THE BEHAVIOR OF THREE NODE POWER NETWORKS

by

A. Araposthathis and P. Varaiya

Memorandum No. UCB/ERL M81/77

30 September, 1981

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

The Behavior of Three Node Power Networks

A. Arapostathis and P. Varaiya
Department of Electrical Engineering and Computer Sciences
and Electronics Research Laboratory
University of California, Berkeley, CA 94720

September 3, 1981

[1] Research supported by U.S. Department of Energy under Contract DE-AS01-78ET29135. For very fruitful discussions the authors thank F. Abdel-Salem, E. Abed, K. Inan, S. Sastry and N. Tsolas.

Abstract

The real power flow equation for a three node network is analyzed in terms of its topological and geometric aspects. It is shown that the set of feasible power injections is convex and to each feasible injection there corresponds a unique stable solution. Certain aspects of the behavior of the associated swing equations are studied. This behavior is quite different from that of a two node network; in particular, the system may not be completely stable even when the damping is made arbitrarily large.

1. *Introduction*

While great advances have been achieved in the numerical solution of the power flow equation, important questions pertaining to the geometric and topological character of the solutions remain unresolved (see [1].)

This study presents a comprehensive analysis of three node power networks in which the transmission network is modeled by the real power flow equation and the generator dynamics by the classical swing equations. Primary attention is devoted towards revealing the geometric and topological structure of the power flow equation.

It is clear from the literature that intuition guiding both load flow and transient stability studies derives largely from the well-understood one dimensional problem, i.e., a two node network. It turns out, however, that even in the "slightly" more complex case of three nodes, the behavior changes dramatically and conjectures that seem reasonable in light of the one dimensional problem are invalid.

Analysis of the power flow equation for three nodes is quite involved and points to the great difficulties to be encountered in the general case. To the extent possible, however, we have used general arguments avoiding the topological properties of the plane.

Section 2 is devoted to the load flow equation. There are two principal results. First, the set of feasible power injectons is convex (Theorem 2.5). Second, to each feasible injection there corresponds a unique stable solution (Theorem 2.4). Surprisingly, the set of stable solutions need not be convex (Theorem 2.3).

Section 3 discusses certain properties of the associated pair of coupled swing equations. The main result is an example where the system is not completely stable no matter how large the damping is. This is in sharp contrast with the behavior of an isolated swing equation. The example is "generic" in the sense that every three node network loses complete stability whenever the power injections become sufficiently large.

2. Global Proprties of the Power Flow Equation

The principal properties of the power flow equation are derived in this section.

2.1 The Load Flow Function

Consider the three node network of Figure 1. Each node is a PV bus, and the third bus is a slack bus. Take the third bus voltage angle as reference and denote the voltage phasor at bus i by:

$$V_i \exp(j\,\vartheta_i),\ i = 0,1,2.$$

By definition,

$$\vartheta_0 \equiv 0.$$

Y_{ij} is the admittance of the lossless transmission line joining i and j. Defining $B_{ij} = V_i V_j Y_{ij}$ we obtain the load flow equation in the form:

$$P_1 = f_1(\vartheta_1,\vartheta_2) = B_1\sin(\vartheta_1) + B_{12}\sin(\vartheta_1-\vartheta_2), \tag{2.1}$$

$$P_2 = f_2(\vartheta_1,\vartheta_2) = B_2\sin(\vartheta_2) + B_{12}\sin(\vartheta_2-\vartheta_1).$$

Throughout this section we assume $B_1 > 0, B_2 > 0, B_{12} > 0$; otherwise, the analysis is trivial in the sense that it reduces to the one dimensional case. Define the vector $\vartheta = (\vartheta_1, \vartheta_2)$. Utilizing the periodicity of function f, we restrict its domain to the set:

$$T^2 = [-\pi, \pi]^2$$

where the end points π and $-\pi$ are identified. The Jacobian of f at ϑ is given by

$$F(\vartheta) = \frac{\partial f}{\partial \vartheta} = \begin{bmatrix} B_1 \cos\vartheta_1 + B_{12}\cos(\vartheta_1 - \vartheta_2) & -B_{12}\cos(\vartheta_1 - \vartheta_2) \\ -B_{12}\cos(\vartheta_1 - \vartheta_2) & B_2 \cos\vartheta_2 + B_{12}\cos(\vartheta_1 - \vartheta_2) \end{bmatrix} \tag{2.2}$$

Following [1] write $F(\vartheta) > 0$ or $F(\vartheta) \geq 0$ according as $F(\vartheta)$ is positive definite or positive semidefinite. Furthermore, $\vartheta \in T^2$ is said to be ***stable*** if $F(\vartheta) > 0$. Denote the stable region by H_s. In addition let H_0 denote the subset of ϑ in T^2 for which $F(\vartheta) \geq 0$ but $F(\vartheta)$ is not strictly positive definite.

Definition 2.1 (see [1]) The ***principal polytope*** H_p is the subset of all $\vartheta \in T^2$ such that $|\vartheta_1| \leq \frac{\pi}{2}$, $|\vartheta_2| \leq \frac{\pi}{2}$ and $|\vartheta_1 - \vartheta_2| \leq \frac{\pi}{2}$.

Also define the following subsets of T^2 :

$$A_1 = \{\vartheta \in T^2 \mid |\vartheta_1| \leq \frac{\pi}{2}, |\vartheta_1 - \vartheta_2| \leq \frac{\pi}{2}\} \tag{2.3a}$$

$$A_2 = \{\vartheta \in T^2 \mid |\vartheta_2| \leq \frac{\pi}{2}, |\vartheta_1 - \vartheta_2| \leq \frac{\pi}{2}\} \tag{2.3b}$$

$$A_3 = \{\vartheta \in T^2 \mid |\vartheta_1| \leq \frac{\pi}{2}, |\vartheta_2| \leq \frac{\pi}{2}\} \tag{2.3c}$$

Denote by H_b the union of these subsets:

$$H_b = A_1 \cup A_2 \cup A_3 \tag{2.4}$$

These sets are exhibited in Figure 2.

2.2 The Geometry of H_s

We begin by establishing an estimate of the stable region.

Theorem 2.1 $H_p \subset H_s \cup H_0 \subset H_b$.

Proof: The first inclusion is proved in [1] for a general network. To prove the second inclusion suppose that ϑ is not in H_b. Then of course ϑ is not in A_3. Hence $\cos\vartheta_1 < 0$, $\cos\vartheta_2 < 0$. Let $u = (1,1)$. Then $\langle u, F(\vartheta)u\rangle = B_1\cos\vartheta_1 + B_2\cos\vartheta_2 < 0$ implying that $F(\vartheta)$ is not positive semidefinite. The assertion follows.

Corollary 2.1

Let $\vartheta \in H_s \cup H_0$. Then for $0 \le \varepsilon < 1$, $\varepsilon\vartheta \in H_s$.

Proof:

Utilizing Lemma 2.5 in [1] it suffices to prove that $H_s \cup H_0 \subset H_\pi$, where H_π is the subset of $\vartheta \in T^2$ such that $|\vartheta_1 - \vartheta_2| \le \pi$. By the previous theorem it is then enough to show that $H_b \subset H_\pi$. By definition of A_1 and A_2, $A_1 \subset H_\pi$, $A_2 \subset H_\pi$. Suppose that $\vartheta \in A_3$. Then $|\vartheta_1| \le \frac{\pi}{2}$, $|\vartheta_2| \le \frac{\pi}{2}$, implying $|\vartheta_1 - \vartheta_2| \le \pi$, so that $A_3 \subset H_\pi$ and the proof is complete.

Corollary 2.1 implies that H_s is connected, that is, the stable region has only one component in contrast with the general case (see [1, Example 1]). Also, $H_s \cup H_0 = \overline{H}_s$, where the bar denotes closure. Hence $H_0 = \partial H_s$ (boundary of H_s). The next result characterizes this boundary.

Theorem 2.2 $H_0 = \partial H_s$ is a connected one dimensional differentiable manifold in T^2.

Proof:

Connectedness follows easily from the previous discussion. To establish that ∂H_s is a manifold, observe that

$$\partial H_s = \{\vartheta \in H_b \mid \det F(\vartheta) = 0\} \tag{2.5}$$

Hence, it suffices to show that $\nabla_\vartheta \det F(\vartheta) \ne 0$ on ∂H_s. For $\vartheta \in T^2$ define the following real functions:

$$\psi_1(\vartheta) = [1,0]\, F(\vartheta) \begin{bmatrix}1\\0\end{bmatrix} = B_1 \cos\vartheta_1 + B_{12}\cos(\vartheta_1-\vartheta_2) \tag{2.6a}$$

$$\psi_2(\vartheta) = [0,1]\, F(\vartheta) \begin{bmatrix}0\\1\end{bmatrix} = B_2 \cos\vartheta_2 + B_{12}\cos(\vartheta_1-\vartheta_2) \tag{2.6b}$$

$$\psi_3(\vartheta) = [1,1]\, F(\vartheta) \begin{bmatrix}1\\1\end{bmatrix} = B_1 \cos\vartheta_1 + B_2\cos\vartheta_2 \tag{2.6c}$$

It is clear that these functions are nonnegative on H_0. Also note that $F(\vartheta)$ never vanishes on T^2 since this would imply $\cos\vartheta_1 = 0$, $\cos\vartheta_2 = 0$, $\cos(\vartheta_1-\vartheta_2) = 0$, an obvious contradiction. Hence, since $[1,0]$, $[0,1]$ and $[1,1]$ are pairwise independent, it follows that at most one of the functions ψ_i may vanish at any $\vartheta \in T^2$. Assume now that $\vartheta \in \partial H_s$ but $\nabla_\vartheta \det F(\vartheta) = 0$. Without loss of generality we can suppose that $\psi_2(\vartheta) > 0$, $\psi_3(\vartheta) > 0$.

We have

$$\nabla_\vartheta \det F(\vartheta) = -\begin{bmatrix} B_1\sin\vartheta_1\psi_2(\vartheta)+B_{12}\sin(\vartheta_1-\vartheta_2)\psi_3(\vartheta) \\ B_2\sin\vartheta_2\psi_1(\vartheta)-B_{12}\sin(\vartheta_1-\vartheta_2)\psi_3(\vartheta)\end{bmatrix} \tag{2.7}$$

Therefore, $\nabla_\vartheta \det F(\vartheta) = 0$ implies

$$B_1\sin\vartheta_1\, \psi_2(\vartheta) + B_{12}\sin(\vartheta_1-\vartheta_2)\, \psi_3(\vartheta) = 0 \tag{2.8a}$$

$$B_2\sin\vartheta_2\, \psi_1(\vartheta) - B_{12}\sin(\vartheta_1-\vartheta_2)\, \psi_3(\vartheta) = 0 \tag{2.8b}$$

Without loss of generality, assume that $\sin\vartheta_1 \geq 0$. From (2.8a) we then have $\sin(\vartheta_1-\vartheta_2) \leq 0$. Since $\vartheta \in \partial H_s \subseteq H_\pi$ it follows that $0 \leq \vartheta_1 \leq \pi$, $-\pi \leq \vartheta_1-\vartheta_2 \leq 0$; thus, $0 \leq \vartheta_2 \leq \pi$ or $\sin\vartheta_2 \geq 0$. This contradicts (2.8) unless $\sin\vartheta_1 = 0$ and $\sin(\vartheta_1 - \vartheta_2) = 0$ in which case ϑ_i is a multiple of π. By Lemma 3 of [1] it is unstable and so ϑ cannot belong to ∂H_s. The proof is complete.

2.3 On the Convexity of H_s.

It was conjectured in [2] that the set of stable solutions H_s is convex. If this were true, one could conclude almost immediately that the stable solution of the load flow function is unique. Here we give a counterexample to this conjecture.

Theorem 2.3 H_s is not generally convex.

Proof: Suppose H_s is convex. Let $\vartheta \in \partial H_s$ be arbitrary. Let $v \in T_\vartheta\, \partial H_s$, the tangent space of ∂H_s at ϑ. Then $\vartheta + tv$, $t \in R$, is a hyperplane (which is a straight line in this context) through ϑ supporting H_s. In addition, since $F(\vartheta)$ has one strictly positive eigenvalue for ϑ in ∂H_s we conclude that for some $\varepsilon > 0$, $\det F(\vartheta+tv) \le 0$ for all $|t| < \varepsilon$. So it must be the case that

$$v^T \frac{\partial^2}{\partial\vartheta^2} (\det F(\vartheta))\, v \le 0. \tag{2.9}$$

We can find explicit expressions for v and $\frac{\partial^2}{\partial\vartheta^2} (\det F(\vartheta))$.

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \frac{\partial^2}{\partial\vartheta^2} (\det F(\vartheta)) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where

$$v_1 = B_{12} \sin(\vartheta_1-\vartheta_2)\, \psi_3(\vartheta) - B_2 \sin \vartheta_2\, \psi_1(\vartheta) \tag{2.10a}$$

$$v_2 = B_{12} \sin(\vartheta_1-\vartheta_2)\, \psi_3(\vartheta) - B_1 \sin \vartheta_1\, \psi_2(\vartheta) \tag{2.10b}$$

$$a_{11} = -B_1 \cos \vartheta_1\, B_{12} \cos(\vartheta_1-\vartheta_2) + 2B_1 \sin\vartheta_1\, B_{12} \sin (\vartheta_1-\vartheta_2) \tag{2.10c}$$

$$a_{22} = -B_2 \cos \vartheta_2\, B_{12} \cos(\vartheta_1 - \vartheta_2) + 2B_2 \sin\vartheta_2\, B_{12} \sin (\vartheta_1-\vartheta_2) \tag{2.10d}$$

$$\begin{aligned} a_{12} = a_{21} = &-B_1 \cos \vartheta_1\, B_2 \cos \vartheta_2 + B_1 \sin\vartheta_1\, B_2 \sin\vartheta_2 \\ &+ B_2 \sin \vartheta_2\, B_{12} \sin(\vartheta_1-\vartheta_2) - B_1 \sin \vartheta_1\, B_{12} \sin (\vartheta_1-\vartheta_2) \end{aligned} \tag{2.10e}$$

Now let $\vartheta = (\vartheta_1,\vartheta_2)$ be given by $\vartheta_1 = 75°$, $\vartheta_2 = -60°$. Consider the following sequence of parameter values $B^n = \{B_1^n, B_2^n, B_{12}^n\}$ given by

$$B_1^n = 2^n \left[\frac{\sqrt{2}}{2} + \frac{1}{2^n}\right] \frac{\sqrt{2}}{4}, \quad B_2^n = \frac{\sqrt{2}}{2} + \frac{1}{2^n}, \quad B_{12}^n = \frac{1}{2}.$$

One can check that $\det F^n(\vartheta) = 0$ for all n, where $F^n(\vartheta)$ denotes the Jacobian of the load flow function with parameters B^n. Furthermore, since $\cos \vartheta_1 > 0$, $\cos \vartheta_2 > 0$ we deduce that $F^n(\vartheta) \ge 0$ for all n and hence $\vartheta \in \partial H_s$. Denoting by v_i^n, a_{ij}^n the values of v_i, a_{ij} (see 2.10) at B^n, ϑ we investigate the asymptotic behavior of v_i^n, a_{ij}^n noting that terms containing $B_1^n \sim 2^{n-2}$ will dominate over the rest.

This leads to

$$v_1^n \sim -2^{n-2}\left[\frac{\sqrt{2}}{4}(1+\sqrt{3})\right]\cos\vartheta_1 \tag{2.11a}$$

$$v_2^n \sim -2^{n-2}\frac{\sqrt{2}}{4}\cos\vartheta_1 \tag{2.11b}$$

$$a_{11}^n \sim 2^{n-2}\frac{\sqrt{2}}{4}\cos\vartheta_1 + 2^{n-1}\frac{\sqrt{2}}{4}\sin\vartheta_1 \tag{2.11c}$$

$$a_{12}^n = a_{21}^n \sim 2^{n-2}\frac{\sqrt{2}}{4}\cos\vartheta_1 - 2^{n-2}\frac{\sqrt{2}}{4}(a+\sqrt{3})\sin\vartheta_1 \tag{2.11d}$$

while a_{22}^n is bounded above. Thus:

$$(v^n)^T\frac{\partial^2}{\partial\vartheta^2}\left[\det F^n(\vartheta)\right]v^n = (v_1^n)^2 a_{11}^n + 2v_1^n v_2^n a_{12}^n + (v_2^n)^2 a_{22}^n \sim 2^{n-6}\sqrt{2},$$

so that for n large enough,

$$(v^n)^T\frac{\partial^2}{\partial\vartheta^2}\left[\det F^n(\vartheta)\right]v^n > 0$$

contrary to the original hypothesis, and the proof is complete.

2.4 On the number of stable solutions

We now seek to establish the uniqueness of the stable solution of the load flow equation. The proof is rather involved, and for reasons of clarity, we first establish two lemmas.

Lemma 2.1 Let $\vartheta, \tilde{\vartheta} \in H_s$ be such that $\vartheta \neq \tilde{\vartheta}$, $f(\vartheta) = f(\tilde{\vartheta})$. Then there exist $\xi \in \overline{H}_s$ and $\tilde{\xi} \in \partial H_s$ such that $f(\xi) = f(\tilde{\xi})$.

Proof: Since $\vartheta, \tilde{\vartheta} \in H_s$, $\det F(\vartheta) \neq 0$, $\det F(\tilde{\vartheta}) \neq 0$, and in view of the inverse function theorem, there exist disjoint neighborhoods V, $\tilde{V}$ of ϑ and $\tilde{\vartheta}$ respectively, and a diffeomorphism $\eta: V \to \tilde{V}$ such that if $x \in V$, then $f(x) = f(\eta(x))$, $x \neq \eta(x)$. Therefore, given the differentiable curve $\gamma(t) = (1-t)\vartheta$, there exists an open interval $[0,\alpha)$ and a differentiable curve $\tilde{\gamma}(t)$ defined on the interval such that $\tilde{\gamma}(0) = \tilde{\vartheta}$ and $f(\tilde{\gamma}(t)) = f(\gamma(t))$, $\tilde{\gamma}(t) \neq \gamma(t)$, $\tilde{\gamma}(t) \in H_s$. Now let $\alpha \in [0,1]$ be the largest such real number. Let $\tilde{\xi} = \lim_{t\to\alpha}\tilde{\gamma}(t)$ and let $\xi = (1-\alpha)\vartheta$. Observe first that $\xi \neq \tilde{\xi}$ since $(1-\alpha)\vartheta \in H_s$ by Corollary 2.1 and hence $F((1-\alpha)\vartheta)$ is nonsingular. Next note that $\alpha < 1$ because if $\alpha = 1$ then $f(\xi) = f(0)$ for some $\xi \in \overline{H}_s$. But then by Corollary 2.1, $\varepsilon\xi \in H_s$ for $\varepsilon \in (0,1)$. Hence,

$$0 = \langle \xi, f(\xi) - f(0) \rangle = \int_0^1 \langle \xi, F(\varepsilon\xi)\, \xi \rangle \, d\varepsilon > 0,$$

which is a contradiction. It remains to show that $\xi \in \partial H_s$. This is clear, for if $\xi \in H_s$ then the definition of $\tilde{\gamma}(t)$ can be extended beyond α, contradicting its maximality. The proof is complete.

Lemma 2.2 Let $\vartheta^0 \in \partial H_s \cap A$ where

$$A = \{\vartheta \in T^2 \mid \frac{\pi}{2} \le \vartheta_1 \le \pi,\ 0 \le \vartheta_2 \le \frac{\pi}{2},\ 0 \le \vartheta_1 - \vartheta_2 \le \frac{\pi}{2}\}.$$

Then,

$$B_1 \sin \vartheta_1^0 + B_2 \sin \vartheta_2^0 \ge B_1,$$

$$B_1 \sin \vartheta_1^0 + B_{12} \sin (\vartheta_1^0 - \vartheta_2^0) \ge B_1.$$

Proof: ∂H_s is a closed connected one dimensional manifold. Furthermore, $(\frac{\pi}{2}, \frac{\pi}{2})$, $(\frac{\pi}{2}, 0)$ are the only points in $\partial A \cap \partial H_s$. This assertion can be verified very easily. One can conclude, then, that there exists a differentiable curve $\gamma(t): R \to A$ such that $\gamma(0) = (\frac{\pi}{2}, \frac{\pi}{2})$, $\gamma(1) = (\frac{\pi}{2}, 0)$, $\gamma(t_0) = \vartheta^0$, $0 \le t_0 \le 1$, and $\gamma(t) \in \partial H_s \cap A$ for all $t \in [0,1]$. Consider the function $\varphi : T^2 \to R$ given by

$$\varphi(\vartheta) = B_1 \sin \vartheta_1 + B_2 \sin \vartheta_2 \tag{2.12}$$

We have

$$\varphi(\gamma(0)) = B_1 + B_2,\ \varphi(\gamma(1)) = B_1 \tag{2.13}$$

Hence it suffices to show that $\varphi(\gamma(t))$ is monotone on $(0,1)$. Consider

$$\frac{d}{dt}\varphi(\gamma(t)) = (1,1)\, F(\gamma(t))\, \dot{\gamma}(t) \tag{2.14}$$

But $\dot{\gamma}(t) \in T_{\gamma(t)}(\partial H_s)$, and $T_\vartheta(\partial H_s)$ is spanned by the vector $v(\vartheta)$ given in (2.10). By direct computation,

$$(1,1)\, F(\vartheta)\, v(\vartheta) = -B_2 \sin \vartheta_2\, B_1 \cos \vartheta_1\, \psi_1(\vartheta) \tag{2.15}$$

$$+ B_1 \sin \vartheta_1 B_2 \cos \vartheta_2 \psi_2(\vartheta) + B_{12} \sin(\vartheta_1 - \vartheta_2)(\psi_3(\vartheta))^2$$

Finally, by inspection, (1.1) $F(\vartheta)\,v(\vartheta) > 0$ on $\partial H_s \cap \text{int}(A)$ and the proof is complete. (The proof of the second inequality follows by symmetry upon interchanging $\vartheta_1 - \vartheta_2$ and ϑ_2.)

We are now ready to prove the main result.

Theorem 2.4 The function f is one-to-one on $\overline{H}_s$.

Proof: By contradiction. Suppose there exist $\vartheta, \tilde{\vartheta}$ in $\overline{H}_s$ with $\vartheta \neq \tilde{\vartheta}$ and $f(\vartheta) = f(\tilde{\vartheta})$. By Lemma 2.1 we can assume $\tilde{\vartheta} \in \partial H_s$, and in view of the symmetry of the load flow function we may assume $\tilde{\vartheta}_1 \geq \frac{\pi}{2}$. Note that since $\tilde{\vartheta} \in \overline{H}_s$ we must necessarily have $0 \leq \tilde{\vartheta}_2 \leq \frac{\pi}{2}$, $0 \leq \tilde{\vartheta}_1 - \tilde{\vartheta}_2 \leq \frac{\pi}{2}$ and Lemma 2.2 implies

$$f_1(\tilde{\vartheta}) = B_1 \sin(\tilde{\vartheta}_1) + B_{12} \sin(\tilde{\vartheta}_1 - \tilde{\vartheta}_2) \geq B_1 \tag{2.16a}$$

$$f_1(\tilde{\vartheta}) + f_2(\tilde{\vartheta}) = B_1 \sin(\tilde{\vartheta}_1) + B_2 \sin(\tilde{\vartheta}_2) \geq B_1. \tag{2.16b}$$

Since $f(\vartheta) = f(\tilde{\vartheta})$ we get

$$f_1(\vartheta) \geq B_1 \tag{2.17a}$$

$$f_1(\vartheta) + f_2(\vartheta) \geq B_1 \tag{2.17b}$$

Observe that (2.17) implies $\sin \vartheta_2 \geq 0$, $\sin(\vartheta_1 - \vartheta_2) \geq 0$, hence, since $\vartheta \in \overline{H}_s$ we must have

$$0 \leq \vartheta_1 \leq \pi,\ 0 \leq \vartheta_2 \leq \frac{\pi}{2},\ 0 \leq \vartheta_1 - \vartheta_2 \leq \frac{\pi}{2}.$$

Now distinguish two cases:

i) $\sin \vartheta_1 \geq \sin \tilde{\vartheta}_1$:

This clearly implies

$$0 \leq \pi - \tilde{\vartheta}_1 \leq \vartheta_1 \leq \tilde{\vartheta}_1 \leq \pi \tag{2.18a}$$

$$0 \leq \vartheta_2 \leq \tilde{\vartheta}_2 \leq \frac{\pi}{2} \tag{2.18b}$$

$$0 \leq \vartheta_1 - \vartheta_2 \leq \tilde{\vartheta}_1 - \tilde{\vartheta}_2 \leq \frac{\pi}{2} \tag{2.18c}$$

Let $u \in R^2$ be given by

$$u = \begin{bmatrix} \tilde{\vartheta}_1 - \vartheta_1 \\ \tilde{\vartheta}_2 - \vartheta_2 \end{bmatrix} \tag{2.20}$$

By direct computation,

$$\begin{aligned}
<u, F(\vartheta + ut)u> &= B_1 \cos(\vartheta_1 + (\tilde{\vartheta}_1 - \vartheta_1)t)(\tilde{\vartheta}_1 - \vartheta_1)^2 \\
&\quad + B_2 \cos(\vartheta_2 + (\tilde{\vartheta}_2 - \vartheta_2)t)(\tilde{\vartheta}_2 - \vartheta_2)^2 \\
&\quad + B_{12} \cos(\vartheta_1 - \vartheta_2 + (\tilde{\vartheta}_1 - \tilde{\vartheta}_2 - \vartheta_1 + \vartheta_2)t)\,(\tilde{\vartheta}_1 - \tilde{\vartheta}_2 - \vartheta_1 + \vartheta_2)^2 \\
&> B_1 \cos(\tilde{\vartheta}_1)(\tilde{\vartheta}_1 - \vartheta_1)^2 + B_2 \cos \tilde{\vartheta}_2\, (\tilde{\vartheta}_2 - \vartheta_2)^2 \\
&\quad + B_{12} \cos(\tilde{\vartheta}_1 - \tilde{\vartheta}_2)\,(\tilde{\vartheta}_1 - \tilde{\vartheta}_2 - \vartheta_1 + \vartheta_2)^2 = <u, F(\tilde{\vartheta})\, u> \geq 0
\end{aligned}$$

for all t in (0,1), and hence,

$$u^T (f(\tilde{\vartheta}) - f(\vartheta)) = \int_0^t <u, F(\vartheta + ut)\, u>\, dt > 0 \tag{2.21}$$

contradicting our initial hypothesis that $f(\tilde{\vartheta}) = f(\vartheta)$.

ii) $\sin \tilde{\vartheta}_1 > \sin \vartheta_1$:

A contradiction here is established in a similar manner and completes the proof.

It is clear that $f(H_s)$ is an open set in R^2. By the continuity of f we also get $\partial f(H_s) \subset f(\partial H_s)$. In fact, we can now also establish that $\partial f(H_s) = f(\partial H_s)$. If this is not the case, then there exists ϑ^0 in ∂H_s such that $f(\vartheta^0) \in f(H_s)$, in violation of Theorem 2.4.

2.5 On the range of f

We seek to characterize the range of the load flow function. We first establish that $f(H_s)$ is a convex set in R^2. The following lemma is useful.

Lemma 2.3 Let C be an open bounded set in R^n such that ∂C is an (n-1) dimensional manifold. Suppose that for any three distinct points x, y, z, in ∂C, $T_x(\partial C) = T_y(\partial C)$, implies $T_z(\partial C) \neq T_x(\partial C)$. Then C is convex.

Proof: By contradiction. Suppose C satisfies the hypothesis of the lemma and C is not convex. Then there exist x_1 and x_2 in C such that $\frac{1}{2}(x_1+x_2)$ is not in C. Define $g: R \rightarrow R^n$ by

$$g(t) = tx_1 + (1-t)x_2. \tag{2.22}$$

Let t_0 in $(0,1)$ be such that $g(t_0) \in \partial C$. Let $x_0 = g(t_0)$ and consider $T_{x_0}(\partial C)$ and a vector w_0 normal to $T_{x_0}(\partial C)$. Define the linear functional λ on $\bar{C}$ by

$$\lambda(x) = \langle x, w_0 \rangle \tag{2.23}$$

Since C is bounded, $\bar{C}$ is compact, there exist points x_m, x_M where λ attains a global minimum and maximum respectively.

Obviously, $x_m \neq x_M$. Furthermore, since

$$\lambda(x_0) = t_0\lambda(x_1) + (1-t_0)\lambda(x_2)$$

and x_1, x_2 are in C, it is evident that $x_0 \neq x_m$, and $x_0 \neq x_M$, while

$$T_{x_m}(\partial C) = T_{x_M}(\partial C) = T_{x_0}(\partial C)$$

contrary to the hypothesis, and the proof is complete.

Lemma 2.4 $\partial f(H_s) = f(\partial H_s)$ is a one-dimensional differentiable manifold in R^2.

Proof: Note that $f\colon \partial H_s \to \partial f(H_s)$ is a continuous bijective map. Thus it suffices to prove that $F(\vartheta)$ is bijective on $T_\vartheta(\partial H_s)$ for all ϑ in ∂H_s. Without loss of generality, assume $\vartheta \in A$ (see Lemma 2.2), i.e., $\frac{\pi}{2} \le \vartheta_1 \le \pi$, $0 \le \vartheta_2 \le \frac{\pi}{2}$, $0 \le \vartheta_1 - \vartheta_2 \le \frac{\pi}{2}$.

We already know from the proof of Lemma 2.2 that $F(\vartheta)v(\vartheta) \neq 0$ for all ϑ in $\partial H_s \cap \mathrm{Int}(A)$ where $v(\vartheta)$ is the vector defined in (2.10) which spans $T_\vartheta(\partial H_s)$. So it remains to show that $F(\vartheta)v(\vartheta) \neq 0$ on $\partial H_s \cap \partial A$, namely, at $(\frac{\pi}{2}, \frac{\pi}{2})$ and $(\frac{\pi}{2}, 0)$.

At $(\frac{\pi}{2}, \frac{\pi}{2})$ we get

$$F(\vartheta)\, v(\vartheta) = \begin{bmatrix} B_{12} & -B_{12} \\ -B_{12} & B_{12} \end{bmatrix} \begin{bmatrix} -B_2B_{12} \\ B_1B_{12} \end{bmatrix} = (B_1+B_2)B_{12}^2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \neq 0.$$

On the other hand at $(\frac{\pi}{2}, 0)$ the assertion follows by inspection from (2.15) and the proof is complete.

Theorem 2.5 $f(H_s)$ is an open convex set in R^2.

Proof: Observe that if $\tilde{\vartheta}$ belongs to ∂H_s, then so does $-\tilde{\vartheta}$ and $T_{f(\tilde{\vartheta})}(\partial f(H_s)) = T_{f(-\tilde{\vartheta})}(\partial f(H_s))$ so that in view of the lemmas above, it suffices to show that there exists no $\vartheta \in \partial H_s$ such that $\vartheta \neq \tilde{\vartheta}$, $\vartheta \neq -\tilde{\vartheta}$ and $F(\vartheta) = \alpha F(\tilde{\vartheta})$ for some $\alpha \neq 0$.

We proceed by contradiction. Let ϑ, $\tilde{\vartheta}$ be as above. Without loss of generality we can assume $\tilde{\vartheta} \in A$, i.e., $\frac{\pi}{2} \leq \tilde{\vartheta}_1 \leq \pi$, $0 \leq \tilde{\vartheta}_2 \leq \frac{\pi}{2}$, $0 \leq \tilde{\vartheta}_1 - \tilde{\vartheta}_2 \leq \frac{\pi}{2}$. Evidently, we must have $\alpha > 0$, so again without loss of generality we can assume $\alpha \geq 1$, as well as $\vartheta_1 \geq 0$. Then from $F(\vartheta) = \alpha F(\tilde{\vartheta})$ we obtain

$$\cos \vartheta_1 = \alpha \cos \tilde{\vartheta}_1 \tag{2.24a}$$

$$\cos \vartheta_2 = \alpha \cos \tilde{\vartheta}_2 \tag{2.24b}$$

$$\cos (\vartheta_1 - \vartheta_2) = \alpha \cos (\tilde{\vartheta}_1 - \tilde{\vartheta}_2). \tag{2.24c}$$

Hence, in view of our assumptions,

$$\cos \vartheta_1 \leq \cos \tilde{\vartheta}_1 \leq 0 \tag{2.25a}$$

$$\cos \vartheta_2 \geq \cos \tilde{\vartheta}_2 \geq 0 \tag{2.25b}$$

$$\cos (\vartheta_1 - \vartheta_2) \geq \cos (\tilde{\vartheta}_1 - \tilde{\vartheta}_2) \geq 0 \tag{2.25c}$$

Since $\vartheta_1 \geq 0$, we obtain from (2.25a)

$$\frac{\pi}{2} \leq \tilde{\vartheta}_1 \leq \vartheta_1 \leq \pi. \tag{2.26a}$$

And from (2.25b) and (2.25c)

$$\vartheta_2 \leq \tilde{\vartheta}_2 \leq \frac{\pi}{2} \tag{2.26b}$$

$$\vartheta_1 - \vartheta_2 \leq \tilde{\vartheta}_1 - \tilde{\vartheta}_2 \leq \frac{\pi}{2}. \tag{2.26c}$$

But adding (2.26b) and (2.26c) gives $\vartheta_1 \leq \tilde{\vartheta}_1$, in violation of (2.26a). The proof is complete.

So far we have established that $f(H_s)$ is a convex set. Notice that $f(\overline{H}_s)$ is thus the convex hull of $f(\partial H_s)$. The question that arises now is whether $f(\overline{H}_s) = f(T^2)$; i.e., whether we can always guarantee the existence of a stable solution. Observe that the point $(\frac{\pi}{2}, \frac{\pi}{2})$ is in ∂H_s, and in addition, the function

$f_1(\vartheta) + f_2(\vartheta)$ attains a global maximum over $f(T^2)$ at $(\frac{\pi}{2}, \frac{\pi}{2})$. Hence, $f(\frac{\pi}{2}, \frac{\pi}{2})$ belongs to $\partial f(T^2)$ and $f(\partial H_s) \cap \partial f(T^2)$ is nonempty.

After this initial observation we proceed to establish the theorem concluding this section.

Theorem 2.6 $f(\overline{H}_s) = f(T^2)$

Proof: By contradiction. Suppose $f(\overline{H}_s) \neq f(T^2)$. From the preceding paragraph,

$$\partial f(\overline{H}_s) \cap \partial f(T^2) \neq \phi. \tag{2.28}$$

On the other hand, since $f(\overline{H}_s) \neq f(T^2)$, and $f(\overline{H}_s)$ is convex, therefore

$$\partial f(T^2) - \partial f(\overline{H}_s) \neq \phi. \tag{2.29}$$

Let H_{0-} denote the set of points in T^2 such that $F(\vartheta) \leq 0$ but $F(\vartheta)$ is not strictly negative definite. It is evident that $\partial f(T^2) - \partial f(\overline{H}_s) \subset f(H_{0-})$. Also, because $f(T^2)$ is connected and $f(\overline{H}_s)$ is convex, it must be the case that

$$\partial f(\overline{H}_s) \cap f(H_{0-}) \neq \phi.$$

Equivalently, since $\partial f(\overline{H}_s) = f(\partial H_s) = f(H_0)$, we must have

$$f(H_0) \cap f(H_{0-}) \neq \phi \tag{2.30}$$

The proof will, therefore, be complete if we show that (2.30) is false. Let $\vartheta \in H_{0-}$, $\tilde{\vartheta} \in H_0$ be arbitrary. It should be evident by now that there is no loss of generality if we assume that $\tilde{\vartheta} \in \mathrm{int}(A)$, where the set A was defined in Lemma 2.2. In other words, we have

$$\frac{\pi}{2} < \tilde{\vartheta}_1 < \pi,\ 0 < \tilde{\vartheta}_2 < \frac{\pi}{2},\ 0 < \tilde{\vartheta}_1 - \tilde{\vartheta}_2 < \frac{\pi}{2} \tag{2.31}$$

Also, since $\vartheta \in H_{0-}$, $\tilde{\vartheta} \in H_0$, we obtain

$$B_1 \cos\vartheta_1 + B_{12} \cos(\vartheta_1 - \vartheta_2) \leq 0 \leq B_1 \cos\tilde{\vartheta}_1 + B_{12} \cos(\tilde{\vartheta}_1 - \tilde{\vartheta}_2) \tag{2.32a}$$

$$B_2 \cos\vartheta_2 + B_{12} \cos(\vartheta_1 - \vartheta_2) \leq 0 \leq B_2 \cos\tilde{\vartheta}_2 + B_{12} \cos(\tilde{\vartheta}_1 - \tilde{\vartheta}_2) \tag{2.32b}$$

$$B_1 \cos\vartheta_1 + B_2 \cos\vartheta_2 \leq 0 \leq B_1 \cos\tilde{\vartheta}_1 + B_2 \cos\tilde{\vartheta}_2. \tag{2.32c}$$

Multiplying (2.32a) and (2.32b) and noticing that $\det F(\vartheta) = \det F(\tilde{\vartheta}) = 0$ we get $\cos^2(\vartheta_1 - \vartheta_2) \leq \cos^2(\tilde{\vartheta}_1 - \tilde{\vartheta}_2)$. Doing the same thing with (2.32a) and (2.32c) as well as with (2.32b) and (2.32c) we finally obtain

$$\cos^2\vartheta_1 \le \cos^2\tilde{\vartheta}_1,\ \cos^2\vartheta_2 \le \cos^2\tilde{\vartheta}_2,\ \cos^2(\vartheta_1-\vartheta_2) \le \cos^2(\tilde{\vartheta}_1-\tilde{\vartheta}_2) \tag{2.33}$$

It must be the case, then, that

$$\sin^2\vartheta_1 \ge \sin^2\tilde{\vartheta}_1,\ \sin^2\vartheta_2 \ge \sin^2\tilde{\vartheta}_2,\ \sin^2(\vartheta_1-\vartheta_2) \ge \sin^2(\tilde{\vartheta}_1-\tilde{\vartheta}_2) \tag{2.34}$$

Now we utilize Lemma 2.2 to obtain

$$B_1 \sin\vartheta_1 + B_{12}\sin(\vartheta_1-\vartheta_2) = B_1\sin\tilde{\vartheta}_1 + B_2\sin(\tilde{\vartheta}_1-\tilde{\vartheta}_2) > B_1 \tag{2.35}$$

$$B_1\sin\vartheta_1 + B_2\sin\vartheta_2 = B_1\sin\tilde{\vartheta}_1 + B_2\sin\tilde{\vartheta}_2 > B_1$$

Hence, $\sin\vartheta_2 > 0$, $\sin(\vartheta_1-\vartheta_2) > 0$, which combined with (2.34) yields

$$\sin\vartheta_2 \ge \sin\tilde{\vartheta}_2,\ \sin(\vartheta_1-\vartheta_2) \ge \sin(\tilde{\vartheta}_1-\tilde{\vartheta}_2) \tag{2.36}$$

Hence, there are only two alternatives: either

$$\sin\vartheta_1 \ge \sin\tilde{\vartheta}_1$$

which immediately yields

$$\vartheta = \tilde{\vartheta}$$

contradicting the hypothesis, or

$$\sin\vartheta_1 \le -\sin\tilde{\vartheta}_1$$

in which case, from (2.31) and (2.36),

$$-\pi < -\tilde{\vartheta}_1 \le \vartheta_1 \le \tilde{\vartheta}_1 - \pi < 0$$

$$0 < \tilde{\vartheta}_2 \le \vartheta_2 \le \pi - \tilde{\vartheta}_2 < \pi$$

so that

$$-2\pi < -\pi-(\tilde{\vartheta}_1-\tilde{\vartheta}_2) \le \vartheta_1-\vartheta_2 \le (\tilde{\vartheta}_1-\tilde{\vartheta}_2-\pi) < 0$$

which yields $\sin(\vartheta_1-\vartheta_2) < \sin(\tilde{\vartheta}_1-\tilde{\vartheta}_2)$ unless

$$\vartheta_1 = -\tilde{\vartheta}_1,\ \vartheta_2 = \pi-\tilde{\vartheta}_2$$

which in turn contradicts (2.35). The proof is complete.

3. Remarks on the Dynamic Behavior

3.1 The Differential Equation

According to the classical model, the motion of the 3-bus power network is governed by the differential equations

$$m_1 \ddot{\vartheta}_1 + d_1 \dot{\vartheta}_1 = P_1 - f_1(\vartheta_1, \vartheta_2) \qquad (3.1)$$

$$m_2 \ddot{\vartheta}_2 + d_2 \dot{\vartheta}_2 = P_2 - f_2(\vartheta_1, \vartheta_2)$$

where the third bus is taken to be a slack bus and the vector $P \doteq (P_1, P_2)$ corresponds to the mechanical power input. It is not difficult to show analytically that (3.1) has at most six equilibrium points in T^2 corresponding to the solutions of

$$P = f(\vartheta) \qquad (3.2)$$

Thus, if P is a regular value of $f(\vartheta)$, i.e., $\det F(\vartheta) \neq 0$, the points in the inverse image $f^{-1}(P)$ must fall under one of the following cases (see Figure 3):

i) for P in R_2, $f^{-1}(P)$ contains two points in T^2, one of which is stable and the other is a saddle with one positive eigenvalue.

ii) for P in R_4, $f^{-1}(P)$ contains four points in T^2, one of which is stable, two are saddles with one positive eigenvalue, and the fourth a saddle with two positive eigenvalues; and finally

iii) for P in R_6, $f^{-1}(P)$ contains six points in T^2, one of which is stable, three are saddle points with one positive eigenvalue, and the rest are saddles with two positive eigenvalues.

In the statements above the eigenvalues refer to the linearization of (3.1) around the equilibrium.

In the study of the dynamics of (3.1) we should expect to encounter saddle connection bifurcations giving rise to much more complex phenomena than in the one-dimensional problem studied in [3]. The next situation is devoted to the study of a certain kind of dynamic bifurcation.

3.2 The Role of Damping in Complete Stability

The motivation of this section stems from the study of the one-dimensional problem as well as the characterization of complete stability to be found in [3].

A dynamical system is said to be completely stable if every trajectory converges to an equilibrium.

Consider the one-dimensional classical equation for a power network:

$$\ddot{\vartheta} + \alpha\,\dot{\vartheta} = \beta - \sin\vartheta \tag{3.3}$$

The global properties of (3.3) are well known (see [3,4]). In particular it is known that if $\beta < 1$; i.e. as long as a stable equilibrium exists, then the differential system is completely stable provided that the damping constant α is large enough. This fact, combined with the role played by damping in the characterization of complete stability in [3], suggests the conjecture that sufficiently large damping should produce complete stability in higher dimensional problems as well. The following example shows that this is false.

Example 3.1 Consider the three node network of Figure 4. The differential equations for such a system are

$$\begin{aligned} \ddot{\vartheta}_1 + d\,\dot{\vartheta}_1 &= (\varepsilon-\varepsilon^2) - \varepsilon\sin\vartheta_1 - \sin(\vartheta_1-\vartheta_2) \\ \ddot{\vartheta}_2 + d\,\dot{\vartheta}_2 &= (\varepsilon-\varepsilon^2) - \varepsilon\sin\vartheta_2 - \sin(\vartheta_1-\vartheta_2) \end{aligned} \tag{3.4}$$

Let K_n be the closed subset of R^4 defined by

$$K_n = \{(\dot{\vartheta}_1,\dot{\vartheta}_2,\vartheta_1,\vartheta_2) \mid \dot{\vartheta}_1=\dot{\vartheta}_2,\ \vartheta_1-\vartheta_2 = 2\pi n\} \tag{3.5}$$

The projections of the K_n on the $(\dot{\vartheta}_1=0,\dot{\vartheta}_2=0)$ plane are shown in Figure 5. Notice that each K_n is an invariant set of the motion of (3.4), and that if $\varepsilon < \frac{4}{5}$, then all the equilibrium points of (3.4) lie in the sets K_n.

By changing coordinates to $z = \vartheta_1-\vartheta_2$, we obtain from (3.4):

$$\ddot{z} + d\dot{z} = -\varepsilon\,(\sin\vartheta_1 - \sin\vartheta_2) - 2\sin z \tag{3.6}$$

It is straightforward to show from (3.6) that there exist $\varepsilon_0 > 0$, $d_0 > 0$ such that the invariant sets K_n are stable for all $\varepsilon < \varepsilon_0$, $d > d_0$. (A closed set is said to be stable if every open neighborhood of the set contains a positively invariant neighborhood.)

Let A_n be the region of attraction of K_n. The A_n are open disjoint sets of R^4 and hence their union does not cover R^4. Thus the complement of this union is a nonempty invariant set and it contains no equilibrium points. Hence a trajectory which starts in this complement can never converge and so the system is not completely stable.

3.3 Complete Stability Revisited

In the example of section 3.2, we showed that arbitrarily large damping need not lead to complete stablity. Upon closer examination of equation (3.4) one can see that for $\varepsilon = 1$ (3.4) not only has six equilibrium points in $T^2 \times R^2$, but it is also completely stable for all positive values of the damping d. So, if we vary ε continuously from $\varepsilon = 1$ to $\varepsilon = 0$, we should expect not only *static* bifurcations, i.e. appearances or disappearances of equilibrium points, to take place, but *dynamic* bifurcations, i.e. saddle connection bifurcations, as well. In Figures 6a and 6b we depict the abstract flow diagrams of (3.4) for $\varepsilon = 1$ and $\varepsilon = \frac{4}{5}$. At $\varepsilon = \frac{4}{5}$ the number of critical points changes from 6 to 2. Notice how the stable manifolds and unstable manifolds fuse together.

The following proposition throws some light on the mechanism of the loss of complete stability.

Proposition 3.1 Consider a three node network such that there are exactly two equilibrium points. Then the system is not completely stable.

Proof: By contradiction. Suppose that the system is completely stable. Let $\vartheta^s = (\vartheta_1^s, \vartheta_2^s)$ and $\vartheta^u = (\vartheta_1^u, \vartheta_2^u)$ be the stable and unstable points respectively. Now, consider the system behavior in R^4. Let $W^u(\vartheta^u)$ be the 1-dimensional unstable manifold of ϑ^u. By the assumption of complete stability, $W^u(\vartheta^u)$ must be bounded. Hence $\overline{W^u(\vartheta^u)}$ is a compact one dimensional manifold with boundary. Let $\vartheta+2\pi(m, n)$ denote the point $(\vartheta_1+2\pi m, \vartheta_2+2\pi n\ )$. The boundary of

$\overline{W^u(\vartheta^u)}$ consists of critical points. It is not hard to show, hence we omit it here, that there is no loss of generality if we assume that

$$\partial\overline{W^u(\vartheta^u)} = \{\vartheta^s + 2\pi(m, n), \vartheta^s + 2\pi(\tilde{m}, \tilde{n})\} \tag{3.7}$$

for some pairs of integers (m, n) and $(\tilde{m}, \tilde{n})$.

Now, let $\vartheta^u_{k,l}$ be the point

$$\vartheta^u_{k,l} = \vartheta^u + 2\pi\,(lm - k\tilde{m}, ln - k\tilde{n}) \tag{3.8}$$

for some integers l and k.

Let

$$N_k = \bigcup_l \overline{W^u(\vartheta^u_{k,l})} \tag{3.9}$$

The sets N_k are closed, invariant and pairwise disjoint. In addition, each N_k is stable, which completes the proof of the proposition.

We conclude with the following theorem which is stated without proof.

Theorem 3.1 Consider the (n+1)-node power network governed by the equation

$$M\ddot{\vartheta} + D\dot{\vartheta} = P - f(\vartheta) \tag{3.10}$$

where M and D are n-dimensional diagonal matrices. Let ϑ_i, i = 1,...,k in T^n be the equilibrium points. Then (3.10) is completely stable if and only if:

i) $W^u(\vartheta_i)$ is bounded in $R^n \times R^n$, i = 1,...,k.

ii) $\bigcup_{v\in\Pi} \bigcup_{i=1}^{k} \overline{W^u(\vartheta_i+v)}$ is a connected set in $R^n \times R^n$ where Π is the set of vectors v all of whose components are multiples of 2π.

References

[1] Arapostathis, A., S. Sastry, P. Varaiya, "Analysis of the Power Flow Equation," *Electrical Power and Energy Systems*, vol 3 (3), 1981, 116-126.

[2] Tavora, C.J., O.J.M. Smith, "Equilibrium Analysis of Power Systems," *IEEE Transactions on Power Apparatus and Systems*, vol. PAS-31, 1972, 1131-

423

1137.

[3] Arapostathis, A., S. Sastry, P. Varaiya, "Global Behavior of Interconnected Power Systems: Part I," ERL Memo M81/3, Electronics Research Lab., Univ. of California, Berkeley, CA.

Figure Captions

Fig 1 The three node network

Fig 2 The sets H_s, H_p

Fig 3 Number of solutions of (3.2)

Fig 4 Network of Example 3.1

Fig 5 The sets K_n

Fig 6a Flow diagram for $\varepsilon = 1$

Fig 6b Flow diagram for $\varepsilon = \frac{4}{5}$

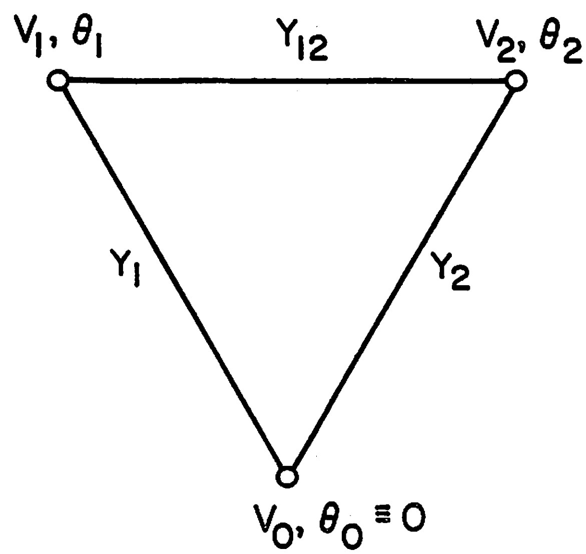


Fig 1. The three node network

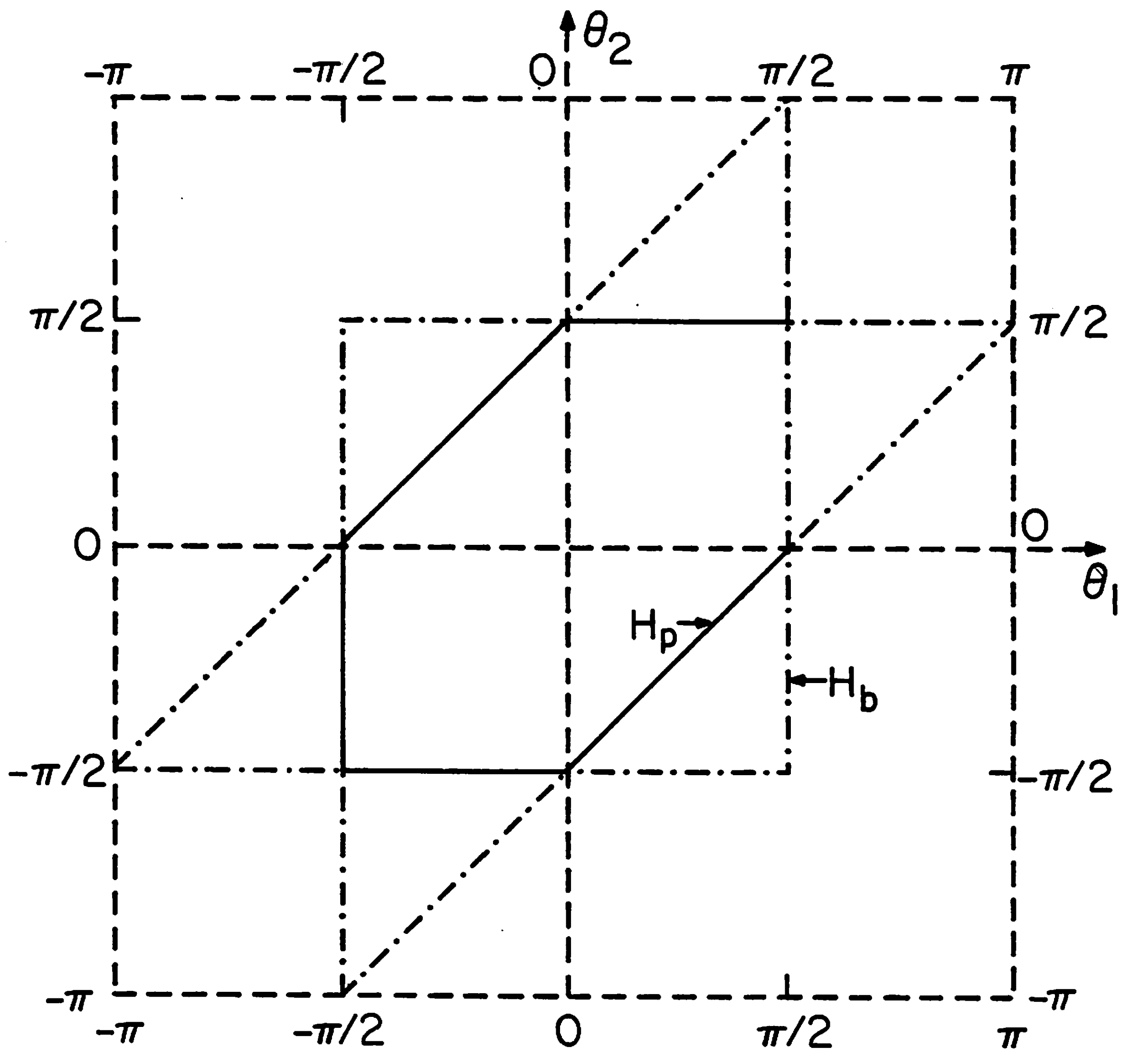


Fig. 2 The sets H_p, H_b

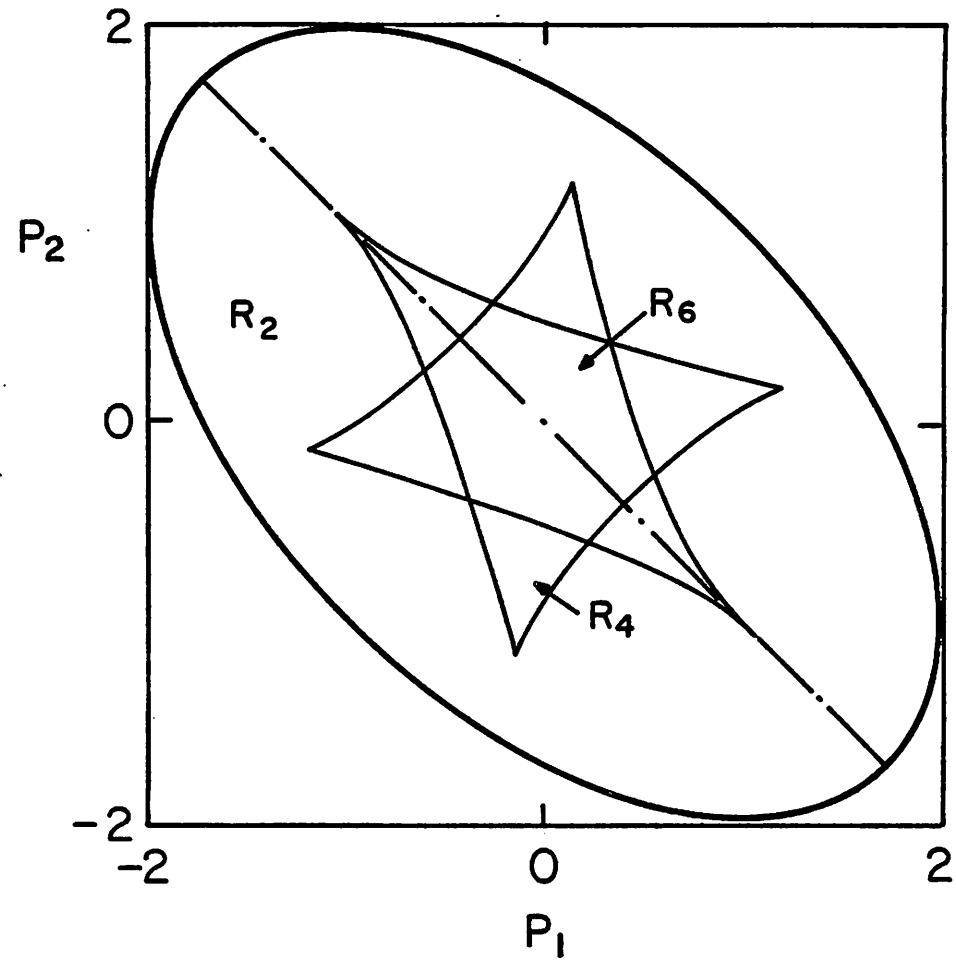
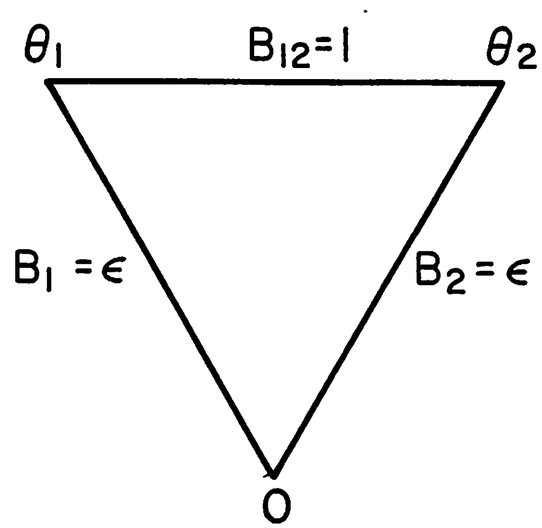


Fig. 3



$P_i=(\epsilon-\epsilon^2)$ $\quad i=1,2$

$M_i=1$ $\quad i=1,2$

$d_i=d$ $\quad i=1,2$

$0<\epsilon<1$

Fig. 4 The network of example 3.1

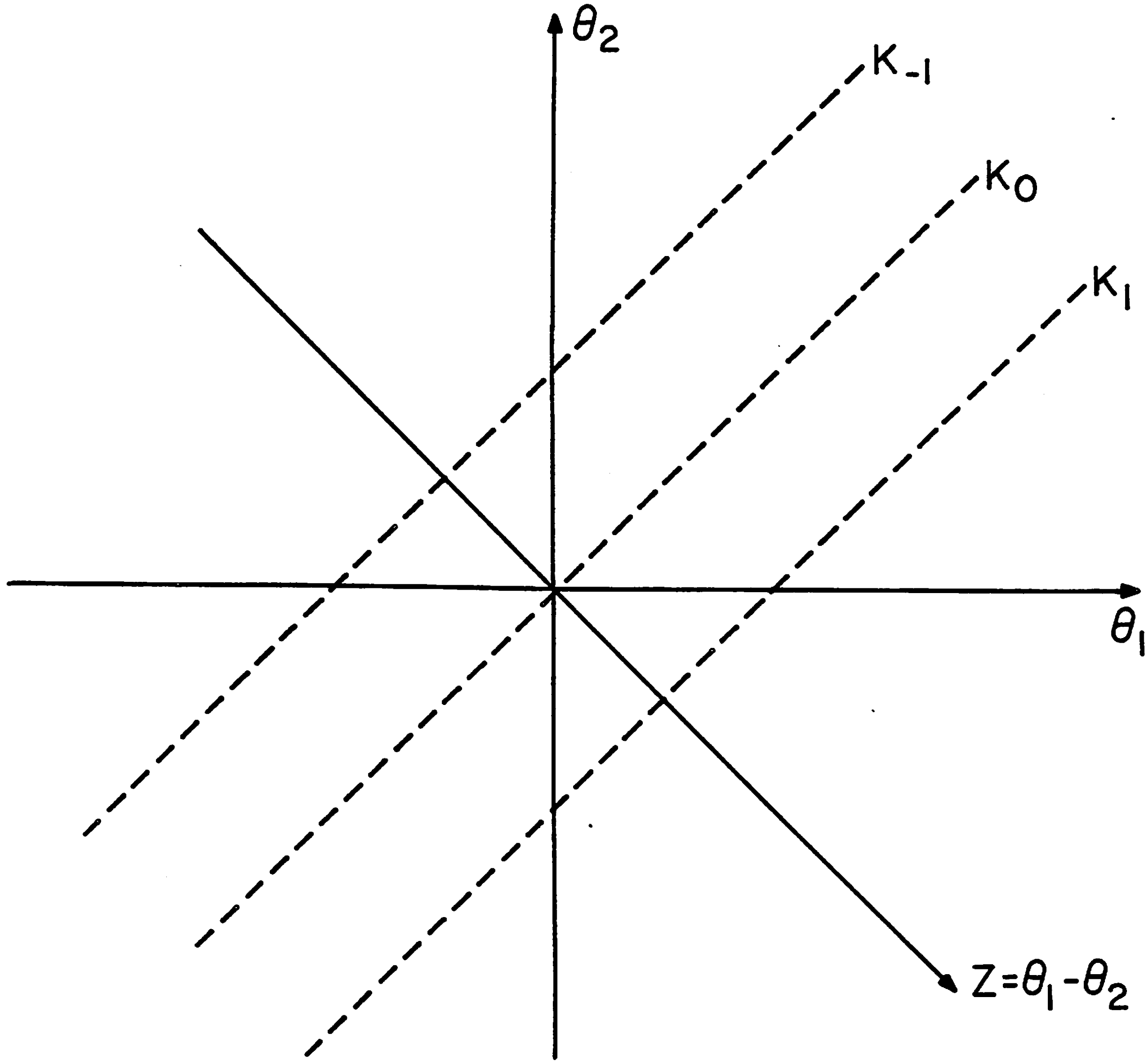


Fig. 5

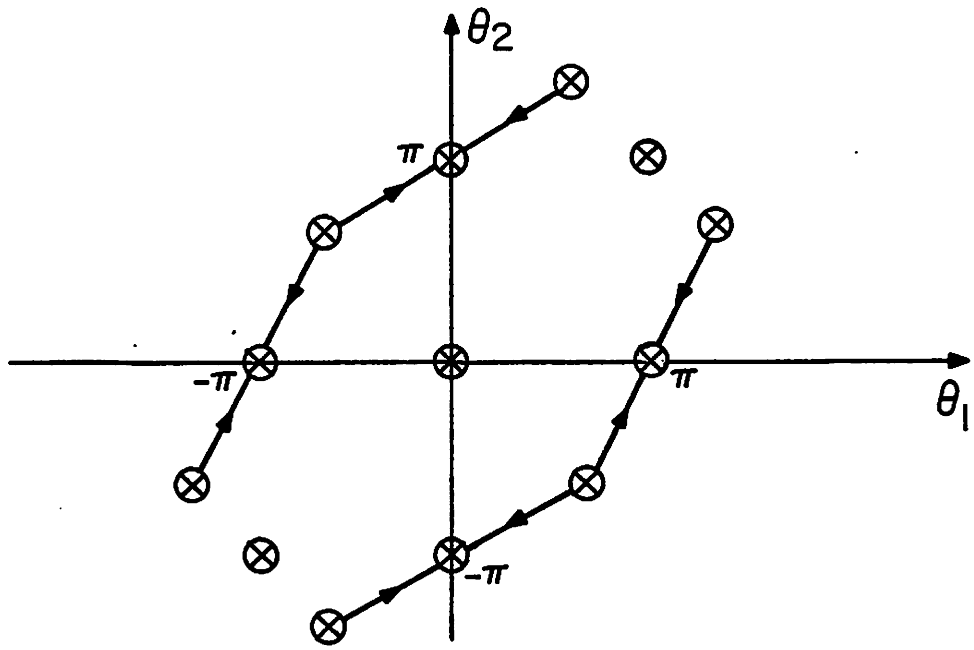


Fig. 6a: $\epsilon = 1$

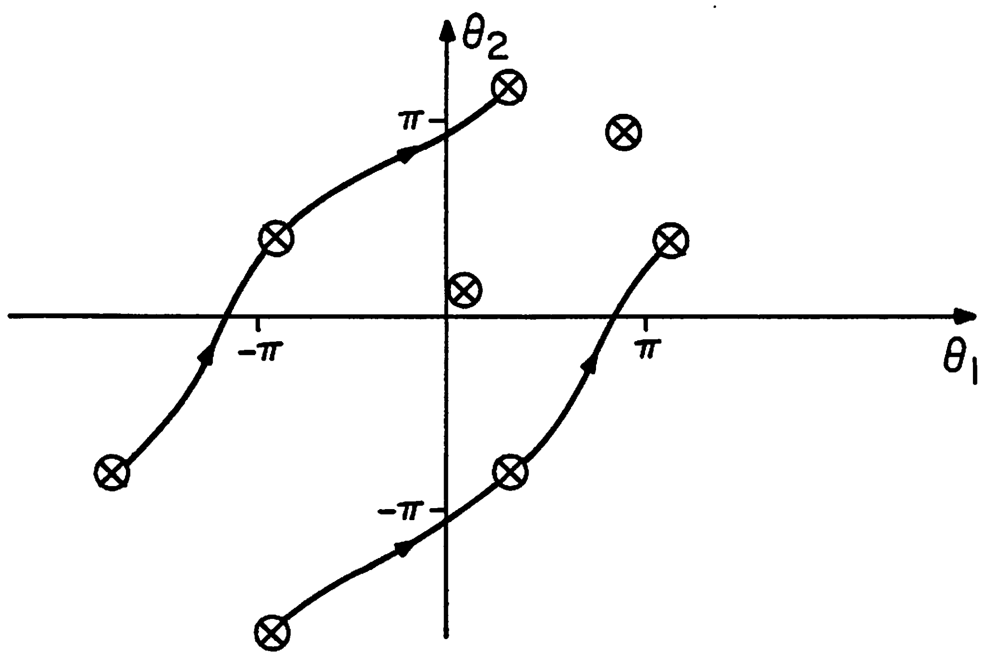


Fig. 6b: $\epsilon = 4/5$