FUZZY RATIONAL CHOICE FUNCTIONS

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Memorandum No. UCB/ERL M81/4

27 January 1981

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ABSTRACT

An approach to choice function theory based on fuzzy set theory is suggested. A number of necessary and sufficient conditions on a fuzzy choice function to be a fuzzy rational choice function of a certain type are established.

Research sponsored by the Department of Energy Contract EC-77-S-01-5105 and the National Science Foundation Grant ENG78-23143.

1. Introduction

This paper is concerned with an approach to choice theory based on fuzzy set theory. Fuzzy rational choice theory turns out to be similar to the crisp one but also has pure fuzzy features.

In general, choice theory considers the following model (see, for example, [1] - [3] and [8]): Let A be a fixed finite set of variants. For each nonempty subset $X \subseteq A$ a nonempty subset $Y \subseteq X$ is chosen in accordance with any rule. In such manner a choice function $Y = C(X)$ is given, which associates with each $X \subseteq A$ its subset $Y \subseteq X$. There are two different methods to describe "entire choice" defined by this way. The first one points out a mechanism of choice whereby part Y is found from X. This method can be called an internal method. The second method indicates the set of all pairs (X,Y) and is called an external method.

All classical choice mechanisms are based on a "pair-dominance" idea. According to it the choice of element $y \in X$ is made as a result of comparisons of this element with any other element $x \in X$. Some given structure on the set A is utilized to make these comparisons, for example, a binary preference relation. The choice functions thus arised have very attractive "rational" properties such as "heritage" and "concordance". One of the main problems in choice theory is a description of characteristic properties of choice functions. These properties, known as "rational choice axioms", emphasize the functions which have an equivalent description in pair-dominance optimization terms.

A general framework of rational choice theory is given in section 2. Our exposition is based on the paper [3].

Section 3 is concerned with fuzzy preferences considered as fuzzy binary relations. The structure of these preferences is studied and various types of fuzzy preferences are defined.

Fuzzy rational choice functions based on fuzzy preference are defined in section 4. Main properties of these functions are established in this section.

It is shown in section 5 that various combinations of characteristic properties emphasize fuzzy choice functions based on fuzzy preferences of certain types.

2. Best variant choice

As it was mentioned above there are two alternative approaches to the theory of best variant choice. The first one is concerned with a mechanism of choice. In this paper only mechanisms based on binary relations are considered. Let R be a binary relation on the set A. We read xRy as "x is preferred to y", i.e. R is assumed to be a preference relation. A choice function based on a preference relation R is defined by

$$Y = C(X) = \{x \in X | xRy \text{ for all } y \in X\}. \quad (2.1)$$

This mechanism is founded on comparisons in pairs of variants. Such "pair-dominance" mechanisms can be regarded as abstract forms of classical optimizational mechanisms based on scalar and vector criteria. Various types of preferences, such as partial orderings, weak orderings etc., define, by (2.1), classes of rational choice functions possessed specific rational properties.

An alternative approach to choice theory considers "characteristic properties" of general choice functions and the main problem here is to describe such combinations of characteristic properties which emphasize exactly the same classes as given by certain pair-dominant mechanisms.

Following [3] let us define main characteristic properties as:

1. Heritage (H): if $X' \subseteq X$, then $C(X') \supseteq C(X) \cap X'$.
2. Strict heritage, or keeping constancy of residual choice (K):

$$\text{if } X' \subseteq X \text{ and } X' \cap C(X) \neq \emptyset, \text{ then } C(X') = C(X) \cap X'.$$

3. Concordance (C): If $X = X' \cup X''$, then $C(X) \supseteq C(X') \cap C(X'')$.

4. Independence of rejecting the outcast variants (O):

   if $C(X) \subseteq X' \subseteq X$, then $C(X') = C(X)$.

Most works concerned with rational choice theory unanimously declare these properties as clearly representing the idea of what is "better" (see [1] - [3] and [8]).

The following theorem represents main classical statements on correspondence between pair-dominance mechanisms and general choice functions.

Theorem 2.1. ([3]). For a choice function to be generable by a choice mechanism (2.1) of 1) an arbitrary preference relation, 2) a weak ordering, and 3) a quasi-transitive ordering it is necessary and sufficient that it satisfies a condition 1) H&C, 2) K, and 3) H&C&O.

Below the statements of this theorem will be extended on fuzzy choice function theory.

3. Fuzzy preferences

Recall that a fuzzy binary relation R on a set A is a fuzzy set with universe AxA and defined by its membership function $R(x,y)$ with a range $[0;1]$.

Definition 3.1. Fuzzy relation R is said to be

1) reflecive if $R(x,x) = 1$ for all $x \in X$;
2) antireflexive if $R(x,x) = 0$ for all $x \in X$;
3) symmetric if $R(x,y) = R(y,x)$ for all $x, y \in X$;
4) antisymmetric if $R(x,y) > 0$ implies $R(y,z) = 0$ for all $x \neq y$;
5) complete if $R(x,y) = 0$ implies $R(y,x) > 0$ for all $x, y \in X$;
6) acyclic if $R(x_i,x_{i+1}) > 0$ for $i = 1,\dots,k-1$ implies $R(x_k,x_1) = 0$ for any sequence $x_1,\dots,x_k$.

7) transitive if $R(x,y) > 0$ and $R(y,z) > 0$ implies $R(x,z) > 0$ for all $x, y, z \in X$.

Remark. It should be noted that our definition of transitivity is different from classical ones (see [4]). Fuzzy set theory admits various possible notions of transitivity (see [5]). Transitivity defined by 7) can be considered as the weakest one in some precise sense (see [6]).

A notion of a strict preference introduced in the following definition plays a significant role in choice theory.

Definition 3.2. Let R be a fuzzy preference. A fuzzy binary relation P_R defined by

$$P_R(x,y) = \begin{cases} R(x,y), & \text{if } R(y,x) = 0, \\ 0 & \text{otherwise,} \end{cases}$$

is said to be a *strict preference*.

Remark. In choice theory aP_Rb iff aRb and $\neg bRa$, i.e. $P_R = R \cap \overline{R^{-1}}$ where $\overline{R^{-1}}$ is a complement of R^{-1} considered as a subset in A x A. Fuzzy set theory admits various possible definitions of a complement (see [7]). Definition 3.2 is based on an *intuitionistic complement* [7] for which we have $P_R = R \cap \overline{R^{-1}}$.

Definition 3.3. A fuzzy preference R is said to be

1) *partial ordering* if it is a reflexive, antisymmetric and transitive fuzzy relation;
2) *chain* if it is a complete partial ordering;
3) *ordering* if it is a reflexive, complete and transitive fuzzy relation;
4) *quasi-transitive preference* if it is a reflexive, complete relation and P_R is a transitive relation.

The following structural properties of fuzzy preference will be used in the next sections.

Proposition 3.1. Let R be an ordering. Then: 1) P_R is a transitive relation, i.e. R is a quasi-transitive preference, 2) $P_R(x,y) > 0$ and $R(y,z) > 0$ imply $P_R(x,z) > 0$, and $R(x,y) > 0$ and $P_R(y,z) > 0$ imply $P_R(x,z) > 0$.

We omit proofs of these statements which are quite similar to crisp ones (see [8]).

4. Fuzzy pair-dominant choice functions

In this section fuzzy choice functions based on fuzzy preferences are defined and their characteristic properties are established.

The following definition is an immediate extension of crisp definition (2.1) and is in accordance with a general approach to a fuzzy decision-making developed in [9]. Let A be a universe of variants and R - a fuzzy preference on A.

Definition 4.1. A *pair-dominant choice function* based on a fuzzy preference R is a mapping which, to each fuzzy set X, assigns a fuzzy subset C_X^R with a membership function

$$C_X^R(x) = \bigwedge_{y \in X} \{R(x,y) \wedge X(x)\} \tag{4.1}$$

(We denote $y \in X$ iff $X(y) > 0$).

One can compare (4.1) with the definition of upper bound in [4].

Note, that C_X^R really is a subset of X. If R and X are crisp sets then definition 4.1 provides the same result as (2.1).

The following lemmas establish general properties of fuzzy pair-cominant mechanisms.

Lemma 4.1. $C_X^R \subseteq C_{carX}^R$, where $carX = \{x \in A : X(x) > 0\}$ is a carrier of X.

Proof follows immediately from (4.1).

Lemma 4.2. C_X^R fulfills the heritage property (H): if $X' \subseteq X$, then $C_{X'}^R \supseteq C_X^R \cap X'$.

Proof. $C_{X'}^R(x) = \bigwedge_{y \in X'} \{R(x,y) \wedge X(x)\}$

$\geq \bigwedge_{y \in X} \{R(x,y) \wedge X'(x) \wedge X(x)\} = C_X^R(x) \wedge X'(x)$, Q.E.D.

Lemma 4.3. C_X^R fulfills the concordance property (C): $C_{X \cup Y}^R \supseteq C_X^R \cap C_Y^R$.

Proof.

$$C_X^R(x) \wedge C_Y^R(x) = [\bigwedge_{u \in X} \{R(x,u) \wedge X(x)\}] \wedge [\bigwedge_{v \in Y} \{R(x,v) \wedge Y(x)\}]$$

$$= \bigwedge_{y \in X \cup Y} \{R(x,y) \wedge X(x) \wedge Y(x)\} \leq \bigwedge_{y \in X \cup Y} \{R(x,y) \wedge (X(x) \vee Y(x))\} = C_{X \cup Y}^R(x),$$

Q.E.D.

Two last lemmas show that a fuzzy pair-dominant choice function fulfills the same properties of heritage and concordance as a crisp one (see theorem 2.1,1)). But, in addition to these properties, it fulfills also very important property (4.2) which has no crisp analogs. The role of this property will be clear from section 5. Here we only note that (4.2) has a quite clear interpretation: if an element x is chosen from a fuzzy set X then it should be chosen from carX and a degree of its belongness to C_X^R does not exceed of that to $C_{\text{car}X}^R$.

The following lemma gives some properties of fuzzy pair-dominant choice mechanisms which follow from main properties of fuzzy preferences.

Lemma 4.4. 1) Let R be a reflexive relation. Then

$$C_{\{x\}}^R = \{x\} \tag{4.3}$$

where {x} is a crisp singleton;

2) Let R be a reflexive and complete relation. Then

$$C^R_{x,y}(u) = \begin{cases} R(x,y) \text{ if } u = x, \\ R(y,x) \text{ if } u = y, \\ 0 \text{ otherwise.} \end{cases}$$

3) Let R be a reflexive, complete relation and P_R is an acyclic relation. Then

$$C^R_X \neq \emptyset \text{ for any } X \neq \emptyset. \tag{4.4}$$

<u>Proof</u>. 1) and 2) follow immediately from (4.1). Note only that they mean nonvoideness of a choice from "small" sets (with cardinality ≤ 2). 3) The proof is quite similar to the crisp one (see [8]).

<u>Corollary</u>. $C^R_X \neq \emptyset$ for $X \neq \emptyset$ if R is an ordering or a quasi-transitive preference.

<u>Proof</u>. Any ordering or quasi-transitive preference is a reflexive complete and acyclic relation.

The main properties of fuzzy pair-dominant choice functions based on particular types of fuzzy preferences are given below.

<u>Lemma 4.5</u>. Let R be a fuzzy ordering. Then

$$\text{if } X' \subseteq X \text{ and } C^R_X \cap X' \neq \emptyset, \text{ then } \operatorname{car}C^R_{X'} = \operatorname{car}C^R_X \cap \operatorname{car}X'. \tag{4.5}$$

<u>Proof</u>. By lemma 4.2 it is sufficient to prove that

$$\operatorname{car}C^R_{X'} \subseteq \operatorname{car}C^R_X \cap \operatorname{car}X'.$$

Let $x \in \operatorname{car}C^R_{X'}$ and $x \notin \operatorname{car}C^R_X \cap \operatorname{car}X'$, i.e. $x \notin \operatorname{car}C^R_X$, because $x \in \operatorname{car}X'$. Since $x \notin \operatorname{car}C^R_X$ there is y such that $R(x,y) = 0$, i.e. $P_R(y,x) > 0$. On the other hand $x \in \operatorname{car}C^R_{X'}$, which implies $y \in \operatorname{car}X \setminus \operatorname{car}X'$. Since

$C_X^R \cap X' \neq \emptyset$ there is z such that $z \in \mathrm{car}X'$ and $z \in \mathrm{car}C_X^R$. Hence, $R(x,z) > 0$ because $x \in \mathrm{car}C_{X'}^R$ and $z \in \mathrm{car}X'$. By proposition 3.1,2) we have $P_R(y,z) > 0$ which implies $R(z,y) = 0$. But $z \in \mathrm{car}C_X^R$ which implies $R(z,y) > 0$. This contradiction completes the proof.

Lemma 4.6. Let R be a fuzzy quasi-transitive preference. Then

$$\text{if } C_X^R \subseteq X' \subseteq X, \text{ then } \mathrm{car}C_{X'}^R = \mathrm{car}C_X^R . \tag{4.6}$$

Proof. By lemma 4.2 it is sufficient to prove that $\mathrm{car}C_{X'}^R \subseteq \mathrm{car}C_X^R$. Let $x \in \mathrm{car}C_{X'}^R$ and $x \notin \mathrm{car}C_X^R$. Since $x \notin \mathrm{car}C_X^R$ there is $y_1 \in X$ such that $P_R(y_1,x) > 0$. We have $y_1 \notin \mathrm{car}C_X^R$ because $x \in \mathrm{car}C_{X'}^R$ which follows $R(x,y) > 0$ for all $y \in X'$. Since $y_1 \in \mathrm{car}C_X^R$ there is y_2 such that $P_R(y_2,y_1) > 0$ which implies $P_R(y_2,x) > 0$, by transitivity of P_R. If $y_2 \notin C_X^R$ then there is y_3 different from y_1 and y_2 such that $P(y_3,x) > 0$ and so on. By finiteness of A we find y such that $P_R(y,x) > 0$ and $y \in \mathrm{car}C_X^R$. But $x \in \mathrm{car}C_{X'}^R$ and $y \in \mathrm{car}X'$, i.e. $R(x,y) > 0$ which contradicts $P_R(y,x) > 0$. Q.E.D.

Note in conclusion that any fuzzy pair-dominant choice function fulfills properties (H) and (C). On the other hand, properties (4.5) and (4.6) are weaker then (K) and (O), respectively, although for crisp sets and preferences they coincide with them. Simple examples show that there are fuzzy orderings and quasi-transitive preferences which do not fulfill (K) and (O). As it is mentioned in [10] it is stipulated by the pair-dominance choice function structure (4.1), "since this functions consider not only the ties between alternatives but also their "power". Having excluded certain alternatives from consideration, we have naturally increased the degree of membership to the fuzzy set C_X^R for other alternatives."

5. Fuzzy Rational Choice

In this section characteristic properties of fuzzy choice functions are defined. Various conjunctions of these properties define certain classes of choice functions based on pair-dominant mechanisms of choice.

Definition 5.1. The following properties of fuzzy choice functions are said to be *characteristic properties*:

1. *Boundedness* (B):

   $C_X \subseteq C_{carX}$;

2. *Heritage* (H):

   if $X' \subseteq X$, then $C_{X'} \supseteq C_X \cap X'$'

3. *Concordance* (C);

   if $X = X' \cup X''$, then $C_X \supseteq C_{X'} \cap C_{X''}$;

4. *Fuzzy strict heritage* (FK);

   if $X' \subseteq X$ and $C_X \cap X' \neq \emptyset$, then $carC_{X'} = carC_X \cap carX'$;

5. *Fuzzy independence of rejecting the outcast variants* (FO):

   if $C_X \subseteq X' \subseteq X$, then $carC_{X'} = carC_X$;

6. *Singleton law* (S):

   $C_{\{x\}} = \{x\}$;

7. *Nonvoideness* (N):

   $C_X \neq \emptyset$ if $X \neq \emptyset$.

Note that properties (B), (K), (O), (S) and (N) are the same as (4.2), (4.5), (4.6), (4.3) and (4.4), respectively. We consider properties (B), (H) and (C) as defined a "rational" fuzzy choice.

Lemma 5.1. Conjunction (B) & (H) implies

$$C_X = C_{carX} \cap X. \tag{5.1}$$

Proof. We have $C_X \supseteq C_{carX} \cap X$, by (H), and $C_X \subseteq C_{carX} \cap X$, by (B), which implies (5.1).

Theorem 5.1. A fuzzy choice function C_X is a fuzzy pair-dominant choice function C_X^R for some fuzzy preference R iff C_X fulfills properties (B), (H) and (C).

Proof. Necessity follows from lemmas 4.1-4.3. To prove sufficiency let us define R by

$$R(x,y) = C_{\{x,y\}}(x) \ .$$

By (H) and (5.1), $X \cap \{x,y\} \subseteq X$ implies

$$C_X \cap X \cap \{x,y\} \subseteq C_{X\cap\{x,y\}} = C_{\{x,y\}} \cap X \cap \{x,y\}$$

or

$$C_X \cap \{x,y\} \subseteq C_{\{x,y\}} \cap X.$$

Hence,

$$C_X(x) \leq C_{\{x,y\}}(x) \wedge X(x)$$

which implies

$$C_X(x) \leq \bigwedge_{y\in X} C_{\{x,y\}} \wedge X(x) = \bigwedge_{y\in X} R(x,y) \wedge X(x) = C_X^R(x).$$

On the other hand,

$$X = \bigcup_{y\in X} (X\cap\{x,y\})$$

which implies, by (5.1) and (C),

$$C_X \supseteq \bigcap_{y\in X} C_{X\cap\{x,y\}} = \bigcap_{y\in X} (C_{\{x,y\}}\cap\{x,y\}\cap X)$$

or

$$C_X(x) \geq \bigwedge_{y \in X} (C_{\{x,y\}}(x) \wedge X(x)) = C_X^R(x), \qquad \text{Q.E.D.}$$

One can compare the statement of theorem 5.1 with statement 1) of theorem 2.1.

By theorem 5.1 the mapping $F : R \to C_X^R$ is a surjection of the set of all fuzzy preferences onto the set of all fuzzy choice functions satisfied properties (B), (H) and (C). This mapping is not a bijection because there are $R_1 \neq R_2$ such that $C_X^{R_1} = C_X^{R_2}$. Let us define $R_1 \sim R_2$ iff $C_X^{R_1} = C_X^{R_2}$. Then $\sim$ is an equivalence relation on the set of all fuzzy preferences. Each fuzzy rational choice function (i.e. that satisfied (B), (H) and (C)) is an image of some class of the relation $\sim$ under the mapping F. The following theorem describes all fuzzy preferences $R \in F^{-1}(C_X)$ for a given C_X.

Theorem 5.2. Let us define a fuzzy preference R_c for any given R by

$$R_c(x,y) = R(x,y) \wedge R(x,x).$$

Then $R \sim R_c$ and $R' \sim R''$ iff $R'_c = R''_c$.

Proof. We have

$$C_X^{R_c}(x) = \bigwedge_{y \in X} \{R_c(x,y) \wedge X(x)\} = \bigwedge_{y \in X} \{R(x,x) \wedge R(x,y) \wedge X(x)\} =$$

$$\bigwedge_{y \in X} \{R(x,y) \wedge X(x)\} = C_X^R(x) \text{ , i.e. } R \sim R_c.$$

Let $R'_c = R''_c$. Then $R' \sim R'_c = R''_c \sim R''$ which implies $R' \sim R''$, by transitivity of $\sim$.

Let $R' \sim R''$, i.e. $C_X^{R''} = C_X^{R''}$. Then

$$R'_c(x,y) = R'(x,y) \wedge R'(x,x) = C_{\{x,y\}}^{R'}(x) = C_{\{x,y\}}^{R''}(x) = R''(x,y) \wedge R''(x,x)$$

$$= R''_c(x,y) \qquad \text{Q.E.D.}$$

One can consider R_c as a "cannonical representative" in the class of $\sim$ which contains R. These cannonical representatives are completely characterized by the property $R_c(x,y) \leq R_c(x,x)$. Note that the latter condition always fulfills for reflexive preferences. Hence, classes of the equivalence relation $\sim$ are singletones for reflexive preferences. We obtain the following

<u>Corollary</u>. The mapping $F : R \to C_X^R$ is a bijection of the set of all fuzzy reflexive relations of preference onto the set of all fuzzy rational choice functions satisfied the property (S).

In general, it is possible for C_X^R to be an empty set for some non-empty X. It was mentioned in section 4 that acyclicity of R implies non-voideness of a choice from non-empty sets. The converse is also true.

<u>Theorem 5.3</u>. Let R be a reflexive complete fuzzy preference. Then C_X^R is a non-empty fuzzy set for all non-empty X iff P_R is an acyclic fuzzy relation.

<u>Proof</u>. The necessity follows from lemma 4.4. Let $C_X^R \neq \emptyset$ for all $X \neq \emptyset$. Suppoe that P_R is not an acyclic relation. Then there is a sequence $x_1,\ldots,x_n$ such that

$$P_R(x_i,x_{i+1}) > 0 \text{ for } i = 1,2,\ldots,n-1, \text{ and}$$
$$P_R(x_n,x_1) > 0. \qquad (5.2)$$

By definition (4.1), we have

$$C^R_{\{x_1,\ldots,x_n\}}(x) = \bigwedge_{i=1}^{n} R(x,x_i) \text{ for } x \in \{x_1,\ldots,x_n\}.$$

We have $R(x_{i+1},x_i) = 0$ for $i = 1,2,\ldots,n-1$ and $R(x_1,x_n) = 0$, by (5.2). Hence, $C^R_{\{x_1,\ldots x_n\}} = \emptyset$. This contradiction completes the proof.

Now we will consider conditions defining the class of fuzzy choice functions which have an equivalent description in terms of a pair-dominant mechanism based on a quisi-transitive fuzzy preference.

Theorem 5.4. A fuzzy choice function C_X is a pair-dominant choice function C_X^R based on a fuzzy quisi-transitive preference R iff it satisfies conditions (B), (H), (C), (FO), (N) and (S).

Proof. The necessity follows from lemmas 4.1 - 4.4 and 4.6. Let C_X fulfills the conditions listed in the theorem. By theorem 5.1 we have $C_X = C_X^R$ for some R. By (S) and (N), R is a reflexive complete relation. Now it is sufficient to show that P_R is a transitive relation, i.e. that if $P_R(x,y) > 0$ and $P_R(y,z) > 0$ then $P_R(x,z) > 0$. Let $X = \{x,y,z\}$. Then

$$C_X^R(t) = R(t,x) \wedge R(t,y) \wedge R(t,z) \text{ for } t \in X.$$

Hence

$$C_X^R(x) = R(x,y) \wedge R(x,z),$$

$$C_X^R(y) = 0, \text{ since } P_R(x,y) > 0, \text{ and}$$

$$C_X^R(z) = 0, \text{ since } P_R(y,z) > 0.$$

By (N), $C_X^R(x) > 0$ which implies $R(x,z) > 0$ and $\text{car}C_X^R = \{x\}$. Let now $X' = \{x,z\}$. Then

$$C_{X'}^R(t) = R(t,x) \wedge R(t,z) \text{ for } t \in \{x,z\}.$$

Hence, $C_{X'}^R(x) = R(x,z)$ and $C_{X'}^R(z) = R(z,x)$. Now, by (FO), $C_X^R \subseteq X' \subseteq X$ implies $\text{car}C_{X'}^R = \text{car}C_X^R = \{x\}$ which implies $R(z,x) = 0$. Hence $P_R(x,z) > 0$,

Q.E.D.

As it follows from theorem 2.1,2) in the crisp case the property (K) is a very strong one. The power of this property also shows itself very clearly in the fuzzy case. Let C_X fulfill conditions (K) and (S). Let X be a crisp set and $x \in C_X$. Then $\{x\} \subseteq X$ and $C_X \cap x \neq \emptyset$. By (K) we have $C_{\{x\}} = C_X \cap \{x\}$, or $C_X(x) = C_{\{x\}}(x) = 1$, by (S), i.e. C_X is a crisp set. From (K) it also immediately follows that

$$C_X = C_{\mathrm{car}X} \cap X. \tag{5.3}$$

Hence, C_X is, essentially, a crisp choice function and coincides with C_X^R for some crisp ordering R, by theorem 2.1 and (5.3). This is a reason, why even fuzzy chains (linear orderings) do not fulfill (K).

On the other hand it was shown in section 4 (lemma 4.5) that fuzzy orderings fulfill the condition (FK) which coincides with (K) for crisp sets and orderings.

We complete this section by the following

Theorem 5.5. A fuzzy choice function C_X is a pair-dominant choice function C_X^R based on a fuzzy ordering R iff it satisfies conditions (B), (H), (C), (FK), (N) and (S).

Proof. The necessity follows from lemmas 4.1 - 4.5. By (B), (H), (C), (N) and (X) we have $C_X = C_X^R$ where R is a fuzzy reflexive complete relation. Let us show that R is a transitive relation, i.e. that $R(x,y) > 0$ and $R(y,z) > 0$ imply $R(x,z) > 0$. Let $X = \{x,y,z\}$. Then

$$C_X^R(x) = R(x,y) \wedge R(x,z)$$

$$C_X^R(y) = R(y,x) \wedge R(y,z), \text{ and}$$

$$C_X^R(z) = R(z,x) \wedge R(z,y), \text{ by reflexivity of R.}$$

Let now $X' = \{x,y\}$. Then

$$C^R_{X'}(x) = R(x,y) \text{ and } C^R_{X'}(y) = R(y,x).$$

If $C^R_X \cap X' = \emptyset$, i.e. $\text{car}C^R_X = \{z\}$, then $R(x,z) = 0$ and $R(y,x) = 0$. By (N), $C^R_X(z) > 0$, which implies $R(z,x) > 0$ and $R(z,y) > 0$. Let $X'' = \{y,z\}$. We have $C^R_{X''}(y) = R(y,z) > 0$ and $C^R_{X''}(z) = R(z,y) > 0$. Since $C^R_X \cap X'' \neq \emptyset$, then, by (FK), $\text{car}C^R_{X''} = \text{car}C^R_X \cap \{y,z\}$. But $\text{car}C^R_{X''} = \{y,z\}$ which contradicts $\text{car}C^R_X = \{z\}$. Hence, $C^R_X \cap X' \neq \emptyset$. Then, by (FK), $\text{car}C^R_{X'} = \text{car}C^R_X \cap X'$. Since $R(x,y) > 0$, then $x \in \text{car}C^R_{X'}$. Hence, $x \in \text{car}C^R_X$ which implies $R(x,z) > 0$. Q.E.D.

6. Conclusion

Fuzzy rational choice theory described above has some characteristic features which distinguish it from the crisp one. The difference is mainly stipulated by the purely fuzzy property (B). For example, there are a lot of "pathological" fuzzy choice functions which fulfill (H) and (C) and do not fulfill (B).

From fuzzy set theory point of view there is a significant difference between classical characteristic properties (H), (C), (O) and (K). The properties (H) and (C) play the same role in both fuzzy and crisp cases. It seems that various transitivity properties which play a great role in classical choice theory, are not so important in fuzzy choice theory (cf. [11] and [12] where some particular choice mechanisms are studied in details).

Only pair-dominant mechanisms based on fuzzy preferences were considered in this paper. It is an interesting problem to study different mechanisms of fuzzy choice based, for example, on fuzzy utility functions [10] and fuzzy hyperrelations [3] in the context of general fuzzy choice theory. We leave this investigation for further publications.

References:

1. Sen, A.K. (1970) "Collective Choice and Social Welfare," San Francisco: Holden-Day.
2. Plott, C.R. (1976) "Axiomatic social choice theory: an overview and interpretation," Amer. J. Polit. Science, v. 20, 511-596.
3. Aizerman, M.A., Malishevski, A.V. (1981) "General theory of best variant choice: some aspects," (to appear in "Trans. on Aut. Control, Bellman Special Issue").
4. Zadeh, L.A. (1971) "Similarity relations and fuzzy orderings," Inf. Sci., 3, 177-200.
5. Goguen, J.A. (1967) "L-fuzzy sets," J. Math. Anal. Appl., 18, 145-174.
6. Ovchinnikov, S.V. (1980) "Transitivity in fuzzy set theory" (unpublished).
7. Ovchinnikov, S.V., "General negations in fuzzy set theory," (to appear in "J. Math. Anal. Appl.").
8. Mirkin, B.G. (1979) "Group choice," New York : Winston.
9. Bellman, R.E., Zadeh, L.A. (1970) "Decision-making in a fuzzy environment," Management Sci., 17, 8141-8164.
10. Ovchinnikov, S.V., "Structure of fuzzy binary relations," (to appear in "Fuzzy Sets and Systems").
11. Orlovsky, S.A. (1978) "Decision-making with a fuzzy preference relation," Fuzzy Sets and Systems, 1, 155-167.
12. Orlovsky, S.A. (1980) "On formalization of a general fuzzy mathematical program problem," Fuzzy Sets and Systems, 3, 311-321.