\omega+\omega-\omega+\omega-\omega)t} \\
+ H_5(j\omega,j\omega,-j\omega,-j\omega,j\omega)\,\frac{A}{2}\frac{A}{2}\frac{\bar{A}}{2}\frac{\bar{A}}{2}\frac{A}{2}\,e^{j(\omega+\omega-\omega-\omega+\omega)t} \\
+ H_5(j\omega,-j\omega,j\omega,j\omega,-j\omega)\,\frac{A}{2}\frac{\bar{A}}{2}\frac{A}{2}\frac{A}{2}\frac{\bar{A}}{2}\,e^{j(\omega-\omega+\omega+\omega-\omega)t} \\
+ H_5(j\omega,-j\omega,j\omega,-j\omega,j\omega)\,\frac{A}{2}\frac{\bar{A}}{2}\frac{A}{2}\frac{\bar{A}}{2}\frac{A}{2}\,e^{j(\omega-\omega+\omega-\omega+\omega)t} \\
+ H_5(j\omega,-j\omega,-j\omega,j\omega,j\omega)\,\frac{A}{2}\frac{\bar{A}}{2}\frac{\bar{A}}{2}\frac{A}{2}\frac{A}{2}\,e^{j(\omega-\omega-\omega+\omega+\omega)t} \\
+ H_5(-j\omega,j\omega,j\omega,j\omega,-j\omega)\,\frac{\bar{A}}{2}\frac{A}{2}\frac{A}{2}\frac{A}{2}\frac{\bar{A}}{2}\,e^{j(-\omega+\omega+\omega+\omega-\omega)t} \\
+ H_5(-j\omega,j\omega,j\omega,-j\omega,j\omega)\,\frac{\bar{A}}{2}\frac{A}{2}\frac{A}{2}\frac{\bar{A}}{2}\frac{A}{2}\,e^{j(-\omega+\omega+\omega-\omega+\omega)t} \\
+ H_5(-j\omega,j\omega,-j\omega,j\omega,j\omega)\,\frac{\bar{A}}{2}\frac{A}{2}\frac{\bar{A}}{2}\frac{A}{2}\frac{A}{2}\,e^{j(-\omega+\omega-\omega+\omega+\omega)t} \\
+ H_5(-j\omega,-j\omega,j\omega,j\omega,j\omega)\,\frac{\bar{A}}{2}\frac{\bar{A}}{2}\frac{A}{2}\frac{A}{2}\frac{A}{2}\,e^{j(-\omega-\omega+\omega+\omega+\omega)t}
\end{array}
\right\}\ \text{5th-order terms}
$$

$$
+ H_7(j\omega,j\omega,j\omega,j\omega,-j\omega,-j\omega,-j\omega)\,\frac{A}{2}\frac{A}{2}\frac{A}{2}\frac{A}{2}\frac{\bar{A}}{2}\frac{\bar{A}}{2}\frac{\bar{A}}{2}\,e^{j(\omega+\omega+\omega+\omega-\omega-\omega-\omega)t}
$$

$$
+ \cdot\cdot\cdot\cdot\cdot \text{ higher-order terms} \tag{4.12}
$$

Since the system in Fig. 9(a) is <u>autonomous</u> (i.e., it has no external forcing functions), there is no loss of generality to choose our time origin such that the oscillation condition $A_z = A$ is a <u>real number</u>. Substituting $A_z = A$ with $\angle A = 0$ into (4.12) and cancelling $\frac{A}{2}\,e^{j\omega t}$ from both sides, we obtain:

$$
1 = H_1(j\omega) + \Omega_1(j\omega)A^2 + \Omega_2(j\omega)A^4 + \cdots + \Omega_n(j\omega)A^{2n} + \cdots \tag{4.13}
$$

where

$$
\Omega_1(j\omega) \triangleq \frac{1}{4}\,\{H_3(j\omega,j\omega,-j\omega) + H_3(-j\omega,j\omega,j\omega) + H_3(j\omega,-j\omega,j\omega)\} \tag{4.14a}
$$

$$
\begin{aligned}
\Omega_2(j\omega) \triangleq \frac{1}{16}\,\{&H_5(j\omega,j\omega,j\omega,-j\omega,-j\omega) + H_5(j\omega,j\omega,-j\omega,j\omega,-j\omega) \\
&+ H_5(j\omega,j\omega,-j\omega,-j\omega,j\omega) + H_5(j\omega,-j\omega,j\omega,j\omega,-j\omega) \\
&+ H_5(j\omega,-j\omega,j\omega,-j\omega,j\omega) + H_5(j\omega,-j\omega,-j\omega,j\omega,j\omega) \\
&+ H_5(-j\omega,j\omega,j\omega,j\omega,-j\omega) + H_5(-j\omega,j\omega,j\omega,-j\omega,j\omega) \\
&+ H_5(-j\omega,j\omega,-j\omega,j\omega,j\omega) + H_5(-j\omega,-j\omega,j\omega,j\omega,j\omega)\}
\end{aligned} \tag{4.14b}
$$

$$
\cdots\cdots
$$

Assuming that $\Omega_n(j\omega) = 0$ for $n > N$, and substituting $\mathcal{H}_1(j\omega) = H_1(j\omega)$ into (4.13), we obtain the determining equation:

$$\boxed{d_N(A,\omega) \triangleq H_1(j\omega) + \Omega_1(j\omega)A^2 + \Omega_2(j\omega)A^4 + \cdots + \Omega_N(j\omega)A^{2N} - 1 = 0} \qquad (4.15)$$

which is precisely (2.18). In the special cases N = 1 and N= 2, we obtain (2.1) and (2.10) respectively.

5. MATHEMATICAL JUSTIFICATION

In this section, we will give a rigorous mathematical proof which justifies the "intuitive approach" used to derive the determining equation in Section 4. In particular, we will present a method for testing whether a solution of the determining equation does indeed imply the existence of a periodic solution having an amplitude $\hat{A}$ and frequency $\hat{\omega}$ closed to the solution. As a bonus, our test will also yield a bound on the approximation error.

A solution of the determining equation $d_N(A,\omega) = 0$ corresponds to an intersection (ω_Q, A_Q) between Re $d_N(A,\omega) = 0$. and Im $d_N(a,\omega) = 0$. Our test consists of constructing a small rectangle Δ about Q which contains the exact solution $(\hat{\omega},\hat{A})$. Our basic strategy is to use degree theory [14] to show that the higher-order terms ($k>N$) neglected in (4.13) in order to arrive at (4.15) does not cause the intersection to leave the rectangle Δ.

A. Modeling the Determining Equation

Equation (4.13) is derived from Fig. 9(b) and is exact if $n \to \infty$. Since (4.15) neglects all terms with $n > N$, let us derive a "symbolic model" which is described exactly by (4.15).

Note that each coefficient $\Omega_n(j\omega)$ in (4.13) is a well-defined algebraic expression generated by the recursive algorithm in Appendix A. Note also that in (4.13) $\Omega_n(j\omega)$ is associated with an amplitude A^{2n}. Hence, neglecting $\Omega_n(j\omega)$ for $n > N$ is equivalent to suppressing all algebraic terms in (4.13) involving $A^{2(N+1)}, A^{2(N+2)}, \cdots$ etc. A review of the recursive algorithm in Appendix A suggests the "symbolic model" shown in Fig. 11 will give precisely (4.15), where T_{2N} is a "symbolic" operator which suppresses all algebraic terms involving A^{2n} with $2n > 2N$. We call this operator "symbolic" to emphasize that unlike the operators $\mathcal{P}$ and $\mathcal{I}-\mathcal{P}$ (which operate on time waveforms and produce time waveforms as outputs), T_{2N} operates on algebraic expressions, such as (4.14), and produces an algebraic expression at its output devoid of higher-order terms $A^{2n+1}, A^{2N+2}, \cdots$.

Similarly, the operator T_{2N+1} suppresses all terms involving $A^{2N+2}, A^{2N+3}, \cdots$ etc.†

The symbolic model in Fig. 11 is introduced here mainly as a conceptual aid in deriving equations which automatically suppress higher-order terms which do not contribute to (4.15). It is not a computer-simulation model.

Since the symbolic model results from keeping only the lower-order terms in A^{2N} or A^{2n+1}, we add a subscript "l" (for lower) to the variables x, y, and z as shown in Fig. 11. Note that

$$x(t) = x_l(t) + x_h(t),\; y(t) = y_l(t) + y_h(t),\; z(t) = z_l(t) + z_h(t) \tag{5.1}$$

where the subscript "h" denotes contributions due to the neglected higher-order terms. Using the recursive algorithm in Appendix A, we can generate a "formal" Volterra series for $x_l(t)$, $y_l(t)$, and $z_l(t)$ in terms of the input u(t). For our present purpose, however, we are mainly interested in $x_l(t)$ due to the input $u(t) = \frac{A}{2} e^{j\omega t} + \frac{\bar{A}}{2} e^{-j\omega t}$ -- the same input used in deriving (4.13) and (4.15):

$$\boxed{\begin{aligned} x_l(t) = {} & \frac{A}{2} e^{j\omega t} + \left(\frac{\bar{A}}{2}\right) e^{-j\omega t} + \left(\frac{A}{2}\right)^2 X_2(j\omega,j\omega)\, e^{j2\omega t} + \cdots \\ & + \left(\frac{A}{2}\right)^3 X_3(j\omega,j\omega,j\omega)\, e^{j3\omega t} + \cdots \\ & + \cdots\cdots\cdots \\ & + \left(\frac{A}{2}\right)^{2N} X_{2N}(j\omega,j\omega,\cdots,j\omega)\, e^{j2N\omega t} \end{aligned}} \tag{5.2}$$

It is important to note that the higher-order transfer functions, $X_2(s_1,s_2)$, $X_3(s_1,s_2,s_3), \cdots$, $X_{2N}(s_1,s_2,\cdots,s_{2N})$ are automatically generated by the recursive algorithm in Appendix A in the process of generating $H_{2N+1}(s_1,s_2,\cdots,s_{2N+1})$. Hence, we can calculate $x_l(t)$ either symbolically or numerically using (5.2).

Observe that if we apply (5.2) as the input to F in Fig. 11, and derive the corresponding expression $T_{2N}(I-P)\, F(x_l(t))$, we would obtain the following identity:

$$x_l = u + T_{2N}(I-P)\, F(x_l) \tag{5.3}$$

In general, $F(x_l(t))$ will contain all harmonics of $e^{j\omega t}$ and all higher-order terms in $\frac{A}{2}$. The operation $(I-P)\, F(x_l(t))$ suppresses the fundamental

†The operator after P is T_{2N+1} and not T_{2N} because we have actually cancelled out an "A" from both sides of (4.12) to obtain (4.13) so that in fact terms involving A^{2N+1} have been included in the derivation of the determining equation (4.15).

component whereas the subsequent operation $T_{2N}(I-P)\ F(x_l(t))$ further suppresses all terms involving $A^{2N+1}, A^{2N+2}, \cdots$, etc.

Corresponding to the input $u(t) = \frac{A}{2} e^{j\omega t} + \frac{\bar{A}}{2} e^{-j\omega t}$, the neglected terms in $x(t)$ is of the form

$$x_h(t) = \sum_{n=-\infty}^{\infty} B_n\ e^{jn\omega t} \tag{5.4}$$

Since T_{2N} suppresses all contributions due to $x_h(t)$, we have

$$T_{2N}(I-P)\ F(x_l) = T_{2N}(I-P)\ F(x_l+x_h) \tag{5.5}$$

On the other hand, Fig. 9(b) shows that

$$x = x_l + x_h = u + (I-P)\ F(x_l+x_h) \tag{5.6}$$

Solving for x_h and making use of (5.3) and (5.6), we obtain

$$x_h = (I-T_{2N})(I-P)\ F(x_l+x_h) \tag{5.7}$$

where I denotes an "algebraic" Identity operator, i.e., it transforms any algebraic expression into itself.

Now decompose F in Fig. 11 into a linear and a nonlinear part:

$$F = F_L + F_{NL} \tag{5.8}$$

Substituting (5.8) into (5.7) and making use of the distributive property of F_L, we obtain

$$x_h = (I-T_{2N})(I-P)\{F_L(x_l) + F_L(x_h) + F_{NL}(x_l+x_h)\} \tag{5.9}$$

Since x_l contains only lower-order terms in A,

$$(I-T_{2N})(I-P)\ F_L(x_l) = 0 \tag{5.10}$$

Since the operator $I-P$ suppresses the first harmonic component, x_h in (5.7) does not contain any first-harmonic component so that we can write

$$(I-T_{2N})(I-P)\ F_L(x_h) = F_L(x_h) \tag{5.11}$$

substituting (5.10) and (5.11) into (5.9), we obtain

$$(I-F_L)x_h = (I-T_{2N})(I-P)\ F_{NL}(x_l+x_h) \tag{5.12}$$

Since F_L is a linear operator, $F_L(Ae^{jk\omega t}) = A\ H_1(jk\omega)e^{jk\omega t}$, where $H_1(s_1)$ is the first term in the Volterra series expansion of F. Assuming

$$\inf_{k\neq\pm 1} |1-H_1(jk\omega)| > 0 \tag{5.13}$$

The operator $(I-F_L)$ can be inverted in the subspace excluding the fundamental harmonic component so that (5.12) can be solved for x_h:

$$\boxed{x_h = (I-F)^{-1}(I-T_{2N})(I-P)\, F_{NL}(x_l+x_h) \triangleq C(x_h)} \tag{5.14}$$

Now for any A and ω, we can calculate $x_l(t)$ from Fig. 11 due to the input $u(t) = \frac{A}{2} e^{j\omega t} + \frac{\bar{A}}{2} e^{-j\omega t}$. For any such $x_l(t)$ -- which depends on A and ω -- (5.14) is a nonlinear operator equation whose solution $x_h(t)$ gives the "correction" due to the neglected higher-order terms. In other words, for any A and ω, the exact solution of the open-loop system in Fig. 2(b), or equivalently Fig. 9(b), is given by $x(t) = x_l(t) + x_h(t)$. We will henceforth call (5.14) the corrector equation.†

B. Existence of Periodic Solution

To prove that the closed loop system in Fig. 2(a) has a periodic solution of frequency ω and fundamental component amplitude A, it suffices to prove the following:

1. For any A and ω, the open-loop system in Fig. 9(b) has a solution due to an input $u(t) = \frac{A}{2} e^{j\omega t} + \frac{\bar{A}}{2} e^{-j\omega t}$.

2. There is a particular $\hat{A}$ and $\hat{\omega}$ such that $PF(\hat{x}(t)) = \hat{u}(t)$, where $\hat{x}(t)$ denotes the exact solution of Fig. 9(b) due to $\hat{u}(t) = \frac{\hat{A}}{2} e^{j\hat{\omega} t} + \frac{\bar{\hat{A}}}{2} e^{-j\hat{\omega} t}$.

In the following theorems, we assume that F is a continuous operator in the sense that the Fourier coefficients of $F(x(t))$ due to $x(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{jn\omega t}$ depend on α_n and ω continuously. We also assume that the Fourier coefficient of the waveform $x_l(t)$ due to $u(t) = \frac{A}{2} e^{j\omega t} + \frac{\bar{A}}{2} e^{-j\omega t}$ depends continuously on A and ω.

Theorem 1. Justification of determining equation

Hypotheses: Suppose the following conditions hold:

1. The determining equation (4.15) has a solution (ω_Q, A_Q). (See Fig. 10). Moreover,§

$$\left|\begin{matrix} \dfrac{\partial \mathrm{Re}\, d_N(A,\omega)}{\partial A} & \dfrac{\partial \mathrm{Re}\, d_N(A,\omega)}{\partial \omega} \\ \dfrac{\partial \mathrm{Im}\, d_N(A,\omega)}{\partial A} & \dfrac{\partial \mathrm{Im}\, d_N(A,\omega)}{\partial \omega} \end{matrix}\right|_{\omega=\omega_Q,\ A=A_Q} \neq 0 \tag{5.15}$$

†The solution of the corrector equation (5.14) is a fixed point of the operator $C(\cdot)$ [11,15].

§Geometrically, (5.15) is equivalent to the condition that the two curves $\mathrm{Re}\, d_N(A,\omega) = 0$ and $\mathrm{Im}\, d_N(A,\omega) = 0$ are not tangent to each other at Q in Fig. 10.

2. There is a closed rectangle Δ containing (ω_Q,A_Q) satisfying the following conditions.

(a) (ω_Q,A_Q) is the only solution of the determining equation (4.15) in Δ.

(b) For all $(\omega,A) \in \Delta$, the corrector equation (5.14) has a solution $x_h(A,\omega)$ which depends continuously on A and ω.

(c) For all (ω,A) on the boundary of the rectangle Δ,[†]

$$|A\, d_N(A,\omega)| > \|(I-T_{2N+1})P\, F(x_l+x_h(A,\omega))\|_1. \tag{5.16}$$

Conclusion:

The single-loop feedback system in Fig. 1 has a periodic solution with frequency $\hat{\omega}$ and fundamental component amplitude $\hat{A}$ inside Δ.

Proof. Hypothesis 2(b) guarantees that the open-loop system in Fig. 9(b) has an exact solution $x(t) = x_l(t) + x_h(t)$ for all $(\omega,A) \in \Delta$ which depends continuously on A and ω. It remains therefore only for us to prove that the exact equation

$$u = PF(x_l+x_h) \tag{5.17}$$

describing the closed-loop system in Fig. 9(a) has a solution. To do this, recast (5.17) as follow:

$$-u + T_{2N+1}\, PF(x_l+x_h) + (I-T_{2N+1})P\, F(x_l+x_h) = 0 \tag{5.18}$$

The first two terms in (5.18) corresponds to $A\, d_N(A,\omega)$ because T_{2N+1} suppresses all terms contributed by x_h. Let the waveform corresponding to the third term in (5.18) be denoted by $\frac{B_N(A,\omega)}{2}\, e^{j\omega t} + \frac{\bar{B}_N(A,\omega)}{2}\, e^{-j\omega t}$. Then the solution of (5.18) is equivalent to that obtained by solving the nonlinear equation

$$f(A,\omega) \triangleq d_N(A,\omega) + \frac{B_N(A,\omega)}{A} = 0 \tag{5.19}$$

where $f(\cdot)$ is a continuous function of A and ω. It follows from Hypotheses 1 and 2(a) that the degree of the mapping $d_N(\cdot)$ in Δ with respect to zero is ± 1 [14]. Moreover, Hypothesis 2(c) guarantees that the "perturbation" due to $B_N(A,\omega)/A$ does not change the degree so that degree of $f(A,\omega)$ is also ± 1. Hence (5.19) has a solution in Δ. This is equivalent to saying that the

[†] $\|x\|_1$ denotes the sum of the magnitude of the Fourier components of $x(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{jn\omega t}$, i.e., $\|x\|_1 \triangleq \sum_{n=-\infty}^{\infty} |\alpha_n|$.

feedback system in Fig. 9(a) has a periodic solution of frequency $\hat{\omega}$ and fundamental amplitude $\hat{A}$ such that $(\hat{\omega},\hat{A}) \in \Delta$. ¤

Remarks. 1. Geometrically speaking, the above proof based on degree theory [14] consists of forming the related equation

$$f_\varepsilon(A,\omega) \triangleq d_N(A,\omega) + \varepsilon\left(\frac{B_N(A,\omega)}{A}\right) = 0 \qquad (5.20)$$

Note that $f_0(A,\omega) = 0$ is precisely the determining equation (4.15), and $f_1(A,\omega) = 0$ is precisely (5.19). As ε varies between 0 and 1, the two curves in Fig. 10 will vary continuously. Hypothesis 2(c) then guarantees that the intersection Q will not leave the rectangle Δ as ε changes from 0 to 1.

2. Theorem 1 provides sufficient conditions for the validity of the method described in Section 2. The hypotheses 2(b) and 2(c), however, are rather complicated to check. The significance of Theorem 1 is therefore mainly theoretical--it serves as a foundation for the method in Section 2.

3. In practice one would resort to Theorem 1 only when the answer is in doubt. In such cases, the following two theorems provide more practical conditions for checking Hypotheses 2(b) and 2(c).

Theorem 2: Checking Hypothesis 2(b)

For each $(\omega_0,A_0) \in \Delta$, the corrector equation (5.14) has a solution $x_h(A_0,\omega_0)$ which depends continuously on A and ω, if it is possible to find a real constant $\gamma > 0$ such that the following hold:

(1) $$\rho \sum_{n=2}^{\infty} \{\|H_n\|'_\infty \cdot \sum_{i=1}^{n} i\binom{n}{i} \cdot [\|x_l\|_1]^{n-i}\, \gamma^{i-1}\} < \alpha \qquad (5.21)$$

(2) $$\rho \sum_{n=2}^{\infty} \|H_n\|'_\infty \{[\|x_l\|_1+\gamma]^n - [\|x_l\|_1]^n\} + \|C(0)\|_1 < \gamma \qquad (5.22)$$

where

$$0 < \alpha < 1 \qquad (5.23a)$$

$$\rho \triangleq \sup_{n\neq\pm1} \left|\frac{1}{1-H_1(jn\omega_0)}\right| \qquad (5.23b)$$

$$\|H_n\|'_\infty \triangleq \sup_{k_1+k_2+\cdots k_n\neq\pm1} |H_n(jk_1\omega_0,jk_2\omega_0,\cdots,jk_n\omega_0)| \qquad (5.23c)$$

and $\binom{n}{i}$ denotes the binomial coefficients. Moreover, if (1) and (2) are satisfied, then the solution is bounded by γ:

$$\|x_h(A_0,\omega_0)\|_1 < \gamma \tag{5.24}$$

<u>Proof</u>. Consider <u>any</u> $x_1 \triangleq x_l + x_{h_1}$ and $x_2 \triangleq x_l + x_{h_2}$ such that $\|x_{h_1}\|_1 < \gamma$ and $\|x_{h_2}\|_1 < \gamma$. Substituting x_1 and x_2 for $x_l + x_h$ in the corrector equation (5.14), we obtain

$$\|C(x_{h_1})-C(x_{h_2})\|_1 \leq \|\rho(I-T_{2N})(I-P)[F_{NL}(x_1)-F_{NL}(x_2)]\|_1$$

$$\leq \rho \sum_{n=2}^{\infty} \|H_n\|'_\infty \|(I-T_{2N})(x_1^n-x_2^n)\|_1 \tag{5.25}$$

Expanding $(x_l+x_{h_1})^n$ and $(x_l+x_{h_2})^n$, we obtain

$$x_1^n - x_2^n = (x_l+x_{h_1})^n - (x_l+x_{h_2})^n$$

$$= \sum_{i=1}^{n} \binom{n}{i}(x_l^{n-i}x_{h_1}^i - x_l^{n-i}x_{h_2}^i) \tag{5.26}$$

Observe that

$$\|x_l^{n-i}x_{h_1}^i-x_l^{n-i}x_{h_2}^i\|_1 \leq \|x_l^{n-i}x_{h_1}^i-x_l^{n-i}x_{h_1}^{i-1}x_{h_2}\|_1$$

$$+ \|x_l^{n-i}x_{h_1}^{i-1}x_{h_2}-x_l^{n-i}x_{h_2}^{i-2}x_{h_2}^2\|_1 + \|x_l^{n-i}x_{h_1}^{i-2}x_{h_2}^2 - \cdots\cdots\|_1$$

$$+ \cdots\cdots\cdots + \|x_l^{n-i}x_{h_1}x_{h_2}^{i-1}-x_l^{n-i}x_{h_2}^i\|_1$$

$$\leq \|x_l^{n-i}x_{h_1}^{i-1}\|_1\|x_{h_1}-x_{h_2}\|_1 + \|x_l^{n-i}x_{h_1}^{i-2}x_{h_2}\|_1\|x_{h_1}-x_{h_2}\|_1$$

$$+ \cdots\cdots + \|x_l^{n-i}x_{h_2}^{i-1}\|_1\|x_{h_1}-x_{h_2}\|_1$$

$$\leq i\|x_l\|_1^{n-i}\gamma^{i-1} \|x_{h_1}-x_{h_2}\|_1 \tag{5.27}$$

Substituting (5.26) - (5.27) into (5.25) and using (5.21), we obtain

$$\|C(x_{h_1})-C(x_{h_2})\|_1 < \alpha\|x_{h_1}-x_{h_2}\|_1 \tag{5.28}$$

Moreover,

$$\|C(x_{h_1})\|_1 = \|(I-F_L)^{-1}(I-T_{2N})(I-P)\, F_{NL}(x_l+x_{h_1})\|_1$$
$$\leq \rho \sum_{n=2}^{\infty} \{\|H_n\|'_\infty \sum_{i=1}^{n} \binom{n}{i} \|x_l\|^{n-i} \|x_{h_1}\|^i\} + \|C(0)\|_1$$
$$\leq \rho \sum_{n=2}^{\infty} \{\|H_n\|'_\infty [(\|x_l\|_1+\gamma)^n - (\|x_l\|_1)^n]\} + \|C(0)\|_1 < \gamma \tag{5.29}$$

Equations (5.28) and (5.29) imply that the operator $C(\cdot)$ is a contraction mapping from the ball of radius γ into itself and so has a fixed point $x_h(A_0,\omega_0)$ [15]. To show that $x_h^* \triangleq x_h(A_0,\omega_0)$ depends on (A,ω) continuously, it suffices to show that x_h^* depends on x_l and ω continuously. Given x_{l_1} and ω_1, let $x_{h_1}^*$ be the corresponding fixed point. Let $x_{h_2}^*$ be the fixed point corresponding to $x_{l_2} = x_{l_1} + \delta_x$ and $\omega_2 = \omega_1 + \delta_\omega$. We want to show that $x_{h_2}^* \to x_{h_1}^*$ as δ_x and $\delta_\omega \to 0$. We will use the notation $C_{x_{l_1},\omega_1}$ to indicate the mapping C with $x_l = x_{l_1}$ and $\omega = \omega_1$. Since C depends on x_l and ω continuously,

$$\delta_c \triangleq \|C_{x_{l_2},\omega_2}(x_{h_1}^*)-C_{x_{l_1},\omega_1}(x_{h_1}^*)\|_1 \to 0 \tag{5.30}$$

as $\delta_x \to 0$ and $\delta_\omega \to 0$.

But $x_{h_1}^*$ is a fixed point of $C_{x_{l_1},\omega_1}$

$$\|C_{x_{l_2},\omega_2}(x_{h_1}^*)-x_{h_1}^*\|_1 = \delta_c \tag{5.31}$$

Since $C_{x_{l_1},\omega_1}$ is a contraction mapping in the ball of radius γ, by continuity, $C_{x_{l_2},\omega_2}$ is also a contraction mapping in the ball of radius $\gamma + \delta_\gamma$ for sufficiently small δ_x and δ_ω. Furthermore, $\delta_\gamma \to 0$ as δ_x and $\delta_\omega \to 0$.

Hence, for sufficienty small δ_x and δ_ω, the fixed point $x_{h_2}^*$ of $C_{x_{l_2},\omega_2}$ satisfies

$$\|x_{h_2}^*-x_{h_1}^*\| < \frac{\delta_c}{1-\alpha_2} \tag{5.32}$$

where α_2 is the Lipschitz constant of $C_{x_{l_2},\omega_2}$. Hence, $x_{h_2}^* \to x_{h_1}^*$ as $\delta_c \to 0$ or as δ_x and $\delta_\omega \to 0$ which implies continuity. ¤

Theorem 3: Checking Hypothesis 2(c)

Given any (ω,A) and $\|x_h^*(A,\omega)\|_1 < \beta$,

$$\|(I-T_{2N+1})P\,F(x_l+x_h^*)\|_1 < \sum_{n=2}^{\infty} \|H_n\|_\infty[(\|x_l\|_1+\beta)^n-(\|x_l\|_1)^n] + \|(I-T_{2N+1})PF(x_l)\|_1 \tag{5.33}$$

where

$$\|H_n\|_\infty \triangleq \sup_{k_1,k_2,\cdots,k_n} |H_n(jk_1\omega,jk_2\omega,\cdots jk_n\omega)| \tag{5.34}$$

Proof. See Appendix B

C. Remarks Concerning Theorems 2 and 3

1. In order to apply Theorem 2, it is necessary to calculate $\|H_n\|'_\infty$, ρ, $\|x_l\|_1$, and $\|C(0)\|_1$ for each $(\omega_0,A_0) \in \Delta$. Although in principle we must check (5.23c) for all possible $k_1+k_2+\cdots+k_n \neq \pm 1$, in most practical oscillators, $|H_n|$ is negligible for large k_i. Hence, $\|H_n\|'_\infty$ can usually be estimated by checking $|H_n|$ for only a few number of "small" $k_1,k_2,\cdots,k_n$.

2. The value of ρ can be estimated by the same procedure.

3. The value of $\|x_l\|_1$ can be calculated from (5.2), which in turn is generated using the recursive algorithm in Appendix A.

4. To calculate $\|C(0)\|_1$, we first generate the algebraic expression for $(I-T_{2N})(I-P)\,F_{NL}(x_l)$ and then substitute it into (5.14). This is the most time-consuming part.

5. The next step is to find the "smallest" $\gamma > 0$ satisfying (5.21) and (5.22). This can be found by a line-search procedure; i.e., starting with an initial guess for γ, reduce it if (5.21)-(5.22) holds (assume $\alpha = 1$). Otherwise, increase it. Our experience shows that γ usually fails to satisfy (5.22) but not (5.21).

6. Using analogous procedure as above, we can also estimate $\|H_n\|_\infty$ for Theorem 3.

D. How to find the Rectangle Δ

With the help of Theorems 2 and 3, we can find a rectangle Δ satisfying Theorem 1 as follow:

(1) Make an initial guess of Δ about (ω_Q,A_Q).

(2) Use Theorem 2 to check hypothesis 2(b) for a reasonable number of sample points $(\omega,A) \in \Delta$.

(3) Use Theorem 3 to check hypothesis 2(c) for a reasonable number of sample points (ω,A) lying in the boundary of Δ. Note that $|A\,d_N(A,\omega)|$ is known and Theorem 3 therefore provides an upper bound for the right-hand-side of (5.16).

The following procedure can be used to obtain a reasonable initial guess for Δ:

(a) Try (ω_Q, A_1), where $A_1 > A_Q$. If Theorems 2-3 are satisfied, decrease A_1. Otherwise, increase A_1.
(b) Try (ω_Q, A_2), where $A_2 < A_Q$. If Theorems 2-3 are satisfied, increase A_2. Otherwise decrease A_2.
(c) Try (ω_1, A_Q), where $\omega_1 > \omega_Q$. If Theorems 2-3 are satisfied, decrease ω_1. Otherwise, increase ω_1.
(d) Try (ω_2, A_Q), where $\omega_2 < \omega_Q$. If Theorems 2-3 are satisfied, increase ω_2. Otherwise, decrease ω_2.
(e) Choose $A_2 \leq A \leq A_1$ and $\omega_2 \leq \omega \leq \omega_1$ as the initial rectangle Δ. Our experience shows that it is usually much harder to find a suitable A_1 than the other 3 points defining Δ.

E. Examples

We have applied the preceding procedure to several examples and in each case obtained a small rectangle Δ about the solution (ω_Q, A_Q) of the associated determining equations. We then compare the results with those obtained by numerical simulation [11]. The following table gives a summary of the results obtained with the earlier Example 4 (Van der Pol Oscillator), Example 6 (Tunnel-diode Oscillator, and Example 7 (Wien-bridge oscillator):

Table 1. Summary of Solutions

	van der Pol oscillator	Tunnel-diode oscillator	Wien-bridge oscillator
solution of the determining equation	$A_Q = 1.998$ $\omega_Q = 0.9975$	$A_Q = 0.301$ $\omega_Q = 99.99\text{x}10^6$	$A_Q = 0.384$ $\omega_Q = 0.996$
the rectangle Δ	$1.95 \leq A \leq 2.05$ $0.992 \leq \omega \leq 1.005$	$0.335 \leq A \leq 0.28$ $98\text{x}10^6 \leq \omega \leq 10.1\text{x}10^6$	$0.37 \leq A \leq 0.42$ $0.985 \leq \omega \leq 1.008$
Classical solution or numerical simulation result	$A = 2$ $\omega = 0.9975$	$A = 0.3$ $\omega = 99.7\text{x}10^6$	$A = 0.385$ $\omega = 0.987$

6. CONCLUDING REMARKS

The determining equation approach presented in this paper is novel in the sense that it represents the first rigorous application of Volterra series to nonlinear oscillation. It is a frequency-domain approach applicable to any single-loop time-invariant nonlinear feedback system with dynamics of any order. Even distributed elements are allowed. The only assumption is that the associated open-loop system has a convergent Volterra series. Although the mathematical proof for validating our approach requires some advanced mathematics, the method itself is simple and requires only algebra.

Like the Krylov, Bogoliubov and Mitropolsky's averaging method, the frequency and amplitude can in principle be calculated to any desired accuracy by choosing a high enough order for the determining equation. However, the most powerful aspects of this approach is often revealed by using only a first-order determining equation (second-order if the nonlinearity is odd symmetric). Higher-order determining equations are extremely complex and are practical only if a computer is used.

It must be emphasized, however, that like the averaging method, the main application of our method is not to calculate an accurate frequency or amplitude. Rather it is most advantageously used to ascertain whether a feedback system will oscillate, and if so, to determine the approximate frequency and amplitude. Such information is most easily obtained with a first-order determining equation. In the event that more accuracy is desired, it is better to resort to a computer-simulation method [11] using the above approximate frequency and amplitude as the initial guess.

Finally, we note that answers obtained using our approach is often more accurate than those obtained by the harmonic balance or describing function approach of the same order. This is because although our approach neglects contributions from all harmonics -- including the fundamental -- those neglected components came from higher-order nonlinearities which are usually small in comparison to those that were retained. Consequently, our approach is somewhat more selective with regards to which components to neglect.

It is also important to note that our determining equation approach is an analytical approach -- in contrast to numerical techniques. Since the determining equation is defined in symbolic form, it is possible to derive design criterion in terms of system parameters. In particular, the sensitivity of the frequency and amplitude to various circuit parameters can be derived in analytic form.

Finally, we remark that our determining equation approach is quite general and is applicable to many related problems in nonlinear mechanics [3-5].

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APPENDIX

A. Recursive Generation of Higher-order transfer Functions

Given the higher-order transfer functions of each element in a system, we can formally generate the overall higher-order transfer functions of that system. A recursive algorithm for generating these transfer functions for nonlinear circuits is given in [10]. In this appendix, we will apply this algorithm to the nonlinear feedback system used in the derivation of determining equations. We will begin with the cascade connection of two nonlinear systems.

A.1. Composition of transfer functions

Let us derive the higher-order output components of a nonlinear system whose input is the output of another nonlinear system and therefore consists also of higher-order components. In particular, consider the cascade connection of two nonlinear systems as shown in Fig. 12. Let

$$u(t) = A_1 e^{p_1 t} + A_2 e^{p_2 t} \tag{A.1}$$

Assume f is such that

$$x(t) = f\big(u(t)\big) = A_1 e^{p_1 t} + A_1 A_2 e^{(p_1+p_2)t} \tag{A.2}$$

That is, the input to h contains one first-order term $(A_1 e^{p_1 t})$ and one second-order term $\left(A_1 A_2 e^{(p_1+p_2)t}\right)$ when the input u of f is given by (A.1). Let

$$h(x) = \frac{d}{dt}(x^2) \tag{A.3}$$

substituting (A.2) into (A.3) we obtain

$$h(x) = h\Big[f\big(u(t)\big)\Big] = \frac{d}{dt}\left(A_1 e^{p_1 t} + A_1 A_2 e^{(p_1+p_2)t}\right)^2$$

$$= \frac{d}{dt}\Big[A_1 e^{p_1 t} A_1 e^{p_1 t} + A_1 e^{p_1 t} A_1 A_2 e^{(p_1+p_2)t} + A_1 A_2 e^{(p_1+p_2)t} A_1 e^{p_1 t} + A_1 A_2 e^{(p_1+p_2)t} A_1 A_2 e^{(p_1+p_2)t}\Big]$$

$$= A_1 A_2 (p_1+p_2) e^{(p_1+p_1)t} + A_1 A_1 A_2 (p_1+p_1+p_2) e^{(p_1+p_1+p_2)t} + A_1 A_2 A_1 (p_1+p_2+p_1) e^{(p_1+p_2+p_1)t}$$
$$+ A_1 A_2 A_1 A_2 (p_1+p_2+p_1+p_2) e^{(p_1+p_2+p_1+p_2)t} \tag{A.4}$$

Equation (A.4) illustrates several facts concerning the higher-order outputs of a nonlinear system contributed by a particular second-order transfer function $\left(H_2(s_1,s_2) = s_1+s_2\right)$. We can generalize these facts to nth-order transfer functions

($n \geq 2$) as follows:

(a) The output generated by an nth-order transfer function will be at least nth-order. Thus, every term in (A.4) is at least second order.

(b) Every nth-order output term generated by the nonlinear part of a system (i.e., transfer functions with order higher than 1) is made of product of input terms with order less than n. This is because any product of two input terms will increase the order by at least one. Thus, in (A.4), the second-order term $\left[A_1A_1(p_1+p_1)e^{(p_1+p_1)t}\right]$ is the result of the product of two first-order terms ($A_1e^{p_1t}$ and itself), multiplied by the second-order transfer function ($H_2(p_1,p_1) = p_1+p_1$).

(c) The contribution of the mth-order transfer function to the nth-order output consists of the sum of <u>all possible</u> products of the following form

$$H_m((s_1+\cdots+s_{k_1}),\cdots,(s_{k_1+\cdots+k_{m-1}+1}+\cdots+s_{k_1+\cdots+k_m}))\,X_{k_1}(x_1,\cdots,s_{k_1})X_{k_2}(s_{k_1+1},\cdots,s_{k_1+k_2})$$

$$X_{k_m}(s_{k_1+k_2+\cdots+k_{m-1}+1},\cdots,s_{k_1+k_2+\cdots+k_m}) \qquad (A.5)$$

where $k_1+k_2+\cdots+k_m = n$, and

$X_{k_1}, X_{k_2},\cdots,X_{k_m}$ are k_1th, k_2th$,\cdots,$ k_mth order term of input. Equation (A.5) means that an nth-order output term is obtained by the product of m input terms with order $k_1,k_2,\cdots,k_m$.

In (A.4), the two 3rd-order terms $\left[A_1A_1A_2(p_1+p_1+p_2)e^{(p_1+p_1+p_2)t}\right.$ and $\left.A_1A_2A_1(p_1+p_2+p_1)e^{(p_1+p_2+p_1)t}\right]$ are the two possible products of a first-order term ($A_1e^{p_1t}$) and a second-order term $\left[A_1A_2e^{(p_1+p_2)t}\right]$, multiplied by the second-order transfer function. Using the notation of (A.5), the two 3rd-order terms can be expressed as

$$Y_3^1(s_1,s_2,s_3) = H_2(s_1,s_2+s_3)\; X_1(s_1)\; X_2(s_2,s_3) \qquad (A.6a)$$

and

$$Y_3^2(s_1,s_2,s_3) = H_2(s_1+s_2,s_3)\; X_2(s_1,s_2)\; X_1(s_3) \qquad (A.6b)$$

We can identify the following terms from (A.6) with corresponding terms from (A.4):

$$X_1(s_1) = A_1e^{s_1t} \qquad (A.7a)$$

$$X_2(s_1,s_2) = A_1A_2e^{(s_1+s_2)t} \quad \text{(A.7b)}$$

$$H_2(s_1,s_2) = s_1 + s_2 \quad \text{(A.7c)}$$

Since there is only one first-order term $A_1e^{p_1t}$ and one second-order term $A_1A_2e^{(p_1+p_2)t}$ in x(t), we can see that $X_1(s_1) = 0$ except when $s_1 = p_1$, and $X_2(s_1,s_2) = 0$ except when $s_1 = p_1$ and $s_2 = p_2$. Hence the only nonzero Y_3^1 and Y_3^2 are $Y_3^1(p_1,p_1p_2)$ and $Y_3^2(p_1,p_2,p_1)$, which are exactly $A_1A_1A_2(p_1+p_1+p_2)e^{(p_1+p_1+p_2)t}$ and $A_1A_2A_1(p_1+p_2+p_1)e^{(p_1+p_2+p_1)t}$.

In general, it is not obvious on how to express all possible products as we did in (A.6a) and (A.6b) for the simple system described by (A.2) and (A.3). The following section will present a recursive procedure for finding such products.

A.2. Generation of higher-order output terms

(a) Notation

A list S is an ordered collection of finite number of objects, denoted by $(s_1,s_2,\cdots,s_n)$. A contiguous part of a list S, i.e. $(s_i,s_{i+1},\cdots,s_{i+k})$, is called a segment of S. A segment of a list is itself a list. A list of segments of the form $((s_1,\cdots,s_{k_1}), (s_{k_1+1},\cdots,s_{k_1+k_2}), \cdots, (s_{k_1+k_2+\cdots+k_{m-1}+1},\cdots,s_n))$ is called a partition of the list $(s_1,s_2,\cdots,s_n)$. For example, let $S = (s_1,s_2,s_3,s_4)$, then (s_1), (s_1,s_2) and (s_2,s_3,s_4) are segments of S and $((s_1),(s_2,s_3),(s_4))$ is a partition of S. Note that the index increases towards the right and each integer occurs exactly once.

(b) Basic Approach

Let $S = (s_1,s_2,\cdots,s_n)$. If we compare the partitions of S with (A.5), we note that tnere is a one-to-one correspondence between all possible forms of (A.5) and all possible partitions of S. For example, if $S = (s_1,s_2,s_3,s_4)$, then the product $H_3(s_1,s_2+s_3,s_4)\ X_1(s_1)\ X_2(s_2,s_3)\ X_1(s_4)$ corresponds to $((s_1),(s_2,s_3),(s_4))$. Thus, <u>the problem of finding all possible products reduces to finding all possible partitions of a list</u>. That is, given a list $S = (s_1,s_2,\cdots,s_n)$, find all partitions of S which contain m segments.

(c) Partitioning Procedure

It is clear that when m = 1 the only possible partition is $((s_1,s_2,\cdots,s_n))$. For m > 1, we will show below that the partitioning can be obtained by solving a series of similar problems but with m reduced to m-1. This <u>partitioning procedure</u> will then be invoked recursively until m = 1.

Consider all possible choices of the first segment in a partition. It may be one of (s_1), $(s_1,s_2),\cdots$, up to $(s_1,s_2,\cdots,s_{n-m+1})$. The reason that it contains at most n-m+1 elements is that the remaining m-1 segments take at least m-1 elements.

For a particular choice of the first segment $S_1 = (s_1,s_2,\cdots, s_i)$, we solve a reduced problem which consists of finding all partitions containing m-1 segments of the remaining list $(s_{i+1},\cdots,s_n)$. This reduced problem is solved by the same partitioning procedure with the number of segments equal to m-1. Solving the reduced problem, we obtain many partitions of the form $(S_2,S_3,\cdots,S_m)$ where $S_2,\cdots,S_m$ are segments. Inserting S_1 into each of these partitions, we obtain partitions of the original list with the first segment equal to S_1.

Repeating the above process for all possible choices of S_1, we obtain all possible partitions.

(d) Illustrative Example

As an example, let $S = (s_1,s_2,s_3,s_4)$ and m=3. Applying the partitioning procedure, the first segment denoted by S_1^3 (the superscript 3 indicates that this is the first of 3 segments) may be either (s_1) or (s_1,s_2).

(1) Let $S_1^3 = (s_1)$, then the reduced problem 2 (2 indicates that this is a problem with m=2) consists of finding all possible partitions of (s_2,s_3,s_4) with 2 segments. Invoking the same partitioning procedure, we find the possible choices of the first segment S_1^2 are (s_2) and (s_2,s_3).

(1.1) For $S_1^2 = (s_2)$, the reduced problem 1 consists of finding partitions of (s_3,s_4) with one segment. The result is clearly $((s_3,s_4))$. Inserting S_1^2 into $((s_3,s_4))$, we obtain $((s_2),(s_3,s_4))$.

(1.2) For $S_1^2 = (s_2,s_3)$, the solution of reduced problem 1 is $((s_4))$. Inserting S_1^2 into it, we obtain $((s_2,s_3),(s_4))$.

Since (s_2) and $(s_2,s3)$ are all the possible choices of S_1^2, we have solved the reduced problem 2 with the pair of partitions $((s_2,(s_3,s_4))$ and $((s_2,s_3),(s_4))$ as its solution.

(2) Inserting S_1^3 into each partition obtained by (1.1) and (1.2), we obtain $((s_1),(s_2),(s_3,s_4))$ and $((s_1),(s_2,s_3),(s_4))$ as the two possible partitions with the first segment equal to (s_1).

(3) Repeating (1) and (2) for the other possible S_1^3, i.e. (s_1,s_2), the reduced problem 2 becomes finding partitions of (s_3,s_4) with 2 segments. The only possible S_1^2 is (s_3) so the only solution is $((s_3),(s_4))$. Inserting S_1^3 into it, we obtain $((s_1,s_2),(s_3),(s_4))$ as the partition.

All together, we obtain $((s_1),(s_2),(s_3,s_4))$, $((s_1),(s_2,s_3),(s_4))$ and $((s_1,s_2),(s_3),(s_4))$ as the three possible partitions.

A.3. Feedback Systems

Consider the systems in Figs. 9(a) and 9(b). We will present an algorithm for obtaining higher-order transfer functions from u to z. Since P is only an ideal filter, it suffices to find the nth-order output $Y_n(s_1,\cdots,s_n)$ at y in Fig. 9(b). The associated higher-order transfer functions can then be trivially obtained by suppressing all terms rejected by the ideal filter.

For simplicity, we will find the higher-order outputs by assuming a unit input amplitude i.e., $u(t) = e^{j\omega t} + e^{-j\omega t}$. Thus, the nth-order output will coincide with the nth-order transfer function.

We first separate F into linear and nonlinear parts F_L and F_{NL}. We will redraw the system as in Fig. 13 where W_n is the nth-order output generated by F_{NL}. By facts (a) and (b) of Section A.1, we know that W_n is generated by second-order to nth-order transfer functions and only input terms with order less than n have an effect on W_n. Thus, W_n can be calculated by the procedure in Section A.2 if all $X_1,X_2,\cdots,X_{n-1}$ are known. Because the linear part F_L does not alter the order of terms, X_n must satisfy the linear subsystem in the dotted box of Fig. 13. Since the output of F_{NL} is at least second-order, we have $W_1 = 0$. Also, all terms in u(t) is considered first order. Hence X_n must satisfy the following equations:†

$$X_1 = U + (I-P)\,F_L(X_1) \qquad \text{(A.8a)}$$

$$X_n = (I-P)\big(W_n+F_L(X_n)\big) \quad (n \geq 2) \qquad \text{(A.8b)}$$

where U is the frequency-domain representation of u(t). We can solve X_1 from (A.8a) and W_n, X_n from (A.8b) recursively by the procedure in Section A.2. Having W_n and X_n, H_n is easily found by

$$H_n = P(W_n+F_LX_n) \qquad \text{(A.9)}$$

Note that the higher-order transfer functions needed for x_l are automatically generated in the process of generating H_{2N+1}.

This algorithm can be implemented on a digital computer either as a "numerical" function which calculates the value of an nth-order transfer function, or as an "operator" which generates the explicit "symbolic" expression for the

† Here we treat F_L as an operator on higher-order terms in the frequency domain. Its input and output relationship is defined by the corresponding input and output relationship in the time domain.

transfer function with the aid of some symbolic algebra management system such as Macsyma [12].† In either case, it should be noted that only one program is necessary for generating all of the transfer functions. With certain data representation technique, the algorithm can be implemented in a straight-forward manner. To improve efficiency, some ad hoc techniques should be included to avoid repeating the same operation.

A.4. Explicit expressions for some higher-order transfer functions

In this section, we will, as an illustration for Section A.3, derive the expression for H_3 and H_5 as given in the paper. An odd-symmetric nonlinearity is assumed for H_5.

(1) Third-order transfer functions

From the algorithm in Section A.3, since X_3 does not contain any first harmonic component we obtain

$$H_3(j\omega,j\omega,-j\omega) = W_3(j\omega,j\omega,-j\omega) \tag{A.10a}$$

Applying the procedure in Section A.2, we obtain

$$W_3(j\omega,j\omega,-j\omega) = H_2(2j\omega,-j\omega)\ X_2(j\omega,j\omega)\ X_1(-j\omega) + H_2(j\omega,0)\ X_1(j\omega)\ X_2(j\omega,-j\omega)$$
$$+ H_3(j\omega,j\omega,-j\omega)\ X_1(j\omega)\ X_1(j\omega)\ X_1(-j\omega) \tag{A.10b}$$

Solving the linear system, we obtain

$$X_2(j\omega,j\omega) = \frac{W_2(j\omega,j\omega)}{1-H_1(2j\omega)} \tag{A.10c}$$

$$X_2(j\omega,-j\omega) = \frac{W_2(j\omega,-j\omega)}{1-H_1(0)} \tag{A.10d}$$

Applying Section A.2,

$$W_2(j\omega,j\omega) = H_2(j\omega,j\omega)\ X_1(j\omega)\ X_1(j\omega) \tag{A.10e}$$

$$W_2(j\omega,-j\omega) = H_2(j\omega,-j\omega)\ X_1(j\omega)\ X_1(-j\omega) \tag{A.10f}$$

Because we assumed $u(t) = e^{j\omega t} + e^{-j\omega t}$ and since *I-P* rejects the first harmonic components, it follows that $X_1(j\omega)$ and $X_1(-j\omega)$ are both equal to 1. Combining (A.10a)-(A.10f), we obtain

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†Macsyma is a system which allows symbolic expressions as its data and performs mathematical operation on them. It also allows the definition of recursive function on these data. Hence the symbolic implementation of our algorithm is feasible.

$$H_3(j\omega,j\omega,-j\omega) = H_2(2j\omega,-j\omega)\,\frac{H_2(j\omega,j\omega)}{1-H_1(2j\omega)} + H_2(j\omega,0)\,\frac{H_2(j\omega,-j\omega)}{1-H_1(0)} + H_3(j\omega,j\omega,-j\omega) \tag{A.10g}$$

Similarly,

$$H_3(j\omega,-j\omega,j\omega) = W_3(j\omega,-j\omega,j\omega)$$

$$= H_2(0,j\omega)\,X_2(j\omega,-j\omega)\,X_1(j\omega) + H_2(j\omega,0)\,X_1(j\omega)\,X_2(-j\omega,j\omega)$$

$$+ H_3(j\omega,-j\omega,j\omega)\,X_1(j\omega)\,X_1(-j\omega)\,X_1(j\omega) \tag{A.11a}$$

$$X_2(-j\omega,j\omega) = \frac{W_2(-j\omega,j\omega)}{1-H_1(0)} \tag{A.11b}$$

$$W_2(-j\omega,j\omega) = H_2(-j\omega,j\omega)\,X_1(-j\omega)\,X_1(j\omega) \tag{A.11c}$$

Combining (A.10) and (A.11), we obtain

$$H_3(j\omega,-j\omega,j\omega) = H_2(0,j\omega)\,\frac{H_2(j\omega,-j\omega)}{1-H_1(0)} + H_2(j\omega,0)\,\frac{H_2(-j\omega,j\omega)}{1-H_1(0)} + H_3(j\omega,-j\omega,j\omega) \tag{A.11d}$$

$$H_3(-j\omega,j\omega,j\omega) = H_2(0,j\omega)\,X_2(-j\omega,j\omega)\,X_1(j\omega) + H_2(-j\omega,2j\omega)\,X_1(-j\omega)\,X_2(j\omega,j\omega)$$

$$+ H_3(-j\omega,j\omega,j\omega)\,X_1(-j\omega)\,X_1(j\omega)\,X_1(j\omega)$$

$$= H_2(0,j\omega)\,\frac{H_2(j\omega,-j\omega)}{1-H_1(0)} + H_2(-j\omega,2j\omega)\,\frac{H_2(j\omega,j\omega)}{1-H_1(2j\omega)} + H_3(-j\omega,j\omega,j\omega) \tag{A.12}$$

(2) Fifth-order transfer function

Assuming that F in Fig. 11(b) is odd symmetric, i.e., H_0, H_2, H_4 are all zero, then there will be no even-order terms. That is, X_2, X_4,$\cdots$ are zero. Also note that X_5 does not contain first-harmonic terms, hence the fifth-order equation is:

$$H_5(jk_1\omega,jk_2\omega,jk_3\omega,jk_4\omega,jk_5\omega)$$

$$= W_5(jk_1\omega,jk_2\omega,jk_3\omega,jk_4\omega,jk_5\omega)$$

$$= H_3\left(j(k_1+k_2+k_3)\omega,jk_4\omega,jk_5\omega\right)\,X_3(jk_1\omega,jk_2\omega,jk_3\omega)\,X_1(jk_4\omega)\,X_1(jk_5\omega)$$

$$+ H_3\left(jk_1\omega,j(k_2+k_3+k_4)\omega,jk_5\omega\right)\,X_1(jk_1\omega)\,X_3(jk_2\omega,jk_3\omega,jk_4\omega)\,X_1(jk_5\omega)$$

$$+ H_3\left(jk_1\omega,jk_2\omega,j(k_3+k_4+k_5)\omega\right)\,X_1(jk_1\omega)\,X_1(jk_2\omega)\,X_3(jk_3\omega,jk_4\omega,jk_5\omega)$$

$$+ H_5\left(jk_1\omega,jk_2\omega,jk_3\omega,jk_4\omega,jk_5\omega\right)\,X_1(jk_1\omega)\,X_1(jk_2\omega)\,X_1(jk_3\omega)\,X_1(jk_4\omega)\,X_1(jk_5\omega) \tag{A.13}$$

where $k_1, k_2, k_3, k_4, k_5 = \pm 1$ and $k_1 + k_2 + k_3 + k_4 + k_5 = 1$

$$X_3(j\ell_1\omega, j\ell_2\omega, j\ell_3\omega) = \begin{cases} \dfrac{W_3(j\ell_1\omega, j\ell_2\omega, j\ell_3\omega)}{1-H_1\left(j(\ell_1+\ell_2+\ell_3)\omega\right)} & \ell_1+\ell_2+\ell_3 \neq \pm 1 \\ 0 & \ell_1+\ell_2+\ell_3 = \pm 1 \end{cases} \tag{A.14}$$

where ℓ_1, ℓ_2, ℓ_3 denote k_1, k_2, k_3, or k_2, k_3, k_4, or k_3, k_4, k_5.

$$W_3(j\ell_1\omega, j\ell_2\omega, j\ell_3\omega) = H_3(j\ell_1\omega, j\ell_2\omega, j\ell_3\omega)\; X_1(j\ell_1\omega)\; X_1(j\ell_2\omega)\; X_1(j\ell_3\omega) \tag{A.15}$$

Combining (A.13), (A.14) and (A.15), we see that the first 3 terms of (A.13) vanish except when there are three consecutive $j\omega$ arguments as in (2.17a), (2.17g) and (2.17j). An additional term of the form (A.14) is added to the fourth term of (A.13) in these three cases. This gives us (2.17a) to (2.17j).

B. Proof of Theorem 3.

$$\|(I-T_{2N+1})\ \mathcal{P}F(x_l+x_h^*)\|_1$$

$$\leq \|(I-T_{2N+1})\ \mathcal{P}[(\sum_{n=2}^{\infty} H_n(x_l+x_h^*)^n\ + H_1x_l\ + H_1x_h^*]\|_1{}^{\dagger} \tag{B.1}$$

Since x_h^* does not contain any first-harmonic component, and since x_l does not contain lower-order terms, they will be anihilated by $(I-T_{2N+1})\mathcal{P}$. Thus, (B.1) becomes

$$\|(I-T_{2N+1})\ \mathcal{P}F(x_l+x_h^*)\|_1 \leq \|(I-T_{2N+1})\mathcal{P}\ \sum_{n=2}^{\infty}\ (x_l+x_h^*)^n\|_1$$

$$\leq \|(I-T_{2N+1})\mathcal{P}\{\sum_{n=2} H_n[(x_l+x_h^*)^n\ -\ x_l^n\ +\ x_l^n]\}\|_1$$

$$\leq \|(I-T_{2N+1})\ \mathcal{P}\{\sum_{n=2}^{\infty}\ \|H_n\|_\infty(\|x_l+x_h^*\ \|^n-\|x_l\|_1^n)\ +\ F(x_l)\}\|_1$$

$$\leq \sum_{n=2}^{\infty}\ \|H_n\|_\infty[(\|x_l\|_1+\beta)^n\ -\ \|x_l\|_1^n]\ +\ \|(I-T_{2N+1})\ \mathcal{P}F(x_l)\|_1 \tag{B.2}$$

This proves Theorem 3. ¤

C. Applications of Theorems 2, and 3.

In this appendix we will use the van der Pol oscillator as an example to illustrate the application of Theorems 2 and 3 to justify the solution of the second-order determining equation.

Since $f(v) = -\frac{1}{3}v^3$, we have

$$H_1(jk_1\omega) = 0 \tag{C.1a}$$

$$H_3(jk_1\omega,jk_2\omega,jk_3\omega) = -\frac{1}{3}\frac{\varepsilon}{-\varepsilon+(j(k_1+k_2+k_3)\omega+\frac{1}{j(k_1+k_2+k_3)\omega}} \tag{C.1b}$$

$$H_5(jk_1\omega,jk_2\omega,jk_3\omega,jk_4\omega,jk_5\omega) = 0 \tag{C.1c}$$

where k_1, k_2, k_3, k_4, k_5 are integers.

† Here the notation H_n represents the output generated by the nth-order transfer function.

C.1. Application of Theorem 2

Given A and ω, we need to find ρ and $\|H_n\|'_\infty$. They can usually be found by choosing a few small numbers for $k_1, \cdots, k_5$. In the case of the van der Pol oscillator, it is clear that $\rho = 1$, $\|H_1\|'_\infty = 0$ and $\|H_5\|'_\infty = 0$. Also, $\|H_3\|'_\infty$ should occur at the smallest possible $k_1 + k_2 + k_3$. In this case, since there is no second harmonic component, it is 3. Thus,

$$\|H_3\|'_\infty = \left| \frac{1}{3} \frac{\varepsilon}{-\varepsilon + j3\omega + \frac{1}{j3\omega}} \right|$$

Applying Appendix A and noting that $H_1 = 0$, we found the only nonzero third-order transfer functions are

$$X_3(j\omega, j\omega, j\omega) = H_3(j\omega, j\omega, j\omega) \tag{C.3}$$

and its complex conjugate. Thus, from (C.3) and (5.2), we obtain

$$x_l \doteq \frac{A_1}{2} e^{j\omega t} + \frac{\bar{A}_1}{2} e^{-j\omega t} + \frac{A_3}{2} e^{3j\omega t} + \frac{\bar{A}_3}{2} e^{-3j\omega t} \tag{C.4}$$

where $A_1 = A$ and $A_3 = \frac{1}{4} H_3(j\omega, j\omega, j\omega) A^3 = -\frac{1}{12} \frac{\varepsilon}{-\varepsilon + j3\omega + \frac{1}{j3\omega}}$.

Clearly, we have

$$\|x_l\|_1 = |A_1| + |A_3|$$

The last term to be found is $C(0)$. In this case, $C(0)$ contains the terms generated by H_3 with order higher than 4 and excludes the first-harmonic component. Note that A_3 is of order 3. From (C.4) it can be seen that these terms are given by

$$6H_3(j\omega, -j\omega, j3\omega)\left(\frac{A_1}{2}\right)\left(\frac{\bar{A}_1}{2}\right)\left(\frac{A_3}{2}\right)e^{j3\omega t} \tag{C.5a}$$

$$3H_3(j\omega, j\omega, j3\omega)\left(\frac{A_1}{2}\right)\left(\frac{A_1}{2}\right)\left(\frac{A_3}{2}\right)e^{j5\omega t} \tag{C.5b}$$

$$3H_3(-j\omega, j3\omega, j3\omega)\left(\frac{\bar{A}_1}{2}\right)\left(\frac{A_3}{2}\right)\left(\frac{A_3}{2}\right)e^{j5\omega t} \tag{C.5c}$$

$$3H_3(j\omega, j3\omega, j3\omega)\left(\frac{A_1}{2}\right)\left(\frac{A_3}{2}\right)\left(\frac{A_3}{2}\right)e^{j7\omega t} \tag{C.5d}$$

$$3H_3(-j3\omega, j3\omega, j3\omega)\left(\frac{\bar{A}_3}{2}\right)\left(\frac{A_3}{2}\right)\left(\frac{A_3}{2}\right)e^{j3\omega t} \tag{C.5e}$$

$$H_3(j3\omega, j3\omega, j3\omega)\left(\frac{A_3}{2}\right)\left(\frac{A_3}{2}\right)\left(\frac{A_3}{2}\right)e^{j9\omega t} \tag{C.5f}$$

and their complex conjugates. The constant coefficient in front of each term results from all possible permutations of the arguments of H_3.

With all these terms, we can check the conditions of Theorem 2 in a straightforward way.

C.3. Application of Theorem 3

We first find $\|H_n\|_\infty$. In this case, only $\|H_3\|_\infty$ is non-zero and occur at $k_1+k_2+k_3 = 1$, i.e.

$$\|H_3\|_\infty = \left|\frac{1}{3}\,\frac{\varepsilon}{-\varepsilon+j\omega+\frac{1}{j\omega}}\right| \tag{C.6}$$

Then, $\|(I-T_{2N+1})\; PF(x_l)\|_1$ can be found by direct substitution of x_l. In this case, $(I-T_{2N+1})\; PF(x_l)$ contains all terms with order higher than 5 and corresponds to the first-harmonic component. They are given by

$$6H_3(j\omega,-j3\omega,j3\omega)\left(\frac{A_1}{2}\right)\left(\frac{\bar{A}_3}{2}\right)\left(\frac{A_3}{2}\right)e^{j\omega t} \tag{C.7}$$

and its complex conjugate.

Since $\|x_l\|_1$ and β have already been found while carrying out Section C.2, we can check (5.33) directly.

The remaining task consists of going through the procedure described in Section 5.D of the paper. This is usually done numerically with the aid of computer.

FIGURE CAPTIONS

Fig. 1. A single-loop nonlinear feedback system.

Fig. 2. (a) A closed-loop nonlinear feedback system

(b) Open-loop nonlinear system

Fig. 3. (a) Circuit containing one 2-terminal nonlinear resistor $\mathcal{R}$.

(b) Equivalent feedback system for voltage-controlled resistor.

(c) Equivalent feedback system for current-controlled resistor.

Fig. 4. (a) Circuit containing one 2-terminal nonlinear inductor $\mathcal{L}$.

(b) Equivalent feedback system for flux-controlled inductor.

(c) Equivalent feedback system for current-controlled inductor.

Fig. 5. (a) Circuit containing one 2-terminal nonlinear capacitor $\mathfrak{C}$.

(b) Equivalent feedback system for charge-controlled capacitor.

(c) Equivalent feedback system for voltage-controlled capacitor.

Fig. 6. Nonlinear RLC circuit described by van der Pol equation.

Fig. 7. Nonlinear RLC circuit described by Duffing's equation.

Fig. 8. (a) Wien-bridge oscillator circuit (b) Controlled-source circuit model of Wien-bridge oscillator (c) Equivalent feedback system.

Fig. 9. (a) Equivalent representation of single-loop feedback system in Fig. 2(a). (b) Associated open-loop system consists of cascade connecting of two subsystems S_1 and S_2.

Fig. 10. Each intersection Q between the two curves Re $d_N(A,\omega) = 0$ and Im $d_N(A,\omega) = 0$ gives a solution of the determining equation $d_N(A,\omega) = 0$.

Fig. 11. The symbolic model used to derive the Nth order determining equation.

Fig. 12. Cascade connection of two systems.

Fig. 13. System decomposed into a linear and nonlinear subsystem.

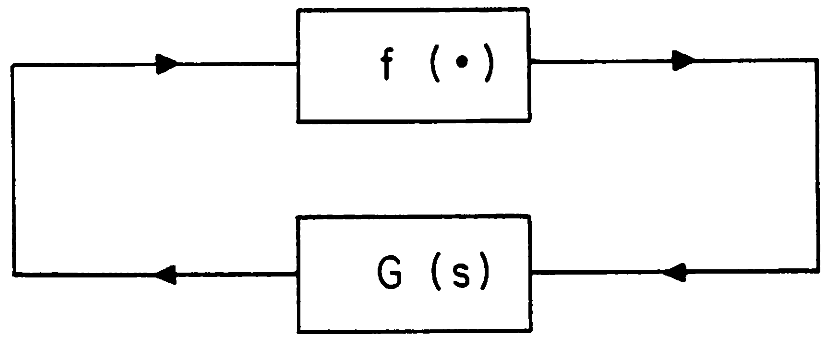
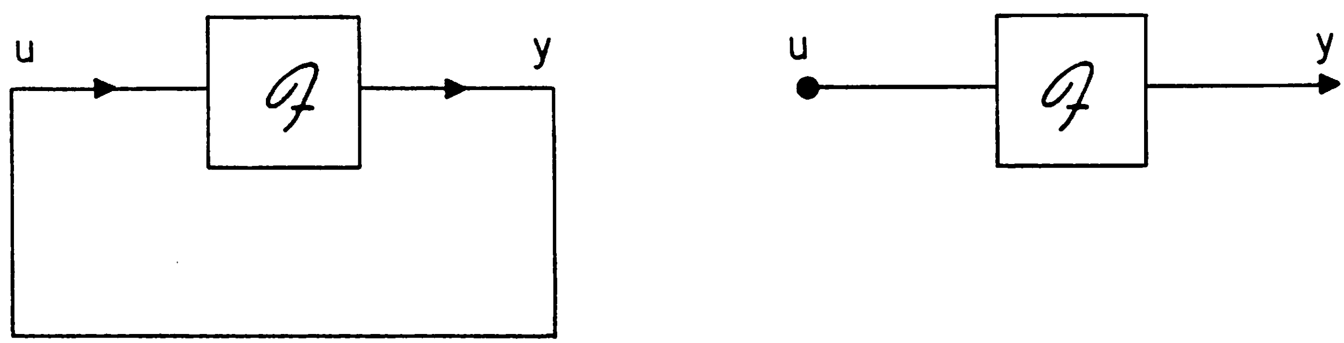
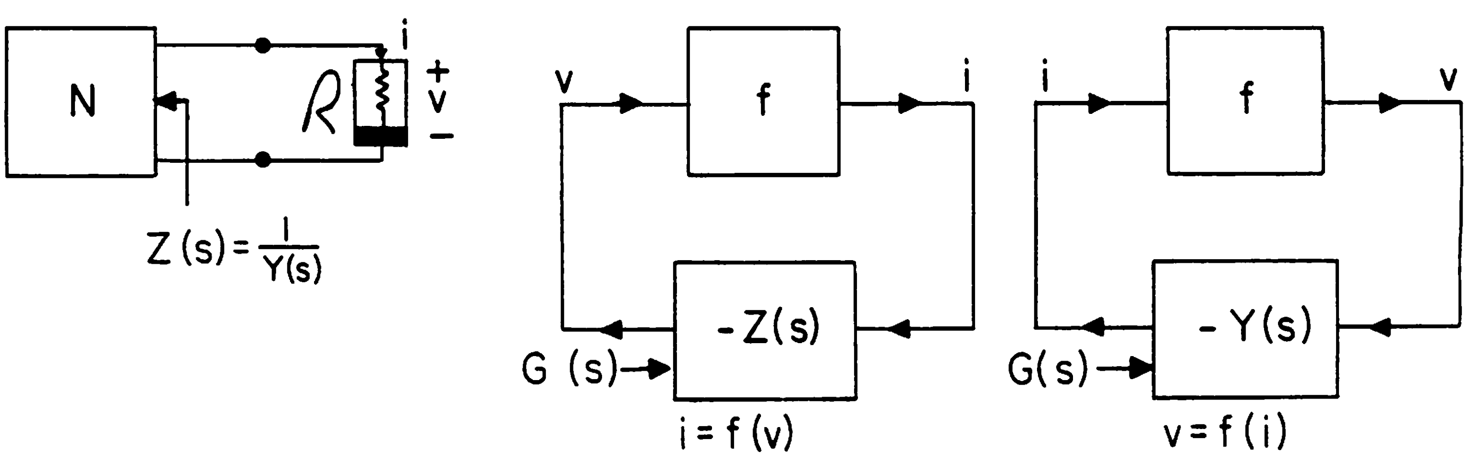


Fig. 1



(a) (b)

Fig. 2



(a) (b) (c)

Fig. 3

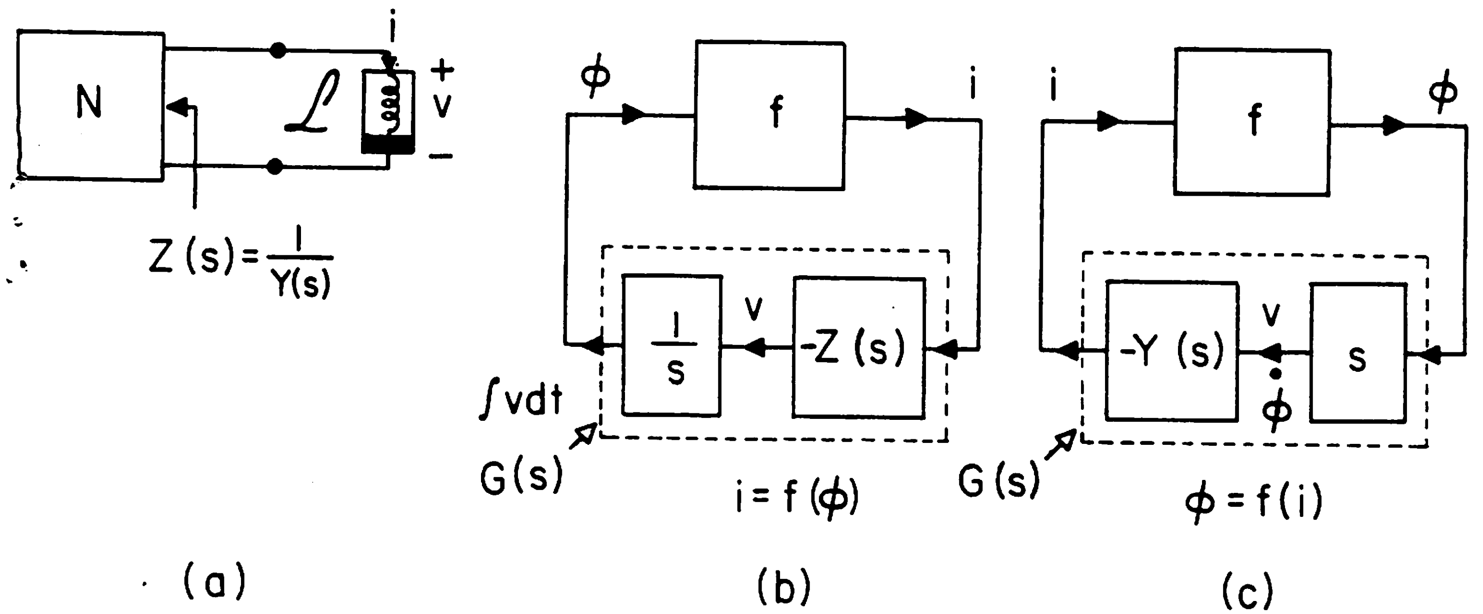
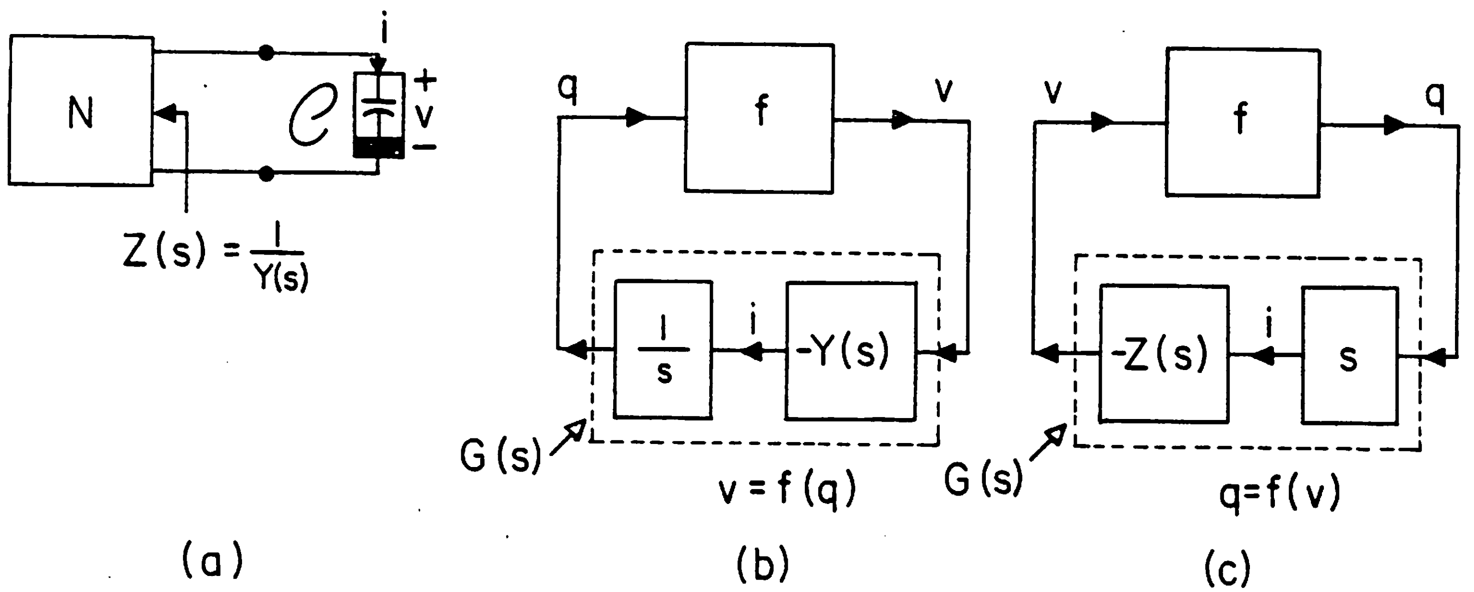

N
$\mathcal{L}$
i
+
v
−
$Z(s)=\frac{1}{Y(s)}$
(a)
ϕ
f
i
v
$\frac{1}{s}$
$-Z(s)$
$\int v\,dt$
G(s)
$i = f(\phi)$
(b)
i
f
ϕ
$-Y(s)$
v
$\dot{\phi}$
s
G(s)
$\phi = f(i)$
(c)

Fig. 4


N
$\mathcal{C}$
i
+
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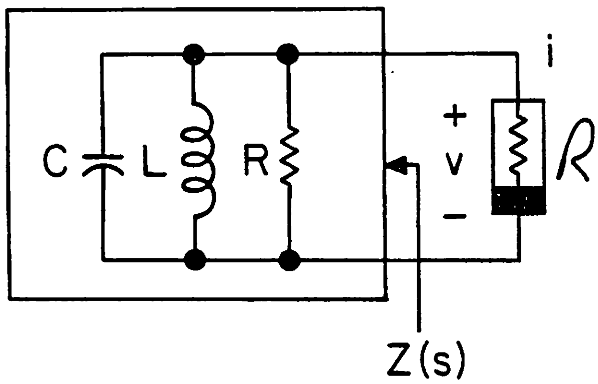
−
$Z(s) = \frac{1}{Y(s)}$
(a)
q
f
v
$\frac{1}{s}$
i
$-Y(s)$
G(s)
$v = f(q)$
(b)
v
f
q
$-Z(s)$
i
s
G(s)
$q=f(v)$
(c)

Fig. 5



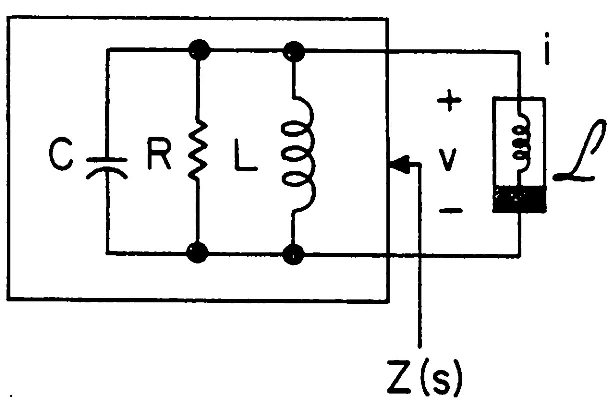
$\mathcal{R}$: $i = -\frac{1}{3} v^3$

$R = 1\,\Omega$

$L = \frac{1}{C} \triangleq -\epsilon$

$Z(s) = \frac{-\epsilon}{-\epsilon + (s + \frac{1}{s})}$

Fig. 6



$\mathcal{L}$: $i = \phi^3$

$R = 1\,\Omega$

$L = \frac{1}{C} \triangleq \epsilon$

$Z(s) = \frac{\epsilon}{\epsilon + (s + \frac{1}{s})}$

Fig. 7

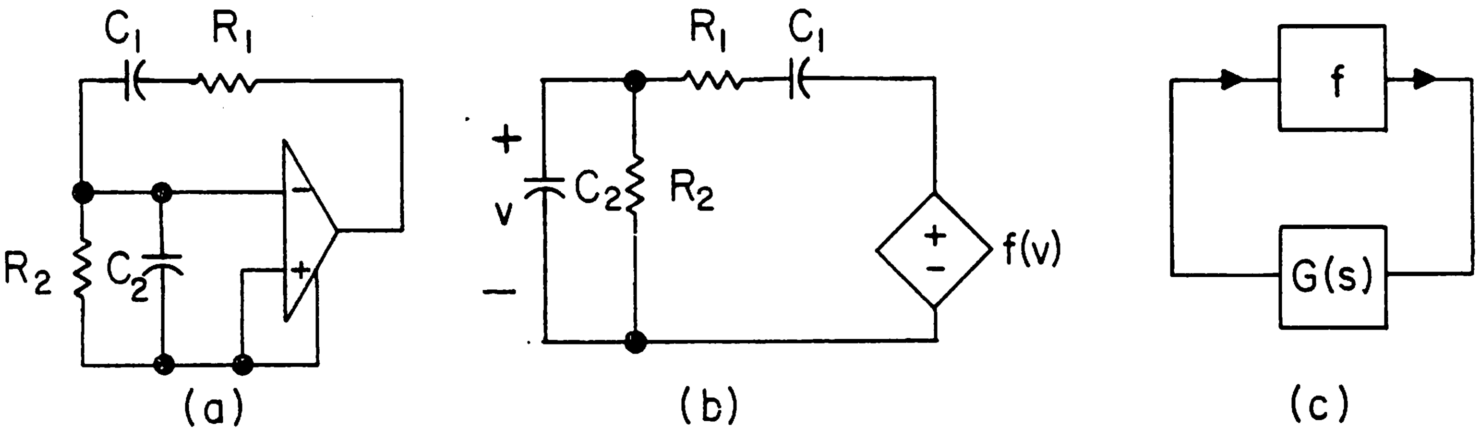


Fig. 8

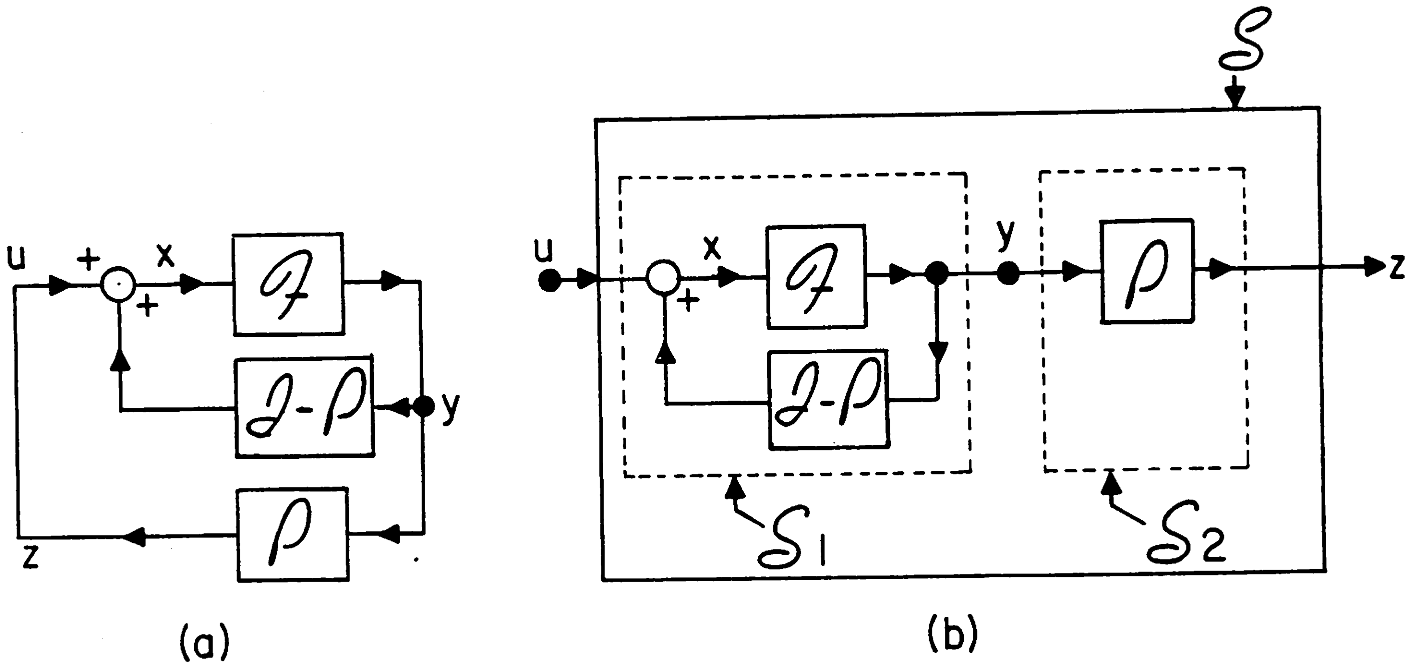

u
+
x
+
y
z
(a)
u
x
+
y
z
(b)

Fig. 9

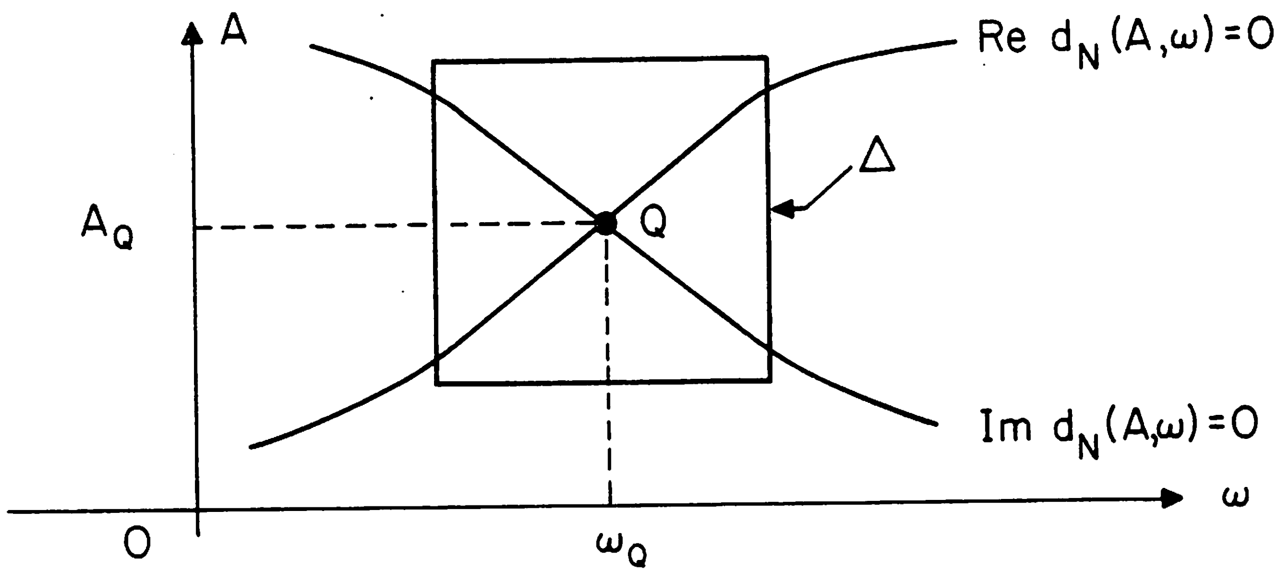
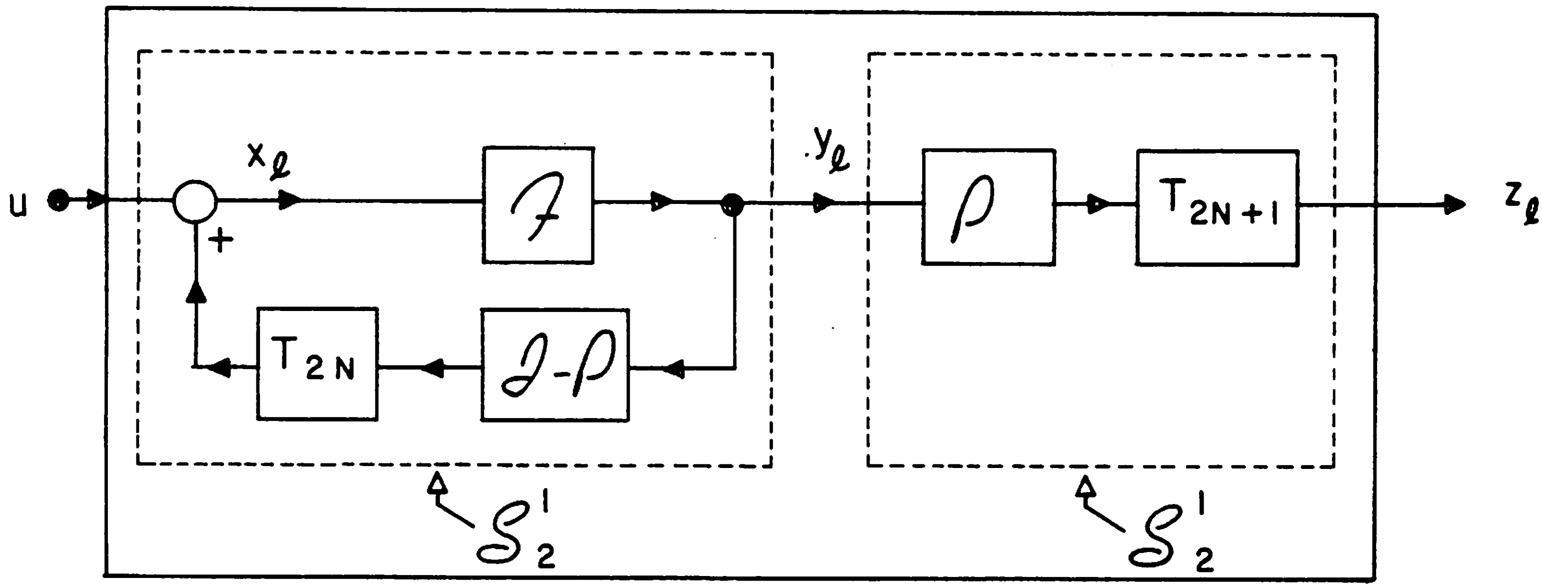

A
Re $d_N(A,\omega)=0$
Δ
A_Q
Q
Im $d_N(A,\omega)=0$
ω
O
ω_Q

Fig. 10



u
x_ℓ
+
$\mathcal{F}$
T_{2N}
$\mathcal{I}-\rho$
y_ℓ
ρ
T_{2N+1}
z_ℓ
$\mathcal{S}_2^1$
$\mathcal{S}_2^1$

Fig.11

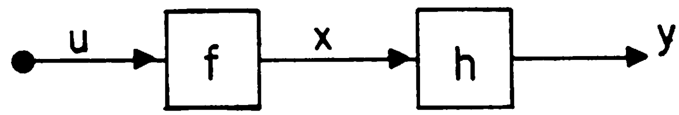


Fig.12

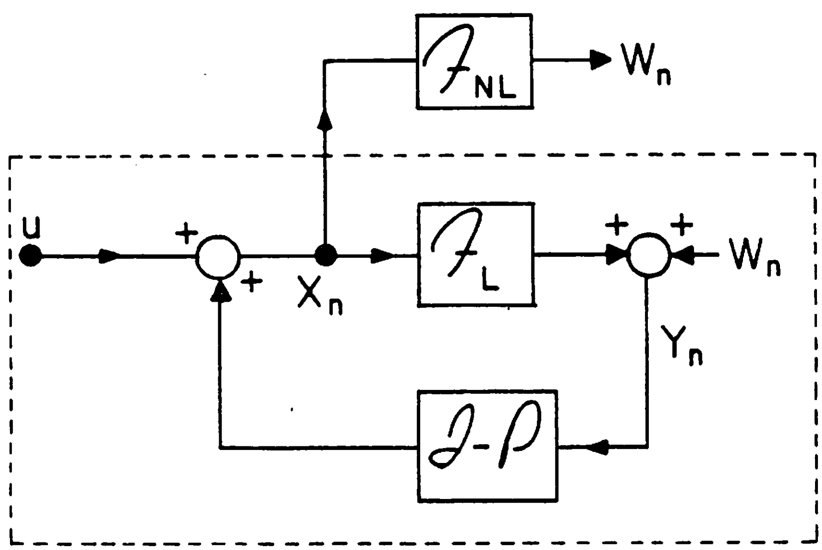


Fig.13