DIAGNOSABILITY OF NONLINEAR CIRCUITS AND SYSTEMS

PART I: THE DC CASE

by

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ABSTRACT

A theory for the diagnosability of nonlinear DC circuits (memoryless systems) is developed. Based on an input-output model, a necessary and sufficient condition for the local diagnosability of the system, which is a rank test on a matrix, is derived. Various ways of reducing the computational complexity of this test are indicated. A sufficient condition for single fault diagnosability, which is much weaker than the necessary and sufficient condition for local diagnosability is also derived. It is also shown that for diagnosable systems, it is possible to pick a finite number of test inputs that are sufficient to diagnose the system. An illustrative example is presented.

Key phrases: nonlinear resistive circuits, input-output model, local diagnosability, single fault diagnosability, Jacobian, test matrix, genericity.

I. INTRODUCTION

During the past decade, considerable research effort has been devoted to the problem of fault diagnosis of analog circuits [1-8,14-15,20-22]. Among the techniques that have approached it as a solvability problem rather than one of estimation, either based on a fault dictionary or statistical techniques, the bulk of the work has been devoted to the diagnosis of linear circuits which are tested at a single frequency (for example [1,2,3]). These techniques are characterized by the following features:

1) The faulty parameter values are (globally) uniquely determined.

2) The number of test points is greater than or equal to the number of parameters that are assumed to be faulty.

3) The diagnosis algorithm is computationally cheap and typically involves solving a set of linear equations.

Some work has been devoted to the multifrequency testing of linear dynamical systems [4,5,6]. A comprehensive theory with the following features was introduced by Sen and Saeks [4]:

1) The faulty parameters values are only <u>locally</u> uniquely determined.

2) The number of test points is much less than the number of parameters.

3) The diagnosis algorithm is computationally expensive and involves solving a set of nonlinear equations.

Some techniques, as for example in [7,8] are intermediate between the two classes described above. In these techniques sufficient conditions are given for the fault diagnosis equations to have globally unique solutions. Also, this is achieved with fewer best points than single frequency testing but these diagnosis algorithms are computationally more expensive. Both [7] and [8] however, restrict themselves to certain classes of linear systems. In particular Navid and Wilson [7] deal with circuits with linear resistors and controlled sources. Their technique is also applicable to nonlinear resistive circuits that can be effectively modelled by their small signal behavior.

In this paper we develop a theory for the fault diagnosis of general nonlinear DC circuits. We are not aware of similar results in the fault diagnosis literature. However some of our results are similar to work in other fields, as for example those of Rothenberg [9] on the identifiability of parametrized probability distribution functions. While the problems being tackled are quite different, the similarity in approach is not surprising, since the essential question being answered in [9] is the same as in fault diagnosis, i.e., one of solvability.

The characteristics of our technique are the same as those of Sen and Saeks [4] for multifrequency testing of linear circuits, which were stated earlier. The diagnosis is based on measurements made with a number of DC inputs. Note that similar to [4], only local uniqueness of the solution of the fault diagnosis equations is guaranteed. The condition for diagnosability that we derive, is therefore, only a necessary condition for globally unique diagnosis. This situation is similar to the one in optimization where solutions are generally guaranteed to satisfy only necessary optimality conditions, i.e., the sequences generated by most optimization algorithms converge to a point that satisfies a necessary condition for optimaltiy.

An important feature of our results is that, like [4], it is split into two distinct parts: (1) conditions for local diagnosability and, (2) the diagnosis algorithm. Our emphasis in this paper is on the former and we deal only briefly with the diagnosis algorithm. The necessary and sufficient condition for local diagnosability that we derive is determined by the system structure and the location of the inputs and outputs of the system. In addition, it does not depend on the choice of test signals, the test algorithm or the faulty parameter values. Such a criterion, being a property solely of the system under consideration, can be used as a design aid to check the testability of a circuit, to choose testpoints and to design "testable" circuits.

Along the same lines as most of the results in the literature, we assume in this paper that, for the systems under consideration, the faults can be adequately represented as variations (possibly large) in a set of parameters, α. We deal with both the case where all the parameters are assumed to be faulty and the single-fault case, i.e., the case where only one of the parameters is assumed to be faulty and the problem is to determine which one and its faulty value.

The paper is organized as follows. In Section II we formulate the system description and introduce some preliminary definitions. In Section III we introduce the concept of local diagnosability and derive a necessary and sufficient condition for the local diagnosability of a parameter point and then extend it to give a condition for the local diagnosability of the system. In Section IV we discuss conditions under which the test for local diagnosability is simpler than the one presented in Section III and we also discuss test input selection. In Section V we deal with single fault diagnosability. In Section VI we discuss a numerical example and present a summary in Section VII. An earlier version of these results was presented in [10].

II. SYSTEM DESCRIPTION

A nonlinear resistive circuit is usually described [11,12] by an algebraic equation of the form

$$g(x,\omega,\alpha) = 0 \tag{2.1}$$

where $x \in \mathbb{R}^m$, $\omega \in \mathbb{R}^{q+p}$, $\alpha \in \mathbb{R}^N$ and $g : \mathbb{R}^m \times \mathbb{R}^{q+p} \times \mathbb{R}^N \to \mathbb{R}^{m+q}$ is assumed to be continuously differentiable. The variables x, α and ω denote, respectively, the internal variables, the parameters and the accessible (input or output) variables. Since, for a meaningful model, once the parameters and the inputs are fixed, the remaining variables are uniquely determined, it follows from the dimensions of the various quantities in (2.1) that p of the components of the vector ω are inputs while the remaining q are outputs. Typically, in networks, there is some flexibility in determining which p components of ω are inputs. Since fault diagnosis is a problem of determining the internal parameter values from input-output measurements, one might imagine that this choice is crucial. We will however establish that, in fact, for our model description this choice is inconsequential.

Note that, the model (2.1) can, more generally, be used to describe the steady state behavior of a nonlinear dynamical circuit that attains a DC steady state when the inputs are constant functions of time. For simplicity, consider a nonlinear dynamical circuit which has uncoupled two-terminal capacitors and inductors and is described by the Sparse Tableau equations [11]. We focus our attention on an inductor, which is described as follows:

$$\dot{\phi}(t) = v(t) \qquad \forall t \geq 0 \tag{2.2a}$$

$$h(\phi(t), i(t), \beta) = 0 \tag{2.2b}$$

where, ϕ, i, v and β, denote respectively, the flux, current, voltage and parameter associated with this inductor and (2.2b) is its constitutive relation. In the DC steady state, (2.2) is replaced by

$$v = 0 \tag{2.3a}$$

$$h(\phi,i,\beta) = 0 \tag{2.3b}$$

The above equation describes an inductor with a steady state flux ϕ, a steady state current i and which is a short circuit in DC.

Since the accessible signals are voltages and currents rather that charges or fluxes, the flux ϕ is not accessible through DC testing, hence the value of the parameter β cannot be determined. On the other hand removing (2.3b) from the DC steady state description of the circuit will in no way affect its consistency since the variable ϕ is not involved in any other equation. Thus, with DC testing of dynamical circuits, capacitor and inductor faults cannot be diagnosed. However, by modelling each capacitor as a parallel combination of the ideal capacitor and an appropriate leakage resistor, and each inductor as a series connection of the ideal inductor and a leakage resistor, open circuits, short circuits and increased leakage in the capacitors and inductors can be determined. Thus the DC approach is capable of diagnosing a dynamical circuit for all the faults in the resistors and the above mentioned faults in the capacitors and inductors. We call such faults static faults.

The above discussion leads us to the following conclusions:

1) For a dynamical circuit, with DC testing, we cannot determine the inductor and capacitor parameters (the values of L and C in the case of linear elements).

2) It is not necessary to assume that the capacitors and inductors are fault-free to diagnose the static faults in the circuit.

3) The circuit equations for the dynamical circuit in DC steady state, with the inductor and capacitor constitutive relations ((2.3b), for example) removed, corresponds to a resistive network obtained from the original network by replacing all inductors with short circuits and all capacitors with open circuits. This reduced resistive network is represented generically, by (2.1). Static faults in the original network can be diagnosed using DC testing if and only if (2.1) is diagnosable.

We will henceforth refer to (2.1) as <u>the system</u>. Let $\Omega \subset \mathbb{R}^{m+q+p+N}$ be the set of all admissible variables of the system, i.e.,

$$\Omega \triangleq \{(x,\omega,\alpha) \in \mathbb{R}^{m+q+p+N} | g(x,\omega,\alpha) = 0\}$$

Let $A \subset \mathbb{R}^N$ be the set of all admissible parameters of the system, i.e.,

$$A \triangleq \{\alpha \in \mathbb{R}^N | \exists (x,\omega) \in \mathbb{R}^{m+q+p} \ni (x,\omega,\alpha) \in \Omega\}$$

For each $\alpha \in A$, let W_α denote the set of all admissible observations i.e.,

$$W_\alpha \triangleq \{\omega \in \mathbb{R}^{p+q} \Big| \exists x \in \mathbb{R}^m \ni (x,\omega,\alpha) \in \Omega\} \quad . \tag{2.4}$$

<u>Definition 2.1</u>: Two parameter points α^1 and $\alpha^2 \in A$ are said to be <u>observationally equivalent</u> if $W_{\alpha^1} = W_{\alpha^2}$. ¤

Our analysis of the diagnosability of the system is based on the assumption that it possesses an input-output model. To this end we have the following.

<u>Definition 2.2</u>: The system is said to have an <u>input-output model</u> $y = f(u,\alpha)$ if there exists a partition $(y,u) \in \mathbb{R}^q \times \mathbb{R}^p$ of ω and two functions $h : \mathbb{R}^p \times \mathbb{R}^N \to \mathbb{R}^m$ and $f : \mathbb{R}^p \times \mathbb{R}^N \to \mathbb{R}^q$ such that

1) $\Omega = \{(h(u,\alpha), (f(u,\alpha),u),\alpha) \in \mathbb{R}^{M+q+p+N} | (u,\alpha) \in \mathbb{R}^p \times \mathbb{R}^N\}$

2) $D_{x,y}g(x,y,u,\alpha)$ is nonsingular $\forall\ (x,y,u,\alpha) \in \Omega$ ¤

Note that in the above definition the variable u is associated with the inputs and the variable y with the outputs. Also, $D_{x,y}g(x,y,u,\alpha)$ denotes the Jacobian of g with respect to x and y evaluated at (x,y,u,α). Other Jacobians are similarly defined. Note that when the system has an input-output model, the parameter space $A = \mathbb{R}^N$ and $\mathbb{R}^p$ may be considered to be the input space. Also, $f(u,\alpha)$ is continuously differentiable with respect to u and α.

We will henceforth assume that the system possesses an input-output model. For a large-scale system, it is not an easy task to determine the input-output model in symbolic form. However, as will become clear in the following sections, we do not require that the symbolic form of the input-output model be known. We only require that given $(\hat{u},\hat{\alpha}) \in \mathbb{R}^p \times \mathbb{R}^N$, $f(\hat{u},\hat{\alpha})$ and $D_\alpha f(\hat{u},\hat{\alpha})$ can be determined. A simulation of (2.1) with $u = \hat{u}$ and $\alpha = \hat{\alpha}$ will give us $f(\hat{u},\hat{\alpha})$ while $D_\alpha f(\hat{u},\hat{\alpha})$ can be determined from the following equation which is due to the implicit function theorem [13].

$$\begin{bmatrix} D_\alpha h(\hat{u},\hat{\alpha}) \\ D_\alpha f(\hat{u},\hat{\alpha}) \end{bmatrix} = -D_{x,y}g(h(\hat{u},\hat{\alpha}),f(\hat{u},\hat{\alpha}),\hat{u},\hat{\alpha})^{-1}\ D_\alpha g(h(\hat{u},\hat{\alpha}),f(\hat{u},\hat{\alpha}),\hat{u},\hat{\alpha}) \quad (2.5)$$

<u>Proposition 2.1</u>: Two parameter points α^1 and $\alpha^2 \in \mathbb{R}^N$ are observationally equivalent if and only if

$$f(u,\alpha^1) = f(u,\alpha^2) \qquad \forall u \in \mathbb{R}^p$$

Proof: The proof follows directly from (2.4) and Definitions 2.1 and 2.2. ¤

The above proposition establishes that the concept of observationally equivalent parameters, which was formulated in Definition 2.1 on the basis of the implicit description (2.1), is completely characterized by any one input-output model. Since our definition of local diagnosability (Definition 3.1) and single fault diagnosability (Definition 5.2) are based on the concept of observationally equivalent parameters, it follows that even for systems that have more than one input-output model, in order to study the diagnosability of such a system, we may restrict our attention to any one input-output model.

III. LOCAL DIAGNOSABILITY

In this section, we first define local diagnosability of a parameter point and local diagnosability of the system. Then, after some preliminary results, we present a theorem that gives a necessary and sufficient condition for local diagnosability of a parameter point. For this theorem, we present both an intuitive explanation and a rigorous proof. We then extend the theorem to give a condition for local diagnosability of the system.

Definition 3.1: A parameter point, $\alpha^0 \in \mathbb{R}^N$, is said to be locally diagnosable if there exists an open neighborhood of α^0 containing no other α which is observationally equivalent to it. We say that the local diagnosability properly holds at a point $\alpha^0 \in \mathbb{R}^N$ if α^0 is locally diagnosable. ¤

In this paper we will often consider generic properties. We make precise the concept of genercity in the following definition.

Definition 3.2: Consider a property as a logic function $\pi(\cdot) : \mathbb{R}^N \to \{T,F\}$ where $\pi(\alpha) = T$ (or F) if π holds (or fails) at $\alpha \in \mathbb{R}^N$. The property is said to be generic in $\mathbb{R}^N$ if there exists $B \subset \mathbb{R}^N$, where B is a closed set with zero Lebesgue measure[1] such that $\pi(\alpha) = F$ only if $\alpha \in B$. ¤

We will sometimes refer to a generic property as a property that holds for almost all parameter points in $\mathbb{R}^N$.

Definition 3.3: The system is said to be locally diagnosable if the local diagnosability property is generic in $\mathbb{R}^N$. ¤

Definition 3.4: Let $M(\alpha)$ be a matrix whose elements are continuous functions of α everywhere in $\mathbb{R}^N$. A parameter point $\alpha^0 \in \mathbb{R}^N$ is said to be a regular point of $M(\alpha)$ if there exists an open neighborhood of α^0 in which $M(\alpha)$ has constant rank. ¤

Definition 3.5: A function $v(\cdot) : \mathbb{R}^p \to \mathbb{R}$ is said to be a weighting function if:

a) $v(u)$ is continuous with respect to u

b) $v(u) > 0 \quad \forall u \in \mathbb{R}^p$ ¤

We now introduce the test matrix

$$R(\alpha) \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} v(u)\, D_\alpha f(u,\alpha)^t\, D_\alpha f(u,\alpha)\, du_1 du_2 \cdots du_p \qquad (3.1)$$

where $D_\alpha f(u,\alpha)^t$ is the transpose of $D_\alpha f(u,\alpha)$. Note that $R(\alpha)$ is a symmetric positive semidefinite matrix in $\mathbb{R}^{N\mathrm{x}N}$.

We now present a key lemma which links the rank of $R(\alpha)$ to the properties of the null space and range space of $D_\alpha f(u,\alpha)$.

[1]Intuitively, "zero volume," as for example, an (N-1)-dim hyperplane in $\mathbb{R}^N$.

Lemma 3.1: Let $v(\cdot)$ be any weighting function such that the matrix function of α, $R(\alpha)$ exists $\forall\alpha \in \mathbb{R}^N$. Consider a partitioning of the vector α as

$$\alpha = \begin{bmatrix}\gamma\\ \delta\end{bmatrix}$$

where $\gamma \in \mathbb{R}^\rho$, $\rho < N$, and $\delta \in \mathbb{R}^{N-\rho}$,

and the associated partitioning

$$R(\alpha) = \begin{bmatrix} R_\gamma(\alpha) & R_{\gamma,\delta}(\alpha) \\ R^t_{\gamma,\delta}(\alpha) & R_\delta(\alpha) \end{bmatrix}$$

where $R_\gamma(\alpha) \in \mathbb{R}^{\rho x\rho}$, $R_\delta \in \mathbb{R}^{(N-\rho)x(N-\rho)}$ and $R_{\gamma,\delta}(\alpha) \in \mathbb{R}^{\rho x(N-\rho)}$. Suppose that

a) α^0 is a regular point of $R(\alpha)$,

b) rank $[R(\alpha^0)] = \rho$,

c) $R_\gamma(\alpha^0)$ is positive definite.

Under these conditions

1) There exists $B(\alpha^0)$, an open neighborhood of α^0, and $L(\alpha)$, a continuous $\rho x(N-\rho)$ matrix function of α, such that

$$[D_\alpha f(u,\alpha)] \begin{bmatrix} L(\alpha) \\ I_{(N-\rho)x(N-\rho)} \end{bmatrix} = 0_{qx(n-\rho)} \quad \forall u \in \mathbb{R}^p, \quad \forall\alpha \in B(\alpha^0).$$

2) $\exists u^i \in \mathbb{R}^p$, $i = 1,\cdots\ell$, $\ell \leq \rho$ such that the first ρ columns of the matrix[2]

$$\mathrm{col}(D_\alpha f(u^1,\alpha^0),\cdots,D_\alpha f(u^\ell,\alpha^0))$$

are linearly independent.

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[2] $\mathrm{col}(A,B) = \begin{bmatrix}A\\ B\end{bmatrix}$.

Proof: Since α^0 is a regular point and the entries of $R(\alpha)$ are continuous, $\exists B(\alpha^0)$ an open neighborhood of α^0, such that $\forall\alpha \in B(\alpha^0)$, the rows of the matrix $[R^t_{\gamma,\delta}(\alpha)\ R_\gamma(\alpha)]$ are linearly dependent on the rows of $[R_\gamma(\alpha)\ R_{\gamma,\delta}(\alpha)]$. Hence $\forall\alpha \in B(\alpha^0)$

$$\begin{bmatrix} R_\gamma(\alpha) & R_{\gamma,\delta}(\alpha) \\ R^t_{\gamma,\delta}(\alpha) & R_\delta(\alpha) \end{bmatrix} \begin{bmatrix} -R_\gamma(\alpha)^{-1}\ R_{\gamma,\delta}(\alpha) \\ I_{(N-\rho)x(N-\rho)} \end{bmatrix} = 0_{Nx(N-\rho)} \tag{3.2}$$

Let,

$$L(\alpha) \triangleq -R_\gamma(\alpha)^{-1}\ R_{\gamma,\delta}(\alpha)$$

Note that $L(\alpha)$ is a continuous function of α. It now follows from (3.2) and the definition of $R(\alpha)$ that

$$D_\alpha f(u,\alpha) \begin{bmatrix} L(\alpha) \\ I \end{bmatrix} = 0_{qx(N-\rho)} \quad \forall u \in \mathbb{R}^p, \quad \forall\alpha \in B(\alpha^0)$$

which is the first result. To prove the second result, we first note that there exists $u^1 \in \mathbb{R}^p$ such that[3] $D_{\alpha_1} f(u^1,\alpha^0) \neq 0$, since if it were not so, it would contradict the positive definiteness of $R_\gamma(\alpha^0)$. Now suppose that for $k < \rho$, we have m inputs, $u^1,\cdots,u^m$, $m \leq k$, such that the matrix

$$H_{m,k} = \begin{bmatrix} D_{\alpha_1} f(u^1,\alpha^0) & \cdots & D_{\alpha_k} f(u^1,\alpha^0) \\ \vdots & & \\ D_{\alpha_1} f(u^m,\alpha^0) & \cdots & D_{\alpha_k} f(u^m,\alpha^0) \end{bmatrix}$$

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[3] α_i is the i-th entry of α.

has full column rank. Hence it has a kxk nonsingular submatrix which can be described as the product $SH_{m,k}$, where S is a kxqm selector matrix that selects k linearly independent rows of $H_{m,k}$. To prove the result, it is enough to show that there exists $u^{m+1} \in \mathbb{R}^p$ such that the matrix $H_{m+1,k+1}$ has full column rank. Suppose that the statement is false. Then there exists no choice of inputs $u^{m+1} \in \mathbb{R}^p$ such that $H_{m+1,k+1}$ has full column rank, i.e., there exists a qxk matrix C(u) such that, $\forall u \in \mathbb{R}^p$,

$$[D_{\alpha_1}(u,\alpha^0)\cdots D_{\alpha_{k+1}}(u,\alpha^0)] = C(u)\ SH_{m,k+1} \tag{3.3}$$

Substituting (3.3) in (3.1) and letting $R^{k+1}(\alpha^0)$ denote the (k+1) x (k+1) principal submatrix of $R(\alpha^0)$, we have

$$R^{k+1} = [SH_{m,k+1}]^t[\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} v(u)\ C(u)^t C(u)\ du_1\cdots du_p][SH_{m,k+1}].$$

Hence its rank is, at most k, which contradicts the positive definiteness of $R_\gamma(\alpha^0)$. Hence the lemma is proved. ¤

We now present a theorem that gives a necessary and sufficient condition for the diagnosability of a regular parameter point.

Theorem 3.1. Let $v(\cdot)$ be any weighting function such that the matrix-valued function of α, $R(\alpha)$, exists $\forall\alpha \in \mathbb{R}^N$, and let α^0 be a regular point[4] of $R(\alpha)$. Under these conditions, the parameter point α^0 is locally diagnosable if and only if $R(\alpha^0)$ is positive definite. ¤

[4]The regularity assumption is crucial. In Theorem 3.2 and Proposition 3.1 we will discuss conditions under which almost all $\alpha \in \mathbb{R}^N$ are regular points.

The following is a plausibility argument for the above theorem. Consider first the case where the input space consists of a finite number of elements $u_1,\cdots,u_k$. Since we can do no better than test the system with all the available inputs, the parameter point α^0 is locally diagnosable if and only if α^0 is a locally unique solution of the concatenated set of input-output equations, i.e., the matrix $\mathrm{col}(D_\alpha f(u^i,\alpha), i = 1,\cdots k)$ has full column rank. Since the rank of a matrix equals that of its transpose multiplied by itself, this is true if and only if the matrix

$$\sum_{i=1}^{k} D_\alpha f(u^i,\alpha^0)^t D_\alpha f(u^i,\alpha^0)$$

has full rank. In the case where the input space consists of a countably infinite number of points, u^i, $i = 1,\cdots,\infty$, the above argument generalizes to the statement that the required condition is that the "matrix" $\mathrm{col}(D_\alpha f(u^i,\alpha), i = 1,\cdots,\infty)$ be full column rank. "Generalizing" the fact that the rank of a matrix equals that of its transpose multiplied by itself, we can compactly represent the required condition as

$$\det\left[\sum_{i=1}^{\infty} v(u^i) D_\alpha f(u^i,\alpha^0)^t D_\alpha f(u^i,\alpha^0)\right] \neq 0 \qquad (3.4)$$

where $v(u^i)$ is a weighting function that has been included to guarantee that the infinite series has a finite sum. The natural extension of (3.4) to the case where we have a continuum of inputs is to replace the summation by an integral, to derive the condition of Theorem 3.1, i.e.,

$$\det R(\alpha^0) = \det\left[\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} v(u) D_\alpha f(u,\alpha)^t D_\alpha f(u,\alpha)du_1,\ldots du_p\right] \neq 0$$

Proof of Theorem 3.1:

If: By the integral form of the mean-value theorem, we have, $\forall u \in \mathbb{R}^p$ and $\forall\alpha$ in any neighborhood of α^0

$$f(u,\alpha) - f(u,\alpha^0) = \int_0^1 D_\alpha f(u,s\alpha+(1-s)\alpha^0)ds\ [\alpha-\alpha^0] \tag{3.5}$$

Suppose now that α^0 is not locally diagnosable. Then there exists an infinite sequence of vectors $\alpha^k \to \alpha^0$, $k = 1,\cdots\infty$, such that $\forall k \in \mathbb{N}$,

$$f(u,\alpha^k) = f(u,\alpha^0) \quad \forall u \in \mathbb{R}^p$$

Using (3.5), we have for all u and k,

$$\int_0^1 D_\alpha f(u,s\alpha^0 + (1-s)\alpha^k)ds a^k = 0$$

where $a^k \in \mathbb{R}^N$ is defined as

$$a^k = \frac{\alpha^k-\alpha^0}{\|\alpha^k-\alpha^0\|}$$

The sequence a^k is an infinite sequence on the unit sphere and therefore it has a limit point a on the unit sphere. As $\alpha^k \to \alpha^0$, a^k approaches a along a subsequence, and in the limit, we have

$$\int_0^1 D_\alpha f(u,\alpha^0)ds\ a = 0 \qquad \forall u \in \mathbb{R}^p$$

i.e.,

$$D_\alpha f(u,\alpha^0)a = 0 \qquad \forall u \in \mathbb{R}^p$$

But this implies that independent of the choice of weighting function,

$$a^t R(\alpha^0)a = 0$$

Hence $R(\alpha^0)$ is not positive definite.

Only If: Suppose that the positive semidefinite matrix $R(\alpha^0)$ is not positive definite, i.e., it has rank $\rho < N$. Without loss of generality, we assume that the $\rho\times\rho$ principal submatrix $R_\gamma(\alpha^0)$ (notation of Lemma 3.1) is positive definite. Since α^0 is a regular point of $R(\alpha)$, we can apply Lemma 3.1. Applying the second result of the lemma, we note that, $\exists u^i \in \mathbb{R}^p$, $i = 1,\cdots,\ell$, $\ell \leq \rho$ such that the first ρ columns of the matrix

$$\mathrm{col}(D_\alpha f(u^1,\alpha^0),\ D_\alpha f(u^2,\alpha^0),\cdots,D_\alpha f(u^\ell,\alpha^0))$$

are linearly independent. Let y^i, $i = 1,\cdots\ell$, be defined by

$$y^i = f(u^i,\alpha^0) \quad i = 1,\cdots\ell \ . \tag{3.6}$$

Let the equations

$$\hat{y} = \hat{F}(\hat{u},\gamma,\delta) \tag{3.7}$$

where $\hat{u} \triangleq \mathrm{col}[u^i, i = 1,\cdots\ell]$, represent a subset of ρ equations of (3.6), whose Jacobian, represented by

$$[D_\gamma\hat{F}(\hat{u},\gamma,\delta)\ D_\delta\hat{F}(\hat{u},\gamma,\delta)]$$

has the property that the $\rho\times\rho$ submatrix $D_\alpha\hat{F}(\hat{u},\gamma^0,\delta^0)$ is nonsingular. Applying the implicit function theorem [13] to (3.7), we conclude that there exists $B(\gamma^0) \subset \mathbb{R}^\rho$ and $B(\delta^0) \subset \mathbb{R}^{N-\rho}$ such that, $\forall\delta \in B(\delta^0)$, (3.7) has a unique solution $\gamma = \psi(\delta) \in B(\gamma^0)$ and that

$$D_\delta\psi(\delta) = -D_\gamma\hat{F}(\hat{u},\psi(\delta),\delta)^{-1}\ D_\delta\hat{F}(\hat{u},\psi(\delta),\delta), \quad \forall\delta \in B(\delta^0) \ . \tag{3.8}$$

We now define a new function $\phi(\cdot,\cdot) : \mathbb{R}^p \times B(\delta^0) \to \mathbb{R}^q$ as

$$\phi(u,\delta) = f(u,\psi(\delta),\delta) \ .$$

Note that,

$$D_\delta\phi(u,\delta) = D_\gamma f(u,\psi(\delta),\delta)\ D_\delta\psi(\delta) + D_\delta f(u,\psi(\delta),\delta)\ . \tag{3.9}$$

We now use the first result in Lemma 3.1, i.e., $\exists\ B(\gamma^0,\delta^0) \subset \mathbb{R}^N$ and $L(\gamma,\delta)$, a continuous $\rho \times (N-\rho)$ matrix-valued function of (γ,δ), such that

$$[D_\gamma f(u,\gamma,\delta)\ D_\delta f(u,\gamma,\delta)] \begin{bmatrix} L(\gamma,\delta) \\ I_{(N-\rho)x(N-\rho)} \end{bmatrix} = 0_{qx(N-\rho)}\ \forall u \in \mathbb{R}^p,\ \forall(\gamma,\delta) \in B(\gamma^0,\delta^0).$$

It follows from this result that $\forall k \in \mathbb{N}$ and $\forall(\gamma,\delta) \in B(\gamma^0,\delta^0)$, rank $\text{col}([D_\gamma f(u^i,\gamma,\delta)\ D_\delta f(u^i,\gamma,\delta)],\ i = 1,\dots,k) \leq \rho$. Assuming, without loss of generality, that $B(\gamma^0) \times B(\delta^0) \subset B(\gamma^0,\delta^0)$, it follows from the above equation and the nonsingularity of $D_\gamma\hat{F}(\hat{u},\psi(\delta),\delta)$, that $\forall u \in \mathbb{R}^p$ and $\forall\ \delta \in B(\delta^0)$, the rows of the matrix

$$[D_\gamma f(u,\psi(\delta),\delta)\ D_\delta f(u,\psi(\delta),\delta)]$$

are linearly dependent on the rows of

$$[D_\gamma\hat{F}(\hat{u},\psi(\delta),\delta)\ D_\delta\hat{F}(\hat{u},\psi(\delta),\delta)],$$

i.e., $\exists$ a matrix $K(u,\delta)$, such that, $\forall u \in \mathbb{R}^p$ and $\forall\ \delta \in B(\delta^0)$,

$$\begin{aligned} D_\gamma f(u,\psi(\delta),\delta) &= K(u,\delta)\ D_\gamma\hat{F}(\hat{u},\psi(\delta),\delta) \\ D_\delta f(u,\psi(\delta),\delta) &= K(u,\delta)\ D_\delta\hat{F}(\hat{u},\psi(\delta),\delta) \end{aligned} \tag{3.10}$$

Substituting (3.8) and (3.10) in (3.9), we conclude that

$$D_\delta\phi(u,\delta) = 0_{qx(N-\rho)} \quad \forall u \in \mathbb{R}^p, \quad \forall\delta \in B(\delta^0)$$

Thus $f(u,\gamma,\delta)$ is constant on the $(N-\rho)$-dimensional manifold defined by $\gamma = \psi(\delta)$, that contains (γ^0,δ^0) and is defined in an open neighborhood around it. Hence α^0 is not locally diagnosable. ¤

Remark 3.1: In the above proof we have established that if $\mathrm{rank}[R(\alpha^0)] = \rho < N$ then there is an $(N-\rho)$ dimensional manifold of observationally equivalent points that contains α^0 and is defined in an open neighborhood of it. Thus,

$$\mu(\alpha^0) \triangleq N\text{-rank}[R(\alpha^0)]$$

is precisely the measure of solvability of the parameter point α^0 discussed in [4] and [15]. Further, in the sequel we will discuss conditions under which $R(\alpha)$ has a generic rank, in which case $\mu(\alpha^0)$ has a generic value μ which is the measure of testability [15] of the system. ¤

Theorem 3.1 gives a necessary and sufficient condition for the local diagnosability of a particular parameter point. If we can show that local diagnosability is a generic property, this result can be used as a condition for the local diagnosability of the system. In Theorem 3.2 we state a condition under which local diagnosability is a generic property. Observe that in both Lemma 3.1 and Theorem 3.1 the results are independent of the choice of weighting function. This is in fact true of all the results we present in this paper. Consequently, from this point on, when we refer to the matrix $R(\alpha)$, we assume that there does exist an appropriate weighting function and will not explicitly state this fact.

Theorem 3.2: Suppose that

$$\rho \triangleq \max_{\alpha\in\mathbb{R}^N} \mathrm{rank}\, R(\alpha)$$

is the generic rank of $R(\alpha)$. Then

1) almost all $\alpha \in \mathbb{R}^N$ are regular points of $R(\alpha)$;

2) the system is locally diagnosable if and only if

$$\rho = N$$ ¤

Note that the generic rank of $R(\alpha)$ can be determined by evaluating its rank for some randomly chosen $\alpha \in \mathbb{R}^N$. In the following two propositions, we present conditions under which the maximum rank of $R(\alpha)$ is its generic rank.

<u>Proposition 3.1</u>: Suppose that $f(u,\alpha)$ is analytic in α. Then,

$$\rho \triangleq \max_{\alpha \in \mathbb{R}^N} \text{ rank } R(\alpha) \ .$$

is the generic rank of $R(\alpha)$.

<u>Proof</u>: It follows from the definition of ρ that there exists $\hat{\alpha} \in \mathbb{R}^N$ and $M(\alpha)$ a $\rho \times \rho$ submatrix of $R(\alpha)$ such that

Thus the determinant of $M(\alpha)$, which is an analytic function of α, is not identically zero. Hence its zero set, say Z, is a closed subset of $\mathbb{R}^N$ with zero Lebesgue measure [16]. It now follows from the definition of ρ that

$$\text{rank } R(\alpha) = \rho \qquad \forall \alpha \in \mathbb{R}^N - Z \ .$$

Hence ρ is the generic rank of $R(\alpha)$. ¤

<u>Proposition 3.2</u>: Suppose that $f(u,\alpha)$ is three times continuously differentiable with respect to α, and let

$$\rho \triangleq \max_{\alpha \in \mathbb{R}^N} \operatorname{rank}[R(\alpha)].$$

Suppose that there exists $M(\alpha)$, a principal $\rho \times \rho$ submatrix of $R(\alpha)$, such that

$$\{\alpha \in \mathbb{R}^N \mid \det M(\alpha) = 0 \text{ and } D_\alpha^2 \det M(\alpha) = 0_{N\times N}\} = \emptyset$$

Under these conditions, ρ is the generic rank of $R(\alpha)$.

Proof: Since $M(\alpha)$ is a principal submatrix of $R(\alpha)$, which is a positive semidefinite matrix, $\det M(\alpha) \geq 0$, $\forall\, \alpha \in \mathbb{R}^N$, hence

$$\det M(\alpha) = 0 \Rightarrow D_\alpha \det M(\alpha) = 0.$$

Let,

$$\psi(\alpha) \triangleq [D_\alpha \det M(\alpha)]^t$$

Consider any $\hat{\alpha}$ such that $\det M(\hat{\alpha}) = 0$. It follows that,

$$\psi(\hat{\alpha}) = 0 \text{ and } D_\alpha \psi(\hat{\alpha}) \neq 0_{N\times N}$$

Without loss of generality, assume that

$$(D_\alpha \psi(\hat{\alpha}))_{11} \neq 0.$$

By the implicit function theorem [13], there exists some $\xi(\cdot)$, such that, in a neighborhood of $\hat{\alpha}$, the solution of $\psi_1(\alpha) = 0$ can be described as

$$(\xi(\alpha_2,\ldots,\alpha_N),\ \alpha_2,\ldots,\alpha_N)$$

which is a local (N-1) dimensional parametrization [13]. Thus

$$Z \triangleq \{\alpha \in \mathbb{R}^N \mid \det M(\alpha) = 0\}$$

is closed and is a subset of an (N-1)-dimensional manifold. Hence Z has zero Lebesgue measure. It now follows from the definition of ρ that

$$\text{rank } R(\alpha) = \rho \qquad \forall \alpha \in \mathbb{R}^N - Z.$$

Hence ρ is the generic rank of $R(\alpha)$. ⌑

IV: SIMPLIFIED TESTS FOR LOCAL DIAGNOSABILITY AND EXISTENCE OF TEST INPUTS

In the preceding section we have discussed conditions under which the test for the local diagnosability of a system reduces to a rank test on the matrix $R(\alpha^0)$ evaluated at a randomly chosen point α^0. Note however, from the definition of $R(\alpha)$ in (3.1), that this can be expensive, especially when p, the number of inputs, is large, since it involves an infinite multidimensional integral. Therefore, we now discuss conditions under which the test for diagnosability is cheaper than the evaluation of (3.1). We first present two simple sufficient conditions.

<u>Corollary 4.1</u>: Let $\tilde{U}$ be an open subset of $\mathbb{R}^p$ and let

$$\tilde{R}(\alpha^0) \triangleq \int_{\tilde{U}} v(u)\; D_\alpha f(u,\alpha^0)^t\; D_\alpha f(u,\alpha^0)\; du_1 \cdots du_p$$

If $\tilde{R}(\alpha^0)$ is positive definite, the parameter point α^0 is locally diagnosable.

Proof: Note that

$$R(\alpha^0) = \tilde{R}(\alpha^0) + \int_{\mathbb{R}^N - \tilde{U}} v(u) D_\alpha f(u,\alpha^0)^t\; D_\alpha f(u,\alpha^0)\; du_1 \ldots du_p$$

Since the two matrices on the right hand side of the above equation are respectively, positive definite and positive semidefinite, $R(\alpha^0)$ is positive definite, hence by Theorem 3.1, α^0 is locally diagnosable.

Proposition 4.1: Suppose that there exist k inputs $u^1,\cdots,u^k$, such that the matrix

$$\mathrm{col}[D_\alpha f(u^1,\alpha^0),\cdots,D_\alpha f(u^k,\alpha^0)]$$

is full column rank. Then the parameter point α^0 is locally diagnosable.

Proof: The proof follows from the Inverse Function Theorem [13] and the definition of local diagnosability. ¤

Our next two propositions are, under suitable conditions, necessary and sufficient conditions for local diagnosability and lead to tests that are computationally cheaper than the evaluation of (3.1).

Proposition 4.2: Suppose that there exists a weighting function $v(\cdot)$ such that $R(\alpha)$ exists $\forall\alpha \in \mathbb{R}^N$, with the additional property

$$\int_{\infty}^{\infty}\cdots\int_{\infty}^{\infty} v(u)\, du_1\cdots du_p = 1$$

Then $\forall\alpha \in \mathbb{R}^N$, for inputs $u^1,u^2,\cdots u^k$, independently sampled from the probability distribution function $v(\cdot)$

$$\lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} D_\alpha f(u^i,\alpha)^t\, D_\alpha f(u^i,\alpha) = R(\alpha) \quad \text{with probability 1} \quad ¤$$

The above proposition is a restatement of the Strong Law of Large Numbers [17]. To paraphrase the proposition, if for a regular point α^0, a Monte Carlo analysis with a sufficiently large statistical sample of inputs, results in a matrix

$$\frac{1}{k}\sum_{i=1}^{k} D_\alpha f(u^i,\alpha^0)^t\, D_\alpha f(u^i,\alpha^0) \tag{4.1}$$

which is singular, then almost surely a regular point α^0 is not locally diagnosable (Theorem 3.1). On the other hand, if for any k, (4.1) is nonsingular, then by Proposition 4.1 , α^0 is locally diagnosable.

To present our final test for local diagnosability, we need some additional notation. Let $\bar{u} \in \mathbb{R}^{N\times p}$ and let $F(\cdot,\cdot) : \mathbb{R}^{N\times p} \times \mathbb{R}^{N} \to \mathbb{R}^{N\times q}$ be defined as

$$F(\cdot,\cdot) = \mathrm{col}[\underbrace{f(\cdot,\cdot),\cdots,f(\cdot,\cdot)}_{N \text{ times}}]$$

<u>Proposition 4.3</u>: Suppose that $f(u,\alpha)$ is analytic in u and α. Then the system is locally diagnosable if and only if

$$\rho \triangleq \max_{(\bar{u},\alpha)\in\mathbb{R}^{N\times p}\times\mathbb{R}^{N}} \operatorname{rank} D_\alpha F(\bar{u},\alpha) = N$$

<u>Proof</u>:

<u>If</u>: We establish, in exactly the same way as in Proposition 3.1, that N is the generic rank of $D_\alpha F(\bar{u},\alpha)$. It now follows from Proposition 4.1 that the system is locally diagnosable.

<u>Only If</u>: Since $f(u,\alpha)$ is analytic with respect to α, almost all $\alpha \in \mathbb{R}^N$ are regular points of $R(\alpha)$. Further since almost all $\alpha \in \mathbb{R}^N$ are locally diagnosable, $\exists\alpha^0$ such that $R(\alpha^0)$ is positive definite (by Theorem 3.1). It follows from Lemma 3.1 that $\exists u^i$, $i = 1,\cdots,\ell \leq N$ such that the matrix

$$\mathrm{col}(D_\alpha f(u^1,\alpha^0),\cdots,D_\alpha f(u^\ell,\alpha^0))$$

has full column rank. Hence for any $\bar{u}^0$ that satisfies

$$\bar{u}_i^0 = u^i, \quad i = 1,\cdots\ell$$

we have

$$\operatorname{rank}[D_\alpha F(\bar{u}^0,\alpha^0)] = N$$

i.e.,

$$\rho = N . \qquad \square$$

Observe that the number of simulations required to test for local diagnosability has been reduced in Proposition 4.2 to a sufficiently large Monte Carlo sample, while in Proposition 4.3, since ρ is the generic rank of $D_\alpha F(\bar{u},\alpha)$, simulations at a randomly chosen $(\bar{u}^0,\alpha^0)$ suffice. We now present two simple examples to illustrate the application of the results developed so far.

Example 4.1: Consider the circuit shown in Fig. 1a with the diode characteristic of Fig. 1b. The system description is given by

$$\begin{aligned} y &= (G+d)u \qquad u \geq 0 \\ y &= Gu \qquad\qquad u < 0 \end{aligned} \tag{4.2}$$

The (possibly faulty) parameters are G and d, i.e., $\alpha = [G,d]^t$.

Note that (4.2) is analytic with respect to α but only continuous with respect to u. We pick $\alpha^0 = [1,1]^t$ as the representative parameter point. Observe that

$$\begin{aligned} D_\alpha(u,\alpha) = &[u \quad u] \qquad u \geq 0 \\ &[u \quad 0] \qquad u < 0 \end{aligned}$$

To determine if this system is locally diagnosable or not, consider

$$\begin{aligned} \tilde{R}(\alpha^0) &= \int_{-1}^{1} D_\alpha(u,\alpha^0)^t \, D_\alpha(u,\alpha^0)du \\ &= \int_{-1}^{0} \begin{bmatrix} u^2 & u^2 \\ u^2 & u^2 \end{bmatrix} du + \int_{0}^{1} \begin{bmatrix} u^2 & 0 \\ 0 & 0 \end{bmatrix} du \\ &= \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 1/3 \end{bmatrix} \end{aligned}$$

In the above computation, the weighting function $v(u) = 1$, $\forall u \in (-1,1)$. Note the $\tilde{R}(\alpha^0)$ is nonsingular. Hence by Corollary 4.1 the parameter point α^0 is locally diagnosable and then, by Proposition 3.1 and Theorem 3.2, the system is locally diagnosable.

Suppose we picked two inputs u^1 and u^2, one greater than zero and the other less than zero, say $u^1 = 3$ and $u^2 = -2$. We observe that

$$\begin{bmatrix} D_\alpha f(u^1,\alpha^0) \\ D_\alpha f(u^2,\alpha^0) \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ -2 & 0 \end{bmatrix}$$

is nonsingular. Hence,by Proposition 4.1,α^0 is locally diagnosable. Note that a Monte Carlo analysis with any appropriate probability distribution function will ascertain the diagnosability of the system since it is guaranteed to eventually pick one input larger than zero and one smaller than zero. Observe finally that since (4.1) is not analytic with respect to u, Proposition 4.3 cannot be applied. In fact (using the notation of Proposition 4.3),

$$\max_{(\bar{u},\alpha)\in\mathbb{R}^2\times\mathbb{R}^2} \operatorname{rank}[D_\alpha F(\bar{u},\alpha)] = 2,$$

is not the generic rank. $D_\alpha F(\bar{u},\alpha)$ does not have a generic rank. ¤

<u>Example 4.2</u>: Consider once again the circuit in Fig. 1a but with the diode characteristics now given by

$$i_d = I_S(e^{av_d}-1)$$

The system is described by the equation

$$y = I_S(e^{au}-1) + Gu \tag{4.3}$$

The parameters are I_S, a and G, i.e., $\alpha = [I_S,a,G]^t$

Note that (4.3) is analytic with respect to u and α. We can therefore apply Proposition 4.3 to test the local diagnosability of the system.

To determine the generic rank of $D_\alpha F(\bar{u},\alpha)$ we pick the point

$$(\bar{u}^0,\alpha^0) = ([-1\ 1\ 2]^t,\ [1\ 1\ 1]^t)$$

and observe that

$$D_\alpha f(\bar{u}^0,\alpha^0) = \begin{bmatrix} -0.632 & -0.368 & -1 \\ 1.718 & 2.718 & 1 \\ 6.389 & 14.778 & 2 \end{bmatrix}$$

is full rank. Hence (4.3) is locally diagnosable. ¤

We next turn to the question of determining the faulty parameter values of a system that is locally diagnosable. Since in this paper we are mainly concerned with diagnosability, our goal here is to show the existence of a finite number of test input signals that are sufficient to diagnose almost all faults. The most straightforward diagnosis algorithm, consists of using ℓ test input signals u^i, $i = 1,\cdots\ell$, measuring the corresponding outputs y^i, $i = 1,\cdots\ell$ and then solving the equation

$$f(u^i,\alpha) - y^i = 0 \qquad\qquad i = 1,\cdots,\ell$$

for the faulty parameter values α, using a stabilized Newton algorithm [18][19]. To apply the algorithm we require that, for almost all $\alpha \in \mathbb{R}^N$,

$$\text{rank}[\text{col}(D_\alpha f(u^i,\alpha)\ i = 1,\cdots\ell)] = N. \tag{4.4}$$

Note that all the tests for local diagnosability developed in this section except for Corollary 4.1 are based on evaluating $f(u,\alpha)$ for a finite number of inputs. Hence the same finite set of inputs that is used to ascertain the local diagnosability of the system is sufficient for the diagnosis algorithm, since (4.4) will be valid for this input set. More generally, we have the following theorem.

Theorem 4.1: Let $f(u,\alpha)$ be analytic with respect to α. Then if there exists α^0 such that $R(\alpha^0)$ is positive definite, it follows that $\exists\ u^i$, $i = 1,\cdots\ell \leq N$ such that, for almost all $\alpha \in \mathbb{R}^N$,

$$\operatorname{rank}[\operatorname{col}(D_\alpha f(u^i,\alpha)\ i = 1,\cdots\ell)] = N$$

Proof: The proof follows from Lemma 3.1 and the analyticity assumption. ¤

Remark 4.1: The proof of Lemma 3.1 suggests a test input selection scheme that is similar to that of Sen and Saeks [4]. Note that we have not addressed the problem of choosing test inputs that are optimal for the numerical well-posedness of the fault diagnosis equations. ¤

Finally, note that if $f(u,\alpha)$ is analytic with respect to u and α, it follows from Proposition 4.3 that any randomly chosen test sequence u^i, $i = 1,\cdots,N$, can diagnose almost all faults.

V. SINGLE FAULT DIAGNOSABILITY

Given a faulty system, in practice it is often true that only a few of the parameters are faulty. In this section we shall study the case of single faults. In linear systems, the problem of locating single faults has been addressed in [20] and [21], while Chen and Saeks [22] discuss numerically efficient algorithms for the diagnosis of single faults using multi-frequency testing.

Definition 5.1: A parameter point $\alpha \in \mathbb{R}^N$ which has only one entry which is not at its nominal value is said to be a single fault. ¤

Assumption 5.1: In this section, we assume that the only possible faults in the system are single faults. ¤

The problem of diagnosis then, is to determine which parameter is faulty and what its faulty value is. Note that we have assumed that all the fault-free parameters are at their nominal value, while in practice the fault-free parameters can be assured to be only within a tolerance region. As will become clear in the sequel, due to analyticity in α (Assumption 5.2), this does not invalidate our results. However, the problem of variations in the fault-free parameters cannot be bypassed when one discusses diagnosis algorithms.

Notation 5.1: In this section, the nominal parameter point is denoted by α^0. ¤

Remark 5.1: As discussed in the introduction, diagnosability is a global uniqueness property. Thus, strictly speaking, we should define a system to be single fault diagnosable if and only if, for almost all single faults, there exists no other possible single fault that is observationally equivalent to it. It is however difficult to derive a necessary and sufficient condition for an arbitrary nonlinear function with such a definition. We therefore make the following definition of single fault diagnosability which is a necessary condition for the definition discussed above. Note that the concept of local diagnosability discussed in the Section III has the same relationship to global diagnosability. ¤

Definition 5.2: The system is said to be single fault diagnosable if $\forall j = 1,\cdots,N$, for almost all[5] $\hat{\alpha}_j \in \mathbb{R}$ and $\forall k = 1,\cdots,N$, $k \neq j$, the following is true.

If M is a 1- or 2-dimensional manifold in the (α_j,α_k)-plane such that all points in M are observationally equivalent and $(\hat{\alpha}_j,\alpha_k^0) \in M$, then

[5] α_j is the j-th entry of α.

$$M \cap \{(\alpha_j^0,\alpha_k) | \alpha_k \in \mathbb{R}\} = \phi$$

$$M \cap \{(\alpha_j,\alpha_k^0) | \alpha_j \in \mathbb{R}\} = \{(\hat{\alpha}_j,\alpha_k^0)\}$$ ¤

Remark 5.2: Note that if we had also included 0-dimensional observationally equivalent manifolds in definition 5.2, then it would be equivalent to the one in Remark 5.1. Pictorially, Definition 5.2 does not preclude a set of isolated observationally equivalent points $\{P_1,P_2,P_3\}$ on the (α_j,α_k)-plane (for some j and k) as shown in Fig. 2, where say, P_1 is the real fault. Observe however, that the condition shown in Fig. 2 is possible even if the system was locally diagnosable. Hence with this definition, we are no worse off than if we required local diagnosability as a precondition to diagnose faults in a system that had only single faults. The advantage of Definition 5.2 however is that a sufficient condition for it, that we will derive, is much weaker than the necessary and sufficient condition for local diagnosability. ¤

We now present a lemma which is a key bridge between Definition 5.2 and the sufficient condition for local diagnosability that we will derive.

Lemma 5.1: If $\forall j = 1,\cdots,N$, for almost all $\hat{\alpha}_j \in \mathbb{R}$ and $\forall k = 1,\cdots,N$, $k \neq j$, $\exists$ $B(\hat{\alpha}_j,\alpha_k^0)$ an open neighborhood of $(\hat{\alpha}_j,\alpha_k^0)$ in the (α_j,α_k)-plane such that no other (α_j,α_k) in that neighborhood is observationally equivalent to it, then, the system is single fault diagnosable. ¤

Notation 5.2: For all $1 \leq j,k \leq N$, we let $R_{jk}(\alpha)$ denote the (j,k)-th entry of the matrix $R(\alpha)$ and $\forall 1 \leq j,k \leq N$, $j \neq k$, we define

$$R^{jk}(\alpha) \triangleq \begin{bmatrix} R_{jj}(\alpha) & R_{jk}(\alpha) \\ R_{jk}(\alpha) & R_{kk}(\alpha) \end{bmatrix}$$

Assumption 5.2: Throughout this section we assume that $f(u,\alpha)$ is analytic with respect to α. As a consequence,

a) $\max_{\alpha \in \mathbb{R}^N} \text{rank } R^{jk}(\alpha) = \text{generic rank } R^{jk}(\alpha) \quad \forall 1 \leq j,k \leq N,\ j \neq k.$

b) $\max_{\alpha \in \mathbb{R}^N} \text{rank } R(\alpha) = \text{generic rank } R(\alpha)$

We will therefore, without loss of generality, assume that the nominal point is a generic point, i.e., in each of the above cases the maximum rank is achieved at α^0.

Theorem 5.1: If $\forall j = 1,\cdots N-1$ and $\forall k = j+1,\cdots,N$ the matrix $R^{jk}(\alpha^0)$ is positive definite, then the system is single fault diagnosable.

Proof: For any pair (α_j,α_k), consider the local diagnosability of the system with only α_j and α_k being considered as parameters. The remaining elements of α are fixed at their nominal values and are now part of the system structure. They will therefore be suppressed in notation. Since $f(u,\alpha_j,\alpha_k)$ is analytic with respect to α_j and α_k and since $R^{jk}(\alpha_j^0,\alpha_k^0)$ is positive definite, for almost all $\hat{\alpha}_j$(resp. $\hat{\alpha}_k$) $\in \mathbb{R}$,

a) $(\hat{\alpha}_j,\alpha_k^0)$(resp. $(\alpha_j^0,\hat{\alpha}_k)$) is a regular point of $R^{jk}(\alpha_j,\alpha_k)$.

b) $R^{jk}(\hat{\alpha}_j,\alpha_k^0)$(resp. $R^{jk}(\alpha_j^0,\hat{\alpha}_k)$) is positive definite.

The result now follows from Definition 3.1, Theorem 3.1 and Lemma 5.1. ¤

Note that results that are similar to Corollary 4.1, and Propositions 4.1, 4.2 and 4.3 can be stated to simplify the sufficiency test for single fault diagnosability. We present one of these propositions. Recall the notation of Proposition 4.3.

Proposition 5.1: Suppose that $f(u,\alpha)$ is analytic in u and α. Then the system is single fault diagnosable if $\exists \bar{u}^0 \in \mathbb{R}^{N\text{x}p}$ such that

$\forall j = 1,\cdots N-1$ and $\forall k = j+1,\cdots N$

$$\mathrm{rank}[D_{\alpha_j} F(\bar{u}^0,\alpha^0)\ D_{\alpha_k} F(\bar{u}^0,\alpha^0)] = 2$$

<u>Proof</u>: The proof is along the lines of Theorem 5.1 except that it uses Proposition 4.3 instead of Theorem 3.1. ¤

We now present an example to illustrate the concepts developed in this section.

<u>Example 5.1</u>: Consider the linear resistive circuit shown in Fig. 3. The circuit is driven by a DC current source, u, and the voltages y_1 and y_2 are the measured outputs. The faulty resistor values are all assumed to be strictly positive. The parameter vector is

$$\alpha = [R_1 R_2 R_3 R_4 R_5]^t$$

and the nominal parameter values are

$$R_1 = 1K,\ R_2 = 2K,\ R_3 = 3K,\ R_4 = 4K,\ R_5 = 5K$$

The system description is given by,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \dfrac{R_3(R_4+R_5)+(R_1+R_2)(R_3+R_4+R_5)}{(R_3+R_4+R_5)} \\ R_2 + \dfrac{(R_4+R_5)R_3R_5}{R_4R_5+R_3R_5+R_3R_4} \end{bmatrix} u \tag{5.1}$$

Since (5.1) is analytic in u and α, we can apply Proposition 5.1. Further since the system is linear, there is no advantage in testing it at more than one input. Therefore, (5.1) is single fault diagnosable if every pair of columns of the matrix $D_\alpha f(u,\alpha)$ evaluated at a test input say 1 mA, and the nominal parameter values is linearly independent. The Jacobian of (5.1) with respect to the parameters evaluated at the nominal parameter values and the given test input is given by the matrix

$$\begin{bmatrix} 1 & 1 & .4167 & .0625 & .0625 \\ 0 & 1 & .4074 & -.1698 & .2105 \end{bmatrix}$$

Note that every pair of columns in the above matrix is linearly independent, hence (5.1) is single-fault diagnosable. Note also that since there are five parameters and only two transfer functions, (5.1) cannot be locally diagnosable.

We next turn our attention to the algorithm for the diagnosis of systems that satisfy the condition of Theorem 5.1. The algorithm is based on test measurements taken with a finite number of test inputs, u^i, $i = 1,\cdots\ell$. It follows from the following theorem that measurements made with only this finite set of inputs preserves the local uniqueness property described in Lemma 5.1 and guaranteed by the conditions of Theorem 5.1.

<u>Theorem 5.2</u>: Suppose that $\forall j = 1,\cdots,N-1$ and $\forall k = j+1,\cdots,N$ the matrix $R^{jk}(\alpha^0)$ is positive definite. Then, $\exists u^i$, $i = 1,\cdot\cdot,\ell$, $\ell \leq \binom{N}{2}$, such that $\forall j = 1,\cdots,N-1$ and $\forall k = j+1,\cdots,N$

$$\text{rank}[\text{col}(D_{\alpha_j} f(u^i,\alpha^0),\ i = 1,\cdot\cdot,\ell) \vdots \text{col}(D_{\alpha_k} f(u^i,\alpha^0), i = 1,\cdot\cdot,\ell)] = 2$$

<u>Proof</u>: The proof follows from Lemma 3.1 by considering each pair (α_j,α_k) separately. ¤

The algorithm that we propose for diagnosis is the search algorithm described by Chen and Saeks [22] for the multi-frequency testing of single faults in linear systems, and is as follows

$\forall k = 1,\cdots,N$ compute

$$C_k \triangleq \min_{\alpha_k \in \mathbb{R}} \|\text{col}(y^i - f(u^i,\alpha_1^0,\cdot\cdot,\alpha_{k-1}^0,\alpha_k,\alpha_{k+1}^0,\cdot\cdot,\alpha_N^0),\ i = 1,\cdot\cdot,\ell)\|$$

The faulty parameter, say α_j, necessarily satisfies the equation

$$C_j = 0$$

while the fault value is the value of α_j for which the zero minimum is achieved.

VI. AN EXAMPLE

We now present a simple example to illustrate the application of our results in a computational environment. Consider the single-stage transistor amplifier with various possible test points, shown in Fig. 4. We use the following Ebers-Moll model for the transistor.

$$i_E = I_{ES}(e^{Kv_{BE}}-1) - \alpha_F I_{ES}(e^{Kv_{BC}}-1)$$

$$i_C = I_{CS}(e^{Kv_{BC}}-1) - \alpha_F I_{ES}(e^{Kv_{BE}}-1)$$

$$K = 38.46\ V^{-1}$$

The parameters of the system are assumed to be

$$= [R_1\ R_2\ R_3\ R_4\ I_{CS}\ I_{ES}\ \alpha_F]^t$$

The nominal values of the resistors are given in Fig. 4 while the nominal values of the transistor parameters are

$$\alpha_F = 0.99$$

$$I_{CS} = 10^{-5}\ \text{mA}$$

$$I_{ES} = 7\text{x}10^{-6}\ \text{mA}.$$

Since all the branch relations are analytic functions, every input-output model, $y = f(u,\alpha)$, is analytic in u and α [16]. We therefore use Proposition 4.3 and Proposition 5.1 to test the local diagnosability and the single fault diagnosability of circuit, respectively. To apply these propositions it suffices to use 7 test input signals. However, to avoid possible numerical difficulties, we used 20 test input signals. The test matrix

$$\mathrm{col}[D_\alpha f(u^i,\alpha^0),\ i = 1,\cdots,20]$$

was created by simulating the circuit at the nominal parameter values and the various test inputs and using the Implicit Function Theorem (2.5). The rank of the test matrix was determined using standard numerical techniques [23]. Recall that

$$\mu \triangleq N - \text{generic rank } [R(\alpha)] .$$

The results are summarized in Table 1. Note that, for entries 4 and 6 in the table, if (R_1+R_2) is considered to be a single parameter, then, the values of μ would be 1 and 0 respectively, and the circuit would be single fault diagnosable.

VII. SUMMARY

In this paper, we have investigated the problem of diagnosability of nonlinear DC circuits (memoryless systems) based on the assumption that the circuit has a global input-output representation. We first consider local diagnosability. In this situation all the parameters are assumed to be faulty. We have derived a necessary and sufficient condition for the local diagnosability of the system which is a rank test on a matrix and indicated various ways of reducing the computational complexity of this test. We then consider single-fault diagnosability and derive a sufficient condition for it which is much weaker than the necessary and sufficient condition for local diagnosability. We have thereby rigorously confirmed the intuitive belief that fewer testpoints are required to diagnose faults in circuits where only one, instead of all the parameters are faulty. We have also shown that for diagnosable systems it is possible to pick a finite number of test inputs that are sufficient to diagnose the system. We have however not addressed the question of choosing these test inputs optimally for the numerical well-posedness of the fault diagnosis equations.

Interestingly, the techniques developed in this paper can be used to derive necessary and sufficient conditions for the local diagnosability of general nonlinear dynamical systems by reformulating the problem in an infinite dimensional setting. We address this problem in Part II of this two-part paper.

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FIGURE CAPTIONS

Fig. 1 (a): Circuit for Examples 4.1 and 4.2.

(b): Diode Characteristics for Example 4.1.

Fig. 2. Isolated, observationally equivalent parameter points.

Fig. 3. Circuit for Example 5.1.

Fig. 4: Single-stage transistor amplifier.

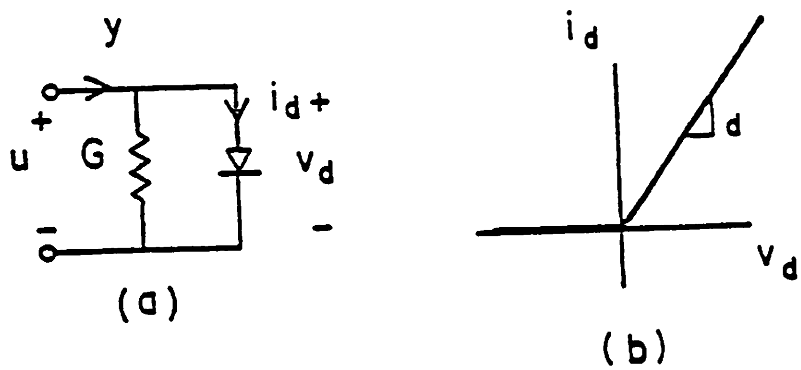


Fig. 1.

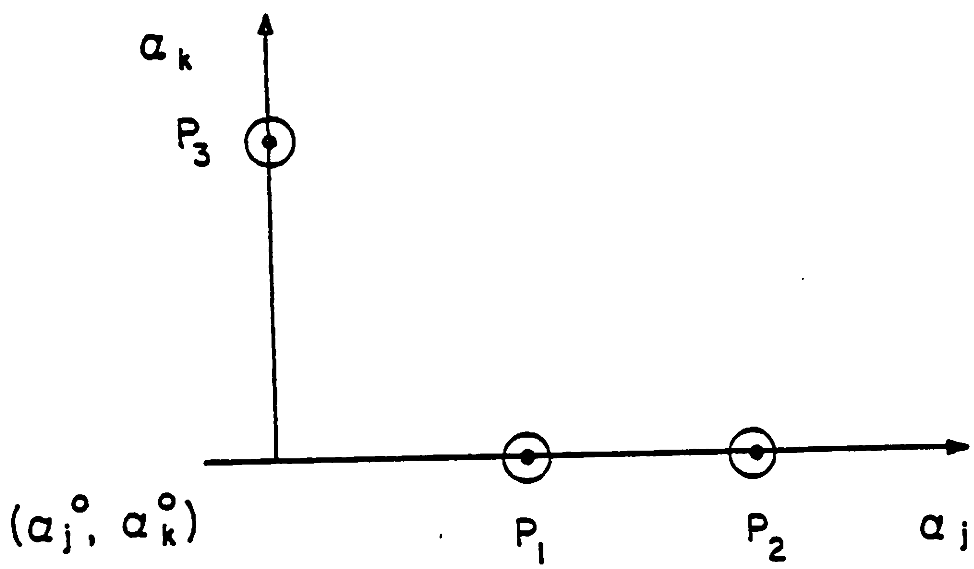
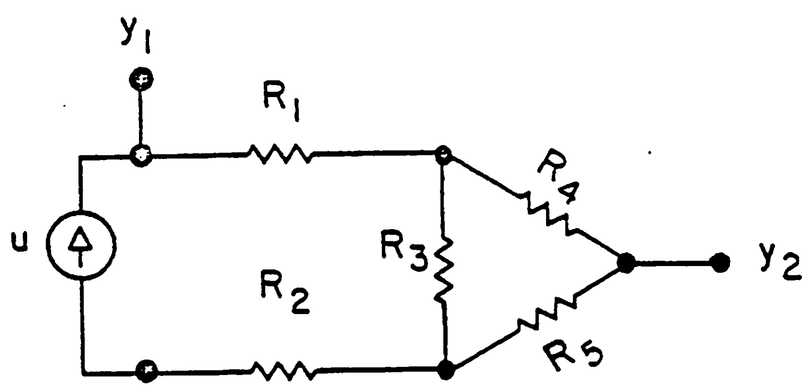
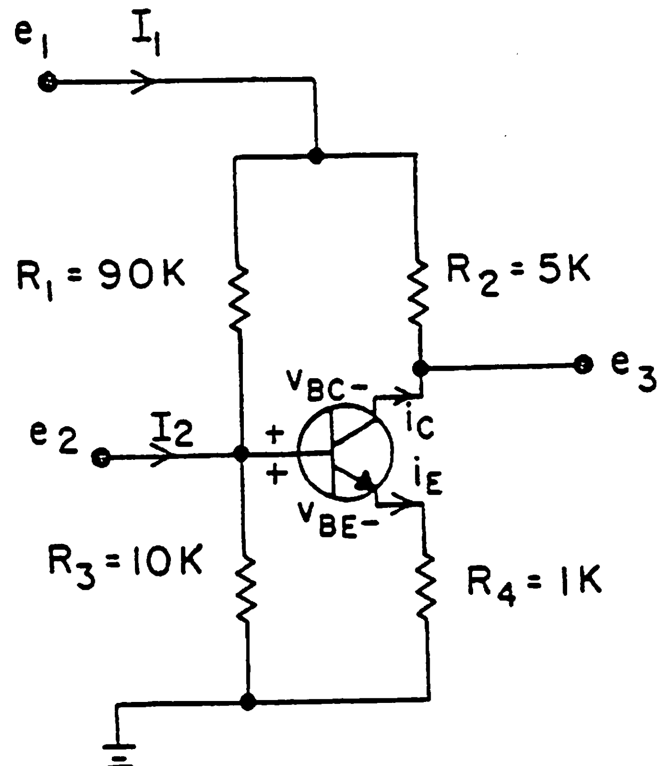


Fig. 2.



y_1
R_1
u
R_2
R_3
R_4
R_5
y_2

Fig. 3.



e_1
I_1
$R_1 = 90K$
$R_2 = 5K$
e_3
v_{BC}
e_2
I_2
i_C
i_E
v_{BE}
$R_3 = 10K$
$R_4 = 1K$

Fig. 4.

TABLE CAPTION

Table 1: Measure of testability of the circuit of Fig. 4 for various input-output combinations.

TABLE 1

No.	Input	Output	μ	Single Fault Diagnosable?
1	I_1	e_1	1	Yes
2	I_1	e_1, e_2	0	Yes
3	I_1	e_1, e_3	0	Yes
4	I_2	e_2	2	Yes*
5	I_2	e_1, e_2	0	Yes
6	I_2	e_2, e_3	1	Yes*

*If (R_1+R_2) is considered to be a single parameter.

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