ENHANCEMENT OF CLASSICAL TRANSPORT PROCESSES
ALONG RESONANCES IN NEAR INTEGRABLE SYSTEMS
WITH MANY DEGREES OF FREEDOM

by

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Memorandum No. UCB/ERL M81/7

25 January 1981

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such systems, the resonances represent "leaks" capable of draining the "trapped" regions of phase space and thereby reducing the average confinement time. Physical applications may be found in studies of high energy storage rings, the magnetic confinement of plasmas, and planetary motion.

Streaming should not be confused with the well known neoclassical diffusion [2]-[4] or Arnold diffusion [5]-[7]. It may be viewed as an extension of the theory of resonant transport [8]-[10] to multidimensional systems. Like the processes mentioned above, streaming is limited to systems with small diffusion rates, where the motion induced by the external transport process is slower than that induced by the resonant libration.

Experimentally, the observed effects of streaming are very similar to those of Arnold diffusion. In confinement systems, particle densities become depressed near the resonance surfaces and "halos" or "tails" appear outside the main body of confined particles. Streaming has been identified as the major cause of particle loss in a two dimensional computer simulation of the colliding beams at SPEAR [11].

2. Description

2.1. A Simple Model

The mechanics of streaming are easily illustrated with a simple model consisting of two uncoupled nonlinear oscillators. The system is integrable and has a well defined two-dimensional action manifold S. The hamiltonian function $H_o(\underline{I})$ depends only on the actions $\underline{I}$, and defines two important families of curves on S:

1) the energy level sets (or contours)

$$H_o(\underline{I}) = h \tag{1}$$

where h is a constant $0<h<\infty$, and

2) the resonance curves

$$\omega_1(I_1)\, m_{k1} + \omega_2(I_2)\, m_{k2} = 0. \tag{2}$$

Here k $= 1,2,...\infty$, $\omega_1 = \partial H_o/\partial I_1$ and $\omega_2 = \partial H_o/\partial I_2$ are the oscillator frequencies, and m_{k1}, m_{k2} are a pair of integers, unique for each k.

These curves are illustrated in fig. 1. The trajectories are confined to two dimensional "invariant tori" defined by fixed values of I_1 and I_2 in the four dimensional phase space. The phase point does not move on the action manifold S.

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1. Introduction

Demands for the long term magnetic confinement of energetic charged particles in both the high energy physics and fusion energy fields have prompted a growing interest in the behavior of near integrable hamiltonian systems. Particular attention has been given to diffusion processes, both those associated with the so-called "intrinsic stochasticity" of the motion and those involving the enhancement of an externally generated noise. This paper describes a process of the later type, resonance "streaming", and presents the formalism necessary for its application to real systems.

Streaming occurs in multi-dimensional [1] nonlinear systems that are close to integrable and subject to an externally generated transport process such as a diffusion or dissipation on the action space. In such a system, integrability is destroyed by the presence of a perturbation in the hamiltonian function. This perturbation, if sufficiently small, does not significantly change the unperturbed motion, but results only in a small bounded oscillation in the action space of the unperturbed system. Although the oscillation itself may be unimportant, it necessitates an averaging procedure when the long term behavior is of interest. Specifically, transport phenomena must be described in terms of the motion of the oscillation-center, rather than of the instantaneous position. When the phase point is outside nonlinear resonance, the oscillation-center transport is almost identical to the unperturbed classical transport. But when the system is resonant, the two can be drastically different in both magnitude and direction. The difference is most pronounced when the direction of the resonant oscillation (refered to henceforth as "resonant libration") is nearly tangent to the resonance surface. In this case the oscillation-center can move rapidly along the resonance surface at a rate that is much greater than, but still proportional to, the classical transport rate. These ideas will be described in greater detail in the following sections.

Streaming is potentially present in any multidimensional system that depends on approximate or adiabatic invariants to confine particles to a particular locality of the phase space. In

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ABSTRACT

When a small perturbation is added to the hamiltonian function of an integrable multi-dimensional nonlinear system, classical transport processes on the action space can be dramatically altered when the phase point is inside a nonlinear resonance. This phenomena, called "streaming" here, can result in the rapid migration of the phase point along the resonance, and a corresponding enhanced transport through the action space. The magnitude of the enhanced transport depends on the direction of the resonant libration induced by the perturbation.

When the two oscillators are weakly coupled (an interaction perturbation is added to the hamiltonian function), the system is no longer stationary on S. But for small perturbations, the motion induced by the coupling is a bounded oscillation about a fixed oscillation-center. The boundedness results from the fact that most of the invariant tori are not destroyed by the coupling, but only distorted somewhat [12-14]. The perturbed tori are shown in fig. 2 where the distortion is seen to be most severe in the region close to a resonance. In fact, an entirely new type of torus appears in this region. These "resonant tori" are nested tubes that run between the leaves of the nonresonant tori. Resonant trajectories spiral around and along these tubes. It will be shown in the next section that the projections of the resonant tori onto S are approximately straight line segments. In an autonomous system such as this, these line segments are always tangent to an energy curve (1).

The survival of the invariant tori under small perturbations allows for the replacement of the old invariants I_1, I_2 by new invariants $<I_1>, <I_2>$. These are just the coordinates of the oscillation-center on S defined as the time average of the actions [15].

2.2. Transport Enhancement

An external process on S is an induced motion on S (random or deterministic, continuous or descrete) that acts in addition to the hamiltonian phase flow. It may be a diffusion, due for example to scattering effects, or a drift, resulting perhaps from a dissipation or accretion of energy. The simplest example of an external process is an instantaneous, or "classical", displacement on S (which may be infinitessimal). Other processes may be constructed from a sequence of such displacements. For an uncoupled system, the long term motion on S may be determined by simply adding up (or integrating) the classical displacements. For a coupled system with a weak external process, such a determination is made by adding up the oscillation-center displacements instead. The two types of displacement are quite different when the phase point is on a resonant torus. The relationship is illustrated in fig. 3.

In fig. 3, the phase point is initially on a resonant torus and the motion on S is an oscillation on a line segment. The oscillation-center is on the resonance curve at the point A. A classical displacement then occurs which takes the phase point from the point **a** to the point **b**. Correspondingly, the oscillation-center jumps from A to B. The magnitude of the oscillation-center displacement is then related to that of the classical displacement by

$$|\mathrm{A}-\mathrm{B}| = |\mathrm{a}-\mathrm{b}|\sin(\alpha)\csc(\phi). \quad (3)$$

If the angle ϕ is very small, $|\mathrm{A}-\mathrm{B}|$ may be much larger than $|\mathbf{a}-\mathbf{b}|$. Furthermore, the oscillation-center displacement is always along the resonance as long as both **a** and **b** are on resonant tori. Thus, any sequence of classical displacements that does not take the system out of resonance will result in a net displacement of the oscillation-center along the resonance, and this net displacement may be much larger in magnitude than the sum of the classical displacements.

The relation (3) is valid for any Riemannian metric on S. However, the angles α and ϕ are metric dependent, and so is the ratio $|\mathbf{A}-\mathbf{B}| / |\mathbf{a}-\mathbf{b}|$. Consequently, the extent to which a particular resonance produces a significant enhancement depends critically on the additional physical considerations from which a convenient metric is chosen.

2.3. Transport Regimes

A resonant torus may be classified into one of three distinct transport regimes. These are approximately equivalent to the well known regimes of neoclassical theory [3].

1) The oscillation-center (or weak transport) regime. For the oscillation-center concept to be valid inside a resonance, the system must remain in the resonance for a time greater than one libration period. This means that the period of resonant libration must be smaller than the time necessary for the classical transport process to move the phase point from the center of the resonance to its edge. The enhancement ratio (3) is only valid in the oscillation-center regime.

2) The classical (or strong transport) regime. If the external process induces a motion on S that is faster than that induced by the resonance, then the presence of the resonance is inconsequential. The transport is said to be "classical" in this case. Although a generic near integrable system has an infinite number of resonances, all but a finite number will fall within the classical regime and can thereby be ignored.

3) The plateau regime. Since the upper limit of 1) does not coincide with the lower limit of 2), there is an intermediate situation. Here the phase point is not in the resonance long enough to complete one libration cycle, but the libration motion that does occur moves the phase point along the resonance a distance greater than the resonance width. In physical systems, transport enhancement is often more important in the plateau regime than in the oscillation-center regime. This is because the plateau regime gets wider as the libration angle ϕ gets smaller (see (52), next section). Thus, the plateau regime is widest in precisely the situation where there is the most enhancement.

2.4. Examples of External Processes

Two well known processes that cannot be built into the hamiltonian function, but may be expressed as sequences of classical displacements, are dissipation and diffusion. In a physical system, dissipation may be due to such things as mechanical friction, "dynamical" friction, or classical radiation. Diffusion might result from particle collisions, quantum radiation, or "quasilinear" scattering from a continuous background spectrum.

2.4.1. Dissipative Processes

A dissipative process is perceived locally as a drift on the unperturbed invariant space S. The presence of a small coupling perturbation allows the phase point to become temporarily trapped in one of the nonlinear resonances. When the phase point is inside such a resonance, the resonant libration is constantly inverting the accumulated drift, effectively cancelling its component perpendicular to the resonance. This phenomena is similar to the phase point trapping that is observed in nonautonomous systems with one degree of freedom [16], where the resonance pumps energy into the system as fast as it is dissipated (see fig. 4). The difference lies in the fact that here the system is autonomous and has no source of energy. The resonance can constrain the system in one direction on S, but cannot effect the rate at which energy is lost. This is seen clearly in fig. 3 where the classical displacement and the oscillation-center displacement connect the same two energy contours. A consequence of this is the fact that the phase point must fall out of the resonance when it approaches a point where the resonance curve is tangent to an energy contour [17], (see fig. 5). Thus, unless it actually intersects an attractor, a nonlinear resonance cannot perpetually "trap" the dissipative system. It can only force a detour on the route to its ultimate destination.

The motion of a phase point under the influence of both dissipation and nonlinear resonance is somewhat analogous to the motion of a frictionless iceboat with a flat sail. The iceboat, like the phase point, is constrained to move in only one direction on the plane. The sail angle relative to the heading of the boat corresponds to the angle of resonant libration ϕ, and the wind velocity plays the role of the drift velocity $\mathbf{v}$. When the maximum velocity of the iceboat is calculated, it is found to correspond exactly to that derived for the resonant oscillation-center (eq. (19) next section). Just as an iceboat can travel considerably faster than the wind (even when close-hauled!), so can the resonant oscillation-center travel more quickly than (and against) the classical drift.

Figure 6 shows two computer generated trajectories plotted on an action space. The system is composed of two weakly coupled nonlinear oscillators with a vertical dissipation (an externally imposed downward velocity which is proportional the vertical coordinate I_2). The attractor is the $I_2 = 0$ line. The first trajectory is initially nonresonant. It drifts down to the resonance, makes a small horizontal jump as it crosses, and then continues on. The direction and magnitude of the horizontal jump depend on the difference between the phases of the two oscillators when the crossing occurs. The second trajectory becomes trapped inside the resonance as it attempts to cross and its oscillation-center is consequently constrained to move along the resonance curve. The libration angle is about 22 degrees and the ratio of resonant to nonresonant drift speed is 2.55 (both of these depend on the scale ratio). The resonant libration is slowly damped by the dissipation and the phase point is drawn toward the center of the resonance (barely perceptable here).

2.4.2. Diffusive Processes

A dissipative process can be described by a vector field on S. A diffusive process, on the other hand, is characterized by a second rank tensor **d**. The components of this tensor are defined by

$$d^{ij} = < v^i v^j >$$

where v^i and v^j are the ith and jth components of a random classical displacement **v**. The diffusion consists of a sequence of such jumps. The probability distribution for this sequence is known and used in the calculation of the average $< >$. The diffusion allows the phase point to wander in and out of the nonlinear resonances. When the phase point is resonant, the oscillation-center diffusion perpendicular to the resonance is zero while the parallel diffusion may be either stronger or weaker than the classical. The enhancement ratio E between the parallel oscillation-center diffusion and the parallel classical diffusion depends, as before, on the metric. It is often convenient, however, to set the components of the metric tensor g^{ij} equal to those of the diffusion tensor d^{ij} (see section 3). This makes the diffusion isotropic and reduces the expression for the enhancement factor to a particularly simple form.

An example of resonant diffusion is shown in fig. 7. The classical diffusion is in the vertical direction only [18]. The phase point starts at the center of the resonance and diffuses slowly out. Meanwhile, its oscillation-center diffuses almost horizontally, along the resonance curve. When the phase point eventually leaves the resonance, the oscillation-center stops its horizontal motion and begins to diffuse vertically, following the classical displacements.

3. Formal Development

A precise description of the streaming phenomena requires the use of curvilinear coordinates (see for example [19]). These make it possible to develop formal expressions without referring to a particular coordinate system or metric.

The system is defined [20] by an autonomous near integrable hamiltonian function on the direct product S × T^n of the n-dimensional torus $T^n = \{\underline{\theta} = (\theta_1, \ldots, \theta_n) mod 2\pi\}$ and a region S of the n-dimensional real vector space $S \subset R^n = \{\underline{I} = (I_1, \ldots, I_n)\}$,

$$H(\underline{I},\underline{\theta}) = H_o(\underline{I}) + \epsilon H_1(\underline{I},\underline{\theta}). \tag{4}$$

The coordinates $\underline{I}$ and $\underline{\theta}$ are the n-component action-angle variables of the unperturbed hamiltonian function H_o. Because the angle space is an n-torus, the small perturbation H_1 can be expanded into a Fourier series in $\underline{\theta}$. Equation (4) can accordingly be rewritten,

$$H(\underline{I},\underline{\theta}) = H_o(\underline{I}) + \epsilon \sum_{k=1}^{\infty} F_k(\underline{I}) e^{i(\underline{m}_k \cdot \underline{\theta})}, \tag{5}$$

where $\{\underline{m}_k = (m_{k1},\ldots,m_{kn})\}$ is the set of all n-tuples of integers and $\underline{m}_k \cdot \underline{\theta}$ is shorthand for $(m_{k1}\theta_1 + m_{k2}\theta_2 + \ldots + m_{kn}\theta_n)$. The equations of motion are,

$$\dot{\underline{I}} = -\epsilon i \sum_{k=1}^{\infty} \underline{m}_k F_k(\underline{I}) e^{i(\underline{m}_k \cdot \underline{\theta})} \tag{6}$$

$$\dot{\underline{\theta}} = \underline{\omega}(\underline{I}) + O(\epsilon). \tag{7}$$

Each of the terms in (6) defines a vector field on S. The direction of the kth vector field is given by the "kth resonance vector" $\mathbf{r}_k$. The components of $\mathbf{r}_k$ in action coordinates are independent of $\underline{I}$ and equal to the integers $\underline{m}_k$,

$$r_k^i = m_{ki}. \tag{8}$$

The resonance vector defines the direction of the motion on S induced by the kth term in (6). The frequencies $\omega_i = \partial H_o / \partial I_i$ define a covector field on S. For each k, there is an additional scalar function on S, the "k-resonance frequency" defined by [21]

$$R(\underline{I}) = r^i \omega_i(\underline{I}) \tag{9}$$

(the subscript "k" is implicit here and in all future references to **r** and those expressions derived from it). The derivative of R is a new covector

$$\nu_i(\underline{I}) = \frac{\partial R(I)}{\partial I_i}, \tag{10}$$

and a pairing with **r** results in a third scalar function, the "effective nonlinearity" ([6] p. 278)

$$M(\underline{I}) = r^i \nu_i(\underline{I}) \tag{11}$$

or

$$M(\underline{I}) = r^i \frac{\partial^2 H_o(\underline{I})}{\partial I_i \partial I_j} r^j. \tag{12}$$

The "k-resonance surface" is defined by

$$R(\underline{I}) = 0. \tag{13}$$

This is the condition for stationary phase in the kth term of (5) when $\epsilon = 0$. Equation (13) identifies the k-resonance surface with those points on S where **r** (and thus the resonant libration), is tangent to an energy level set $H_o = h$.

The normal to the k-resonance surface is defined by

$$n^i = g^{ij}(\underline{I})\ \nu_j(\underline{I}), \tag{14}$$

where g^{ij} is an arbitrary Riemannian metric tensor. This tensor defines an angle ϕ between **r** and **n**,

$$M = g_{ij} n^i r^j = |\mathbf{r}||\mathbf{n}|\sin(\phi), \tag{15}$$

where the norms $|\mathbf{r}|$ and $|\mathbf{n}|$ are given by

$$|\mathbf{r}|^2 = g_{ij} r^i r^j \tag{16}$$

$$|\mathbf{n}|^2 = g^{ij} \nu_i \nu_j. \tag{17}$$

For a given classical displacement **v** on S, the oscillation-center displacement **V** differs from **v** by a vector parallel to **r** (see figs. 3 and 8). Since **V** has no component perpendicular to the resonance surface, any enhancement that occurs must be along the "enhancement line" defined by the projection of **r** onto the tangent to the resonance surface.

The relationship between **v** and **V** is evident from fig. 8. The magnitude of the projection of **v** onto the direction defined by **n** is

$$\mathrm{v_n} = \frac{\mathrm{v}^i n^j}{|\mathbf{n}|} g_{ij}. \tag{18}$$

The difference between **v** and **V** is then

$$\mathbf{v} - \mathbf{V} = \frac{\mathbf{r}}{|\mathbf{r}|} \mathrm{v_n} \csc(\phi). \tag{19}$$

Using (15),(16) and (17), this can be rewritten

$$V^i = \mathrm{v}^j \left(\delta_j^i - \frac{\nu_j r^i}{M} \right). \tag{20}$$

This expression depends only on metric-independent tensors, and is applicable to systems with n degrees of freedom. It may be applied directly to systems with a uniform (or locally uniform) drift.

Although the strength of the perturbation ϵ does not affect the enhancement directly, it does affect the resonance width and libration frequency. Thus, as ϵ goes to zero, resonances occupy a smaller and smaller portion of the phase space. They eventually disappear (relative to the external process) into the classical regime.

With a diffusive process, the linear transformation used in (20) can also be used to define an oscillation-center diffusion tensor D^{ij} in terms of a classical diffusion tensor d^{lm}

$$D^{ij} = d^{lm}\left(\delta_l^i - \frac{\nu_l r^i}{M}\right)\left(\delta_m^j - \frac{\nu_m r^j}{M}\right). \tag{21}$$

Any increase in the diffusion rate **D** over the classical rate **d** will appear in the direction parallel to the enhancement line. It is useful therefore to project the two diffusion tensors onto the enhancement line and compare them. The projections are scalar diffusion coefficients that depend on the metric. Their ratio E compares the enhanced diffusion to the classical diffusion in the same direction. The definition of E is

$$E = \frac{<A^2>}{<B^2>} \tag{22}$$

where A is the magnitude of the projection of an oscillation-center displacement **V** onto the enhancement line

$$A = \csc(\phi)\tan(\phi)\mathrm{v}^i g_{ij}\left(\frac{r^j}{|\mathbf{r}|} - \frac{n^j}{|\mathbf{n}|}\csc(\phi)\right), \tag{23}$$

and B is the corresponding magnitude for the classical displacement **v**

$$B = \tan(\phi)\mathrm{v}^i g_{ij}\left(\frac{r^j}{|\mathbf{r}|}\csc(\phi) - \frac{n^j}{|\mathbf{n}|}\right). \tag{24}$$

The diffusion tensor **d** is related to the classical displacement **v** by

$$d^{ij} = <\mathrm{v}^i\mathrm{v}^j>. \tag{25}$$

Using (25) and (15), the mean squared values of A and B are

$$<A^2> = \sec^2(\phi)\, d^{ij} g_{im} g_{jl}\left\{\frac{r^m r^l}{|\mathbf{r}|^2} - \frac{r^m n^l}{M} - \frac{n^m r^l}{M} + \frac{n^m n^l|\mathbf{r}|^2}{M^2}\right\}, \tag{26}$$

$$<B^2> = \tan^2(\phi)\, d^{ij} g_{im} g_{jl}\left\{\frac{n^m n^l}{|\mathbf{n}|^2} - \frac{r^m n^l}{M} - \frac{n^m r^l}{M} + \frac{r^m r^l|\mathbf{n}|^2}{M^2}\right\}. \tag{27}$$

When (26) and (27) are substituted into (22), the resulting expression is rather cumbersome. It can be simplified considerably however, by choosing a particular metric. This metric, called the "standard" metric here, is defined by setting the components of the metric tensor equal to those of the diffusion tensor

$$g^{ij} = d^{ij}. \tag{28}$$

With this definition, the diffusion is isotropic on S. Using the metric identity

$$g^{ij} g_{im} = \delta^j_m, \tag{29}$$

and (28), the ratio of (26) to (27) reduces to

$$E = \csc^2(\phi) = \frac{|\mathbf{r}|^2 |\mathbf{n}|^2}{M^2}, \tag{30}$$

where

$$|\mathbf{r}|^2 = d_{ij} r^i r^j \tag{31}$$

$$|\mathbf{n}|^2 = d^{ij} \nu_i \nu_j. \tag{32}$$

The choice of metric (28), means that the oscillation centers of an ensemble of points with the same nonresonant initial conditions on S will diffuse isotropically under **d**. The oscillation-centers of a similar ensemble, starting inside a resonance, will by (21) diffuse isotropically in all directions except two. They will not expand at all in the direction perpendicular to the resonance surface, and they will expand faster by a factor csc (ϕ) in the direction along the enhancement line. For practical applications then, (30) is very useful in determining the relative dangers of the different resonances.

4. The Analysis of a Physical System

There a four basic steps in analyzing a given system for streaming effects.

1) Find a reasonable analytic (hamiltonian) model.

2) Determine which resonances are not in the classical regime.

3) Discard those that give insignificant enhancement factors.

4) Examine the enhancement directions of those remaining. Do they lead to and from interesting places? For example, can they transport particles to a limiter, a loss cone, or an escape trajectory?

4.1. Finding a Model

For a physical system to exhibit "near integrable" behavior, a faithful analytic model $H(\underline{I},\underline{\theta})$ must satisfy certain properties. The model must consist of an unperturbed function $H_o(\underline{I})$ which depends only on the actions, and a small perturbation $\epsilon H_1(\underline{I},\underline{\theta})$ which must be periodic in $\underline{\theta}$. The function $H_o(\underline{I},\underline{\theta})$ may be linear in $\underline{I}$ as long as the average of $H_1(\underline{I},\underline{\theta})$ over all θ is nonzero and nonlinear in $\underline{I}$ (for further details, see [22]). To be sure that invariant tori do indeed exist, it is advisable to check another condition of the KAM theorum [13]; the determinant of the Hessian matrix of H_o must be nonzero at the point of interest

$$\det\left[\frac{\partial^2 H_o}{\partial I_i \partial I_j}\right] \neq 0. \tag{33}$$

4.2. Resonance Classification

At any point on S, the nth resonance of the system may be classified according to the scheme described in sec. 2.3. To do this, it is necessary to know the width w of the resonant libration and the libration frequency f. Expressions for these quantities are derived in a number of works (see for example [5] or [6]). They are

$$w = 4|\mathbf{r}|\left(\frac{\epsilon \mathrm{F}}{M}\right)^{1/2} \tag{34}$$

and

$$f = (\epsilon \mathrm{F} M)^{1/2}, \tag{35}$$

where F is the resonance amplitude from (5). The definition of the effective resonance width Δ is dependent on whether the external process is diffusive or dissipative.

For a diffusive process, this definition is (see fig. 9a)

$$\Delta = |w \sin(\phi)| \tag{36}$$

or

$$\Delta = \left|\frac{w\ M}{|\mathbf{n}||\mathbf{r}|}\right| = 4\left|\frac{(\epsilon \mathrm{F} M)^{1/2}}{|\mathbf{n}|}\right| \tag{37}$$

where $|\mathbf{n}|$ is determined by the standard metric (32). The time it takes for the classical diffusion to move the phase point across the resonance is

$$t = \frac{\Delta^2}{2} = 8\left|\frac{\epsilon \mathrm{F} M}{|\mathbf{n}|^2}\right|. \tag{38}$$

Therefore, the resonance is in the oscillation-center regime if

$$t > \frac{2\pi}{f} \tag{39}$$

or

$$\frac{8(\epsilon F M)^{3/2}}{|\mathbf{n}|^2} > 2\pi. \tag{40}$$

The definition of the classical regime depends on the *rms* speed of the resonant libration. This is simply

$$s = \frac{w\,f}{\sqrt{8}}. \tag{41}$$

The classical regime is given then by

$$t\,s < \Delta \tag{42}$$

or

$$\frac{8(\epsilon F M)^{3/2}}{|\mathbf{n}|^2} < \frac{\sqrt{8}M}{|\mathbf{n}||\mathbf{r}|}. \tag{43}$$

Finally, the plateau regime is defined by the reverse of the two inequalities (40) and (43),

$$2\pi > \frac{8(\epsilon F M)^{3/2}}{|\mathbf{n}|^2} > \sqrt{8}\sin(\phi). \tag{44}$$

For a dissipative process with an approximately uniform drift velocity v across the resonance, the definition of Δ is (see fig. 12b)

$$\Delta = |w \sin(\phi)\csc(\alpha)| \tag{45}$$

or

$$\Delta = \left| 4\left(\frac{\epsilon F}{M}\right)^{1/2} \frac{M}{\nu_{,}\mathbf{v}^{i}} |\mathbf{v}| \right| \tag{46}$$

where α is the angle between the classical velocity v and the resonance surface. The time it takes for the phase point to drift classically across the resonance is

$$t = \frac{\Delta}{|\mathbf{v}|} = \left| 4\frac{\sqrt{\epsilon F M}}{\nu_{,}\mathbf{v}^{i}} \right|. \tag{47}$$

Using (39), the oscillation center regime for this case is

$$\frac{4\epsilon F M}{\nu_i v^i} > 2\pi. \tag{48}$$

The classical regime is

$$s < |\mathbf{v}| \tag{49}$$

or

$$\frac{4\epsilon F M}{|\mathbf{v}||\mathbf{n}|} < \frac{\sqrt{8}M}{|\mathbf{n}||\mathbf{r}|}. \tag{50}$$

The plateau regime for the dissipative case is thus

$$2\pi > \frac{4\epsilon F M}{v^i \nu_i} > \frac{4\epsilon F M}{|\mathbf{v}||M|} > \sqrt{8}\sin(\phi). \tag{51}$$

Since the definition of the plateau regime depends on the metric, the significance of enhanced motion for "crossing" trajectories (illustrated by the nonresonant trajectory in fig. 6) is dependent entirely on physical considerations beyond those already specified. In cases where both dissipation and diffusion exist together, it is usually convenient to use the standard metric.

In a typical Fourier spectrum, the resonance amplitudes eventually fall off exponentially with increasing $|\mathbf{r}|$. This means, from (44) and (51) that there is a cutoff value of $|\mathbf{r}|$ above which all resonances are in the classical regime. It should be noted that although the enhanced transport on a single resonance is proportional to the classical transport, a reduction of the classical transport, say of d^{ij}, extends the enhanced regimes, (oscillation-center and plateau) to higher values of $|\mathbf{r}|$. Since more resonances are now available for streaming, the overall transport is not necessarily reduced.

4.3. Enhancement Magnitude

Even when a resonance is well inside the oscillation-center regime, it does not necessarily provide enhanced transport. For a diffusive process, the level of enhancement is determined by (30),(31) and (32). The enhanced diffusion along the resonance is E times the classical diffusion in the same direction.

4.4. Enhancement Direction

In systems with only two degrees of freedom, the resonance surface is a curve on S, and the enhanced transport can proceed in only one direction. But when n>2 (when the resonance surface has more than one dimension), this direction is determined by the enhancement line, which in turn, is defined by the vector

$$e = \frac{\mathbf{r}}{|\mathbf{r}|}\sec(\phi) - \frac{\mathbf{n}}{|\mathbf{n}|}\tan(\phi). \tag{52}$$

This is the direction given by the projection of **r** onto the resonance surface.

5. Conclusion

A small amount of diffusion or dissipation in a near integrable hamiltonian system may be redirected and amplified in the regions of phase space occupied by the nonlinear resonances. In particular, the resonances may provide corriders along which phase points can "stream" rapidly from one region of phase space to another.

Given a particular hamiltonian function and a classical transport process on the action space, it is possible to determine on which resonances the streaming occurs, how strong it is, and in which direction it proceeds.

Streaming is only one of several transport mechanisms associated with near integrable systems. Its importance relative to the other mechanisms depends rather critically on the perturbation strength ϵ, and the magnitude of the classical diffusion **d** (or classical drift **v**). If **d** is very small, Arnold diffusion will be more important than streaming. If **d** is very large, the resonances will be in the classical regime and there will be no enhancement at all. The conditions on ϵ are similar. If ϵ is very large, resonances will overlap, destroying the invariant tori. If ϵ is very small, the resonances will again be in the classical regime with no enhancement effects.

There is some analytical and computational evidence for streaming effects in colliding beam machines [11,24]. Conclusive experimental corroboration has yet to be made, but past phenomenological studies of the beam-beam interaction (see for example [25]), appear to be at least compatible with the above theory. Studies of resonance transport in magnetic mirror machines [9,10] indicate that streaming phenomena may also occur there. However, whether or not the multidimensional model described above is superior in this application to the conventional (one dimensional) model is not clear at this time.

ACKNOWLEDGEMENTS

I would like to thank A. J. Lichtenberg and M. A. Lieberman for their suggestions and guidance in the preparation of this paper. This work was supported by the Office of Naval Research, grant ONR-N00014-79-C-0674 and the National Science Foundation, grant NSF-ENG-78-26372.

NOTES AND REFERENCES

[1] If the external noise is not conservative, i.e. if it can change the total energy of the system, then streaming can occur with only two degrees of freedom. Otherwise, three degrees of freedom are necessary.

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[15] The degree to which $\langle I_1 \rangle$ and $\langle I_2 \rangle$ are indeed invariant depends on the length of time T over which the averages $\langle\,\rangle$ are defined. However, the variations in these quantities generally decrease exponentially with 1/T.

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[17] Such a point is called "quasi-isochronous" by Chirikov [5], pp. 283.

[18] Since the determinant of **d** is zero, it cannot, in this case, be used to define the metric.

[19] E. G. Harris. **Introduction to Modern Theoretical Physics,** John Wiley & Sons, New York (1975) pp. 22.

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[21] Identical upper and lower indices within a single term imply a sum of n terms according to the usual (Einstein) convention.

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FIGURE CAPTIONS

1. Resonance curves (lines) and energy contours (ellipses) in two dimensional action space. The hamiltonian function for this example is $H_o(\underline{I}) = I_1^2+(6I_2)^2$. The resonance labels are the values of m_1 where $\omega_1 m_1+\omega_2 = 0$. The small square is expanded in detail in fig. 3.

2. Invariant tori in the three dimensional energy surface. By definition, a trajectory remains on a particular torus for all time. At a resonance, the coupling perturbation creates tubular "resonant" tori that run between the leaves of the nonresonant tori.

3. Oscillation-center displacement inside a resonance. The phase point, originally oscillating about the point A, is instantaneously displaced from a to b. The oscillation center jumps from A to B. The ratio of the magnitudes of the two displacements is $|A-B|/|a-b| = \sin(\alpha)\csc(\phi)$.

4. Dissipative trapping in one dimension. A charged particle moves under the influence of a traveling electrostatic wave (with phase velocity $v_\phi = 1$) and friction. The particle is initially traveling in the same direction as the wave but faster. When the friction reduces the particle's velocity to v_ϕ, the particle becomes trapped and spirals into the resonance. This trajectory is somewhat exceptional since for most initial conditions, the particle will not be trapped as it crosses the resonance.

5. Quasi-isochronous points. When the elliptical contours of constant energy (fig. 1) are distorted to produce inflection points, quasi-isochronous points appear. Here three resonances are shown, labeled by the ratios m_1/m_2. The 1/1 resonance forms a separatrix. The resonances inside the separatrix form closed curves. The quasi-isochronous points are the intersections between these closed curves and the dotted line. The hamiltonian function represented here is $H = I_1^2+I_2^2+1.5\exp\{[(I_1-1)^2+(I_2-1)^2]/.562\}$. The energy contours are labeled with the values of H.

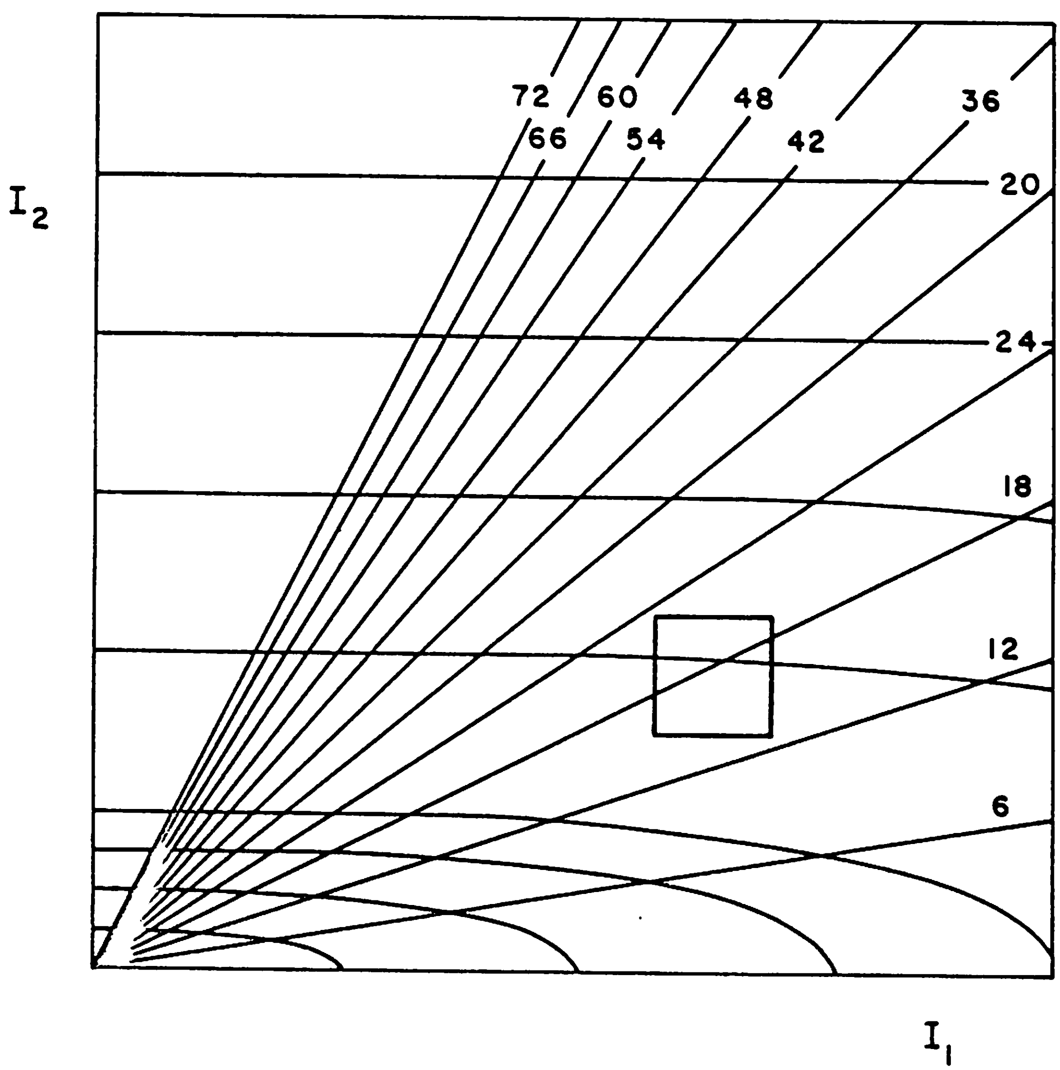
6. Dissipative trapping in two dimensions. Two trajectories are shown in action space. The system consists of two nonlinear oscillators with a weak coupling $H = I_1^2+I_2^2+10^{-5}\cos(\theta_1-\theta_2)$ and a superimposed weak dissipation $\dot{I}_2 = -I_2\times10^{-5}$. The first phase point begins at i_1 and descends to f_1, crossing the coupling resonance $\omega_1 = \omega_2$ on its way. The second phase point descends from i_2 but instead of crossing the resonance, becomes trapped (as in fig. 4). The time intervals for the two trajectories are the same. The dashed lines show the energy contours at the initial and final positions.

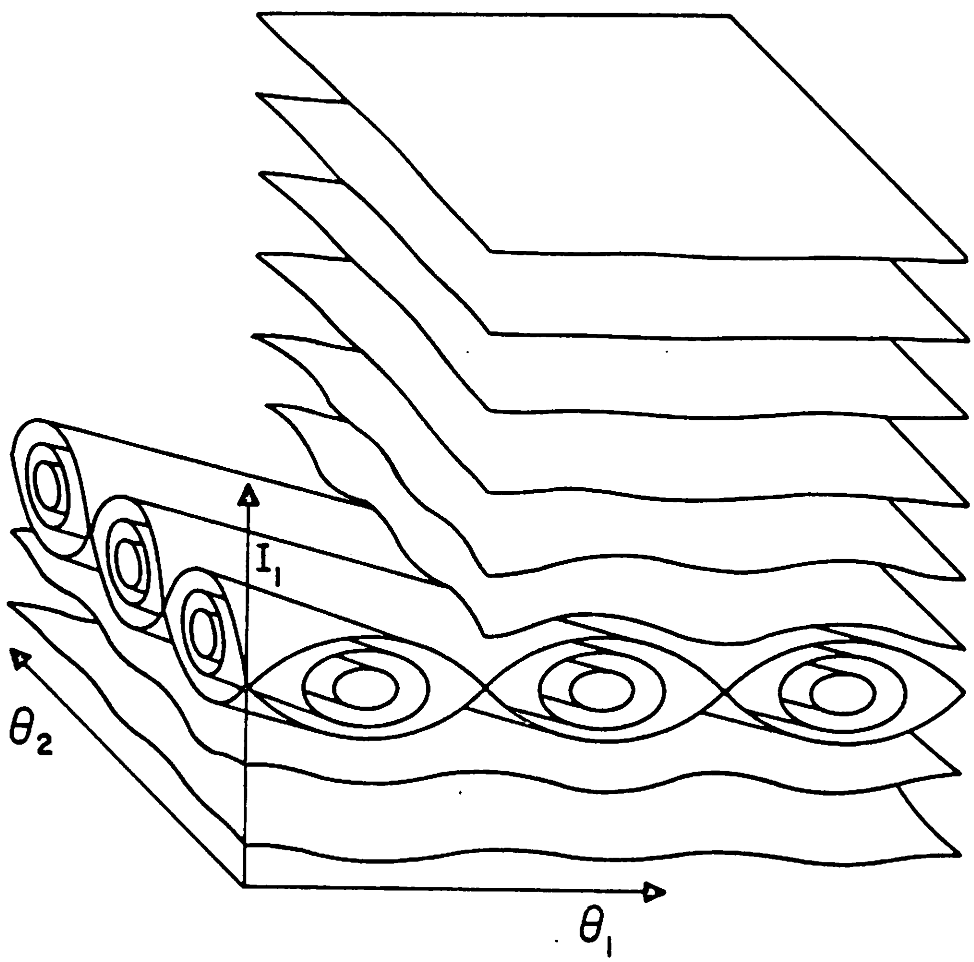
7. Resonance diffusion in two dimensions. The hamiltonian function is the same as in fig. 6. The external process is now a vertical diffusion consisting of small steps $\Delta I_2 = 6\times10^{-5}\sin(rn)$ where rn is a random number between zero and 2π, and the time between jumps is $\Delta t=1$. The trajectory shown is that of the oscillation-center since the

position of the phase point is averaged over successive time intervals of length $T = 500$. Successive averaged positions are connected by line segments. The initial position i is at the center of the resonance $\omega_1 = \omega_2$. The particle eventually diffuses out of and away from the resonance, ending the run at f.

8. Displacements in three dimensions. The space shown is the action space to a three dimensional action manifold S. The horizontal plane is the resonance surface. The vertical plane is defined by the two vectors $\mathbf{r}_k$ and $\mathbf{n}_k$. The intersection between these planes is the "enhancement line". The oscillation-center displacement $\mathbf{V}$ differs from the classical displacement $\mathbf{v}$ by a vector parallel to $\mathbf{r}_k$. The magnitude of this vector is $|\mathbf{V}-\mathbf{v}| = |\mathbf{V}_p-\mathbf{v}_p| = v_n \csc(\phi)$ where $\mathbf{V}_p$ and $\mathbf{v}_p$ are the projections of $\mathbf{V}$ and $\mathbf{v}$ onto the vertical plane. The length v_n is the magnitude of the projection of $\mathbf{v}$ onto $\mathbf{n}_k$.

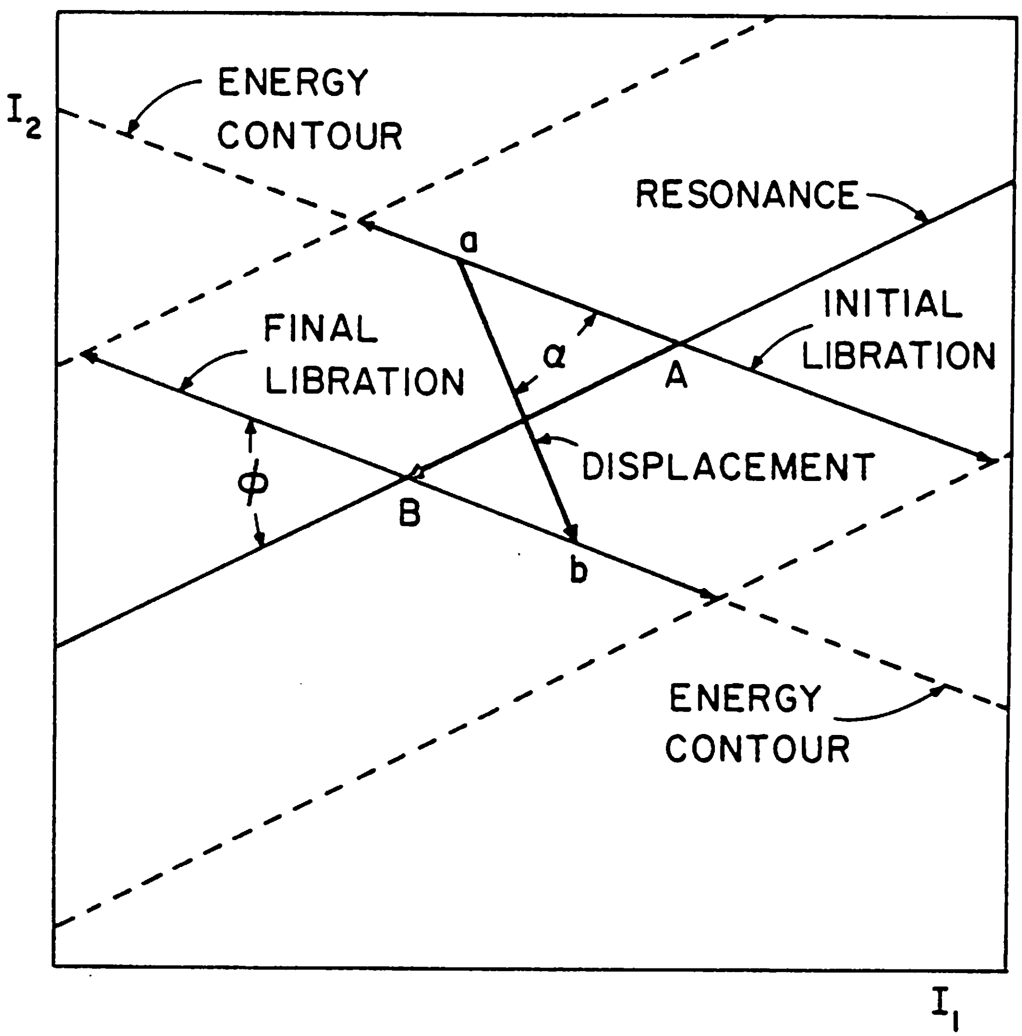
9. Resonance widths. a) With diffusion, the width of the resonance Δ is the perpendicular width defined relative to the standard metric. b) With dissipation, the resonance width is that seen by a drifting phase point, ie. the width parallel to the dissipative drift $\mathbf{v}$.


72
60
48
36
66
54
42
20
24
18
12
6
I_2
I_1

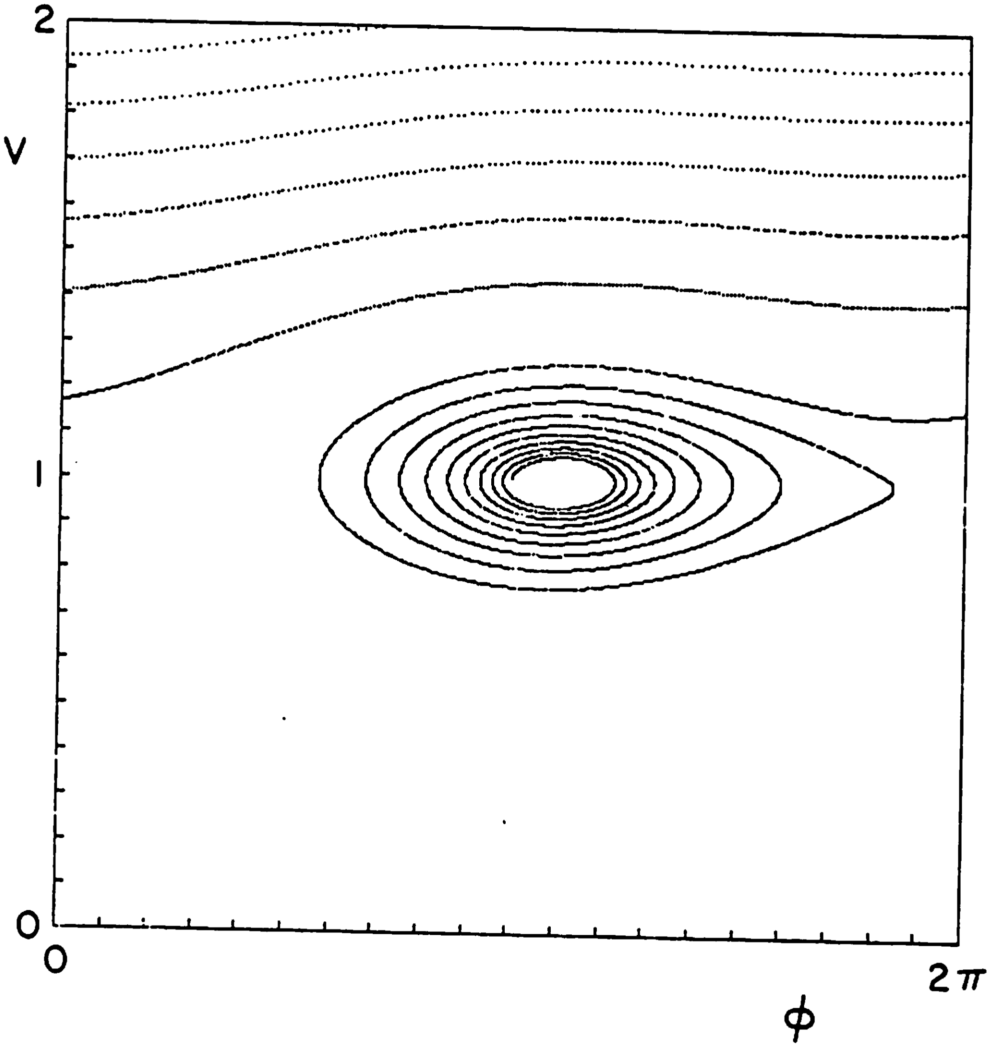


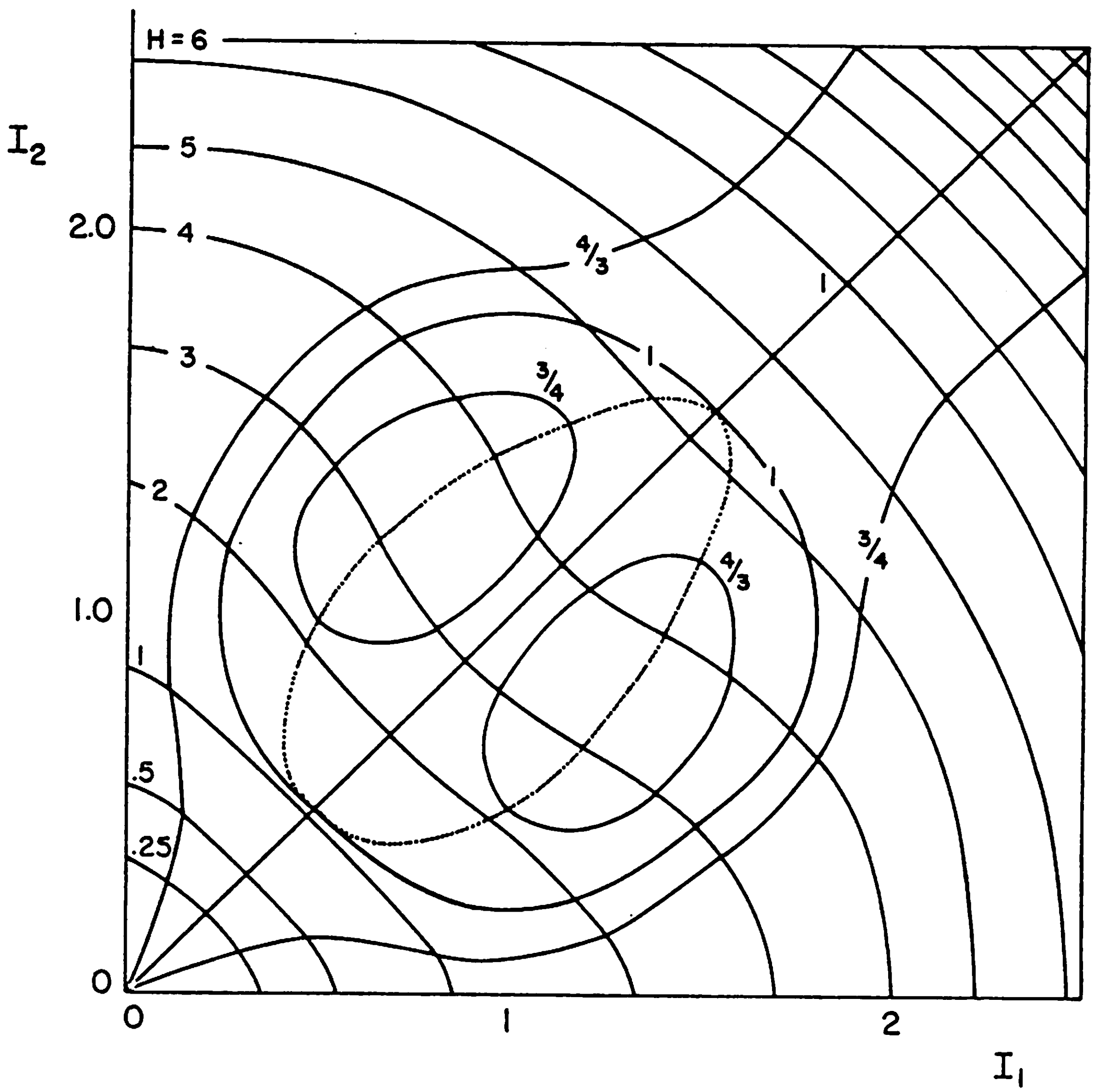
I_1
θ_2
θ_1

FIGURE 3

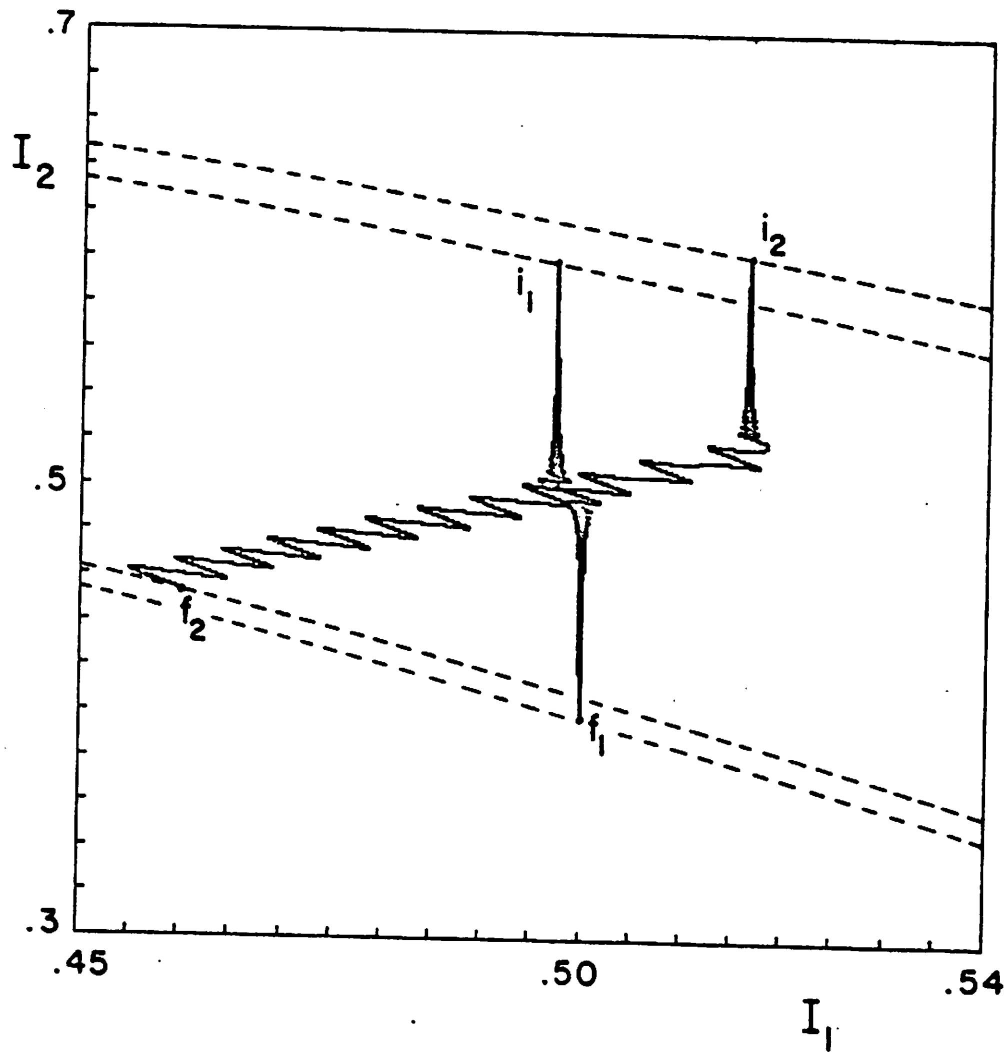


I_2
ENERGY CONTOUR
RESONANCE
a
INITIAL LIBRATION
FINAL LIBRATION
α
A
DISPLACEMENT
φ
B
b
ENERGY CONTOUR
I_1

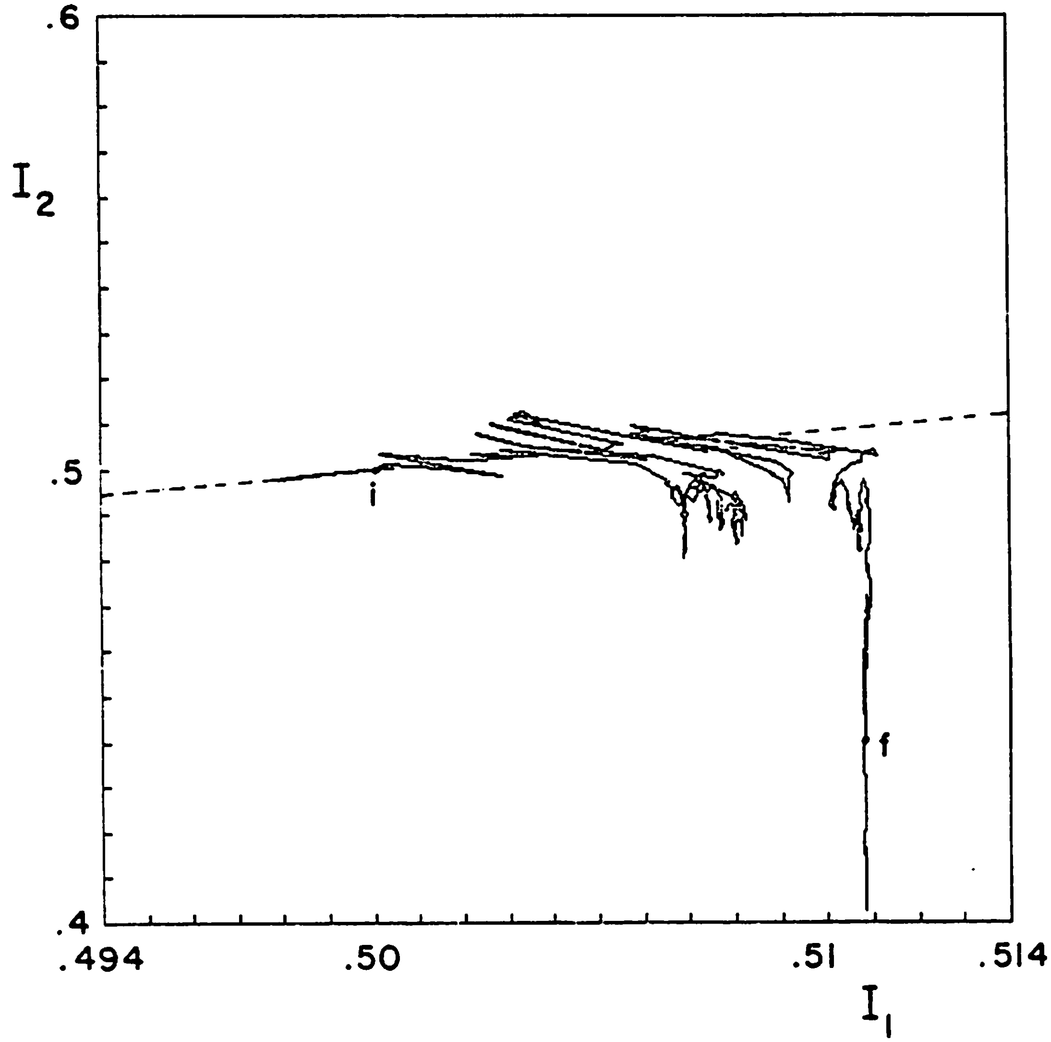

2
V
1
0
0
2π
φ

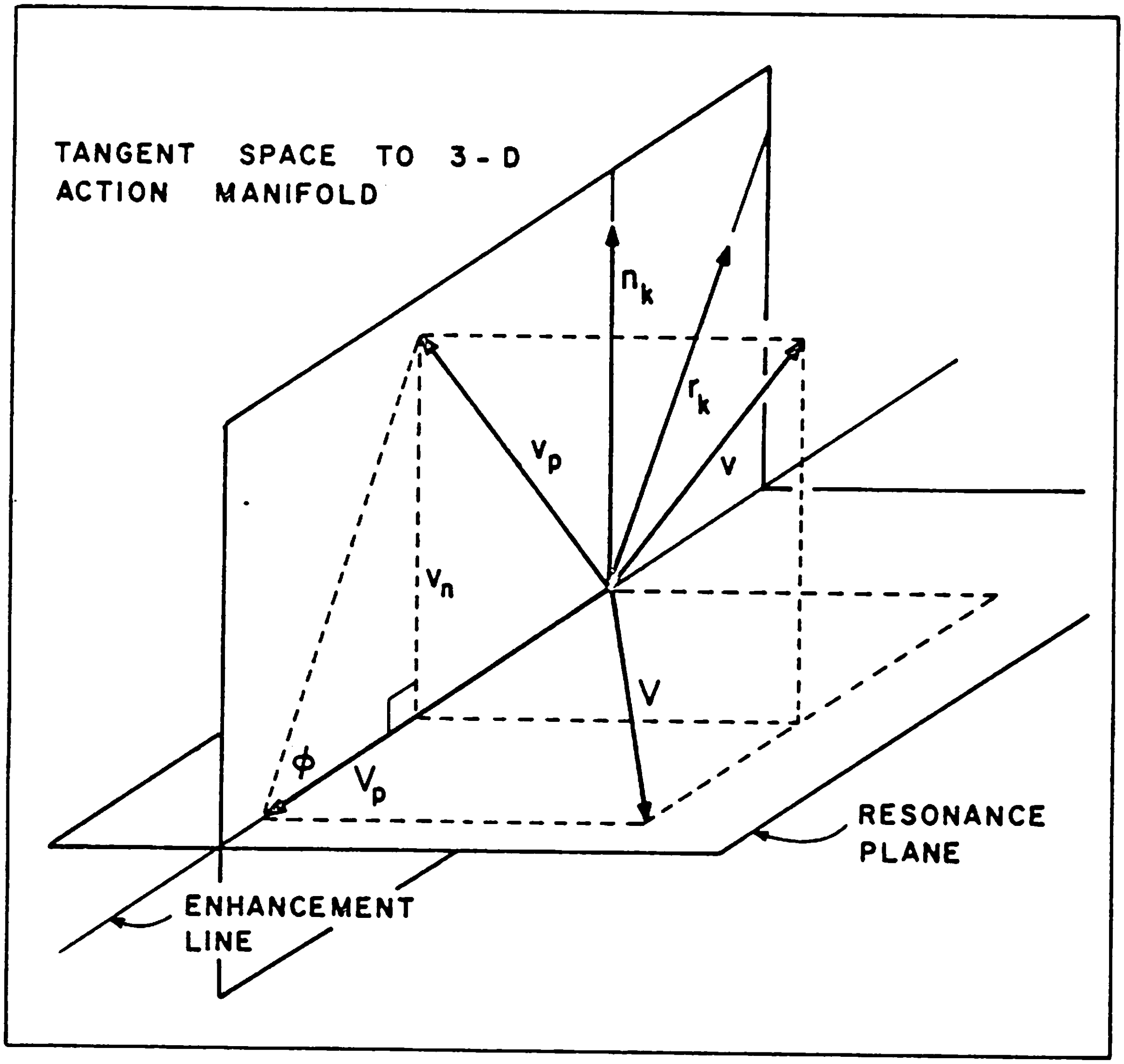


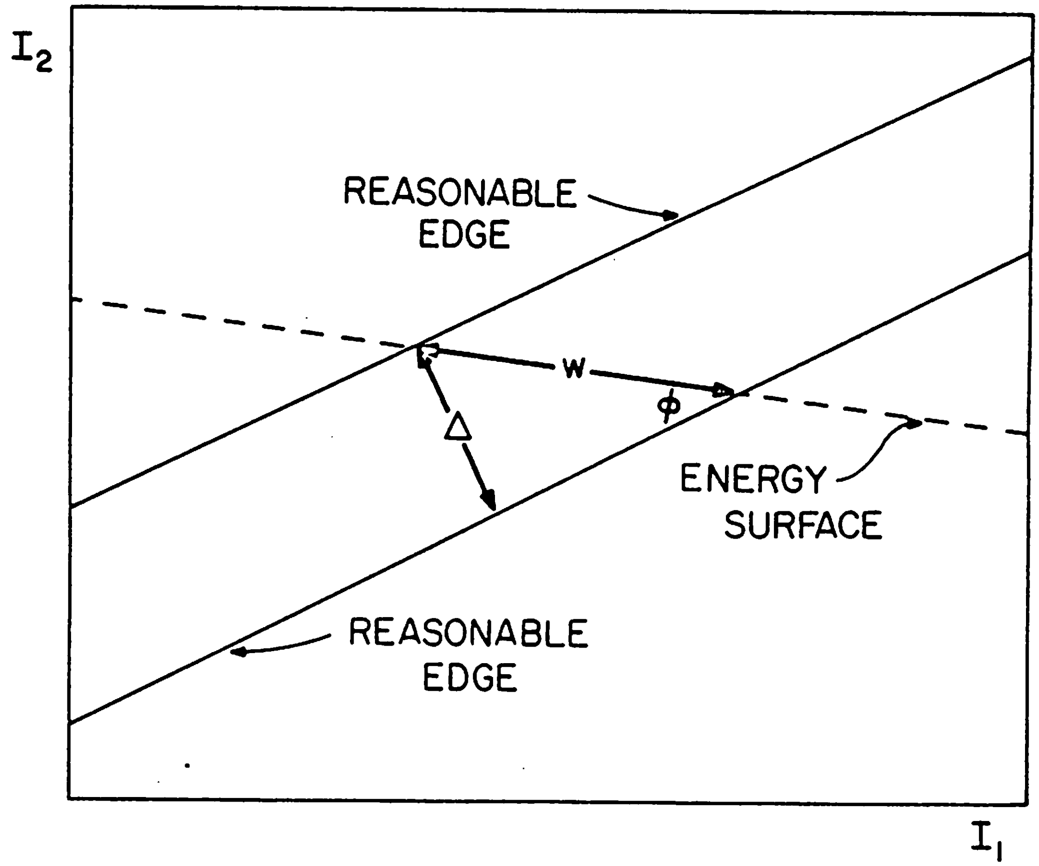
H = 6
5
4
3
2
1
.5
.25
2.0
1.0
0
0
1
2
I_2
I_1
4/3
1
3/4
1
1
4/3
3/4



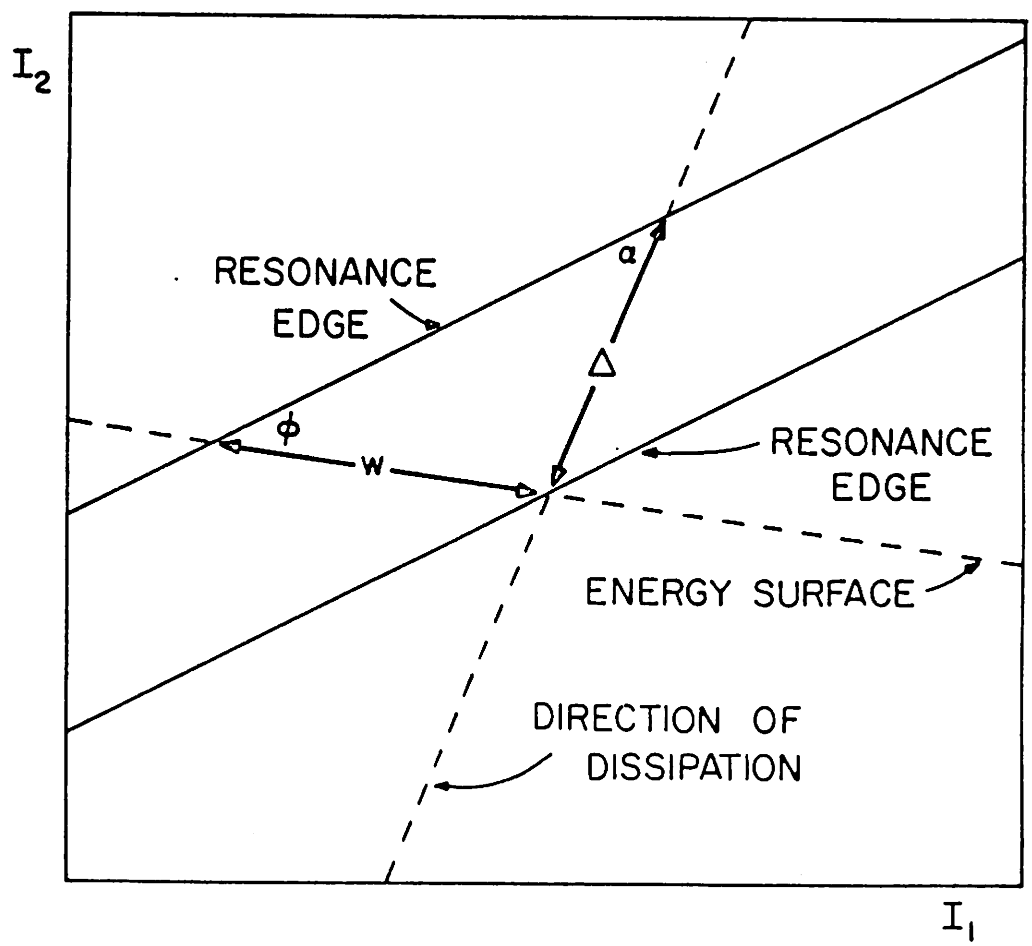
.7
.5
.3
I_2
i_1
i_2
f_1
f_2
.45
.50
.54
I_1


.6
.5
.4
I_2
.494
.50
.51
.514
I_1
i
f


TANGENT SPACE TO 3-D ACTION MANIFOLD
n_k
r_k
v_p
v
v_n
V
ϕ
V_p
RESONANCE PLANE
ENHANCEMENT LINE



I_2
REASONABLE EDGE
W
Δ
φ
ENERGY SURFACE
REASONABLE EDGE
I_1



I_2
RESONANCE EDGE
α
Δ
φ
W
RESONANCE EDGE
ENERGY SURFACE
DIRECTION OF DISSIPATION
I_1