NECESSARY AND SUFFICIENT CONDITION FOR ROBUST STABILITY OF LINEAR DISTRIBUTED FEEDBACK SYSTEMS

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Abstract

Considering linear time-invariant distributed feedback systems, we derive a necessary and sufficient condition for robust stability with respect to plant perturbations belonging to a specified ball. The conclusion is shown to exhibit design limitations imposed by plant uncertainties.

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1. Introduction

One of the main purposes of feedback is to reduce the sensitivity of the closed-loop system to changes in the plant (Horowitz 1963). A considerable amount of the literature is devoted to the effect of perturbations on system stability and/or system performance (see e.g., Åstrom 1980, Cruz 1972, Cruz et al. 1981, Desoer et al. 1977, Desoer 1978, Desoer and Wang 1980, Doyle 1979, Doyle and Stein 1981, Postlethwaite et al. 1981, Safanov 1980, Safanov et al. 1981, Sandell 1979, Willems 1971, Zames 1981). Some of these references consider nonlinear systems as well as linear systems. However, most of the results are restricted to "stable" perturbations. It is only recently that Doyle and Stein (1981) stated a necessary and sufficient condition for robust stability in the case of linear lumped systems. We have difficulties with their proof. (See Appendix A).

In this paper, we study the robust stability of linear time-invariant distributed multivariable systems with more general feedback configurations (than unity-feedback). Using efficient formulations of classical results (Rudin 1974), we obtain a simple derivation of a necessary and sufficient condition for robust stability over a class of additive plant perturbations. The robust stability requirement clearly exhibits the limitations imposed on feedback system design by plant uncertainties.

Organization: Section 2 describes the system under consideration. In Section 3, stability is defined and a set of equivalent stability tests is obtained. In Section 4, the necessary and sufficient condition for robust stability is derived and design considerations are included. In Section 5, robust stability for a different feedback configuration is discussed.

Special notations and definitions

For $\sigma \in \mathbb{R}$, (typically $\sigma > 0$), $\mathbb{C}_{\sigma+}$ denotes the closed right half plane $\mathrm{Re}(s) \geq \sigma$. $f \in A(\sigma)$ iff $f(t) = f_a(t) + \sum_0^\infty f_i \delta(t-t_i)$ where $f_a : \mathbb{R} \to \mathbb{R}$, with $f_a(t) = 0$ for $t < 0$, $t \mapsto f_a(t) \exp(-\sigma t) \in L_1$; $t_0 = 0$, $t_i > 0$, $\forall i > 0$; $\forall i$, $f_i \in \mathbb{R}$ and $i \mapsto f_i \exp(-\sigma t_i) \in \ell_1$. $f \in A_-(\sigma)$ iff, for some $\sigma_1 < \sigma$, $f \in A(\sigma_1)$. $\hat{f}$ denotes the Laplace transform of f. $\hat{A}_-(\sigma) := \{\hat{f} : f \in A_-(\sigma)\}$. $\hat{A}_-^\infty(\sigma)$, ($\hat{A}_{-,o}(\sigma)$, resp.), denotes the subset of $\hat{A}_-(\sigma)$ consisting of those $\hat{f}$ that are bounded away from zero at infinity in $\mathbb{C}_{\sigma_+}$, ($\hat{f}$ that go to zero at infinity in $\mathbb{C}_{\sigma+}$, resp.). $\hat{\mathcal{B}}(\sigma) := [\hat{A}_-(\sigma)][\hat{A}_-^\infty(\sigma)]^{-1}$, the commutative algebra of fractions $\hat{g} = \hat{n}/\hat{d}$ where $\hat{n} \in \hat{A}_-(\sigma)$ and $\hat{d} \in \hat{A}_-^\infty(\sigma)$ (Callier and Desoer 1978, 1979, 1980a)(for the general technique, see e.g., Jacobson (1980, Sec. 7.2), and Bourbaki (1970, Chap. II, Sec. 2)). $\hat{\mathcal{B}}_o(\sigma) := [\hat{A}_{-,o}(\sigma)]\,[\hat{A}_-^\infty(\sigma)]^{-1}$. Let $H \in \hat{\mathcal{B}}(\sigma)^{m\times n}$, $N_r D_r^{-1}$, ($D_\ell^{-1} N_\ell$, resp.) is called a <u>right-coprime factorization</u> (r.c.f.) (<u>left coprime factorization</u> (l.c.f.), resp.)[1] of H if and only if

(i) N_r and D_r (N_ℓ and D_ℓ, resp.) have all their elements in $\hat{A}_-(\sigma)$ and $\det D_r$ ($\det D_\ell$, resp.) $\in \hat{A}_-^\infty(\sigma)$;

(ii) $H = N_r D_r^{-1}$ ($H = D_\ell^{-1} N_\ell$, resp.);

(iii) (N_r, D_r) are <u>right coprime</u> (r.c.), i.e., $\exists\, U_r \in \hat{A}_-(\sigma)^{n\times m}$ and $V_r \in \hat{A}_-(\sigma)^{n\times n}$ s.t.

$$U_r N_r + V_r D_r = I_n$$

((iii)' (D_ℓ, N_ℓ) are <u>left coprime</u> (l.c.), i.e., $\exists\, U_\ell \in \hat{A}_-(\sigma)^{n\times m}$ and $V_\ell \in \hat{A}_-(\sigma)^{m\times m}$ s.t.

$$N_\ell U_\ell + D_\ell V_\ell = I_m, \text{ resp.}).$$

Given a r.c.f. $H = N_r D_r^{-1}$ (a l.c.f. $H = D_\ell^{-1} N_\ell$, resp.), $\chi_H := \det D_r$, ($\det D_\ell$, resp.), $n_H^+ :=$ the number of $\mathbb{C}_{\sigma_+}$-zeros of $\det D_r$ ($\det D_\ell$, resp.), counting multiplicities. $A_- := A_-(0)$, $\hat{\mathcal{B}} := \hat{\mathcal{B}}(0)$, etc. Let $A \in \mathbb{C}^{m\times n}$, $\sigma_{max}[A]$, ($\sigma_{min}[A]$, resp.) := the largest (smallest, resp.) singular

[1] $\forall H \in \hat{\mathcal{B}}(\sigma)^{m\times n}$, algorithms are available to obtain both r.c.f. and l.c.f. (Callier and Desoer 1980b, Vidyasagar et al. 1980).

value of A (Stewart 1973). ∎

2. System Description

Consider the linear time-invariant multivariable feedback system $S(P,C,F)$, consisting of a plant, a precompensator and a feedback compensator with transfer function matrices P, C and F. Let the plant P be subjected to additive perturbation ΔP thus becoming the additively perturbed plant $\tilde{P} := P + \Delta P$. Denote by $S(\tilde{P},C,F)$ the resulting additively perturbed feedback system (see Fig. 1). We impose the following assumptions on P, C, F and ΔP:

(I) For some $\sigma_o < 0$, $P \in \hat{\mathcal{B}}_o(\sigma_o)^{n_o x n_i}$, $\Delta P \in \hat{\mathcal{B}}_o(\sigma_o)^{n_o x n_i}$, $C \in \hat{\mathcal{B}}(\sigma_o)^{n_i x n_o}$ and $F \in \hat{\mathcal{B}}(\sigma_o)^{n_o x n_o}$. (2.1)

∎

(II) $N_{\tilde{p}r}D_{\tilde{p}r}^{-1}$ is a r.c.f. of $\tilde{P}$; $D_{c\ell}^{-1}N_{c\ell}$ is a l.c.f. of C; and $N_{fr}D_{fr}^{-1}$ is a r.c.f. of F. (2.2)

∎

In terms of ξ_1, ξ_2 and ξ_3 defined in Fig. 1, the system $S(\tilde{P},C,F)$ is described by

$$D(\Delta P)\xi = N_\ell u, \quad N_r(\Delta P)\xi = y, \tag{2.5}$$

where $\xi := [\xi_1^T \vdots \xi_2^T \vdots \xi_3^T]^T$, $u := [u_1^T \vdots u_2^T \vdots u_3^T]^T$, $y := [y_1^T \vdots y_2^T \vdots y_3^T]^T$, and[2]

$$D(\Delta P) := \left[\begin{array}{c|c|c} -I_{n_i} & D_{\tilde{p}r} & \\ \hline & -N_{\tilde{p}r} & D_{fr} \\ \hline D_{c\ell} & & N_{c\ell}N_{fr} \end{array}\right] \tag{2.6}$$

[2]Throughout this paper, all the unfilled blocks in a matrix have all their elements equal to zero.

$$N_\ell := \left[\begin{array}{c:c:c} & I_{n_i} & \\ \hdashline & & I_{n_o} \\ \hdashline N_{c\ell} & & \end{array}\right], \quad N_r(\Delta P) := \left[\begin{array}{c:c:c} I_{n_i} & & \\ \hdashline & N_{\tilde{p}r} & \\ \hdashline & & N_{fr} \end{array}\right] \tag{2.7}$$

From (2.5), $H_{yu}(\Delta P)$: $u \mapsto y$, the I/O map of the system $S(\tilde{P},C,F)$, is given by

$$H_{yu}(\Delta P) = N_r(\Delta P)\ D(\Delta P)^{-1}\ N_\ell \ . \tag{2.10}$$

<u>Comments</u>:

(a) With assumption (2.2), inspection of (2.6) and (2.7) shows that

$$(N_r(\Delta P),\ D(\Delta P)) \text{ are r.c.}; \quad (D(\Delta P),\ N_\ell) \text{ are l.c.} \tag{2.13}$$

(b) Direct calculation using (2.6) gives

$$\det D(\Delta P) = \eta\ \chi_{\tilde{p}} \cdot \chi_C \cdot \chi_F \cdot \det[I_{n_o} + \tilde{P}CF] \tag{2.16}$$

where $\eta = 1$ or -1 depending on the size of the matrices involved. From (2.16), it is easy to show that (2.1) and (2.2) imply[3]

$$\det D(\Delta P) \in \hat{A}_-^\infty(\sigma_o) \ . \tag{2.17}$$

and hence, $D(\Delta P)^{-1} \in \hat{B}(\sigma_o)^{(2n_i+n_o)\times(2n_i+n_o)}$. Consequently,

$$H_{yu}(\Delta P) \in \hat{B}(\sigma_o)^{(n_i+2n_o)\times(n_i+2n_o)} ,$$

i.e., the system $S(\tilde{P},C,F)$ is <u>well-posed</u>. ✖

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[3]Indeed: (i) (2.1) implies $\det[I_{n_o} + \tilde{P}CF(s)] \to 1$ as $|s| \to \infty$ in $\mathbb{C}_{\sigma_o+}$; (ii) (2.2) and the definition of coprime factorizations imply $\det D_{\tilde{p}r}$ $\det D_{c\ell}$, $\det D_{fr} \in \hat{A}_-^\infty(\sigma_o)$; (iii) From(2.6), $\det D(\Delta P) \in \hat{A}_-(\sigma_o)$. The claim (2.17) follows from (i), (ii), (iii) and (2.16).

3. System Stability

In this section, stability is defined and a set of equivalent necessary and sufficient conditions for the stability of the system $S(\tilde{P},C,F)$ are derived.

Definition 3.1.

Given any $\sigma < 0$, we say that the system $S(\tilde{P},C,F)$ of Fig. 1 is $A_-(\sigma)$-stable if and only if

$$H_{yu}(\Delta P) \in \hat{A}_-(\sigma)^{(n_i+2n_o)\times(n_i+2n_o)}$$ ∎

Theorem 3.2. (Stability Tests)

Consider the system $S(\tilde{P},C,F)$ shown in Fig. 1 satisfying (2.1) and (2.2). U.t.c., we have the following equivalences:

(I) $H_{yu}(\Delta P) \in \hat{A}_-(\sigma_o)^{(n_i+2n_o)\times(n_i+2n_o)}$; (3.11)

(II) $\det D(\Delta P)$ has an inverse in $\hat{A}_-(\sigma_o)$; (3.12)

(III) $\det D(\Delta P)$ has no $\mathbb{C}_{\sigma_o+}$-zeros ; (3.13)

(IV) the Nyquist diagram[4] of $\det[I_{n_o} + \tilde{P}CF]$
(i) does not go through the origin, and
(ii) encircles the origin $n^+_{\tilde{P}} + n^+_C + n^+_F$ times counter-clockwise. (3.14)

∎

[4] More precisely, the image under the map $s \mapsto \det[I_{n_o} + \tilde{P}CF(s)]$ of the clockwise contour $D(\sigma_o;R)$, with R arbitrarily large, consisting of (i) a straight line segment starting from $(\sigma_o,-jR)$ to (σ_o,jR), (except for the usual infinitesimal left indentations at the zeros of $\chi_{\tilde{P}} \cdot \chi_C \cdot \chi_F$ with real part equal to σ_o), and (ii) the semi-circle in $\mathbb{C}_{\sigma_o+}$ joining these points.

Comment: The equivalence "(3.11) $\Leftrightarrow$ (3.13)" is a slight extension of the result for unity-feedback systems (Theorem 3.1(i) in Callier and Desoer 1980b). Note that the present method of proof is more efficient. ∎

Proof of Theorem 3.2.

(3.11) $\Leftrightarrow$ (3.12). Consider (2.6)-(2.10).

$\Leftarrow$. Since $\hat{A}_-(\sigma_0)$ is a commutative ring, (3.12) holds if and only if $D(\Delta P)^{-1} \in \hat{A}_-(\sigma_0)^{(2n_i+n_o)x(2n_i+n_o)}$ (MacLane and Birkoff 1979); then, (3.12)$\Rightarrow$(3.11) by the closure properties of $\hat{A}_-(\sigma_0)$.

$\Rightarrow$. Condition (2.13) implies that $\exists$ U_r, V_r, U_ℓ and V_ℓ, all with elements in $\hat{A}_-(\sigma_0)$, such that

$$U_r N_r(\Delta P) + V_r D(\Delta P) = I_{2n_i+n_o} , \qquad (3.21)$$

$$N_\ell U_\ell + D(\Delta P) V_\ell = I_{2n_i+n_o} . \qquad (3.22)$$

Post multiply (3.21) by $D(\Delta P)^{-1} N_\ell U_\ell$, premultiply (3.22) by $D(\Delta P)^{-1}$ and add:

$$D(\Delta P)^{-1} = U_r H_{yu}(\Delta P) U_\ell + V_r N_\ell U_\ell + V_\ell \qquad (3.23)$$

Equation (3.23), the closure properties of $\hat{A}_-(\sigma_0)$ and (3.11) give $D(\Delta P)^{-1} \in \hat{A}_-(\sigma_0)^{(2n_i+n_o)x(2n_i+n_o)}$. Hence, (3.12) follows.

(3.12) $\Leftrightarrow$ (3.13). From (2.17), det $D_{pr}(\Delta P)$ is bounded away from zero at infinity in $\mathbb{C}_{\sigma_0+}$; hence, (3.13) is equivalent to

$$\inf_{s \in \mathbb{C}_{\sigma_0+}} |\det D(\Delta P)(s)| > 0 . \qquad (3.24)$$

Since (3.24) is equivalent to (3.12) (Desoer and Vidyasagar 1975, Appendix D) (Hille and Phillips 1957), the equivalence "(3.12)$\Leftrightarrow$(3.13)" follows.

$(3.13) \Leftrightarrow (3.14)$. Recall that

$$\det D(\Delta P) = \eta \cdot \chi_{\tilde{P}} \cdot \chi_C \cdot \chi_F \ \det[I_{n_o} + \tilde{P}CF] \tag{2.16}$$

where $\eta = 1$ or -1. Since (i) $\det D(\Delta P)$, $\chi_{\tilde{P}}$, χ_C and χ_F are analytic in $\mathbb{C}_{\sigma_o+}$ and on the contour $D(\sigma_o;R)$, (ii) $\det[I_{n_o} + \tilde{P}CF]$ is meromorphic in $\mathbb{C}_{\sigma_o+}$ and analytic on $D(\sigma_o;R)$, and (iii) all the factors in (2.16) are bounded away from zero at infinity at $\mathbb{C}_{\sigma_o+}$, (hence all have only a finite number of $\mathbb{C}_{\sigma_o+}$-zeros), it follows that (3.13) holds

$\Leftrightarrow$

(a) each of the $\mathbb{C}_{\sigma_o+}$-zeros of $\chi_{\tilde{P}} \cdot \chi_C \cdot \chi_F$ is precisely cancelled by a pole of $\det[I_{n_o} + \tilde{P}CF]$ with the same multiplicty,
(b) $\det[I_{n_o} + \tilde{P}CF]$ has no $\mathbb{C}_{\sigma_o+}$-poles except at those $\mathbb{C}_{\sigma_o+}$-zeros of $\chi_{\tilde{P}} \cdot \chi_C \cdot \chi_F$,
(c) $\det[I_{n_o} + \tilde{P}CF]$ has no $\mathbb{C}_{\sigma_o+}$-zeros;

$\Leftrightarrow$

(a) the $\mathbb{C}_{\sigma_o+}$-poles of $\det[I_{n_o} + \tilde{P}CF]$ are exactly the $\mathbb{C}_{\sigma_o+}$-zeros of $\chi_{\tilde{P}} \cdot \chi_C \cdot \chi_F$ with the same multiplicities,
(b) $\det[I_{n_o} + \tilde{P}CF]$ has no $\mathbb{C}_{\sigma_o+}$-zeros;

$\Leftrightarrow$

(3.14) holds,

where the last equivalence follows by the "argument principle" (Dieudonné, 1969, (9.17.2)), (Rudin 1974, Theorem 10.42). ∎

4. Robust Stability

We obtain here a necessary and sufficient condition for the robust stability of the system $S(P,C,F)$ under a class of perturbations.

For simplicity, we consider $\sigma_o = 0$. (If $\sigma_o < 0$, the arguments are the same, but the calculations are more cluttered). Let $\mathcal{D}$, the class of all allowable additive plant perturbations, be described by

$$\mathcal{D} := \{\Delta P : \text{(a) } \Delta P \in \hat{\mathcal{B}}_o^{n_o \times n_i} ; \tag{4.5}$$

$$\text{(b) } \sigma_{max}[P(j\omega)] < \ell_a(\omega), \ \forall \omega \in \mathbb{R}_+ ; \tag{4.6}$$

$$\text{(c) } n^+_{\tilde{P}} = n^+_P .\} \tag{4.7}$$

where $\omega \mapsto \ell_a(\omega)$ is a given "tolerance" function satisfying

(i) $\omega \mapsto \ell_a(\omega)$, mapping $\mathbb{R}_+$ into $\mathbb{R}_+ \backslash \{0\}$, is continuous; (4.8)

(ii) $\exists k \in \mathbb{N}^*$ such that $\ell_a(\omega)\, \omega^k > 1$ for all ω sufficiently large. (4.9)

Note that the "size" condition (4.6) implies that

$$\forall \Delta P \in \mathcal{D}, \ \Delta P \text{ has no poles on the } j\omega\text{-axis.} \tag{4.10}$$

Theorem 4.1. (Necessary and Sufficient Condition for Robust Stability)

Consider the system $S(\tilde{P},C,F)$ shown in Fig. 1 satisfying (2.1) and (2.2). Let $\sigma_o = 0$ and let $\mathcal{D}$ be described by (4.5)-(4.9). U.t.c., if

$$S(P,C,F) \text{ is } \mathcal{A}_-\text{-stable,} \tag{4.21}$$

then

$$\forall \Delta P \in \mathcal{D}, \ S(\tilde{P},C,F) \text{ is } \mathcal{A}_-\text{-stable;} \tag{4.22}$$

$\Leftrightarrow$

$$\sigma_{max}[CF(I_{n_o}+PCF)^{-1}(j\omega)] \le 1/\ell_a(\omega), \ \ \forall \omega \in \mathbb{R}_+ . \tag{4.23}$$

∎

Remarks 4.1.1:

(a) The importance of this theorem is that it gives a necessary and sufficient condition for robust stability over a class of perturbations.

This class of uncertainties should be viewed as a ball of uncertainty around the nominal plant P.

(b) Neither P, nor C, nor F, nor the perturbation ΔP is assumed to be stable. It is only required that $n^+_{\tilde{P}} = n^+_P$, i.e., "the unstable poles of $\tilde{P} = P + \Delta P$ are obtained by moving around the unstable poles of P."

(c) The theorem also exhibits the fact that plant uncertainties put bounds on the achievable benefits of feedback (see also Postlethwaite et al. 1981, Zames 1981). For example, to achieve large desensitization over wide bandwidth (note that with $Q := CF(I_{n_o}+PCF)^{-1}$, $(I_{n_o}+PCF)^{-1} = I_{n_o}-PQ$), or to significantly modify the open-loop dynamics, a "large" Q is required at and above the edge of the plant-passband. However, stability under perturbations imposes the requirement $\sigma_{max}[Q(j\omega)] \leq 1/\ell_a(\omega)$, $\forall\omega \in \mathbb{R}_+$, thus limiting what can be achieved in this connection. This requirement may be more restrictive than that resulting from noise requirements (see, e.g., Desoer and Chen 1981, Sec. V, Equation (28)).

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Proof of Theorem 4.1:

For simplicity, we assume that $\chi_P \cdot \chi_C \cdot \chi_F$ has no zeros on the $j\omega$-axis. (Hence, by (4.10), $\forall\Delta P \in \mathcal{D}$, $\chi_{\tilde{P}} \cdot \chi_C \cdot \chi_F$ has no zeros on the $j\omega$-axis). Let $I := I_{n_o}$.

(4.22) ⇒ (4.23). By (3.14), condition (4.22) implies that

$$\det[I + \tilde{P}CF(j\omega)] \neq 0, \quad \forall\omega \in \mathbb{R}_+, \quad \forall\Delta P \in \mathcal{D} . \tag{4.31}$$

Now, (4.21) implies that, on $\mathbb{R}$, $\det[I + PCF(j\omega)] \neq 0$ and is bounded; thus (4.31) holds if and only if

$$\det[I + \Delta P\, Q(j\omega)] \neq 0, \quad \forall\omega \in \mathbb{R}_+, \quad \forall\Delta P \in \mathcal{D} , \tag{4.32}$$

where

$$Q := CF(I+PCF)^{-1} . \tag{4.33}$$

We prove "(4.32)⇒(4.23)" by contradiction. Suppose that (4.23) fails, equivalently, $\exists \omega_1 \in \mathbb{R}_+$ s.t.

$$\sigma_1 := \sigma_{max}[Q(j\omega_1)] > \frac{1}{\ell_a(\omega_1)} . \tag{4.34}$$

Let $U \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \ddots \end{bmatrix} V^*$ be a singular-value decomposition of $Q(j\omega_1)$, where $U := (u_{ij})_{n_i x n_i} \in \mathbb{C}^{n_i x n_i}$ and $V := (v_{ij})_{n_o x n_o} \in \mathbb{C}^{n_o x n_o}$ are <u>unitary</u> matrices. Choose

$$\Delta P_1(s) := \begin{bmatrix} \alpha_1(s) \\ \vdots \\ \alpha_{n_o}(s) \end{bmatrix} (-1/\sigma_1)\ h_q(s)^{k'}\ [\beta_1(s),\cdots,\beta_{n_i}(s)] \tag{4.35}$$

where $k' \in \mathbb{N}^*$,

$$h_q(s) := 1/[1 + (\frac{s}{\omega_1} + \frac{\omega_1}{s})q] \quad \text{with} \quad q > 0 \quad , \tag{4.36}$$

$$\alpha_i(s) := \frac{s}{\omega_1}\ Im(v_{i1}) + Re(v_{i1}), \qquad \text{for } i = 1,\cdots,n_o,$$

$$\beta_i(s) := -\frac{s}{\omega_1}\ Im(u_{i1}) + Re(u_{i1}), \quad \text{for } i = 1,\cdots,n_i .$$

We claim that, for some $k' \in \mathbb{N}^*$ and some $q > 0$, $\Delta P_1 \in \mathcal{D}$. Indeed:

(i) By choosing $k' \geq 3$, ΔP_1 is a strictly proper exponentially stable rational matrix; hence (4.5) and (4.7) hold.

(ii) Setting $s = j\omega_1$ in (4.35) gives

$$\Delta P_1(j\omega_1) = \begin{bmatrix} v_{11} \\ \vdots \\ v_{n_o 1} \end{bmatrix} (-1/\sigma_1)\ [u_{11}^*,\cdots,u_{n_i 1}^*] = V \left[\begin{array}{c|c} \frac{-1}{\sigma_1} & \\ \hline & \end{array}\right] U^* \tag{4.38}$$

and hence, using (4.34),

$$\sigma_{max}[\Delta P_1(j\omega_1)] = \frac{1}{\sigma_1} < \ell_a(\omega_1) \ . \tag{4.39}$$

With (4.39), by assumptions (4.8) and (4.9) on $\omega \mapsto \ell_a(\omega)$ and using the characteristics of h_q, we can choose $k' \in \mathbb{N}^*$ and $q > 0$, both sufficiently large, such that (4.6) holds. Thus, by (i) and (ii), $\Delta P_1 \in \mathcal{D}$.
By (4.38),

$$\det[I + \Delta P_1 Q(j\omega_1)] = \det\{I + V \operatorname{diag}[-1,0,\cdots,0]\, V^*\}$$

$$= \det V \cdot \det\{\operatorname{diag}[0,1,\cdots,1]\} \det V^* = 0;$$

hence, (4.32) is violated for $\Delta P_1 \in \mathcal{D}$ at $\omega = \omega_1$, and by contradiction, (4.32) implies (4.23).

Since we have established "(4.22) $\Rightarrow$ (4.31) $\Leftrightarrow$ (4.32) $\Rightarrow$ (4.23),"
we have proved "(4.22) $\Rightarrow$ (4.23)."

$\Leftarrow$. Since the assumption (4.21) and (4.23) do not involve ΔP, we will consider a <u>fixed</u> but <u>arbitrary</u> $\Delta P \in \mathcal{D}$ and show that (4.21) and (4.23) imply

$$S(\tilde{P},C,F) \text{ is } A_-\text{-stable} \tag{4.45}$$

Hence the conclusion (4.22) follows by the arbitrariness of $\Delta P \in \mathcal{D}$.

Since $\forall A \in \mathbb{C}^{m\times n}$, $\sigma_{max}[A] = \sup_{\|z\|_2=1} \|Az\|_2$, $\sigma_{min}[A] = \inf_{\|z\|_2=1} \|Az\|_2$, we have $\forall \omega \in \mathbb{R}_+$, $\forall \varepsilon \in [0,1]$,

$$\begin{aligned}\sigma_{min}[I + \varepsilon\Delta P\, Q(j\omega)] &= \inf_{\|z\|_2=1} \|[I + \varepsilon\Delta P\, Q(j\omega)]z\|_2 \\ &\geq \inf_{\|z\|_2=1} \{\|z\|_2 - \varepsilon\|\Delta P\, Q(j\omega)z\|_2\} = 1 - \varepsilon\, \sigma_{min}[\Delta P\, Q(j\omega)] \\ &\geq 1 - \sigma_{min}[\Delta P\, Q(j\omega)] \geq 1 - \sigma_{max}[\Delta P\, Q(j\omega)] \\ &\geq 1 - \sigma_{max}[\Delta P(j\omega)]\, \sigma_{max}[Q(j\omega)] \\ &> 1 - \ell_a(\omega)\, \sigma_{max}[Q(j\omega)] \ .\end{aligned} \tag{4.46}$$

where we used (4.6) to obtain (4.46). From (4.46), assumption (4.23) implies

$$\sigma_{min}[I + \varepsilon\Delta P\, Q(j\omega)] > 0, \quad \forall\omega \in \mathbb{R}_+, \quad \forall\varepsilon \in [0,1] \quad . \tag{4.47}$$

Since, by (4.5), ΔP goes to zero as $|\omega| \to \infty$, and, by (4.21), $Q : u_3 \mapsto (-y_1)$ is bounded on the $j\omega$-axis, $\sigma_{min}[I + \varepsilon\Delta PQ(j\omega)] \to 1$ as $|\omega| \to \infty$. Hence (4.47) implies that $\exists\ \delta_1 > 0$ such that

$$|\det[I + \varepsilon\Delta P\, Q(j\omega)]| > \delta_1 > 0, \quad \forall\omega \in \mathbb{R}_+, \quad \forall\varepsilon \in [0,1] \ . \tag{4.48}$$

Similarly, $\exists\ \delta_2 > 0$ such that $|\det[I + PCF(j\omega)]| > \delta_2 > 0$, $\forall\omega \in \mathbb{R}_+$; hence, (4.48) is equivalent to

$$|\det[I + (P+\varepsilon\Delta P)CF(j\omega)]| > \delta > 0, \quad \forall\omega \in \mathbb{R}_+, \quad \forall\varepsilon \in [0,1], \tag{4.49}$$

where $\delta := \delta_1\delta_2$.

Define $N_0 : \omega \mapsto \det[I + PCF(j\omega)]$ and $N_1 : \omega \mapsto \det[I + \tilde{P}CF(j\omega)]$, both mapping $\overline{\mathbb{R}}$ into $\mathbb{C}$. (Note that, by continuity, $N_i(\infty) = N_i(-\infty) = 1$, for $i = 1,2$). Thus, N_0 and N_1 are closed bounded curves. Now, let $\Omega := \mathbb{C}\backslash\overline{B(0;\delta)}$, ($\Omega$ is a connected open set), we claim that

$$\underline{N_0 \text{ and } N_1 \text{ are } \Omega\text{-homotopic cycles.}} \tag{4.54}$$

Indeed:

(i) The map $h : (\omega,\varepsilon) \mapsto \det[I + (P+\varepsilon\Delta P)CF(j\omega)]$ is <u>continuous</u> on $\overline{\mathbb{R}} \times [0,1]$ and, by (4.49), h maps $\overline{\mathbb{R}} \times [0,1]$ into Ω with $h(\cdot,0) = N_0(\cdot)$, $h(\cdot,1) = N_1(\cdot)$, and $\forall\varepsilon \in [0,1]$, $h(\infty,\varepsilon) = h(-\infty,\varepsilon) = 1$; hence the <u>closed</u> curves N_0 and N_1 are <u>Ω-homotopic</u> (Rudin 1974, p. 239).

(ii) By assumption, P, C, F and ΔP, all with elements in $\hat{\mathcal{B}}$, have no poles on the $j\omega$-axis, the maps N_0 and N_1 are analytic in $\mathbb{R}$; thus N_0 and N_1 are C^1 on $\overline{\mathbb{R}}$. Our claim (4.54) follows from (i) and (ii) above.

Since $0 \notin \Omega$, N_0 and N_1 have the same index with respect to the origin (Rudin 1974, Theorem 10.40). Thus

$$\left.\begin{array}{l}\text{the number of encirclements of } N_0, \\ \text{the Nyquist diagram of } \det[I+PCF], \\ \text{around the origin;}\end{array}\right\} = \left\{\begin{array}{l}\text{the number of encirclements of } N_1, \\ \text{the Nyquist diagram of } \det[I+\tilde{P}CF], \\ \text{around the origin.}\end{array}\right. \quad (4.56)$$

By (4.7), (4.21) and Theorem 3.2, we conclude "(4.56) $\Leftrightarrow$ (4.45)." Then, by the sequence of implications above, we proved "(4.23) $\Rightarrow$ (4.45)" for the $\Delta P \in \mathcal{D}$ chosen. ■

The necessary and sufficient condition of Theorem 4.1 can also be formulated in terms of multiplicative plant perturbations.

Corollary 4.2

Consider the multiplicatively perturbed system $S(P',C,F)$ with $P' := (I_{n_o}+M)P$. Let $P \in \hat{B}_o^{n_o \times n_i}$, $C \in \hat{B}^{n_i \times n_o}$, $F \in \hat{B}^{n_o \times n_o}$ and $M \in \mathcal{M}$ where, for a given tolerance function $\omega \mapsto \ell_m(\omega)$ satisfying conditions similar to (4.8) and (4.9), $\mathcal{M}$ is described by

$\mathcal{M} := \{M:$ (a) elements of $M \in [\hat{A}_-]\,[\hat{A}_-\backslash\{0\}]^{-1}$;[5]

(b) $MP \in \hat{B}_o^{n_o \times n_i}$;

(c) $\sigma_{max}[M(j\omega)] < \ell_m(\omega)$, $\forall \omega \in \mathbb{R}_+$;

(d) $n^+_{p'} = n^+_p$.$\}$

U.t.c., if

$S(P,C,F)$ is A_--stable.

then

$\forall M \in \mathcal{M}$, $S(P',C,F)$ is A_--stable;

$\Leftrightarrow$

$$\sigma_{max}[PCF(I_{n_o}+PCF)^{-1}(j\omega)] \leq 1/\ell_m(\omega), \quad \forall \omega \in \mathbb{R}_+ .$$ ■

[5] Condition (a) says that all elements of M must belong to the field of fractions of the commutative domain $\hat{A}_-$.

Remarks 4.2.1:

(a) $M \in \mathcal{M}$ may not be proper; however, both P and P' must be strictly proper.

(b) For lumped unity-feedback systems, Corollary 4.2 is the result stated in Doyle and Stein (1981). ∎

Proof of Corollary 4.2:

The proof follows the same lines as that of Theorem 4.1 with the following substitutions: $\Delta P \leftarrow M$, $\mathcal{D} \leftarrow \mathcal{M}$, $Q \leftarrow H := PCF(I+PCF)^{-1}$, $\Delta P_1 \leftarrow M_1$ where

$$\left.\begin{aligned} \alpha_i(s) &:= |v_{i1}| \prod_{j=1}^{a_i} \frac{s+x_{ij}}{s-x_{ij}} \\ \beta_i(s) &:= |u_{i1}| \prod_{j=1}^{b_i} \frac{s+y_{ij}}{s-y_{ij}} \end{aligned}\right\} \quad i = 1,\cdots,n_o,$$

and the a_i's, b_i's $\in \mathbb{N}^*$ and the x_{ij}'s, y_{ij}'s $\in \mathring{\mathbb{C}}_-$ are such that

(i) $\alpha_i(j\omega_1) = v_{i1}$, $\beta_i(j\omega_1) = u_{i1}^*$;

(ii) $(I+M_1)$ does not cancel any $\mathbb{C}_+$-pole of P. ∎

5. Another Configuration

The method used in Section 4 applies to more general configurations, for example, the feedback configuration used in Åstrom (1980), Chen et al. (1981), and Pernebo (1981). More precisely, we now consider the distributed system $\overline{S}(\tilde{P},[\Pi \vdots F])$ as shown in Fig. 2, where

(i) the additively perturbed system $\tilde{P} := P + \Delta P$, with both P and $\Delta P \in \hat{\mathcal{B}}_o^{n_o \times n_i}$, has a r.c.f. $\tilde{P} = N_{\tilde{p}r} D_{\tilde{p}r}^{-1}$, and (5.1)

(ii) the two-input one-output compensator $[\Pi \vdots F] \in \hat{\mathcal{B}}^{n_i \times 2n_o}$ has a l.c.f. $[\Pi \vdots F] = D_{c\ell}^{-1} [N_{\pi\ell} \vdots N_{f\ell}]$. (5.2)

We assume that the controller is realized so that $D_{c\ell}^{-1}$ lies inside the feedback-loop (see Fig. 2). Note that (5.2) implies that $N_{\pi\ell}$ and $N_{f\ell}$ have elements in $\hat{A}_-$, and that $D_{c\ell}$ can be chosen rational.

Then using analysis similar to that for the system $S(\tilde{P},C,F)$, we easily obtain the following:

(I) $H_{\bar{y}\bar{u}}(\Delta P) : [u_1^T \vdots u_2^T \vdots v_1^T]^T \mapsto [y_1^T \vdots y_2^T]^T$, the I/O map of the system $\bar{S}(\tilde{P},[\Pi \vdots F])$, is given by

$$H_{\bar{y}\bar{u}}(\Delta P) = \bar{N}_r(\Delta P)\, \bar{D}(\Delta P)^{-1}\, \bar{N}_\ell$$

where

$$\bar{D}(\Delta P) := \left[\begin{array}{c|c} D_{c\ell} & N_{f\ell} N_{\tilde{p}r} \\ \hline -I_{n_i} & D_{\tilde{p}r} \end{array}\right],$$

$$\bar{N}_r(\Delta P) := \left[\begin{array}{c|c} I_{n_i} & \\ \hline & N_{\tilde{p}r} \end{array}\right], \quad \bar{N}_\ell := \left[\begin{array}{c|c|c} N_{f\ell} & & N_{\pi\ell} \\ \hline & I_{n_i} & \end{array}\right].$$

(II) $(\bar{N}_r(\Delta P), \bar{D}(\Delta P))$ are r.c.; $(\bar{D}(\Delta P), \bar{N}_\ell)$ are l.c.

(III) $\det \bar{D}(\Delta P) = \eta\, \chi_{\tilde{p}} \cdot \chi_{[\Pi \vdots F]} \cdot \det[I_{n_o} + \tilde{P}F]$

where $\eta = 1$ or -1, depending on the size of the matrices involved.

(IV) The system $\bar{S}(\tilde{P},[\Pi \vdots F])$ is A_--stable;

$\Leftrightarrow$ (by definition)

$$H_{\bar{y}\bar{u}}(\Delta P) \in A_-^{(n_i+n_o)\times(n_i+2n_o)};$$

$\Leftrightarrow$

$$\det \bar{D}(\Delta P) \text{ has no zeros in } \mathbb{C}_+;$$

$\Leftrightarrow$

the Nyquist diagram of $\det[I_{n_o} + \tilde{P}F]$

(i) does not go through the origin, and

(ii) encircles the origin $n^+_{\tilde{p}} + n^+_{[\Pi \vdots F]}$ times counterclockwise.

∎

By reasoning as in Section 4, we can prove

Theorem 5.1.

Consider the system $\overline{S}(\tilde{P},[\Pi \vdots F])$ shown in Fig. 2 satisfying (5.1) and (5.2). Let $\mathcal{D}$ be described by (4.5)-(4.9). U.t.c., if

$\overline{S}(P,[\Pi \vdots F])$ is A_--stable,

then

$\forall \Delta P \in \mathcal{D}$, $\overline{S}(\tilde{P},[\Pi \vdots F])$ is A_--stable;

$\Leftrightarrow$

$$\sigma_{max}[F(I_{n_o}+PF)^{-1}(j\omega)] \leq 1/\ell_a(\omega), \quad \forall \omega \in \mathbb{R}_+$$

∎

6. Conclusion

For efficient engineering design it is important to know when sufficient conditions are, in fact, necessary. For this reason, it is worthwhile to have a detailed and convincing proof of such conditions for as large a class of systems as possible. A little thought shows that the method and the results of this paper apply -- with the usual changes from s to z, etc. -- to discrete-time systems whether lumped or distributed (For a description of discrete distributed systems see Cheng and Desoer 1980). ∎

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Appendix A: Difficulties with [Doy. 2, pp. 6-7]

(1) For closed-loop stability, the number of encirclements of det[I+GK] must be equal to - {the number of unstable modes of G + the number of unstable modes of K}, (and not the number of unstable modes of GK, as stated in [Doy. 2, p. 7]).

(2) {the number of unstable modes of G = the number of unstable modes of G} does not imply

{the number of unstable modes of GK = the number of unstable modes of G'K}, as stated in [Doy. 2, p. 7].

(3) How do we know that the set of all L's s.t.

$$\bar{\sigma}[L(j\omega)] < \ell_m(\omega) \ , \quad \forall \omega \in \mathbb{R}_+ \tag{13}$$

and s.t.

$$\text{the number of } \mathbb{C}_+\text{-poles of } (I+L)G = \text{the number of } \mathbb{C}_+\text{-poles of } G \tag{13a}$$

is a connected set so that the "warping" argument above condition (14) of [Doy. 2, p. 7] holds?

(4) Consider condition (14) of [Doy. 2, p. 7]

$$0 < \underline{\sigma}[I + [I + \varepsilon L(\bar{s})]G(\bar{s})K(\bar{s})]$$

$\forall \varepsilon \in [0,1]$, $\forall \bar{s} \in$ D-contour, $\forall L(s)$ s.t. $\bar{\sigma}[L(j\omega)] < \ell_n(\omega)$, $\forall \omega \in \mathbb{R}_+$.

Suppose now that, with $\varepsilon = 1$, G and G' have the same number of $\mathbb{C}_+$-poles and that there is a $\mathbb{C}_+$-pole of G' which is not a pole of G. Simple calculations show that, $\forall \varepsilon \in (0,1)$, the number of $\mathbb{C}_+$-poles of $G' = (I+\varepsilon L)G$ and G' will not remain the same, thus violating condition (13a) above. For example: let $G = \frac{1}{s-1}$, $G' = \frac{1}{s-1-\delta} = (1 + \frac{\delta}{s-1-\delta}) \frac{1}{s-1}$, with $\delta > -1$; then $L = \frac{\delta}{s-1-\delta}$ and $(I+\varepsilon L)G = [1 + \frac{\varepsilon\delta}{s-1-\delta}] \frac{1}{s-1} = \frac{s-1-(1-\varepsilon)\delta}{(s-1)(s-1-\delta)}$. Thus for $\varepsilon = 0$, $G' = G$ has one $\mathbb{C}_+$-pole at 1;

for $0 < \varepsilon < 1$, G' has <u>two</u> $\mathbb{C}_+$-poles at 1 and $1+\delta$;

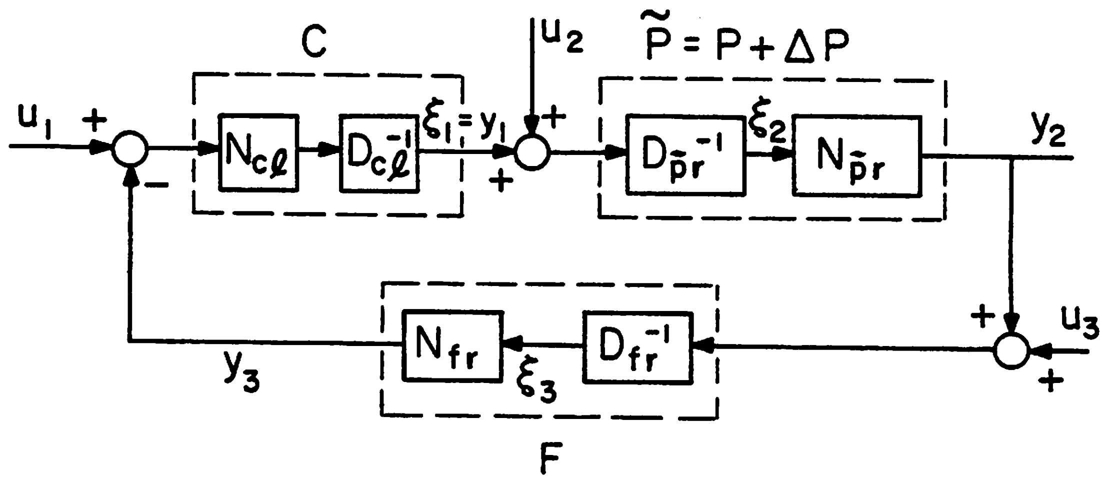
for $\varepsilon = 1$, G' has one $\mathbb{C}_+$-poles at $1+\delta$.

(5) The equivalence of (16) and (17) in [Doy. 2, p. 7] is obvious if L is <u>only</u> to satisfy (13). However, the L under consideration must not only be <u>rational</u>, but also satisfy (13) <u>and</u> (13a). ¤

Figure Caption

Fig. 1. The system $S(\tilde{P},C,F)$.

Fig. 2. The system $\bar{S}(\tilde{P},[\Pi\vdots F])$.


C
u_2
$\tilde{P} = P + \Delta P$
u_1
+
−
$N_{c\ell}$
$D_{c\ell}^{-1}$
$\xi_1 = y_1$
+
+
$D_{\tilde{p}r}^{-1}$
ξ_2
$N_{\tilde{p}r}$
y_2
+
u_3
+
N_{fr}
ξ_3
D_{fr}^{-1}
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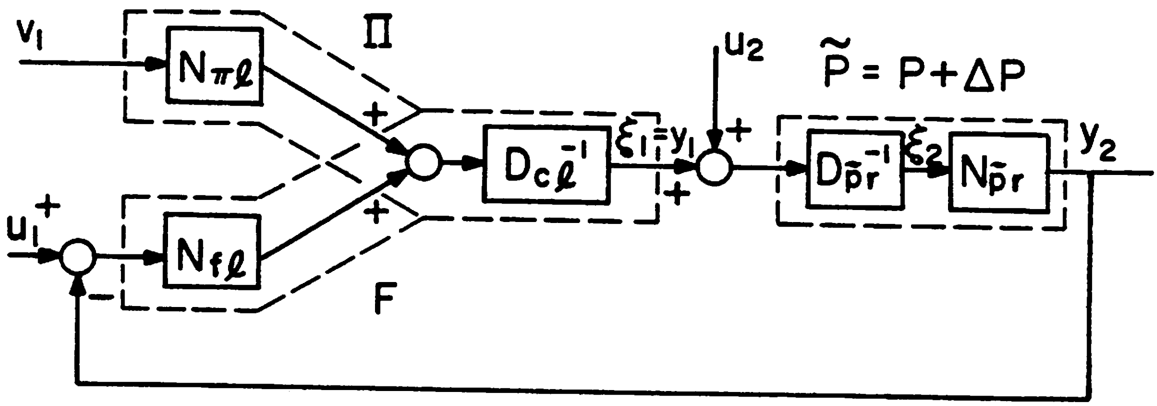
F



v_1
$N_{\pi\ell}$
Π
u_2
$\tilde{P} = P + \Delta P$
+
$D_{c\ell}^{-1}$
$\xi_1 = y_1$
+
$D_{\tilde{p}r}^{-1}$
ξ_2
$N_{\tilde{p}r}$
y_2
+
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u_1
+
$N_{f\ell}$
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