THE (p+q)-PORT TRANSFORMER

by

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ABSTRACT

Two approaches for realizing a *(p+q)-port transformer* using operational amplifiers are presented. Unlike iron-core transformers, our realization is valid from *dc* to a relatively high frequency limited only by the op amp's frequency response. The stability and limitation of the two approaches are analyzed and compared. Examples are given to illustrate two unique and indispensable applications of the (p+q)-port transformer: *Synthesis of nonlinear n-ports* and *nonlinear programming*.

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1. Introduction

A (p+q)-port transformer is a multiport resistive element described by the following constitutive relation:*

$$\begin{bmatrix} \mathbf{v}_a \\ \mathbf{i}_b \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{K}^T \\ -\mathbf{K} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{i}_a \\ \mathbf{v}_b \end{bmatrix} \qquad (1)$$

where

$$\mathbf{v}_a = \left[v_{a1}, v_{a2}, \ldots, v_{ap} \right]^T \qquad \mathbf{i}_a = \left[i_{a1}, i_{a2}, \ldots, i_{ap} \right]^T$$

are p-vectors,

$$\mathbf{v}_b = \left[v_{b1}, v_{b2}, \ldots, v_{bq} \right]^T \qquad \mathbf{i}_b = \left[i_{b1}, i_{b2}, \ldots, i_{bq} \right]^T$$

are q-vectors, and

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1p} \\ k_{21} & k_{22} & \cdots & k_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ k_{q1} & k_{q2} & \cdots & k_{qp} \end{bmatrix} \qquad (2)$$

is a (p×q) real matrix, called the turns-ratio matrix.

Using traditional two-port ideal transformers, a (p+q)-port transformer can be expressed as a combination of p×q ideal transformers, as depicted in Fig.1 . For simplicity, Fig.1 can be symbolized as in Fig.2 and, using vector's notation, can be simplified further as shown in Fig.3 .

The (p+q)-port transformer is a very interesting and important multiport element in circuit theory. It has attractive features by itself, and also it is an indispensable building block in many circuit synthesis and simulation problems. Here we list some interesting facts related to the (p+q)-port transformer:

(1) The (p+q)-port transformer is the *only* multiport element which is both *reciprocal* and *anti-recipocal*. In addition, it is non-energic,lossless and passive.

* In this paper, unless otherwise stated, we use bold lower-case letters to denote vectors, and bold capital letters to denote matrices.

(2) The *connection n-port* [1] is a special case of a (p+q)-port transformer.

(3) Every linear anti-reciprocal n-port resistor can be synthesized by gyrators and a (p+q)-port transformer.

(4) Every ***reciprocal n-port resistor*** represented by a continuous n-dimensional piecewise-linear function can be realized by using only ***two-terminal*** piecewise-linear resistors and a (p+q)-port transformer[2]. Besides, every nonlinear ***lossless*** n-port capacitor and inductor can be realized by a nonlinear ***reciprocal n-port resistor*** and mutators[3]. Therefore the (p+q)-port transformer is also important for realizing n-port capacitors and inductors.

(5) A large class of nonlinear programming problems can be simulated by using (p+q)-port transformers in addition to voltage sources, current sources,resistors and ideal diodes[4,5].

As depicted in Fig.1, a (p+q)-port transformer can be synthesized by combining "p×q" ordinary two-port transformers together. But, to physically build a (p+q)-port transformer using iron cores would be very cumbersome and the device would also be very bulky. It would require very meticulous winding of the coils on the transformer core and very precise physical positioning of the coils to ensure the proper turns ratio and coupling. In this paper, we developed two approaches using operational amplifiers for realizing a (p+q)-port transformer. Besides overcoming the above objections, our realizations have several additional advantages:

(1) They are compatible with modern integrated-circuit technology and can therefore be fabricated in module form.

(2) In contrast to the iron-core transformer, they operate not only at ac , but also at dc. This is a very useful property. As will be seen in *Section 6.2*, without this property, the (p+q)-port transformer can not be used to solve nonlinear programming problems.

(3) The turns-ratio matrix **K** can be easily modified.

However, since operational amplifiers are used, other problems such as detortion and oscillation could occur and must be suppressed through careful design. In addition, the operational amplifiers will restrict the range of operating frequency, signal level, load and source properties, etc.

In *Section 2* we present a "direct approach" for realizing a (p+q)-port transformer. This approach realizes (1) directly. In *Section 3* we present another approach--"scattering-matrix approach", which, instead of realizing (1) directly, realizes the scattering matrix corresponding to (1). Both approaches apply to grounded (p+q)-port transformers. If a floating signal is desired, the grounded-to-floating converting technique presented in *Section 4* may be used. In *Section 5* a comparison between the two approaches is given. Using the concept of "absolute stability"[6,7], extensive experimental and computer stability analysis have been made and the results are summarized. In the last section we present two application examples: a piecewise-linear n-port resistor synthesis problem and a nonlinear programming problem. The two essentially different examples show the wide variety of applications of (p+q)-port transformers.

2. Circuit realization: Direct approach

For simplicity, we begin by considering a traditional (1+1)-port transformer (Fig.4) whose constitutive relations are

$$\begin{bmatrix} v_a \\ i_b \end{bmatrix} = \begin{bmatrix} 0 & k \\ -k & 0 \end{bmatrix} \begin{bmatrix} i_a \\ v_b \end{bmatrix} \tag{3}$$

where k is a real number.

The circuit in Fig.5 realizes this constitutive relations for any $k \geq 0$. To analyse the function of this circuit, we divide it into two parts. The lower part of this circuit (below the broken line) fulfills the voltage constraint $v_a = kv_b$ while the upper part (above the broken line) fulfills the current constraint $i_b = -ki_a$. To see this, use the *ideal model* (Fig.6) for the op amps. In Fig.6, the left part is a nullator described by

$$v = 0\,, \qquad i = 0$$

The right part of Fig.6 is a norator, whose constitutive relation consists of every points in the v-i plane.

In the lower part of Fig.5, we have

$$v_6 = v_a\,, \quad v_8 = v_b\,, \quad v_9 = v_{10}$$

$$v_9 = \frac{v_6}{k+1}, \quad v_{10} = \frac{kv_8}{k+1}$$

Therefore,

$$v_a = kv_b$$

In the upper part, we have

$$v_2 = v_a - Ri_a\,, \quad v_3 = v_7\,, \quad v_3 = \frac{v_2+v_4}{2}$$

$$v_4 = v_b - \frac{Ri_b}{k}, \quad v_7 = \frac{v_a+v_b}{2}$$

which follows that

$$i_b = -ki_a$$

To realize a (1+1)-port transformer with *negative* values of k, we need only a slight modification as shown in Fig.7 . The analysis of the function of this circuit is similar to that of Fig.5 .

To realize a (1+q)-port transformer with the following constitutive relation

$$\begin{bmatrix} v_a \\ i_{b1} \\ i_{b2} \\ \cdot \\ \cdot \\ i_{bq} \end{bmatrix} = \begin{bmatrix} 0 & k_{11} & k_{21} & \cdots & k_{q1} \\ -k_{11} & 0 & 0 & \cdots & 0 \\ -k_{21} & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -k_{q1} & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} i_a \\ v_{b1} \\ v_{b2} \\ \cdot \\ \cdot \\ v_{bq} \end{bmatrix} \tag{4}$$

where

$$k_{11}, k_{21}, \ldots, k_{q1} \geq 0$$

we extend the circuit in Fig.5 to the circuit in Fig.8 . The left part of this circuit is basically the same as the left part of Fig.5 , except that the value of resistor R_v should be readjusted from kR to $R\sum_{j=1}^{q} k_{j1}$. The right part of this circuit consists of "q" similar blocks: $N_{b1}, N_{b2}, \ldots, N_{bq}$. Each block has the same structure. Using the ideal op amp model, and the same reasoning as before, it is easy to show that this circuit realizes (4) exactly.

If any k_{i1} in (4) is negative, the modification similar to those shown in Fig.7 should be made on the corresponding N_{bi} block.

Finally, to form a (p+q)-port transformer we connect "p" (1+q)-port transformers together. Each of the "p" blocks $N_1, N_2, \ldots, N_q$ corresponds to a column of (2). When they are connected in parallel across the q-port as shown in Fig.9, the whole network N of Fig.9 realizes (1). Thus the realization of a (p+q)-port transformer has been completed.

3. Circuit realization: Scattering-matrix approach

Any linear passive n-port N may be completely specified by an arbitrarily assigned *reference* resistance matrix*

$$\mathbf{R} = \begin{bmatrix} r_1 & \cdot & \cdot & \cdot & 0 \\ \cdot & r_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & r_n \end{bmatrix} \tag{5}$$

and an associated *scattering matrix*

$$\mathbf{S} = \begin{bmatrix} s_{11} & s_{12} & \cdot & \cdot & s_{1n} \\ s_{21} & s_{22} & \cdot & \cdot & s_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{n1} & s_{n2} & \cdot & \cdot & s_{nn} \end{bmatrix} \tag{6}$$

In this case N is characterized by:

$$\mathbf{b} = \mathbf{S}\,\mathbf{a} \tag{7}$$

where **a** and **b** are defined as:

$$\mathbf{a} = \frac{1}{2} \begin{bmatrix} \frac{1}{\sqrt{r_1}}(v_1 + r_1 i_1) \\ \frac{1}{\sqrt{r_2}}(v_2 + r_2 i_2) \\ \cdot \\ \cdot \\ \frac{1}{\sqrt{r_n}}(v_n + r_n i_n) \end{bmatrix} \tag{8}$$

$$\mathbf{b} = \frac{1}{2} \begin{bmatrix} \frac{1}{\sqrt{r_1}}(v_1 - r_1 i_1) \\ \frac{1}{\sqrt{r_2}}(v_2 - r_2 i_2) \\ \cdot \\ \cdot \\ \frac{1}{\sqrt{r_n}}(v_n - r_n i_n) \end{bmatrix} \tag{9}$$

Equation (7) can be realized by the circuit shown in Fig.10[8]. Here, the block M is the circuit realization of

$$v_{in_m} = \sqrt{r_m} \sum_{j=1}^{n} s_{mj} \frac{1}{\sqrt{r_j}} v_{out_j} \qquad m = 1, 2, \ldots, n \tag{10}$$

* The components $r_1, r_2, \ldots, r_n$ of $\mathbf{R}$ are usually called the *port reference or normalization* numbers.

For a linear resistive n-port N such as a (p+q)-port transformer, the elements of S are real scalars. Therefore in this case v_{in_m} (m=1,2,...,n) can be synthesized using a combination of inverting amplifiers, non-inverting amplifiers, and summing circuits, all of which are easily designed using op amps and resistors.

To apply this approach to a (p+q)-port transformer, we first convert (1) to the corresponding expression (7). From *Section 2* we have seen that a (p+q)-port transformer can be synthesized by connecting "p" (1+q)-port transformers together. Also, for another reason that will be made clear soon, we will start by considering a (1+q)-port transformer with the constitutive relation (4). From [3] we have the following general formula for transforming a given n-port representation (e. g., hybrid matrix H in (4)) into *any other* representation (e. g., scattering matrix S).*

$$\Lambda' = [(\alpha'\mathbf{a}+\beta'\mathbf{c})\Lambda+(\alpha'\mathbf{b}+\beta'\mathbf{d})][(\gamma'\mathbf{a}+\delta'\mathbf{c})\Lambda+(\gamma'\mathbf{b}+\delta'\mathbf{d})]^{-1} \tag{11}$$

In this particular case, we have

$$\Lambda = \mathrm{H} = \begin{bmatrix} 0 & k_{11} & k_{21} & \cdot\cdot & k_{q1} \\ -k_{11} & 0 & 0 & \cdot\cdot & 0 \\ -k_{21} & 0 & 0 & \cdot\cdot & 0 \\ \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ -k_{q1} & 0 & 0 & \cdot\cdot & 0 \end{bmatrix} \tag{12}$$

$$\Lambda' = \mathbf{S} \tag{13}$$

$$\mathbf{a} = \mathbf{d} = \begin{bmatrix} 1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 \end{bmatrix} \tag{14}$$

$$\mathbf{b} = \mathbf{c} = \begin{bmatrix} 0 & 0 & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & \cdot & \cdot & 1 \end{bmatrix} \tag{15}$$

$$\alpha' = \gamma' = \frac{1}{2}\mathbf{R}^{-0.5} \tag{16}$$

$$\beta' = -\frac{1}{2}\mathbf{R}^{0.5} \tag{17}$$

$$\delta' = \frac{1}{2}\mathbf{R}^{0.5} \tag{18}$$

* Here we adopted the same notatins as in [3].

$$\delta' = \frac{1}{2}\mathrm{R}^{0.5} \tag{18}$$

where

$$\mathrm{R}^{0.5} = \begin{bmatrix} \sqrt{r_1} & 0 & \cdot\cdot & 0 \\ 0 & \sqrt{r_2} & \cdot\cdot & 0 \\ \cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot & \cdot\cdot & \cdot \\ 0 & 0 & \cdot\cdot & \sqrt{r_{q+1}} \end{bmatrix} \tag{19}$$

$$\mathrm{R}^{-0.5} = \begin{bmatrix} \frac{1}{\sqrt{r_1}} & 0 & \cdot\cdot & 0 \\ 0 & \frac{1}{\sqrt{r_2}} & \cdot\cdot & 0 \\ \cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot & \cdot\cdot & \cdot \\ 0 & 0 & \cdot\cdot & \frac{1}{\sqrt{r_{q+1}}} \end{bmatrix} \tag{20}$$

Substituting (12)-(20) into (11), we get

$$\mathbf{S} = \begin{bmatrix} -\frac{\sqrt{r_1}}{2} & \frac{k_{11}}{2\sqrt{r_1}} & \frac{k_{21}}{2\sqrt{r_1}} & \cdot\cdot & \frac{k_{q1}}{2\sqrt{r_1}} \\ \frac{k_{11}\sqrt{r_2}}{2} & \frac{1}{2\sqrt{r_2}} & 0 & \cdot\cdot & 0 \\ \frac{k_{21}\sqrt{r_3}}{2} & 0 & \frac{1}{2\sqrt{r_3}} & \cdot\cdot & 0 \\ \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ \frac{k_{q1}\sqrt{r_{q+1}}}{2} & 0 & 0 & \cdot\cdot & \frac{1}{2\sqrt{r_{q+1}}} \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{r_1}}{2} & \frac{k_{11}}{2\sqrt{r_1}} & \frac{k_{21}}{2\sqrt{r_1}} & \cdot\cdot & \frac{k_{q1}}{2\sqrt{r_1}} \\ \frac{-k_{11}\sqrt{r_2}}{2} & \frac{1}{2\sqrt{r_2}} & 0 & \cdot\cdot & 0 \\ \frac{-k_{21}\sqrt{r_3}}{2} & 0 & \frac{1}{2\sqrt{r_3}} & \cdot\cdot & 0 \\ \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ \frac{-k_{q1}\sqrt{r_{q+1}}}{2} & 0 & 0 & \cdot\cdot & \frac{1}{2\sqrt{r_{q+1}}} \end{bmatrix}^{-1} \tag{21}$$

In general every element of S will be a function of $(r_1, r_2, \ldots, r_{q+1}; k_{11}, k_{21}, \ldots, k_{q1})$. However, if we choose

$$r_2 = \frac{r_1}{k_{11}^2},\ r_3 = \frac{r_1}{k_{21}^2},\ \cdots,\ r_{q+1} = \frac{r_1}{k_{q1}^2} \tag{22}$$

as the port normalization numbers, then (21) can be simplified dramatically as follow:

$$\mathbf{S} = \frac{1}{q+1}\begin{bmatrix} q-1 & 2 & 2 & . & . & 2 \\ 2 & q-1 & -2 & . & . & -2 \\ 2 & -2 & q-1 & . & . & -2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 2 & -2 & -2 & . & . & q-1 \end{bmatrix} \tag{23}$$

which is independent from $(r_1, r_2, ..., r_{q+1}; k_{11}, k_{21}, ..., k_{q1})$. This property will prove most convenient in building and adjusting the transformer. This is the reason why we synthesize the (1+q)-port transformer first.

The scattering-matrix approach for realizing the constitutive relation (1) can now be summarized as follows:

(1) Partition the matrix (2) into "p" columns. For each column i (i=1,2,...,p) do steps (2) and (3).

(2) Choose an appropriate value of r_i (for example, 1KΩ or 10KΩ). Choose the elements of (5) as follows:

$$\mathbf{R}(i) = \begin{bmatrix} r_i & 0 & . & . & 0 \\ 0 & \frac{r_i}{k_{1i}^2} & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & \frac{r_i}{k_{qi}^2} \end{bmatrix} \tag{24}$$

(3) Using (23) and (24), build a (1+q)-port transformer, as indicated in Fig.11 . Depending on the signs and values of each k_{ji} (j=1,2,...,q;i=1,2,...,p), the circuit inside the M_i block will have different structures. In *Appendix 1* we give an example of the actual circuit realization of a (1+2)-port transformer.

(4) Connect these "p" (1+q)-ports together as shown in Fig.9 to form the complete (p+q)-port transformer.

4. Grounded-to-floating input conversion

If a floating input voltage rather than a grounded voltage is desired, the grounded-to-floating convertors (Fig.12) can be used[9]. This is basically a differential-to-single-ended convertor modified such that the port current constraints are also satisfied.

5. A comparison of the two approaches

Both approaches realize the constitutive relation (1). Their frequency range and voltage/current range are similar. Both depends on the op amps in the circuit.

Structurally the direct approach is better. It has the following advantages:

(1) It uses much fewer op amps and the resulting circuit is much simpler.

(2) The turns-ratio matrix **K** is easier to adjust in this approach than in the other. To adjust a single element k_{ij} of $\mathbf{K}$, we need only adjust two resistors (see Fig.8). In the scattering-matrix approach, to adjust an element k_{ij}, usually many resistors have to be changed. Besides, in the direct approach, for each element k_{ij} of $\mathbf{K}$, one resistor controls the voltage ratio, and the other controls the current ratio. Since the voltage ratio and the corresponding current ratio need not be equal, the direct approach can be easily extended to realize the following more general constitutive relation:

$$\begin{bmatrix} \mathbf{v}_a \\ \mathbf{i}_b \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{K}_v \\ \mathbf{K}_i & 0 \end{bmatrix} \begin{bmatrix} \mathbf{i}_a \\ \mathbf{v}_b \end{bmatrix} \tag{25}$$

Here, the (p×q) matrix $\mathbf{K}_v$ and the (q×p) matrix $\mathbf{K}_i$ are independent and include

$$\mathbf{K}_v = -\mathbf{K}_i^T$$

as a special case. This property provides the possibility of using the direct approach to realize a wider variety of n-port resistors.

(3) The direct approach results in a unified circuit structure for different values of p and q, while the other does not. Therefore the former is more suitable to be integrated in module form.

But, concerning the load ranges, the scattering-matrix approach has its own advantage too. Since both approaches are based on op amps, it is a crucial problem to avoid oscillations. To determine whether an active n-port will oscillate or not under certain termination conditions, we apply the following concept of ***absolute stability*** [6,7]:

Definition:

A two-port N is said to be ***absolutely stable*** if all natural frequencies of the terminated two-port (Fig.13) have negative real parts for any pair of passive terminations y_1 and y_2.

If the active two-port in question has a **Y**-matrix representation, the criteria for absolute stability are the so called Llewellyn's conditions:

$$g_{11} = \mathrm{Re}[y_{11}(j\omega_0)] > 0$$

$$g_{22} = \mathrm{Re}[y_{22}(j\omega_0)] > 0 \qquad (26)$$

$$\eta = \frac{2g_{11}g_{22} - \mathrm{Re}(y_{12}y_{21})}{|y_{12}y_{21}|} > 1$$

where η is called the stability parameter. *A two-port is absolutely stable if and only if Llewellyn's conditions are satisfied for all ω ($0<\omega<\infty$).*

For **Z**, **H** and **G** parameters the correspoding Llewellyn's conditions have similar forms[10].

For simplicity we will analyse and compare the absolute stability of both approaches for the case p=q=1 only.

The (1+1)-port transformer of the direct approach is shown in Fig.5 . For each op amp we use a single-pole model (Fig.14). For the op amp A_1, A_2 and A_3 (LM301) we choose the following parameters:

$$a = 200000\,, \quad r = 1K\Omega\,, \quad c = 25\mu f$$

Since the op amp A_4 (LM310), used in the voltage follower, has a much better frequency response than the other op amps, we simply choose

$$a = 1\,, \quad r = c = 0$$

Using these models for each op amp in Fig.5, we can easily check Llewellyn's condition using a standard computer simulation program, such as SPICE. A computer output sample for the (1+1)-port transformer realized by the direct approach with $k=1$ is given in Table 1.

Table 1
A computer output checking Llewellyn's
conditions for the (1+1)-port transformer
realized by the direct approach

f,Hz	g_{11}	g_{22}	η
0	1.667e+01	1.667e+01	1.000e+00
1	1.639e+01	1.639e+01	1.000e+00
10	6.146e+00	6.146e+00	1.000e+00
100	9.680e-02	9.616e-02	1.000e+00
1K	1.073e-03	4.722e-04	1.000e+00
10K	1.097e-04	-4.894e-04	0.9989e+00
100K	1.001e-04	-4.707e-04	0.8928e+00
1M	1.002e-04	5.001e-05	1.724e+00
10M	1.000e-04	9.999e-05	3.058e+04
100M	1.001e-04	1.000e-04	3.047e+08

Note that at some frequencies (e.g. f=10KHz and f=100KHz) Llewellyn's conditions are not satisfied ($g_{22}<0$). Hance this circuit is not absolutely stable. It is potentially unstable for some passive terminations.

Next we would like to know for which terminations this circuit will be unstable. Since the most commonly used terminations are resistors, we choose two passive linear resistors as terminations to find out for which ranges (if any) the circuit would be unstable. When we connect two resistors at the two ports as shown in Fig.15, the total admittance seen from $1-1'$ is

$$Y_1 = \frac{1}{R_1} + \frac{\frac{y_{11}}{R_2} + \Delta y}{\frac{1}{R_2} + y_{22}} \qquad (27)$$

where $\Delta y = y_{11}y_{22} - y_{12}y_{21}$. If $\mathrm{Re}(Y_1(j\omega_0))<0$, this circuit will oscillate when a current source is applied across $1-1'$. This means that the natural frequencies of the augmented two-port are not restricted to the open-left half complex frequency plane. Therefore, whenever Y_1 of (27) has negative real part at any ω ($0 \le \omega < \infty$) for some R_1 and R_2, the two-port is unstable for the particular terminations R_1 and R_2. Likewise, the total admittance seen from $2-2'$ is

$$Y_2 = \frac{1}{R_2} + \frac{\frac{y_{22}}{R_1} + \Delta y}{\frac{1}{R_1} + y_{11}} \tag{28}$$

If $\text{Re}(Y_2)<0$, the augmented two-port is unstable. Using the values of $y_{11}, y_{12}, y_{21}, y_{22}$ calculated from a computer program (the real parts of y_{11} and y_{22} are listed in *Table 1* for 10 sampled frequencies) for different values of R_1 and R_2 (0.01 Ω ,0.1 Ω ,1 Ω ,10 Ω ,100 Ω ,1K Ω ,10K Ω ,100K Ω ,1M Ω and 10M Ω), Y_1 and Y_2 have been computed using (27) and (28). Whenever $\text{Re}(Y_1)<0$ and/or $\text{Re}(Y_2)<0$ the computer prints out a symbol indicating oscillations. Fig.16 is a computer output for the (1+1)-port transformer with $k=1$.

These results have also been verified experimentally. In the laboratory, for different values of R_1 and R_2 (short circuit, 1 Ω ,10 Ω ,100 Ω ,1K Ω ,10K Ω ,100K Ω ,1M Ω and open circuit) we made measurements for the circuit in Fig.15. For some values of R_1 and R_2 oscillations across R_1 and R_2 have been observed. However, because of the discrepancy among different op amps, including op amps having identical model numbers, oscillations have been observed for different values of R_1 and R_2. Fig.17 gives one pattern of experimental results. Fig.17 is very close to Fig.16.

We repeat the same analysis for the (1+1)-port transformer realized by the scattering-matrix approach. A computer output checking Llewellyn's conditions is shown in *Table 2*. Note that Llewellyn's conditions are almost always satisfied, except for some frequencies where the stability parameter η is equal to 1. However, as shown in *Appendix 2*, when $\eta=1$, no resistive terminations can make the two-port unstable. Hence, if we restrict the terminations to resistors, the two-port is actually absolutely stable. In the laboratory, we did the same measurements as in the direct approach. In agreement with the computational results, no oscillations were observed experimentally. Therefore, with respect to absolute stability the scattering-matrix approach is better. However, for many practical applications, the termination connected to the q-port is a voltage source, which has very low output impedance. Hence, R_2 is near to zero. According to Fig.16 and Fig.17, the (1+1)-port transformer realized by the direct approach is also stable in this case.

Table 2

A computer output checking Llewellyn's conditions for the (1+1)-port transformer realized by the scattering-matrix approach

f,Hz	g_{11}	g_{22}	η
0	5.714e+00	5.714e+00	1.000e+00
1	5.613e+00	5.613e+00	1.000e+00
10	2.038e+00	2.038e+00	1.000e+00
100	3.151e-02	3.151e-02	1.000e+00
1K	3.495e-04	3.495e-04	1.000e+00
10K	3.657e-04	3.657e-04	1.000e+00
100K	3.385e-05	3.385e-05	1.001e+00
1M	7.340e-05	7.340e-05	3.321e+00
10M	1.000e-04	1.000e-04	1.952e+03
100M	1.000e-04	1.000e-04	1.941e+06

On the other hand, although the (1+1)-port transformer realized by the scattering approach is stable for any pair of passive resistive terminations, there is no guarantee that it will also be stable for active terminations. Actually we have observed oscillations when some active resistors were connected to it. As will be seen in the next section, two-terminal active resistors are needed for the n-port piecewise-linear resistor synthesis problem. Fortunately the (p+q)-port transformer realized by the direct approach is stable for this situation.

6. Applications of the (p+q)-port transformer

6.1. Piecewise-linear n-port resistor synthesis

As pointed out in[2], every reciprocal n-port resistor represented by a continuous *n-dimentional piecewise-linear function*

$$i_1 = a_1+b_{1_1}v_1+\ldots+b_{1_n}v_n+\sum_{k=1}^{m_1} g_{1k}\,|\alpha_{1k_1}v_1+\ldots+\alpha_{1k_n}v_n-\beta_{1k}|$$

$$\vdots \qquad (29)$$

$$i_n = a_n + b_{n_1} v_1 + \ldots + b_{n_n} v_n + \sum_{k=1}^{m_n} g_{nk} \,|\, \alpha_{nk_1} v_1 + \ldots + \alpha_{nk_n} v_n - \beta_{nk} \,|$$

is realizable by a circuit containing only 2-terminal piecewise-linear resistors and a (p+q)-port transformer with q=n. An explicit circuit realization is given in [2] without physically building it. Here we present a numerical example which was confirmed by experimental results:

Suppose we wish to realize a 2-port resistor described by:

$$i_1 = -\frac{1}{4} + \frac{v_1}{2} + \frac{v_2}{4} - \frac{1}{2}\left|v_1 + \frac{v_1}{2} - \frac{1}{2}\right| + \frac{1}{4}\left|\frac{1}{2}v_1 + v_2 + \frac{1}{2}\right| \tag{30}$$

$$i_2 = -\frac{1}{8} + \frac{v_1}{4} + \frac{v_2}{8} - \frac{1}{4}\left|v_1 + \frac{v_1}{2} - \frac{1}{2}\right| + \frac{1}{2}\left|\frac{1}{2}v_1 + v_2 + \frac{1}{2}\right| \tag{31}$$

which is reciprocal according to[2].

The basic building blocks besides the (p+q)-port transformer consist of 2-terminal voltage-controlled resistors described by

$$i = a + bv + g\,|v - \beta| \tag{32}$$

where a, b, g and β are real numbers.

In order to realize (30) and (31), let us try to use two such resistors, i.e.

$$R_1: \quad i = a_1 + b_1 v + g_1 |v - \beta_1| \tag{33}$$

$$R_2: \quad i = a_2 + b_2 v + g_2 |v - \beta_2| \tag{34}$$

and a (2+2)-port transformer, whose turns-ratio matrix is

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \tag{35}$$

If we connect the two resistors across the p-ports and two voltage sources across the q-ports as shown in Fig.18, we would obtain

$$i_{R_1} = a_1 + b_1(k_{11}v_1 + k_{21}v_2) + g_1 |k_{11}v_1 + k_{21}v_2 - \beta_1| \tag{36}$$

$$i_{R_2} = a_2 + b_2(k_{12}v_1 + k_{22}v_2) + g_2 |k_{12}v_1 + k_{22}v_2 - \beta_2| \tag{37}$$

and

$$i_1 = k_{11} i_{R_1} + k_{12} i_{R_2}$$

$$= (k_{11}a_1+k_{12}a_2)+(k_{11}^2b_1+k_{12}^2b_2)v_1+(k_{11}k_{21}b_1+k_{12}k_{22}b_2)v_2 \tag{38}$$

$$+k_{11}g_1|k_{11}v_1+k_{21}v_2-\beta_1|+k_{12}g_2|k_{12}v_1+k_{22}v_2-\beta_2|$$

$$i_2 = k_{21}i_{R_1}+k_{22}i_{R_2}$$

$$= (k_{21}a_1+k_{22}a_2)+(k_{11}k_{21}b_1+k_{12}k_{22}b_2)v_1+(k_{21}^2b_1+k_{22}^2b_2)v_2 \tag{39}$$

$$+k_{21}g_1|k_{11}v_1+k_{21}v_2-\beta_1|+k_{22}g_2|k_{12}v_1+k_{22}v_2-\beta_2|$$

Comparing (38) and (39) with (30) and (31), we can determine all coefficients in (33), (34), and (35). They are:

$$k_{11} = 1\,,\ k_{12} = \frac{1}{2},\ k_{21} = \frac{1}{2},\ k_{22} = 1$$

$$a_1 = -\frac{1}{4},\ a_2 = 0\,,\ b_1 = \frac{1}{2},\ b_2 = 0$$

$$g_1 = -\frac{1}{2},\ g_2 = \frac{1}{2},\ \beta_1 = \frac{1}{2},\ \beta_2 = -\frac{1}{2}$$

To build R_1 and R_2, we adopted the technique from[9]. The resulting oscillograph pictures of characteristics of R_1 and R_2 are shown in Fig.19.

The final circuit is obtained by connecting R_1 and R_2 across a (2+2)-port transformer having a turns-ratio matrix

$$\mathbf{K} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \tag{40}$$

as shown in Fig.18. While v_2 is kept constant, the v_1–i_1 characteristics are measured. Similarly, the v_2–i_2 characteristics are measured while v_1 is kept constant. These pictures are shown in Fig.20 and Fig.21. The experimental results agree closely with the theoretical predictions.

6.2. Nonlinear programming

A general nonlinear programming problem can be stated as follow:

*Minimize a scalar objective function $\varphi(\mathbf{v})$ subject to the inequality constraints:**

* $\mathbf{v} \geq 0$ means $v_j \geq 0$ for all j.

$$\mathbf{v} \geq 0 \tag{41}$$

$$\mathbf{f}(\mathbf{v}) \geq 0 \tag{42}$$

where **v** *is a q-vector and* **f** *is a p-vector.*

If φ is a *concave* function and each f_i is a *convex* function, the program is called *concave program.* Among the class of concave programs, there are two specially interesting subclasses: linear program and quadratic program. Actually the former is only a special case of the latter. For the *quadratic programming problem,* the objective function is:

$$\boxed{\varphi(\mathbf{v}) = \mathbf{j}^T\mathbf{v} + \frac{1}{2}\mathbf{v}^T\mathbf{G}\mathbf{v}} \tag{44}$$

where **j** is a q-vector and **G** is a q×q positive semi-define *symmetric* real matrix.** The constraints are:

$$\boxed{\mathbf{v} \geq 0\,, \qquad \mathbf{f}(\mathbf{v}) = \mathbf{A}\mathbf{v} - \mathbf{e} \geq 0} \tag{45}$$

where **e** is a p-vector and **A** is a p×q matrix.

Using circuit simulation, we can give (44) a very good physical interpretation. The circuit of Fig.22 consists of a (p+q)-port transformer, voltage sources, current sources, ideal diodes and resistors. The bold lines denote repeated circuits.

Clearly, this circuit satisfies all the constraints of (45). Every item of (44) and (45) has a counterpart in this circuit: **f** represents the voltages across the diodes $\mathbf{D}_1$, **v** represents the voltages across the diodes $\mathbf{D}_2$, **j** represents the current sources, **e** represents the voltage sources, **G** is the symmetric conductance matrix and **A** is the transpose of the turns-ratio matrix **K** of the (p+q)-port transformer. In *Appendix 3,* we show that $\varphi(\mathbf{v})$ of (44) is just the *total co-content* of this circuit. It follows from the stationary co-content theorem[3,11] that the operating point of this circuit is a stationary point of the total co-content. Moreover, since $\varphi(\mathbf{v})$ in (44) is concave and each f_i in (45) is convex, $\varphi(\mathbf{v})$ has a *global* minimum which coincides with the unique stationary point of φ. Therefore *the solution of this circuit gives the minimum of (44).*

We have also verified this conclusion experimentally. To realize the ideal diodes and current sources, we adopted the techniques from [9]. For example,

** When $\mathbf{G} = 0$, it reduces to a linear program.

Fig.23 gives the circuit realization of a "biased" ideal diode and the oscillograph picture of its $v-i$ characteristic.

To be specific, suppose the function to be minimized is

$$\varphi(\mathbf{v}) = v_1+2v_2+\frac{1}{2}[v_1\ v_2]\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$=v_1+2v_2+v_1^2-v_1v_2+v_2^2 \tag{46}$$

subject to the constraints

$$v_1 \geq 0$$

$$v_2 \geq 0$$

$$f_1=v_1+\frac{1}{2}v_2-1\geq 0 \tag{47}$$

$$f_2=\frac{1}{2}v_1+v_2-2\geq 0$$

The physical circuit simulating (46)-(47) is shown in Fig.24.* The current sources and resistances have been normalized to I_0 and R_0 where

$$I_0=0.1mA \qquad R_0=10K\Omega$$

The mathematical solutions of this problem are:

$$v_1 = \frac{8}{7}= 1.143\,, \qquad v_2 = \frac{10}{7}= 1.429$$

The experimental results are

$$v_1 = 1.17V\,, \qquad v_2 = 1.46V$$

The slight discrepancy in the answers is due to inperfections of the circuit components.

*For the complete circuit, see *Appendix 4*.

7. Appendix

7.1. Appendix 1: An example of a scattering-matrix approach (1+2)-port transformer

Suppose the turns-ratio matrix **K** is given as

$$\mathbf{K} = \begin{bmatrix} k_{11} \\ k_{21} \end{bmatrix} \tag{A1}$$

Substituting $q=2$ into (23) for $n=3$, we obtain the scattering matrix

$$\mathbf{S} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} \tag{A2}$$

We choose

$$\mathbf{R} = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix} = \begin{bmatrix} r_0 & 0 & 0 \\ 0 & \frac{r_0}{k_{11}^2} & 0 \\ 0 & 0 & \frac{r_0}{k_{21}^2} \end{bmatrix} \tag{A3}$$

Substituting (A2) and (A3) into (10) yields

$$v_{in_1} = \frac{1}{3}v_{out_1} + \frac{2k_{11}}{3}v_{out_2} + \frac{2k_{21}}{3}v_{out_3} \tag{A4}$$

$$v_{in_2} = \frac{2}{3k_{11}}v_{out_1} + \frac{1}{3}v_{out_2} - \frac{2k_{21}}{3k_{11}}v_{out_3} \tag{A5}$$

$$v_{in_3} = \frac{2}{3k_{21}}v_{out_1} - \frac{2k_{11}}{3k_{21}}v_{out_2} + \frac{1}{3}v_{out_3} \tag{A6}$$

Equations (A4), (A5) and (A6) can be realized by the circuits in Fig.A1 . The (1+2)-port transformer is then synthesized as shown in Fig.A2 .

7.2. Appendix 2: Stability analysis for the case $\eta = 1$

Suppose the **Y** parameters of the two-port N and the two terminations evaluated at frequency $j\omega_0$ are:

$$y_{11} = g_{11}+jb_{11}$$

$$y_{12} = g_{12}+jb_{12}$$

$$y_{21} = g_{21}+jb_{21}$$

$$y_{22} = g_{22}+jb_{22}$$

$$y_1 = g_1+jb_1$$

$$y_2 = g_2+jb_2$$

Define:

$$M = \mathrm{Re}[y_{12}y_{21}]$$

$$N = \mathrm{Im}[y_{12}y_{21}]$$

$$L = |y_{12}y_{21}|$$

$$\alpha = g_{22} - \frac{M}{2g_{11}}$$

$$\beta = b_{22} - \frac{N}{2g_{11}}$$

$$\gamma = \frac{L}{2g_{11}}$$

Then we have

$$\eta = \frac{2g_{11}g_{22}-M}{L} = \frac{g_{22}-\dfrac{M}{2g_{11}}}{\dfrac{L}{2g_{11}}} = \frac{\alpha}{\gamma}$$

Therefore $\eta = 1$ means $\alpha = \gamma$. When port 2–2$'$ is terminated by the admittance y_2, the input admittance Y_{11} (Fig.A3) can be expressed as:

$$Y_{11} = y_{11} - \frac{y_{12}y_{21}}{y_{22}+y_2}$$

and hence the real part of Y_{11} can be written as:

$$G_{11} = g_{11} - \frac{M(g_2+g_{22})+N(b_2+b_{22})}{(g_2+g_{22})^2+(b_2+b_{22})^2} \tag{A7}$$

If we choose

$$y_2(j\omega_0) = 0+j(\frac{N}{2g_{11}} - b_{22})$$

then from (A7) we get

$$G_{11} = g_{11} - \frac{Mg_{22}+\frac{N^2}{2g_{11}}}{g_{22}^2 + (\frac{N}{2g_{11}})^2} = \frac{g_{11}}{g_{22}^2 + (\frac{N}{2g_{11}})^2}(\alpha^2-\gamma^2) = 0$$

Since now $Y_{11} = 0+jB_{11}$ is purely reactive, we can choose a passive admittance $y_1 = 0-jB_{11}$ to let the over-all driving-point admittance $Y'_{11}(j\omega_0) = 0$. Hence Z'_{11} has a pole at $j\omega_0$ and therefore the circuit is potentially unstable. On the other hand, if we choose any y_2 with $g_2>0$, then from (A7) we will get $G_{11}>0$. Therefore there is no passive y_1 which can render $Y'_{11} = 0$. We conclude therefore that only reactive terminations y_1 and y_2 can render the circuit unstable when $\eta=1$.

7.3. Appendix 3: Total co-content of the circuit in Fig.22

The stationary co-content theorem states[3,11]:

In a network N containing only voltage sources and voltage-controlled elements (including current sources), a twig voltage vector $\mathbf{v}_T$ is an operating point of N if and only if $\mathbf{v}_T$ is a stationary point of the total co-content function $\overline{G}(\mathbf{v}_T)$, where $\overline{G}$ is defined as the sum of the co-contents of all elements in N except voltage sources.

In the circuit of Fig.22, voltages of all branches can be determined in terms of $\mathbf{v}$ and $\mathbf{e}$. Since the components of $\mathbf{e}$ are constants, $\mathbf{v}$ is a *complete set* of variables[1,11]. The total co-content is therefore a function of $\mathbf{v}$. In this circuit there are five types of elements: voltage source, current source, ideal diode, linear resistor and (p+q)-port transformer.

An ideal diode ($v>0, i=0; v=0, i<0$) is neither voltage-controlled nor current-controlled. But if we connect a small resistor r in series with it, it will become voltage-controlled. Its co-content will be $\overline{G}_d = \frac{rv^2}{2}$. In the ideal situation, $r\to 0$, hence $\overline{G}_d \to 0$.

A (p+q)-port transformer consists of "$(p \times q)$" 2-port ideal transformers. Even though the 2-port ideal transformer (Fig.4) is not a voltage-controlled element, it has a well defined co-content; namely,

$$\overline{G}_t = \int_0^{v_a} i_a dv_a + \int_0^{v_b} i_b dv_b = \int_0^{v_b} k i_a dv_b - \int_0^{v_b} k i_a dv_b = 0$$

Hence the co-content of the (p+q)-port transformer is also zero. Therefore the total co-content is contributed only by current sources and resistors. The co-content of the current source j is:

$$\overline{G}_j(v) = \int_0^{v} j dv = jv \tag{A8}$$

The co-content of the resistor r (or conductor g) is:

$$\overline{G}_g(v) = \int_0^{v} i dv = \frac{gv^2}{2} \tag{A9}$$

Summing all co-contents given by (A8) and (A9) and using a vector notation, we obtain the *total co-content:*

$$\overline{G}(\mathbf{v}) = \mathbf{j}^T \mathbf{v} + \frac{1}{2} \mathbf{v}^T \mathbf{G} \mathbf{v}$$

which is (44).

7.4. Appendix 4: Complete circuit realization of Fig.24

A complete realization of the circuit in Fig.24 is shown in Fig.A4. The circuit inside broken lines is a (2+2)-port transformer realized by the direct approach with $k_{11} = k_{22} = 1$, $k_{12} = k_{21} = 0.5$. The other parts contain ideal diodes, biased ideal diodes, current sources and resistances. All dc voltage sources in this circuit (+1V,+2V,-10V,-12V) can be obtained from the +15V and -15V sources using circuits similar to the dc source in Fig.23(b).

8. References

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Figure captions

Fig.1 A (p+q)-port transformer.

Fig.2 Simplified notation of a (p+q)-port transformer.

Fig.3 Vector's notation of a (p+q)-port transformer.

Fig.4 A (1+1)-port transformer.

Fig.5 Circuit realizing a (1+1)-port transformer.

Fig.6 Operational amplifier and its ideal model.

Fig.7 Circuit realizing a (1+1)-port transformer with negative k.

Fig.8 Circuit realizing a (1+q)-port transformer.

Fig.9 Block diagram of a (p+q)-port transformer formed by interconnection of "p" (1+q)-port transformers.

Fig.10 Circuit realizing the scattering matrix of a n-port, where M is the realization of:

$$v_{in_m} = \sqrt{r_m} \sum_{j=1}^{n} s_{mj} \frac{1}{\sqrt{r_j}} v_{out_j}, \qquad m = 1,2,\dots,n$$

Fig.11 Scattering-matrix approach realization of a (1+q)-port transformer, where M_i is the realization of:

$$v_{in_{mi}} = \sqrt{r_{mi}} \sum_{j=1}^{q+1} s_{mj_i} \frac{1}{\sqrt{r_{ji}}} v_{out_{ji}}, \qquad m = 1,2,\dots,q+1$$

Fig.12 A grounded-to-floating converter.

Fig.13 A terminated two-port.

Fig.14 A single-pole op amp model.

Fig.15 A terminated (1+1)-port transformer.

Fig.16 Computer output indicating unstable ranges of loads.

Fig.17 Experimental results. Area where oscillations have been observed are shown hatched.

Fig.18 The connection of sources and loads across a (2+2)-port transformer.

Fig.19 The v-i oscillograph pictures of R_1 and R_2.
Horizontal scale: 2V/div , Vertical scale: 0.2mA/div .

$$(a)\ R_1 : i = -\frac{1}{4} + \frac{v}{2} - \frac{1}{2}\left|v - \frac{1}{2}\right|, \qquad (b)\ R_2 : i = \frac{1}{2}\left|v + \frac{1}{2}\right|$$

Fig. 20 v_1–i_1 characteristic measured by holding v_2 constant.
Horizontal scale: 2V/div , Vertical scale: 0.2mA/div .

(a) $v_2 = -\frac{1}{2}$.

The theoretical result from (30) is $i_1 = -\frac{3}{8} + \frac{v_1}{2} - \frac{1}{2}\left|v_1 - \frac{3}{4}\right| + \frac{1}{4}\left|\frac{v_1}{2}\right|$

(b) $v_2 = 0$.

The theoretical result from (30) is $i_1 = -\frac{1}{4} + \frac{v_1}{2} - \frac{1}{2}\left|v_1 - \frac{1}{2}\right| + \frac{1}{4}\left|\frac{v_1}{2} + \frac{1}{2}\right|$

(c) $v_2 = \frac{1}{2}$.

The theoretical result from (30) is $i_1 = -\frac{1}{8} + \frac{v_1}{2} - \frac{1}{2}\left|v_1 - \frac{1}{4}\right| + \frac{1}{4}\left|\frac{v_1}{2} + 1\right|$

Fig. 21 v_2–i_2 characteristic measured by holding v_1 constant.
Horizontal scale: 2V/div , Vertical scale: 0.2mA/div .

(a) $v_1 = -\frac{1}{2}$.

The theoretical result from (31) is $i_2 = -\frac{1}{4} + \frac{v_2}{8} - \frac{1}{4}\left|\frac{v_2}{2} - 1\right| + \frac{1}{2}\left|v_2 + \frac{1}{4}\right|$

(b) $v_1 = 0$.

The theoretical result from (31) is $i_2 = -\frac{1}{8} + \frac{v_2}{8} - \frac{1}{4}\left|\frac{v_2}{2} - \frac{1}{2}\right| + \frac{1}{2}\left|v_2 + \frac{1}{2}\right|$

(c) $v_1 = \frac{1}{2}$.

The theoretical result from (31) is $i_2 = \frac{v_2}{8} - \frac{1}{4}\left|\frac{v_2}{2}\right| + \frac{1}{2}\left|v_2 + \frac{3}{4}\right|$

Fig. 22 Circuit simulating a quadratic program.

Fig. 23
(a) A biased ideal diode to be realized.
(b) Circuit realization of (a).
(c) v-i characteristics of the circuit in (b).

Fig.24 Circuit for solving the quadratic-programming problem defined by (46) and (47).

Fig.A1
(a) Circuit realizing Equation (A4).
(b) Circuit realizing Equation (A5).
(c) Circuit realizing Equation (A6).

Fig.A2 The complete circuit of a (1+2)-port transformer realized by the scattering matrix approach.

Fig.A3 A terminated two-port.

Fig.A4 The complete circuit of Fig.24.

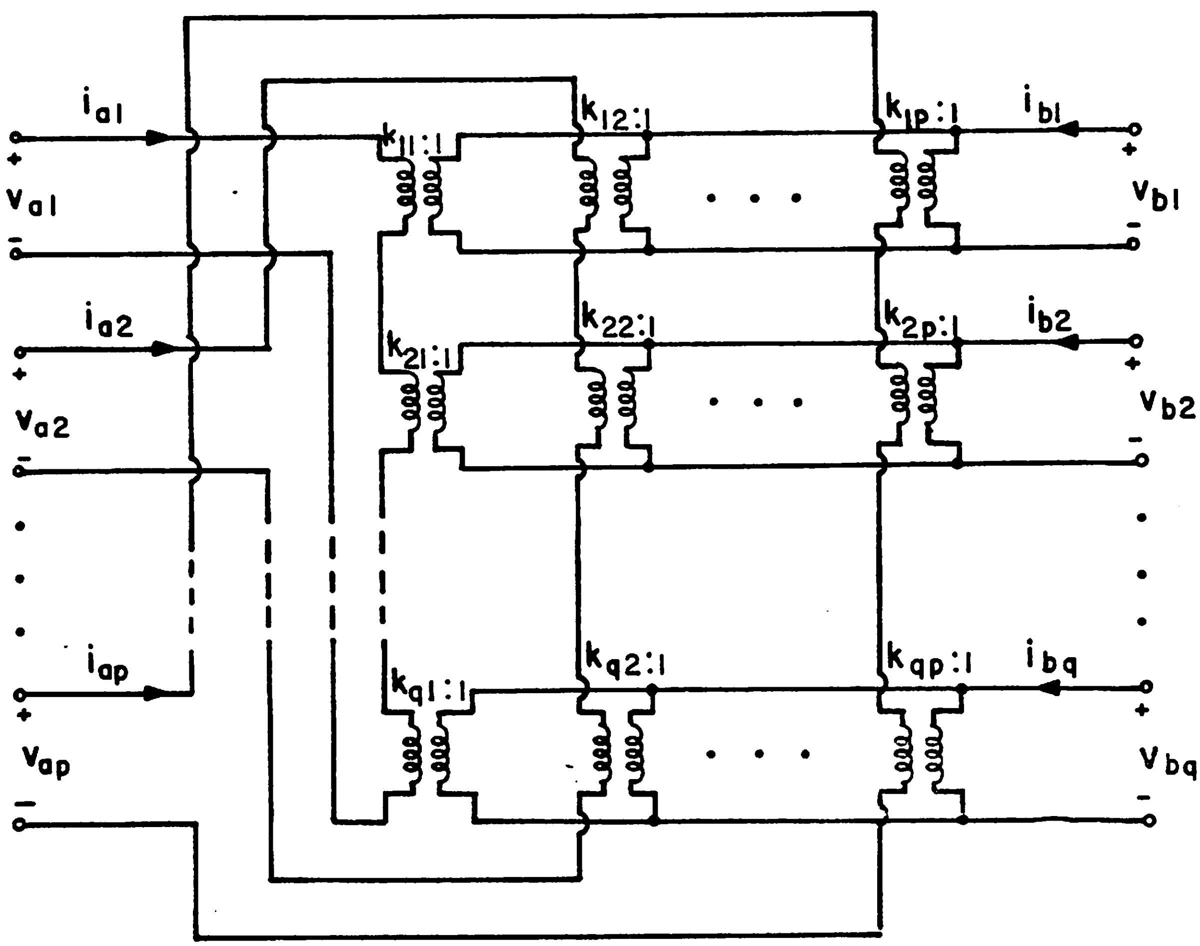


Fig. 1

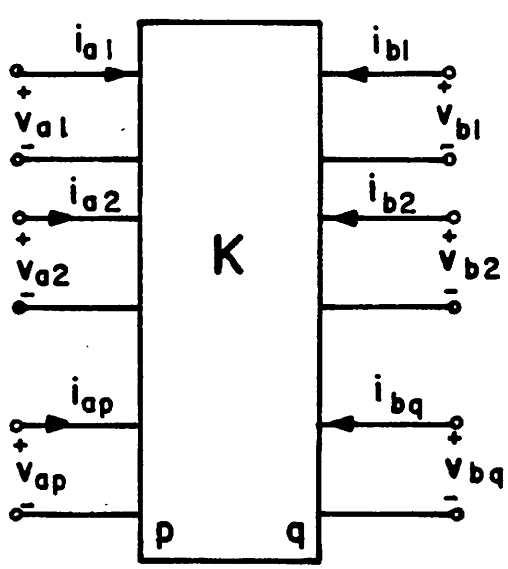


Fig. 2

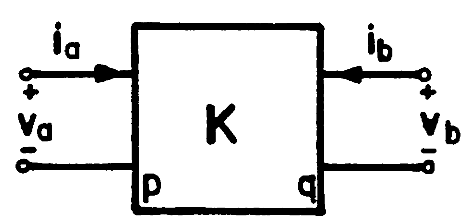


Fig. 3

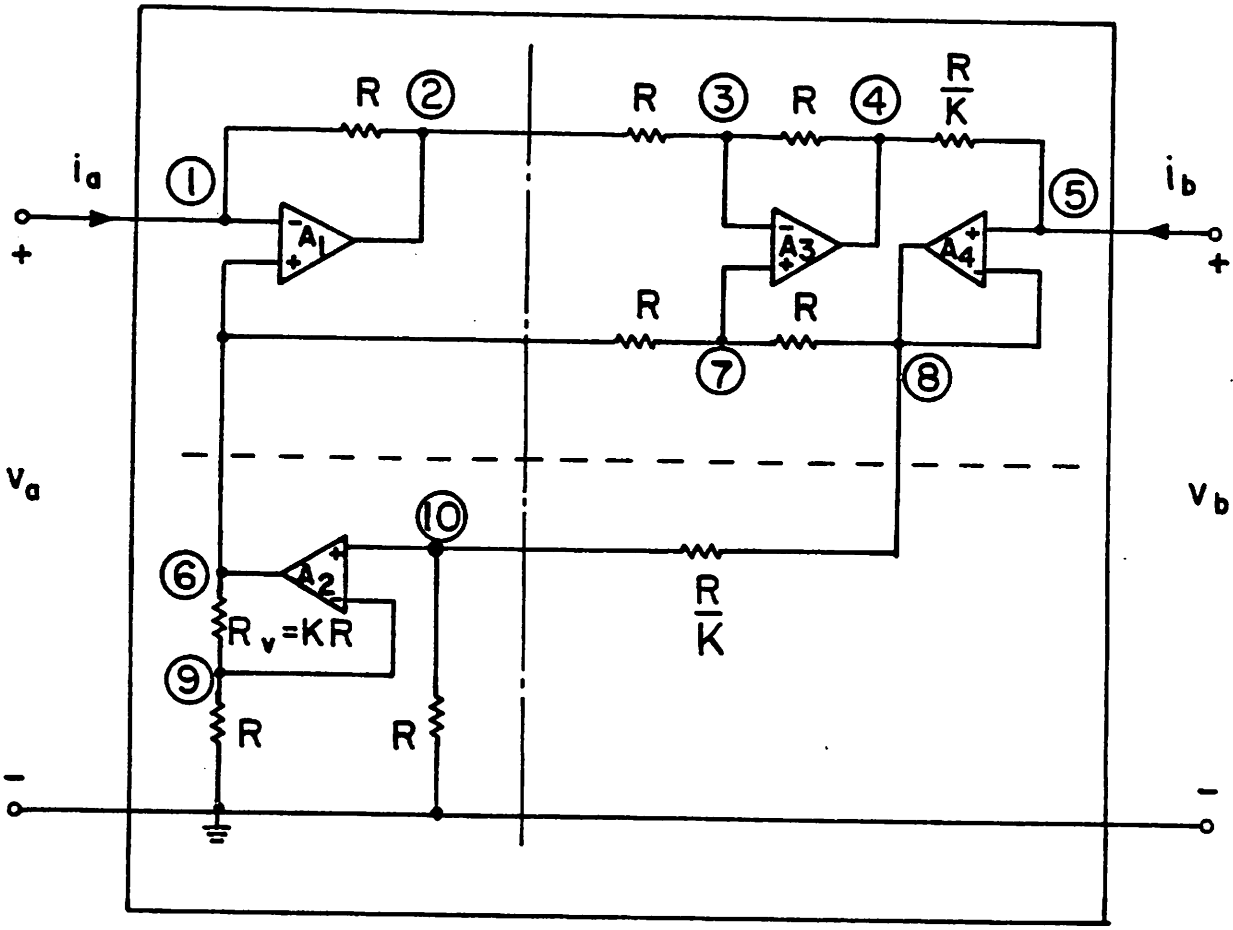


Fig. 5

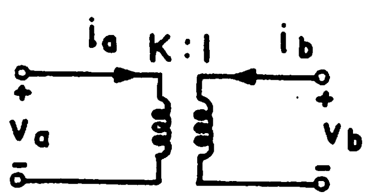


Fig. 4

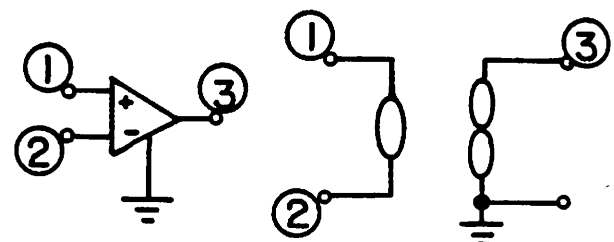


Fig. 6

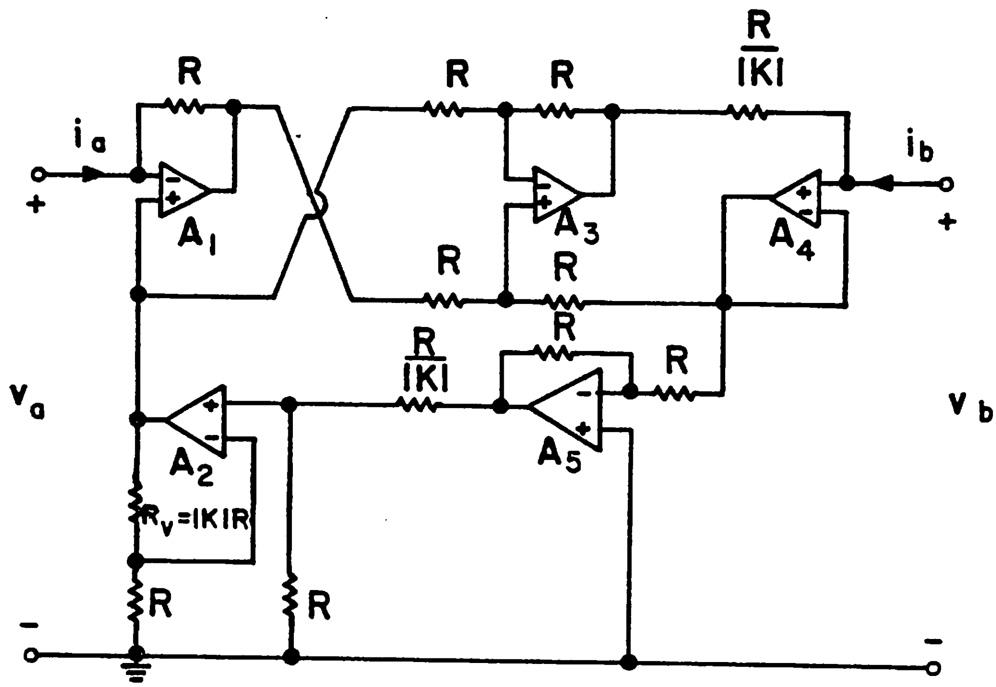
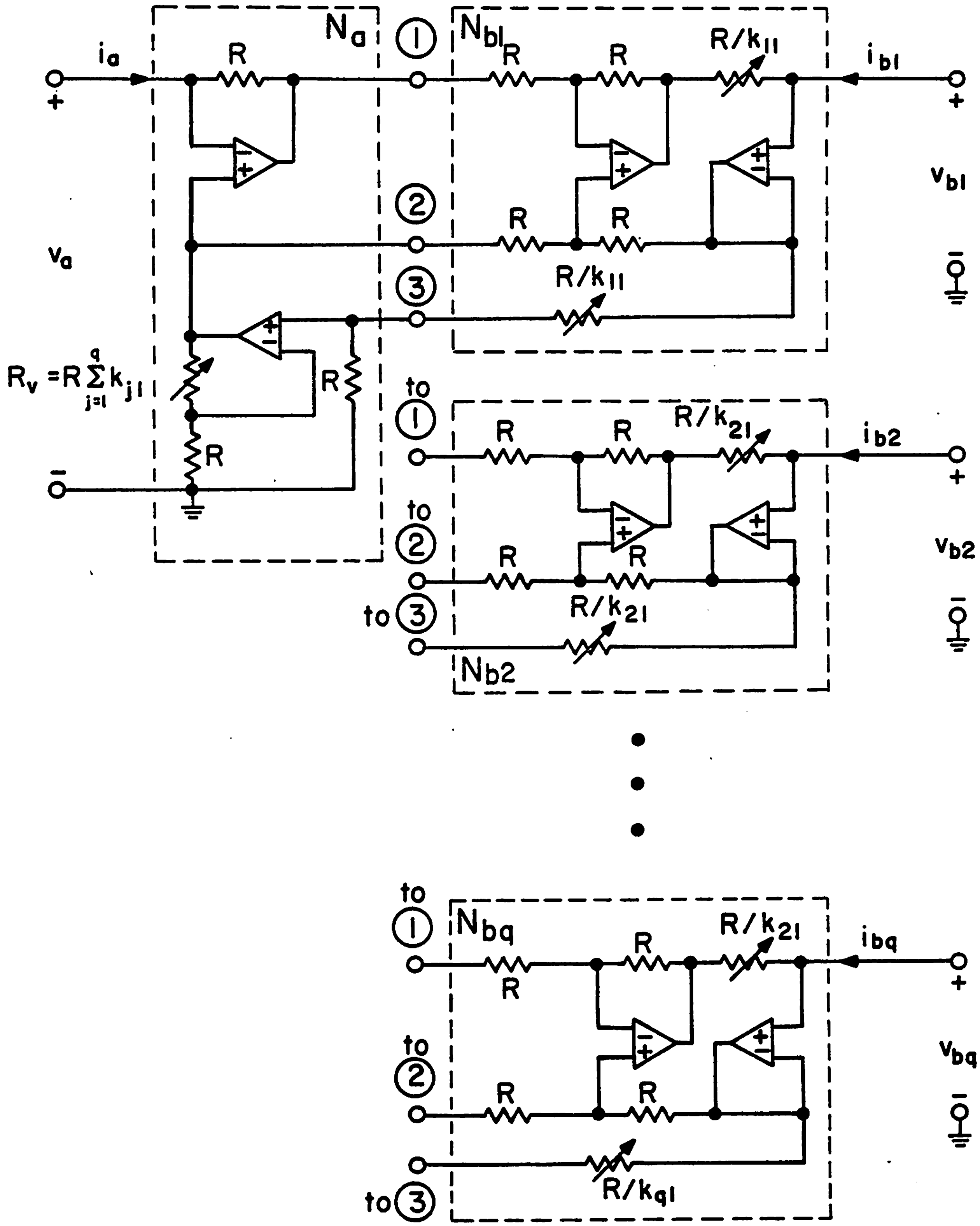
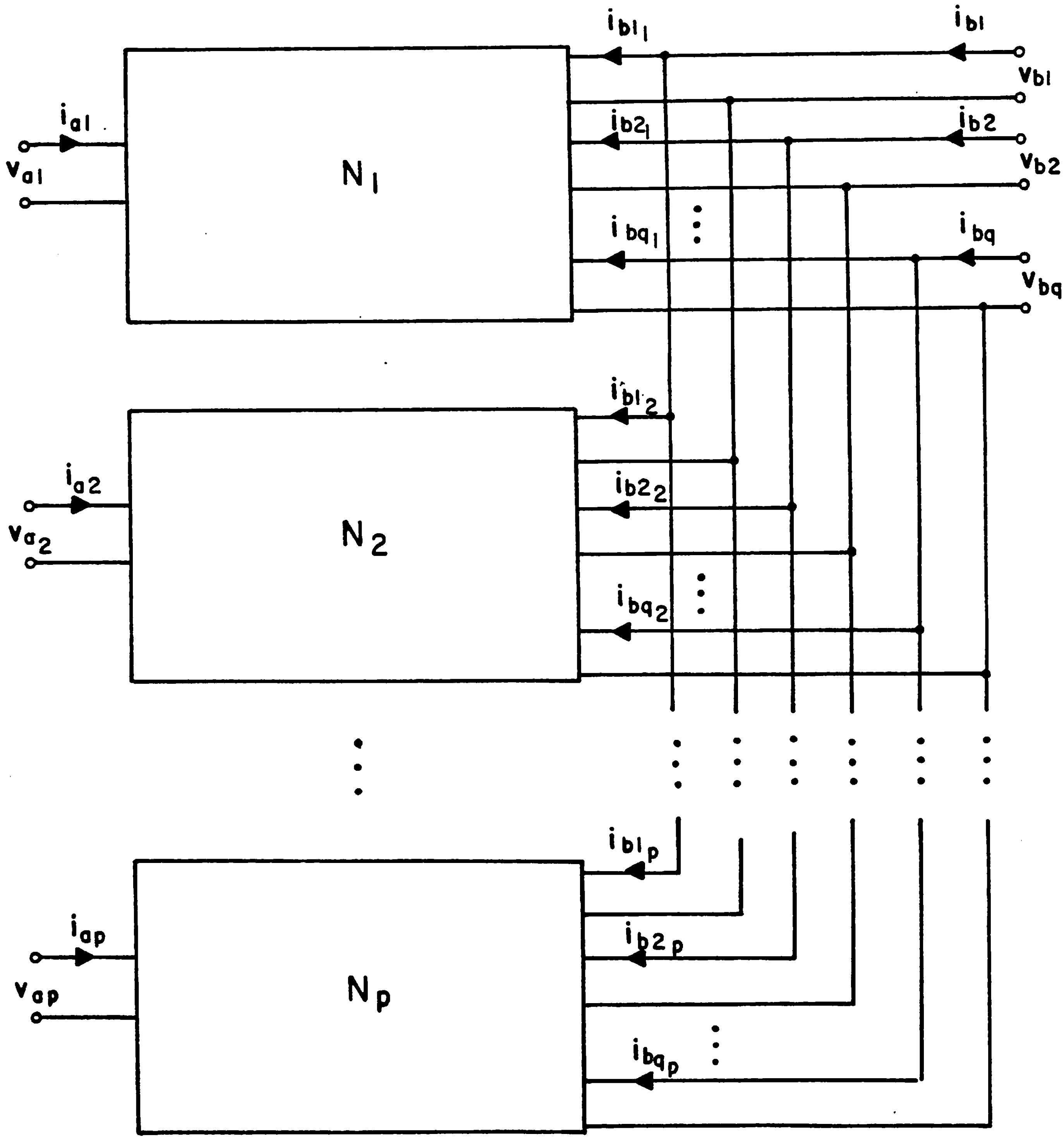


Fig. 7



Na
Nb1
Nb2
Nbq
ia
ib1
ib2
ibq
va
vb1
vb2
vbq
R
R/k11
R/k21
R/kq1
Rv = R Σ(j=1 to q) kj1
to
1
2
3
+

Fig. 8



i_{b1_1}
i_{b1}
v_{b1}
i_{a1}
i_{b2_1}
i_{b2}
v_{a1}
N_1
v_{b2}
i_{bq_1}
i_{bq}
v_{bq}
i_{b1_2}
i_{a2}
i_{b2_2}
v_{a_2}
N_2
i_{bq_2}
i_{b1_p}
i_{ap}
i_{b2_p}
v_{ap}
N_p
i_{bq_p}

Fig. 9

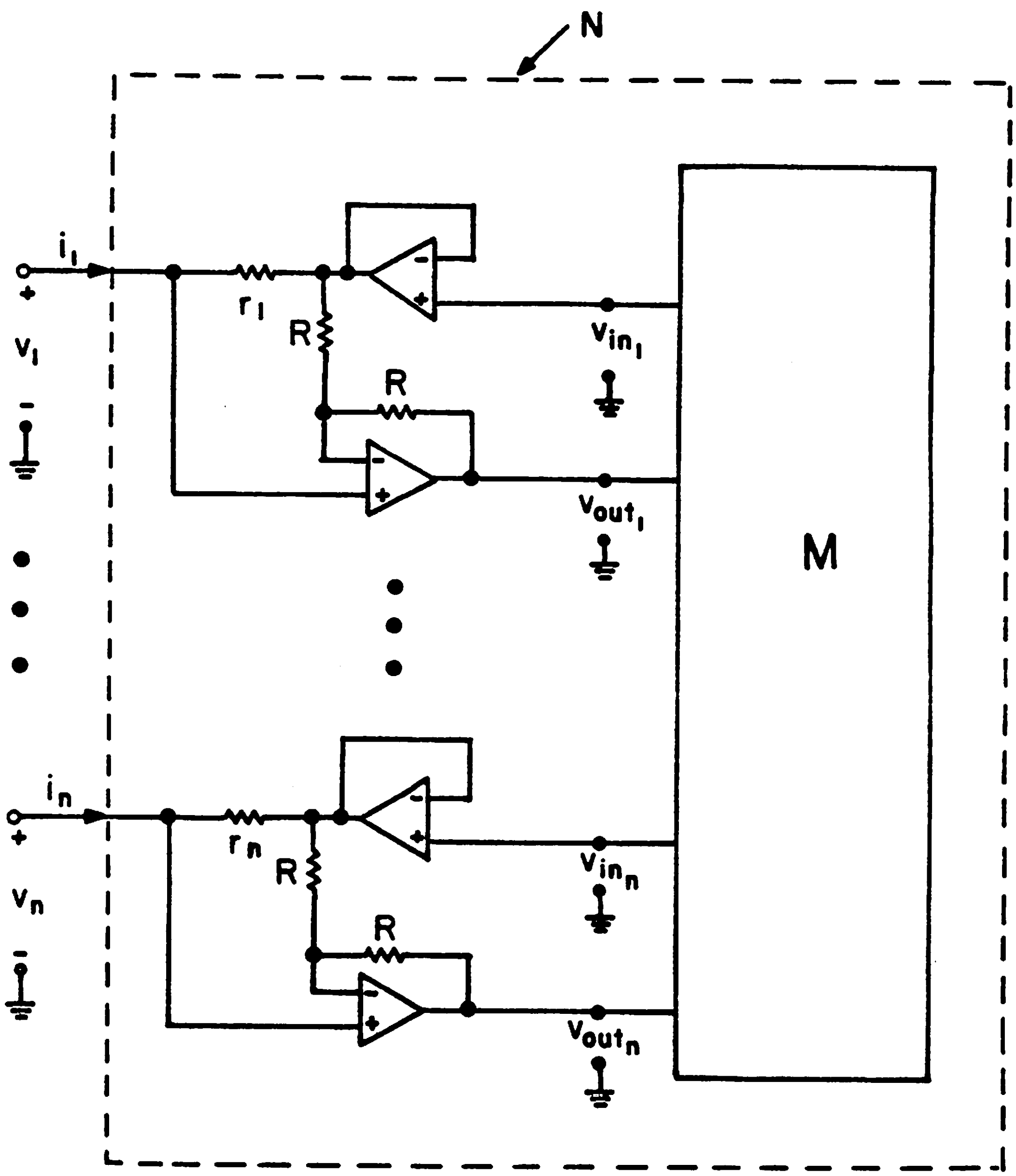
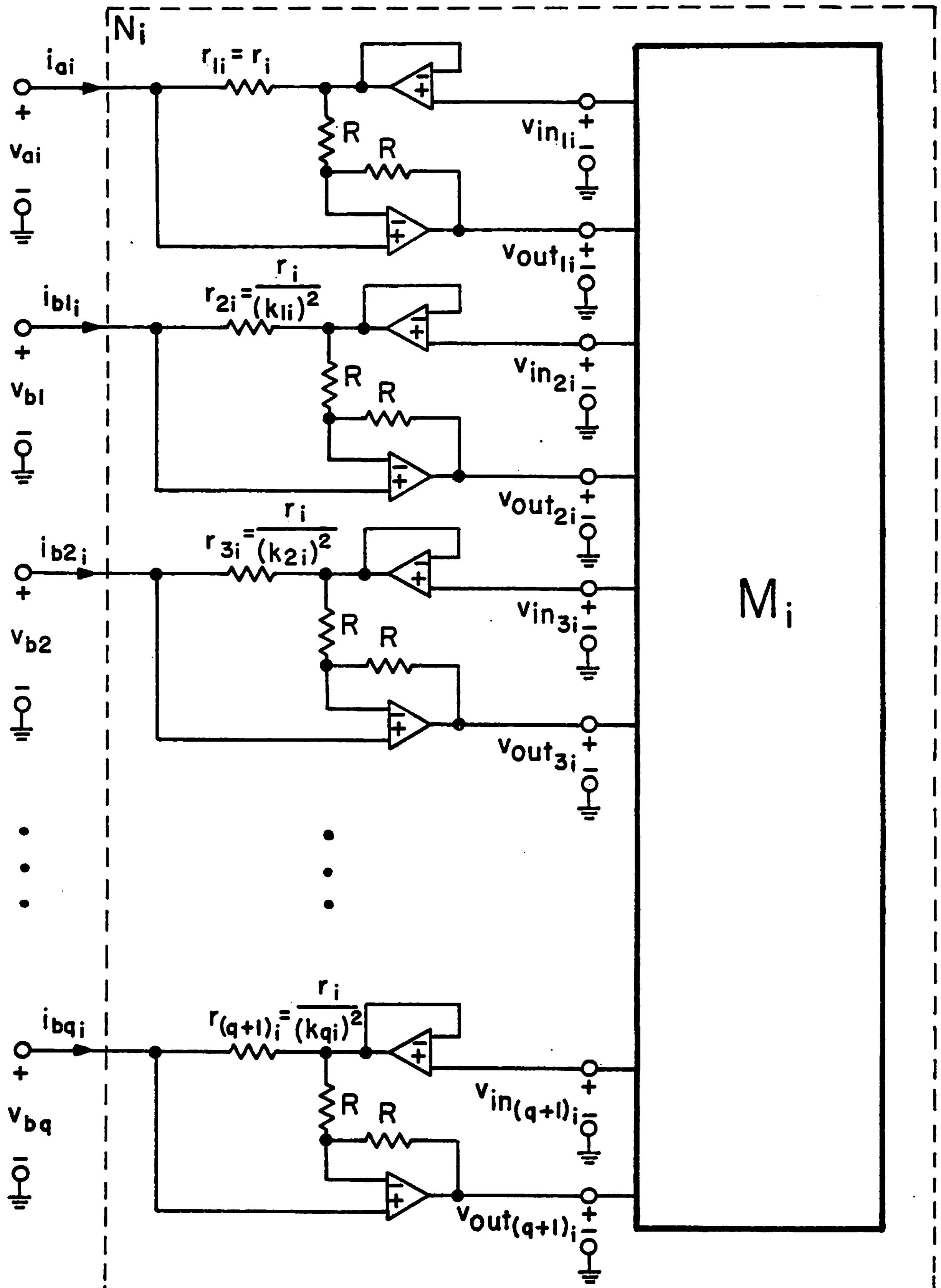


Fig. 10



Ni
iai
vai
r1i = ri
R
R
vin1i
vout1i
ib1i
vb1
r2i = ri/(k1i)^2
vin2i
vout2i
ib2i
vb2
r3i = ri/(k2i)^2
vin3i
vout3i
ibqi
vbq
r(q+1)i = ri/(kqi)^2
vin(q+1)i
vout(q+1)i
Mi

Fig. 11

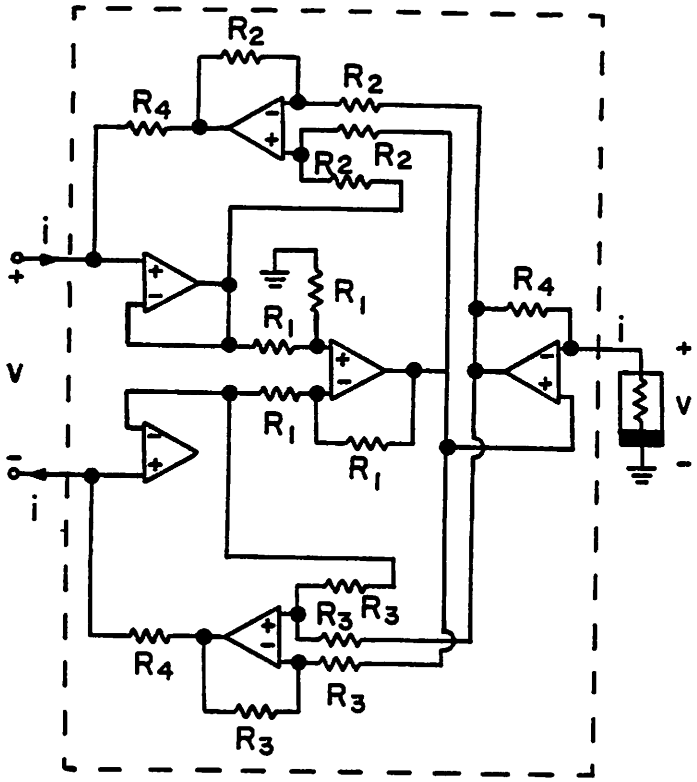


Fig. 12

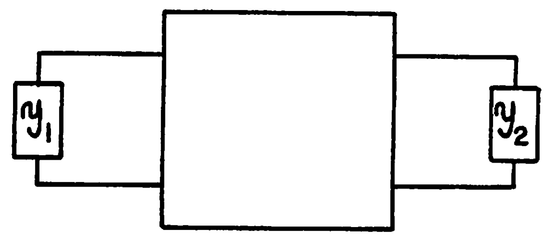


Fig. 13

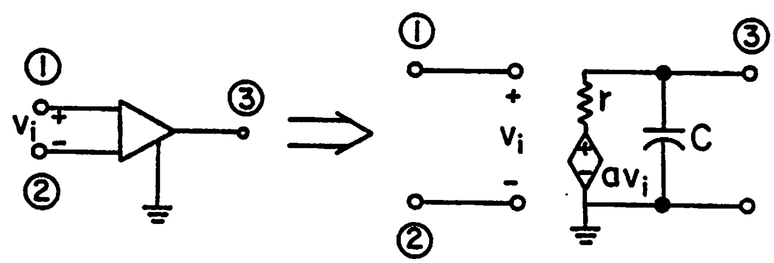


Fig. 14

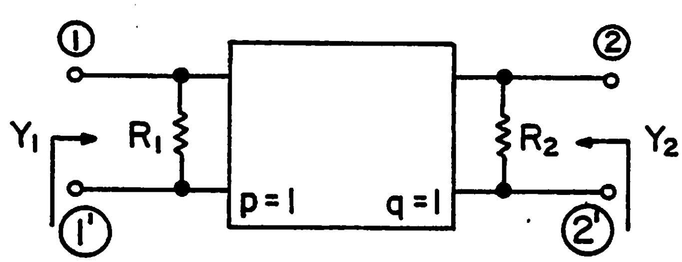


Fig. 15

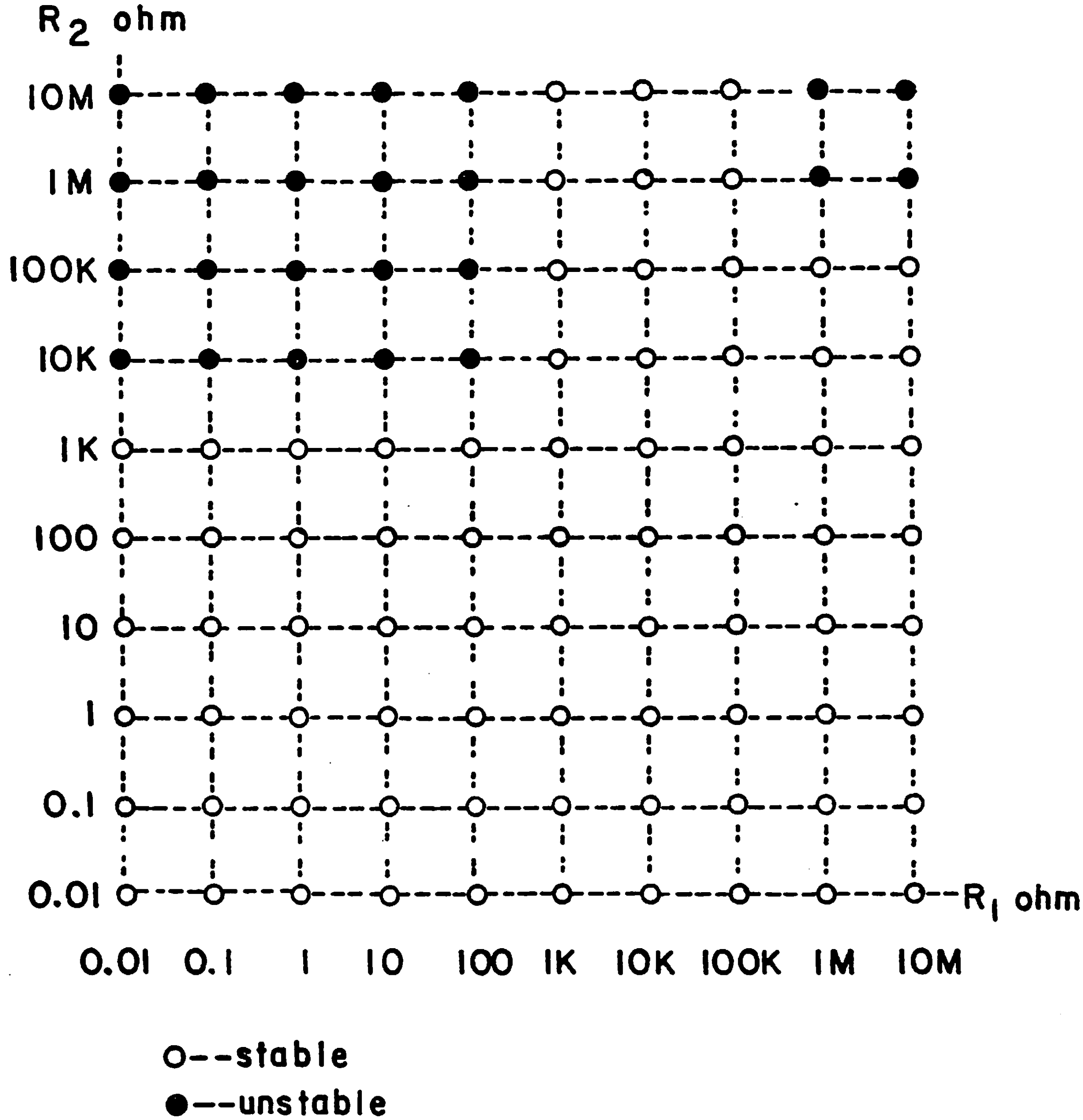
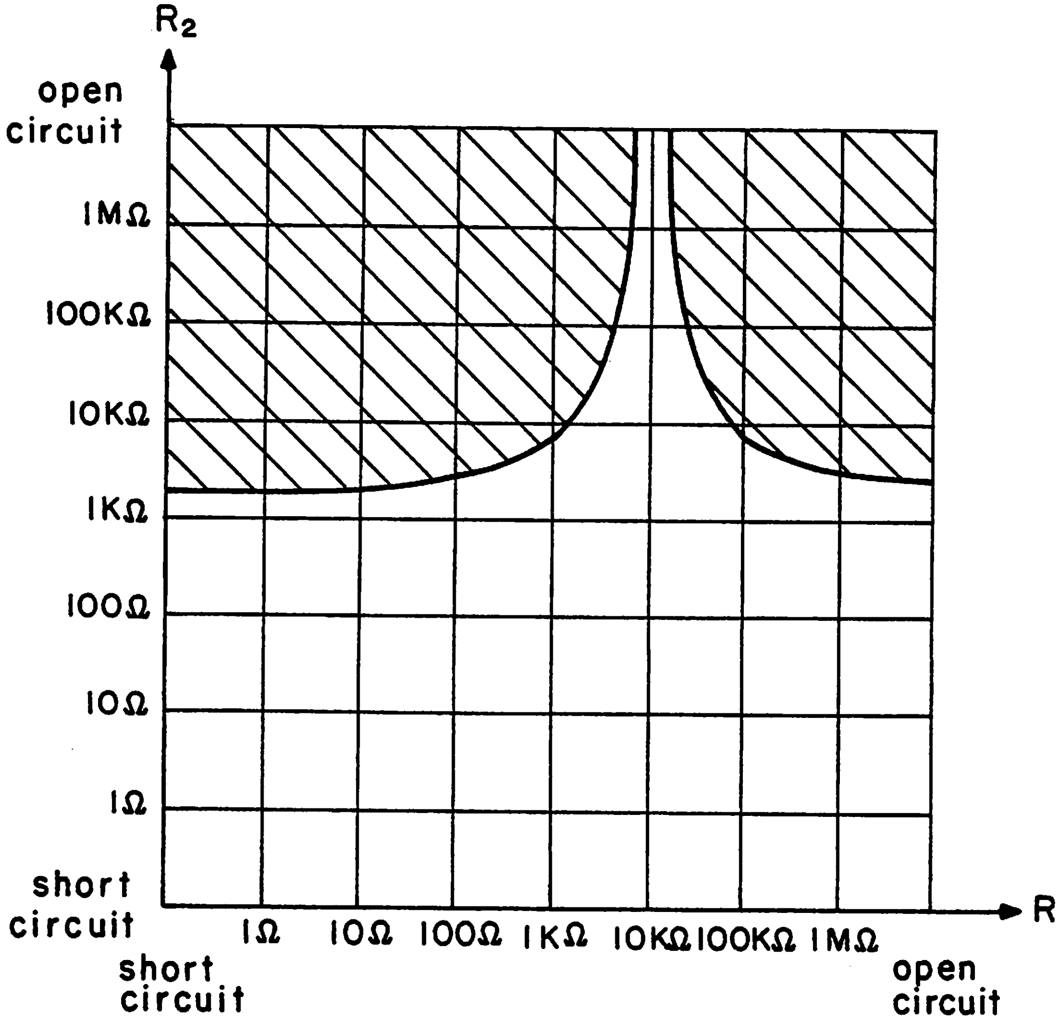
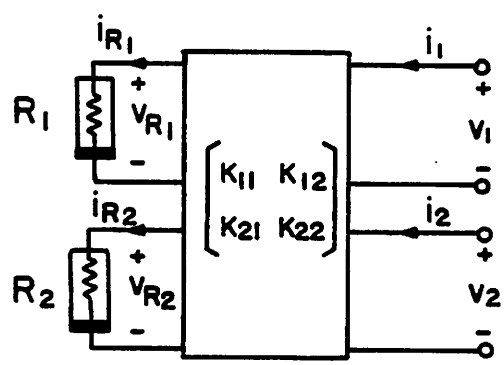

R2 ohm
10M
1M
100K
10K
1K
100
10
1
0.1
0.01
R1 ohm
0.01
0.1
1
10
100
1K
10K
100K
1M
10M
o--stable
●--unstable

Fig.16



R₂
open circuit
1MΩ
100KΩ
10KΩ
1KΩ
100Ω
10Ω
1Ω
short circuit
1Ω
10Ω
100Ω
1KΩ
10KΩ
100KΩ
1MΩ
short circuit
open circuit
R₁

Fig. 17



i_{R_1}
R_1
V_{R_1}
i_{R_2}
R_2
V_{R_2}
K_{11} K_{12}
K_{21} K_{22}
i_1
v_1
i_2
v_2

Fig. 18

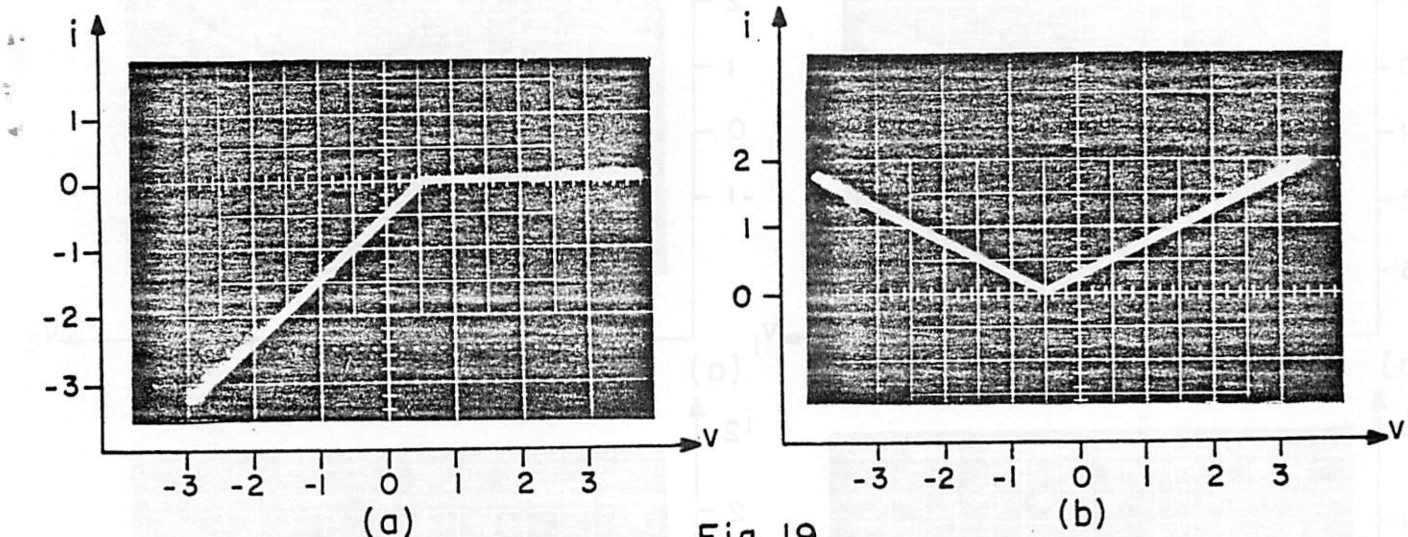


Fig. 19

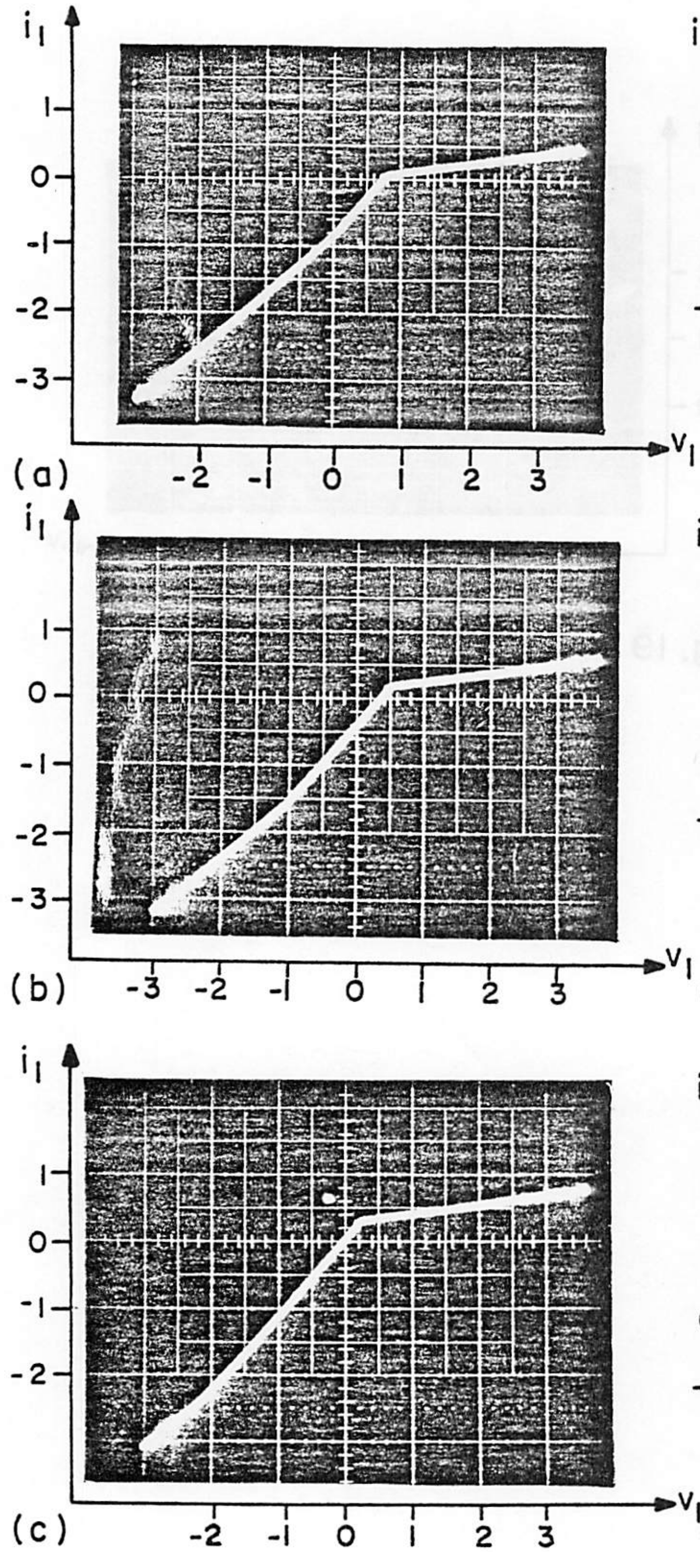


Fig. 20

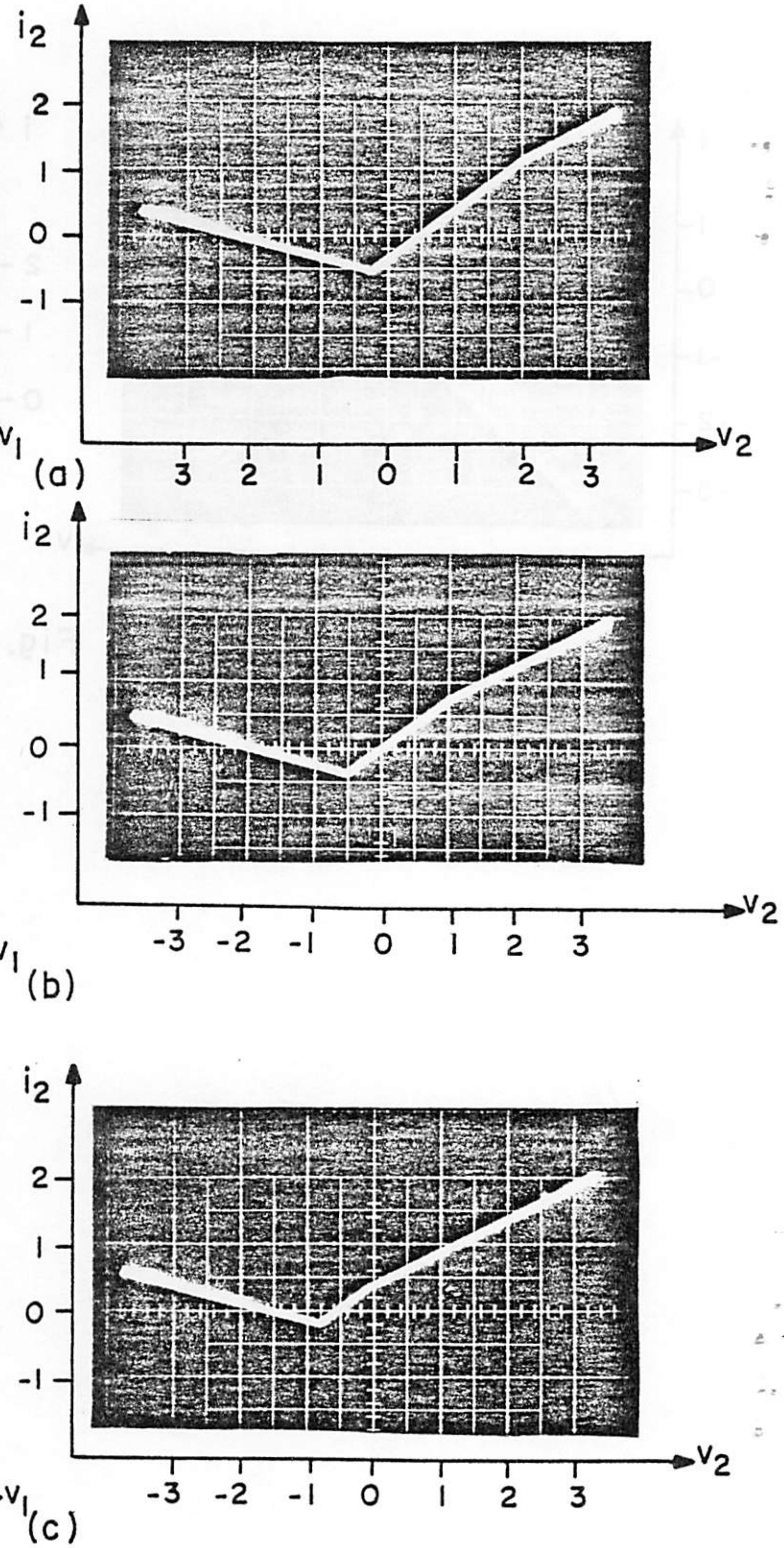


Fig. 21

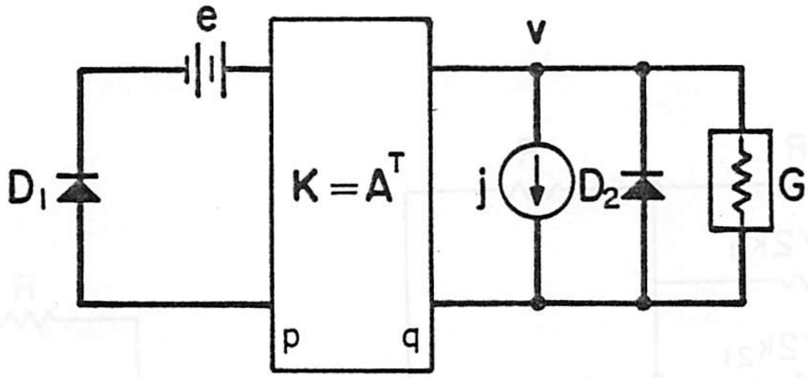


Fig. 22

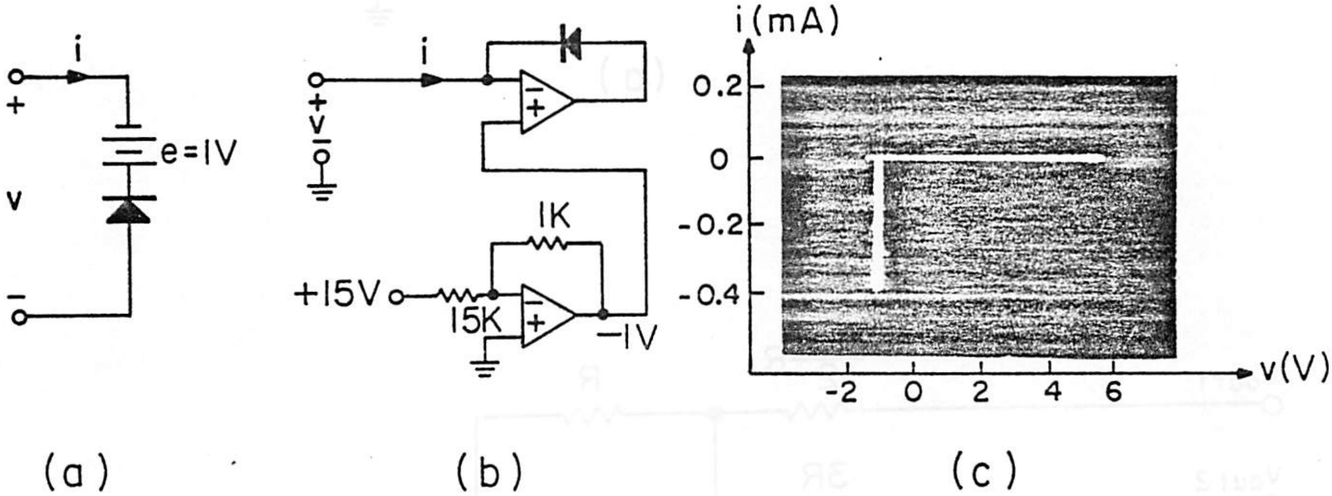


Fig. 23

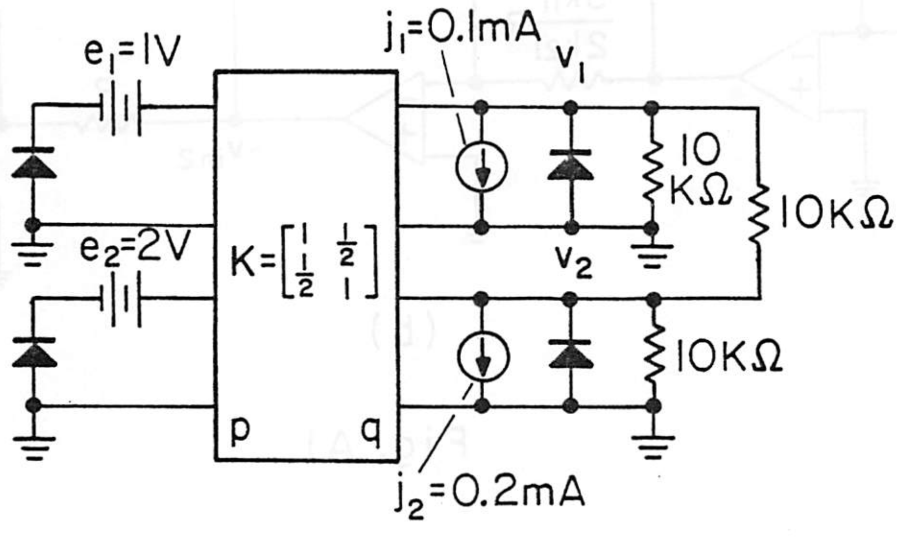
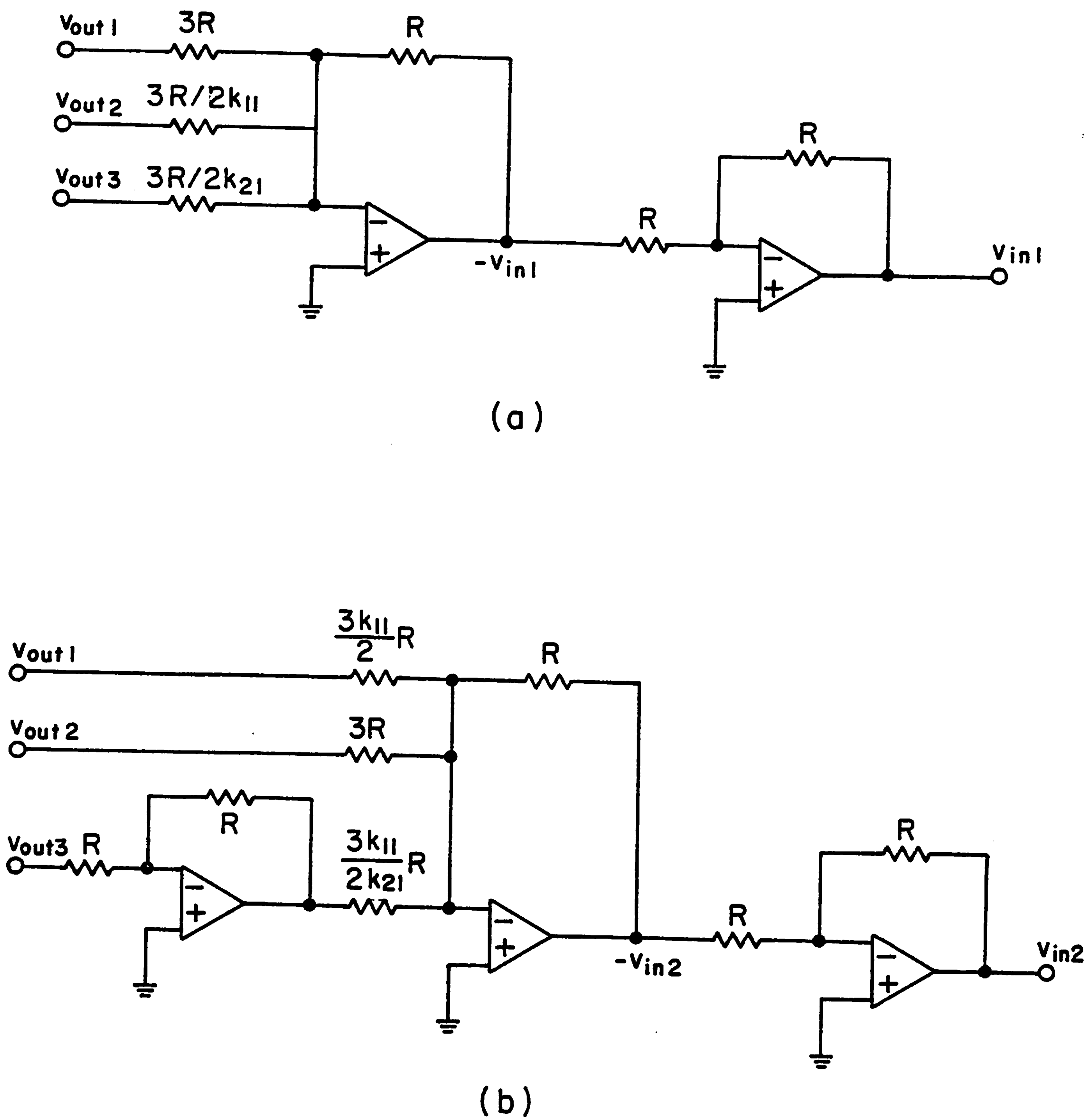
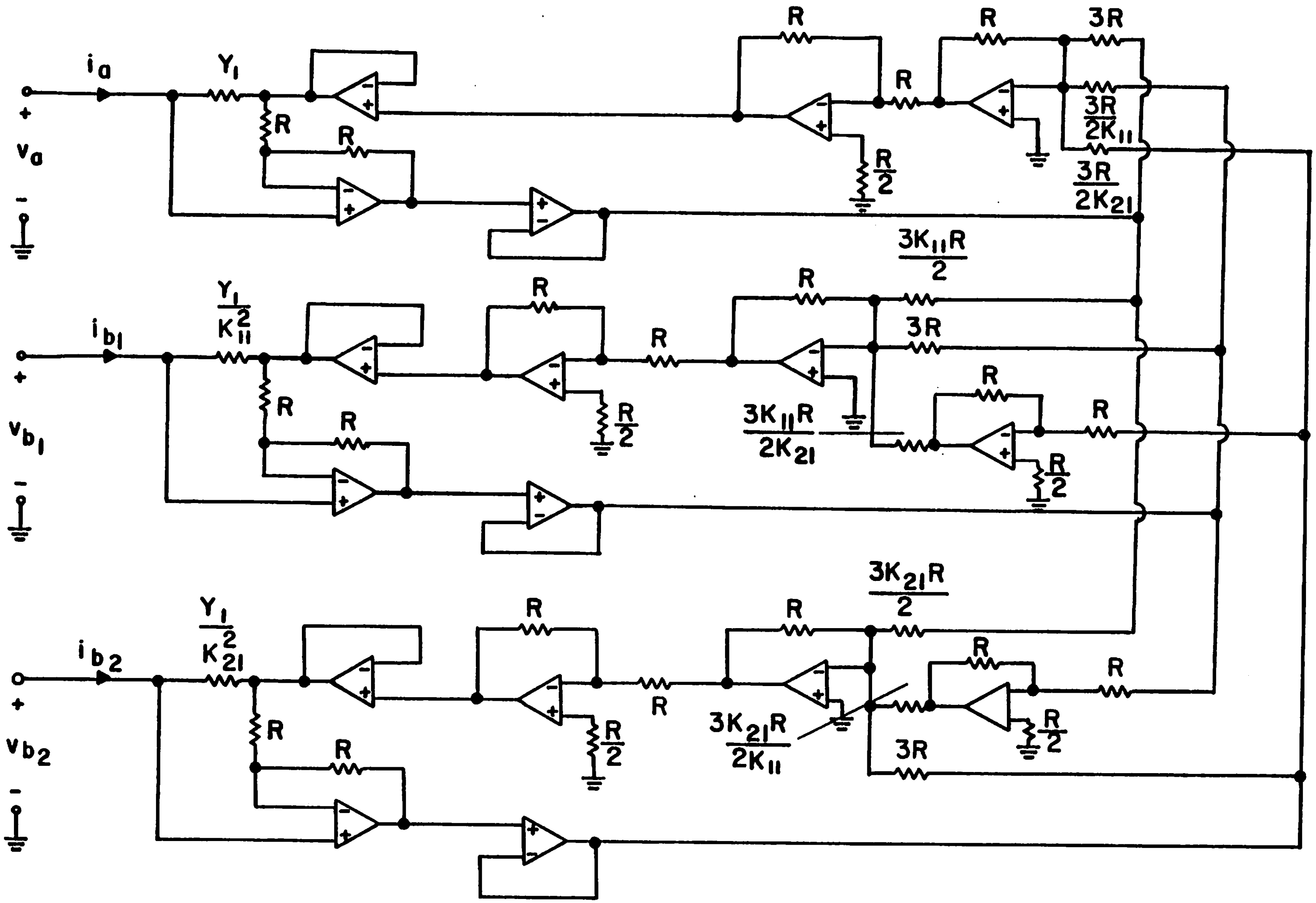


Fig. 24



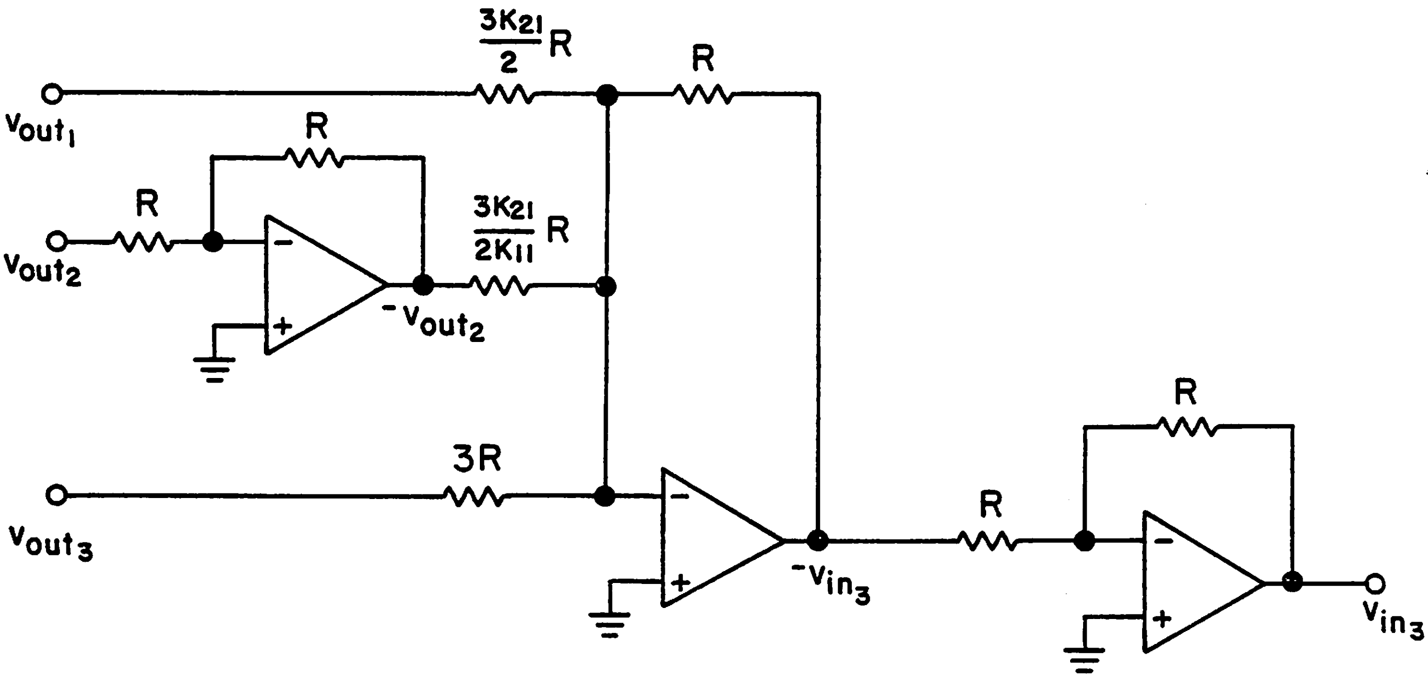
Vout 1
3R
R
Vout 2
3R/2k11
Vout 3
3R/2k21
-Vin1
R
R
Vin1
(a)
Vout 1
3k11/2 R
R
Vout 2
3R
Vout3
R
R
3k11/2k21 R
-Vin2
R
R
Vin2
(b)

Fig. A1



i_a
Y_1
v_a
+
-
R
R
R
R
R
3R
$\frac{3R}{2K_{11}}$
$\frac{R}{2}$
$\frac{3R}{2K_{21}}$
$\frac{3K_{11}R}{2}$
i_{b1}
$\frac{Y_1}{K_{11}^2}$
v_{b1}
3R
$\frac{3K_{11}R}{2K_{21}}$
$\frac{R}{2}$
$\frac{3K_{21}R}{2}$
i_{b2}
$\frac{Y_1}{K_{21}^2}$
v_{b2}
$\frac{3K_{21}R}{2K_{11}}$
3R

Fig. A2



(c)

Fig. A1

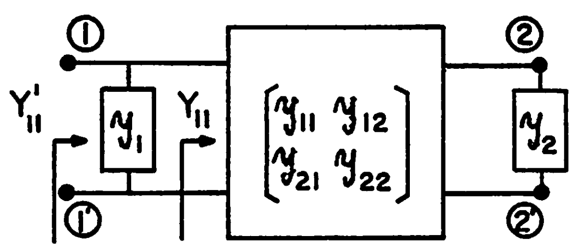


Fig. A3

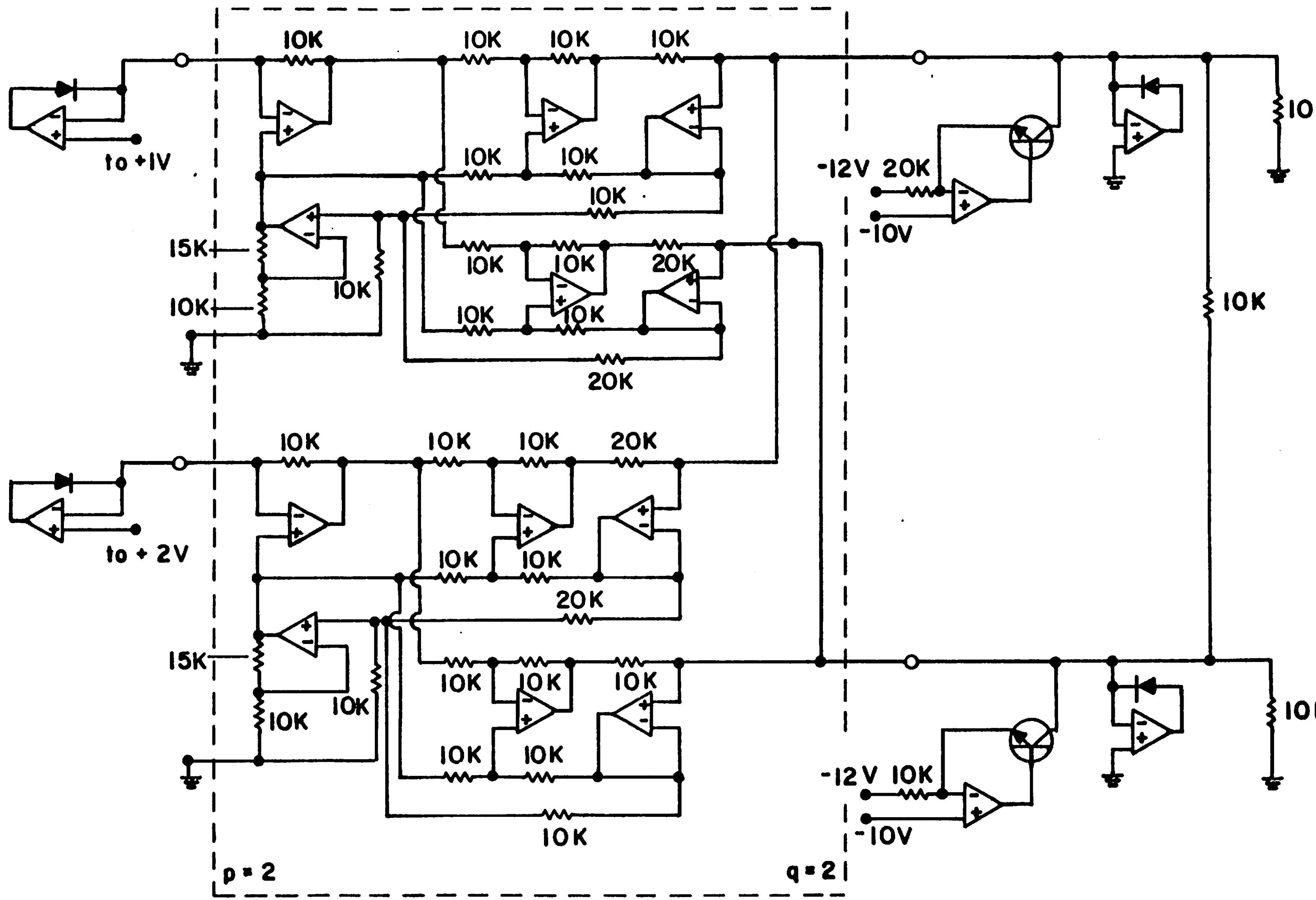

10K
10K
10K
10K
to +1V
10K
10K
10K
15K
10K
10K
10K
20K
10K
10K
10K
20K
-12V 20K
-10V
10K
10K
10K
10K
10K
10K
20K
to + 2V
10K
10K
20K
15K
10K
10K
10K
10K
10K
10K
10K
10K
-12V 10K
-10V
10K
p = 2
q = 2

Fig. A4