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by

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ABSTRACT

Structures of similarity relations and fuzzy weak orderings are studies in detail in view of applications to measurements in a fuzzy environment.

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1. INTRODUCTION

A binary relation language is an important tool of the theory of measurements (see, for example, book [4]). Specifically, the theory of nonimal and ordinal scales is based on theories of equivalence relations and weak orderings. These binary relations have a simple structure which can be described as follows (bearing in mind a context of the measurement theory). Let M be a finite set of empirical objects and $\mathbb{R}$ denotes the set of real numbers. If f is a mapping $f: M \to \mathbb{R}$ (a scale), then the inverse image of a diagonal relation on $\mathbb{R}$ is an equivalence relation I on M which is a kernel of a mapping f. Classes of equivalence relation I form a partition of M and each class is an inverse image of some $u \in M$ and vice versa. Moreover, any equivalence relations on M is a kernel of some mapping f. Further, let L denote a natural linear ordering on real numbers. Then an inverse image of L with respect to a mapping $f: M \to \mathbb{R}$ is a weak ordering over M, i.e., a reflexive, complete and transitive binary relation on M. Any such a relation R defines an ordered partition of M; elements of this partition are classes of equivalence relation $I = R^{-1} \cap R$, these classes are inverse images of elements from f(M) and ordered in accordance with L. Again, any ordering on M may be obtained in this way.

The aim of this paper is to extend this framework keeping in mind applications to measurements in a fuzzy environment. For technical reasons we will consider a finite subset $N = f(M) \subset \mathbb{R}$ instead of the set of all reals.

Section 2 deals with coverings and resemblance relations in fuzzy set theory. The notions of a covering and a resemblance relation are important and useful generalizations of partitions and equivalence relations, for they yield models for mechanisms of "likeness."

Let M be a finite set of objects and N a finite set of their attributes, such that any object $a \in M$ has at least one attribute $i \in N$. If P_i denote the subset of all $a \in M$ that possess an attribute i, then, obviously,

$$M = \bigcup_{i \in N} P_i \tag{1.1}$$

More generally, elements of the set N may be regarded as "names" of attributes and subsets P_i as "models" of these attributes. Any family of subsets of M, which fulfills (1.1), is called a covering of M. Thus, attributes form a covering of the set of objects. Conversely, if covering (1.1) of the set M is given, one can consider P_i as an attribute: "belongness to P_i" with name i. In this sense, there is a one-to-one correspondence between families of attributes and coverings.

This framework provides a very important mechanism of resemblance relations. Namely, we say that two objects are resembled if they have a common attribute. Formally, this notion of likeness can be described in the following way. Let R be a binary relation on M defined as

$$xRy \text{ if only if there exist } i \in N \text{ such that } x, y \in P_i \tag{1.2}$$

Then R is a reflexive and symmetric binary relation. Such relations are called resemblance relations. It is easy to see that, generally speaking, a resemblance relation defined by (1.2) is not necessarily a transitive relation. Note that non-transitivity usually arises from comparisons by means of different parameters or attributes.

The notion of resemblance relation provides a more abstract description of likeness than that of covering language. Let R be a resemblance relation on the set M, i.e., any reflexive symmetric relation on M. We say that x resembles y if and only if xRy. Properties of reflexivity and symmetry provide the most common properties of likeness. Nevertheless, it turns out that resemblance relations give a description of likeness which is equivalent to the covering one. Namely, for any given resemblance relation there is a covering which generates this relation by (1.2). Thus, we have two equivalent mechanisms of likeness which fuzzy extensions will be studied in Section 2.

There is a very important particular case of the framework described above when each object of M has exactly one attribute in N. This case is called a classification problem. In addition to (1.1) we have a property

$$P_i \cap P_j = \phi \quad \text{for} \quad i \neq j \tag{1.3}$$

in this case. Coverings satisfying (1.3) are said to be partitions. It is easy to verify that a resemblance relation defined by (1.2) for partitions is a transitive relation, i.e., an equivalence relation. Such relations provide a proper mathematical model for a common notion of "sameness." We have classical dual descriptions of classifications in terms of partitions and equivalence relations in this case. An extension of this framework on fuzzy set theory will be studied in Section 3. Note that the term "similarity relation" will be used below instead of equivalence relation (see [5]).

Fuzzy surjective mappings are defined and studied in Section 4. It is shown that similarity relations can be described as inverse images of crisp diagonals under fuzzy mappings as well as inverse images of fuzzy diagonals under crisp mappings.

Fuzzy weak orderings are a subject of study in Section 5. Necessary and sufficient conditions are established for a quasi-inverse image of a linear ordering on N to be a fuzzy weak ordering on M in terms of surjective mappings.

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2. COVERINGS AND RESEMBLANCE RELATIONS

Let U be a fuzzy set with a universe M.

Definition 2.1. A family $\Sigma = \{P_i\}_{i \in N}$ of fuzzy sets with a common universe M is said to be a covering of the set U if and only if

$$U = \bigcup_{i \in N} P_i .$$

Below we suppose that M and N are finite sets. Definition 2.1 is a natural extension of (1.1). In accordance with Section 1 the set N could be regarded as a set of attributes. Then one can say that $P_i(x)$ is a degree of certainty with which an object x has an attribute i. In this context P_i is considered as a fuzzy subset of objects which have an attribute i.

The following definition presents a natural extension of (1.2).

Definition 2.2. A fuzzy binary relation defined by

$$R_\Sigma(x,y) = \bigvee_{i \in N} P_i(x) \wedge P_i(y) \qquad (2.1)$$

is said to be a fuzzy relation associated to Σ.

Lemma 2.1. Any relation R_Σ fulfills the following properties:

1) $R_\Sigma (x,y) = R_\Sigma(y,x)$ for all $x,y \in M$; (2.2)

2) $R_\Sigma (x,y) \le R_\Sigma(x,x) \wedge R_\Sigma(y,y)$ for all $x,y \in M$; (2.3)

3) $R_\Sigma (x,x) = U(x)$ for all $x \in M$. (2.4)

Proof. (2.2) and (2.4) are evident. We have $P_i(x) \leq U(x)$ and $P_i(y) \leq U(y)$ which implies

$$P_i(x) \wedge P_i(y) \leq U(x) \wedge U(y) = R_\Sigma(x,x) \wedge R_\Sigma(y,y)$$

for all $i \in N$. Hence,

$$R_\Sigma(x,y) = \bigvee_{i \in N} \{P_i(x) \wedge P_i(y)\} \leq R_\Sigma(x,x) \wedge R_\Sigma(y,y). \quad \text{¤}$$

By (2.2), R_Σ is a symmetrical relation. Note that (2.3) is fulfilled for reflexive relations. This property can be regarded as a weak reflexivity (see [6]). We consider fuzzy relations satisfying properties (2.2), (2.3) and (2.4) as analogous to crisp resemblance relations.

Definition 2.3. A fuzzy binary relation is said to be a resemblance relation on a fuzzy set U if and only if it fulfills properties (2.2), (2.3) and (2.4).

It follows from Lemma 2.1 that any fuzzy binary relation associated to a covering is a resemblance relation. The following theorem shows that the converse is also true (see also [6] where the same result is proved independently in a different context).

Theorem 2.1. Let R be a resemblance relation on a fuzzy set U. There is a covering Σ such that $R = R_\Sigma$.

Proof. A fuzzy set K is said to be a pre-class of R if and only if $K(x) \wedge K(y) \leq R(x,y)$ for all $x,y \in M$. The set of all pre-classes of R is an inductive poset (see [2], p. 192). Maximal elements of this poset are called classes of R. Let N denote the set of all classes. We define a family of fuzzy sets by

$$K_{a,b}(x) = \begin{cases} R(a,b), & \text{if } x=a \text{ or } x=b, \\ 0, & \text{otherwise} \end{cases}$$

for all $a,b \in M$. Then $K_{a,b}$ is a pre-class of R for any $a,b \in M$ by (2.2) and (2.3). If P_i $(i \in N)$ is any class of R which contains $K_{a,b}$ then $P_i(a) \wedge P_i(b) = R(a,b)$. Hence,

$$\bigvee_{i \in N} \{P_i(x) \wedge P_i(y)\} = R(x,y)$$

for all $x,y \in M$, which implies

$$\bigvee_{i \in N} P_i(x) = U(x) ,$$

since R is a relation on U, i.e., $\Sigma = \{P_i\}_{i \in N}$ is a covering of U such that $R = R_\Sigma$. ¤

Thus, for each covering Σ of a fuzzy set U there is a resemblance relation R_Σ on U associated to Σ by (2.1) and, conversely, for each resemblance relation R on U there is a covering Σ of U such that $R = R_\Sigma$. Usually, it is possible that $R_{\Sigma_1} = R_{\Sigma_2}$ for different coverings Σ_1 and Σ_2. Let us consider the following

Example 2.1*.

Let $M = \{x_1, x_2, x_3\}$ and $U = M$. Let us consider the two following coverings of U:

$\Sigma_1 =$

	P_1	P_2	P_3
x_1	1	α	α
x_2	α	1	γ
x_3	β	γ	1

and

$\Sigma_2 =$

	P_1	P_2	P_3
x_1	1	α	β
x_2	α	1	α
x_3	β	γ	1

*From a different point of view an analogous example is examined in [1].

where $\alpha < \beta < \gamma$. It is easy to verify that

$R_{\Sigma_1} = R_{\Sigma_2} =$

	x_1	x_2	x_3
x_1	1	α	β
x_2	α	1	γ
x_3	β	γ	1

Since $R = R_{\Sigma_1} = R_{\Sigma_2}$ is a resemblance relation, it is possible to calculate all its classes. It turns out that they form a covering

$\Sigma_3 =$

	P_1	P_2	P_3	P_4
x_1	1	α	α	β
x_2	α	1	γ	α
x_3	β	γ	1	1

Hence, we have at least three different coverings Σ_1, Σ_2 and Σ_3 such that $R_{\Sigma_1} = R_{\Sigma_2} = R_{\Sigma_3}$.

If R is a resemblance relation on U and Σ is a covering such that $R = R_\Sigma$ then, obviously, each element of Σ is a pre-class of R. Therefore, classes of R form a covering which is a maximal one among coverings Σ possessed a property $R = R_\Sigma$. This covering will be denoted by Σ_R. Then we have $R_{\Sigma_R} = R$, but, generally speaking, $\Sigma_{R_\Sigma} \neq \Sigma$.

<u>Example 2.2.</u>

For coverings from the previous example we have $\Sigma_{R_{\Sigma_1}} = \Sigma_{R_{\Sigma_2}} = \Sigma_3$, and $\Sigma_{R_{\Sigma_3}} = \Sigma_3$ too. Obviously, Σ_3 contains both Σ_1 and Σ_2.

Any covering $\Sigma = \{P_i\}_{i \in N}$ admits an interpretation as a fuzzy correspondence. Namely, let us consider a fuzzy correspondence $F_\Sigma : M \to N$ defined by its membership function

$$F_\Sigma(x,i) = P_i(x).$$

In the classical case a resemblance relation R_F on M based on a correspondence F is a kernel of F, i.e., $R_F = F^{-1} \circ F$. The latter formula can be extended on the fuzzy case by a $(\wedge,\vee)$-composition rule which gives $R_{F_\Sigma}(x,y)=(F_\Sigma^{-1} \circ F_\Sigma)(x,y) = \bigvee_i \{F_\Sigma(x,u) \wedge F_\Sigma(y,i)\} = \bigvee_i \{P_i(x) \wedge P_i(y)\} = R_\Sigma(x,y)$, i.e., the same result as (2.1).

On the other hand for a given fuzzy correspondence $F:M \to N$ one can consider a covering $\Sigma_F = \{F(x,u)\}_{i \in N}$ of a fuzzy set $U(x) = \bigvee_{i \in N} F(x,u)$. Thus, coverings and fuzzy correspondences provide equivalent descriptions of resemblance mechanisms.

3. SIMILARITY RELATIONS, PARTITIONS AND QUOTIENT-SETS

There is an important particular case of fuzzy resemblance relations, namely, similarity relations. These relations were introduced in [5] and studied, for example, in [1] and [3].

Definition 3.1. A fuzzy binary relation S on M is said to be a similarity relation on U if and only if it is a symmetric and transitive relation and $S(x,x) = U(x)$ for all $x \in M$.

Recall that a fuzzy relation S is said to be transitive if and only if

$$S(x,y) \wedge S(y,z) \leq S(x,z)$$

for all $x,y,z \in M$.

Since a similarity relation S is a symmetric and transitive one we have $S(x,y) = S(x,y) \wedge S(y,x) \leq S(x,x)$. In the same way $S(x,y) \leq S(y,y)$ which implies that S has weak reflexivity property (2.3). Thus, similarity relations are a particular case of resemblance relations.

Let Σ be a covering of U. According to Section 2 a fuzzy binary relation R_Σ associated to Σ is a resemblance relation. Is this relation a similarity relation? The following theorem gives an answer.

Theorem 3.1. Let $\Sigma = \{P_i\}_{i \in N}$ be a covering of U. A fuzzy binary relation R_Σ is a similarity relation if and only if for each pair $i,j \in N$ and each pair $x,y \in M$ there is $k \in N$ such that

$$h_{ij} \wedge P_i(x) \wedge P_j(y) \leq P_k(x) \wedge P_k(y) \tag{3.1}$$

where $h_{ij} = \bigvee_{x \in M} \{P_i(x) \wedge P_j(x)\}$ is a height of $P_i \cap P_j$.

Proof. Let Σ fulfills (3.1). It is necessary to prove only transitivity of R_Σ. We have

$$R_\Sigma(x,y) \wedge R_\Sigma(y,z) = \left[\bigvee_{i \in N} \{P_i(x) \wedge P_i(y)\}\right] \wedge \left[\bigvee_{i \in N} \{P_i(y) \wedge P_i(z)\}\right]$$

$$= \bigvee_{i,j \in N} \{P_i(x) \wedge P_i(y) \wedge P_j(y) \wedge P_j(z)\} \leq \bigvee_{i,j \in N} \{h_{ij} \wedge P_i(x) \wedge P_j(x)\}$$

$$\leq \bigvee_{k \in N} \{P_k(x) \wedge P_k(z)\} = R_\Sigma(x,z), \text{ by (3.1)}.$$

Hence, R_Σ is a transitive relation.

Now, let R_Σ be a similarity relation associated to a given covering Σ. Since R_Σ is transitive, we have

$$R_\Sigma(x,t) \wedge R_\Sigma(t,y) \leq R_\Sigma(x,y)$$

for any $t \in M$, which implies, as above,

$$\bigvee_{i,j \in N} \{P_i(x) \wedge P_i(t) \wedge P_j(t) \wedge P_j(y)\} \leq \bigvee_k \{P_k(x) \wedge P_k(y)\}.$$

Hence, for given pairs $i,j \in N$ and $x,y \in M$ there is $k \in N$ such that

$$P_i(x) \wedge P_i(t) \wedge P_j(t) \wedge P_j(y) \leq P_k(x) \wedge P_k(y)$$

for any $t \in M$, which implies

$$h_{ij} \wedge P_i(x) \wedge P_j(y) = \bigvee_{t \in M} \{P_i(x) \wedge P_i(t) \wedge P_j(t) \wedge P_j(y)\}$$

$$\leq P_k(x) \wedge P_k(y) \text{ .} \qquad \square$$

There is another necessary and sufficient condition on Σ for R_Σ to be a similarity relation which involves the notion of α-level-set. Recall (see [5]) that a α-level-set of a fuzzy set A is a crisp set

$$A^\alpha = \{ x \in M | A(x) \geq \alpha\}, \ \alpha \in [0,1].$$

Let $\Sigma = \{P_i\}_{i \in N}$ be a covering of a fuzzy set U. Then $\Sigma^\alpha = \{P_i^\alpha\}_{i \in N}$ is, obvioulsy, a crisp covering of U^α for all $\alpha \in [0,1]$.

Theorem 3.2. A resemblance relation R_Σ associated to Σ is a similarity relation on U if and only for all $\alpha \in [0,1]$, each pair $i,j \in N$ and each pair $x,y \in M$ such that $x \in P_i^\alpha$, $y \in P_j^\alpha$ and $P_i^\alpha \cap P_j^\alpha \neq 0$ there is $k \in N$ such that $x,y \in P_k^\alpha$.

Proof. Let Σ fulifl1s conditions of the theorem and $\alpha \in [0,1]$ be any given number. We have

$xR_\Sigma^\alpha y$ if and only if $R_\Sigma(x,y) \geq \alpha$ if and only if
$\bigvee_{i \in N} \{P_i(x) \wedge P_i(y)\} \geq \alpha$ if and only if
there is i such that $P_i(x) \wedge P_i(y) \geq \alpha$ if and only if
there is i such that $P_i(x) \geq \alpha$ and $P_i(y) \geq \alpha$ if and only if
there is i such that $x,y \in P_i^\alpha$.

It is easy to verify now that R_Σ is a transitive relation on the set U^α. Let $x,y,z \in M$ and $\alpha = R_\Sigma(x,y) \wedge R_\Sigma(y,z)$. Then $xR_\Sigma^\alpha y$ and $yR_\Sigma^\alpha z$ which imply $xR_\Sigma^\alpha z$, by transitivity of R_Σ. Hence,

$$R_\Sigma(x,z) \geq \alpha = R_\Sigma(x,y) \wedge R_\Sigma(y,z),$$

i.e., R is a similarity relation on U.

Conversely, let R_Σ be a similarlity relation on U and $\alpha \in [0,1]$. Let, also, $P_i^\alpha \cap P_j^\alpha \neq 0$ and $x \in P_i^\alpha$, $y \in P_j^\alpha$. Since $P_i^\alpha \cap P_j^\alpha \neq 0$, there is $z \in P_i^\alpha \cap P_j^\alpha$. We have $xR_\Sigma z$, because $x,z \in P_i^\alpha$, and $zR_\Sigma y$, because $z,y \in P_j^\alpha$. R_Σ^α is a transitive crisp relation, since R_Σ is a similarity relation. Hence, $xR_\Sigma^\alpha y$ which implies $x,y \in P_k^\alpha$ for some $k \in N$. ¤

Theorems 3.1 and 3.2 give an internal description of those coverings which generate, by (2.1), similarity relations.

As in the general case, it is also possible that different coverings generate the same similarity relation. Let S be a similarity relation on U. It was mentioned in Section 2 that there is a unique maximal covering Σ such that $S = R_\Sigma$. Elements of this covering are classes of S. Classes of similarity relations admit a very simple description. They turn out to be classical <u>similarity classes</u> which were introduced in original Zadeh's work ([5]).

<u>Theorem 3.3</u>. Any class of a similarity relation S is a similarity class [a] for some $a \in M$.

<u>Proof</u>. Recall (see [5]) that a similarity class [a] is a fuzzy set with a membership function $[a](x) = S(a,x)$. We have

$$[a](x) \wedge [a](y) = S(x,a) \wedge S(a,y) \leq S(x,y)$$

because of symmetry and transitivity of S. Hence, each [a] is a pre-class of S. Let P be a class of S. Denote $\bar{a}$ an element such that $P(x) \leq P(\bar{a})$ for all $x \in M$. Since P is a class we have

$$P(x) = P(x) \wedge P(\bar{a}) \leq S(x,\bar{a}) = [\bar{a}]\,(x).$$

But it is possible only if $P = [\bar{a}]$, since $[\bar{a}]$ is a pre-class of S. ¤

Generally speaking, a converse theorem is not true. For example, let us consider a similarity relation S defined by

S =

	x_1	x_2	x_3
x_1	1	1	α
x_2	1	1	α
x_3	α	α	α

on the set $U = \{ (x_1,1), (x_2,1), (x_3,\alpha) \}$, where $0 < \alpha < 1$. There are two similarity classes in this case, namely,

$$[x_1] = [x_2] = \{(x_1,1), (x_2,1), (x_3,\alpha)\} \quad \text{and}$$
$$[x_3] = \{(x_1,\alpha), (x_2,\alpha), (x_3,\alpha)\} .$$

But there is only one class of S, namely, $P = \{(x_1,1), (x_2,1), (x_3,\alpha)\}$. Note that $[x_3] \subset P$.

Nevertheless, there is an important particular case when the converse statement is true.

Theorem 3.4. If S is a reflexive similarity relation then each similarity class [a] is a class of S.

Proof. Let us suppose that there is a class P of S which contents a pre-class [a], i.e., $P(x) \geq [a]\,(x)$ for all $x \in M$. Then $P(a) \geq [a]\,(a) = 1$

which implies $P(a) = 1$. Since P is a class, we have

$$P(x) = P(x) \wedge P(a) \leq S(a,x) = [a](x)$$

which implies $P = [a]$. ¤

Corollary 3.1. Classes of reflexive similarity relations are exactly their similarity classes.

In the general case, any class of a similarity relation S is a maximal similarity class, i.e., a similarity class which is not contained in any other. Since similarity classes are known as soon as S is known it is easy to determine classes of S.

Coverings, which elements are similarity classes of some similarity relation, are very important in similarity relation theory.

Definition 3.2. A covering $\Pi = \{P_i\}_{i \in N}$ is said to be a partition if and only if there is a similarity relation S such that Π is a set of all similarity classes of S.

Fuzzy partitions, thus defined, admit an independent description in internal terms. Namely, let us define a family of crisp sets $\{\Pi_i\}_{i \in N}$ by

$$\Pi_i = \{x \mid P_i(x) = h(P_i) = U(x)\}$$

for any given covering $\Pi = \{P_i\}_{i \in N}$. If $\{\Pi_i\}_{i \in N}$ is a crisp partition of M then for each $a \in M$ there is a unique i such that $a \in \Pi_i$ and for each $i \in N$ there is a such that $a \in \Pi_i$. We denote $[a] = P_i$ if and only if $a \in \Pi_i$ in this case. We also use a notation $\Pi_{[a]}$ for Π_i if $a \in \Pi_i$.

Theorem 3.5. (See also [3]) A covering Π is a partition if and only if

1) $\{\Pi_i\}_{i\in N}$ is a crisp partition of M, and

2) $h([a]\cap[b]) = [a]\,(b)\wedge[b]\,(a)$. (3.2)

Proof. 1) Let Π be a fuzzy partition, i.e., there is a similarity relation S such that Π is a set of all similarity classes [a] of S. We have

$$\begin{aligned}\Pi_{[a]} &= \{x\,|\,[a](x) = h([a]) = U(x)\}\\ &= \{x\,|\,S(a,x) = \bigvee_{u\in M} S(a,u) = S(x,x)\}\\ &= \{x\,|\,S(a,x) = S(a,a) = S(x,x)\}. \end{aligned} \tag{3.3}$$

$\Pi_{[a]} \neq 0$, since $a\in\Pi_{[a]}$. Let us suppose that $x\in\Pi_{[a]}\cap\Pi_{[b]}$. Then, by (3.3),

$$S(a,x) = S(a,a) = S(x,x) = S(b,b) = S(b,x). \tag{3.4}$$

We have

$$\begin{aligned}[a](t) &= S(a,t) \geq S(a,b)\wedge S(b,t) \geq S(a,x)\wedge S(x,b)\wedge S(b,t)\\ &= S(b,b)\wedge S(b,t) = S(b,t) = [b](t),\end{aligned}$$

by (3.4), symmetry and transitivity of S. In the same way, $[b](t) \geq [a](t)$ which implies $[a]=[b]$. Hence, $\{\Pi_{[a]}\}_{[a]\in N}$ is a crisp partition of M.

Further, we have

$$\begin{aligned}h([a]\wedge[b]) &= \bigvee_{x\in M} [a](x)\wedge[b](x) = \bigvee_{x\in M} S(a,x)\wedge S(x,b)\\ &= S(a,b) = S(a,b)\wedge S(b,a) = [a](b)\wedge[b](a),\end{aligned}$$

by transitivity and symmetry of S.

2) Let Π be a covering satisfying conditions of the theorem. We define $S(x,y)=[x](y)$. Then, by (3.2),

$$[x](t)\wedge[y](t) \leq [x](y)\wedge[y](x), \text{ for each } t. \tag{3.5}$$

Substituting $t = x$ and $t = y$, we obtain, respectively, $[y](x) \leq [x](y)$ and $[x](y) \leq [y](x)$. Hence, $[x](y) = [y](x)$, i.e., S is a symmetric relation. By (3.5) and symmetry of S we also have

$$S(x,y)\wedge S(y,z) = [x](y)\wedge[z](y) \leq [x](z)\wedge[z](x) = S(x,z),$$

i.e., S is a transitive relation. By definition of $\Pi_{[x]}$ we have $S(x,x) = [x](x) = U(x)$. Hence, S is a similarity relation on U. ¤

Note that partitions are defined whereby similarity classes but not classes. The following example illustrates the difference between these cases.

Example 3.1.

Let S be again a similarity relation defined by

$$S = \begin{array}{c|c|c|c|} & x_1 & x_2 & x_3 \\ \hline x_1 & 1 & 1 & \alpha \\ \hline x_2 & 1 & 1 & \alpha \\ \hline x_3 & \alpha & \alpha & \alpha \\ \hline \end{array}, \quad 0 < \alpha < 1 .$$

There is a unique class $P = \{(x_1,1), (x_2,1), (x_3,\alpha)\}$ of S. On the other hand a partition Π defined by S has two elements: $P = [x_1] = [x_2]$ and $[x_3] = \{(x_1,\alpha), (x_2,\alpha), (x_3,\alpha)\}$.

The notions of quotient-set and cannonical mapping are very important in classical set theory. Our previous exposition permits to give a proper extension of these notions on fuzzy set theory.

Defintion 3.3. Let S be a similarity relation on the set U with universe M and N a set of all similarity classes of S. A fuzzy set H with universe N defined by

$$H([a]) = \bigvee_{x \in M} [a](x)$$

is said to be a fuzzy quotient-set of U with respect to S and denoted $H = U/S$. A fuzzy mapping $\Pi: M \rightarrow N$ defined by

$$\Pi(x,[a]) = [a](x)$$

is said to be a canonical mapping.

It is easy to verify that Π is a well-defined mapping. The following theorem establishes some common properties of notions introduced.

Theorem 3.6. 1) $\Pi(U) = H$, i.e., H is an image of U with respect to Π;
2) $\Pi^{-1}(H) = U$, i.e., U is an inverse image of H with respect to Π;
3) $\Pi^{-1}([a])=[a]$, i.e., an inverse image of a fuzzy singleton [a] in H is a fuzzy subset [a] in M;
4) $S = \Pi^{-1} \circ \Pi$, i.e., S is a kernel of Π.

Proof. 1) $(\Pi(U))([a]) = \bigvee_{x \in M} \{\Pi(x,[a]) \wedge U(x)\}$

$$= \bigvee_{x \in M} \{S(x,a) \wedge S(x,x)\} = \bigvee_{x \in M} S(x,a) = \bigvee_{x \in M} [a](x) = H([a]).$$

2) $(\Pi^{-1}(H))(x) = \bigvee_{[a] \in N} \{\Pi^{-1}(x,[a]) \wedge H([a])\}$

$$= \bigvee_{[a] \in N} \{[a](x) \wedge \bigvee_{u \in M} [a](u)\} = \bigvee_{[a] \in N} S(a,x) = S(x,x) = U(x).$$

3) By definition, a fuzzy singleton [a] in H is a fuzzy set [a] with a membership function

$$[a]([x]) = \begin{cases} H([a]), \text{ if } [x]=[a], \\ \\ 0 \quad\quad , \text{ otherwise} \end{cases}$$

We have

$$(\Pi^{-1}([a]))(x) = \bigvee_{[t]\in N} \{\Pi(x,[t])\wedge[a]([t])\} = \Pi(x,[a])\wedge H([a])$$

$$= S(a,x) \wedge \bigvee_{u\in M} S(a,u) = S(a,x) = [a](x).$$

4) $$(\Pi^{-1} \circ \Pi)(x,y) = \bigvee_{[t]\in N} \{\Pi(x,[t])\wedge\Pi(y,[t])\}$$

$$= \bigvee_{[t]\in N} \{S(x,t)\wedge S(t,y)\} = S(x,y).$$ ¤

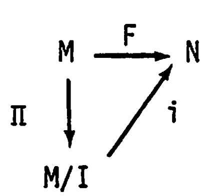
For any given partition $\Pi = \{P_i\}_{i\in N}$ of U one can consider a fuzzy set H with a membership function

$$H(i) = h(P_i)$$

Then, by Theorem 3.5, H is a quotient-set of U with respect to a proper similarity relation S. Thus, fuzzy partition theory presented above is quite analogous to a crisp one.

4. SIMILARITY RELATIONS AND FUZZY MAPPINGS

There is an important relationship between equivalence relations and surjective mappings in a classical set theory. Namely, let $F:M\rightarrow N$ be a (crisp) mapping from M onto N. Then there is an equivalence relation I on M such that the following diagram



is commutative, i.e., $F = i \circ \pi$, where π is a canonical mapping and i is an isomorphism. Roughly speaking, any crisp mapping onto may be regarded as a canonical mapping.

A particular class of fuzzy mappings onto is studied in this section; these mappings may be considered as fuzzy canonical mappings (within to isomorphisms). The reader is refered to [3] for definitions of fuzzy mappings and their compositions.

We begin with establishing some additional properties of canonical mappings.

Definition 4.1. A similarity relation Δ on X is said to be a diagonal on X iff $\Delta(x,y) = 1$ implies $x = y$.

Remark. A diagonal on X may be regarded as a fuzzy relation of equality.

Lemma 4.1. Let $\Pi: M \to M/S$ be a canonical mapping. Then $\Pi \circ \Pi^{-1}$ is a diagonal on M/S.

Proof. Obviously, $\Pi \circ \Pi^{-1}$ is a symmetric and reflexive fuzzy binary relation. Further, we have

$$(\Pi \circ \Pi^{-1})([x],[y]) = \bigvee_a \{\Pi(a,[x]) \wedge \Pi(a,[y])\}$$

$$= \bigvee_a \{S(a,x) \wedge S(a,y)\} = S(x,y),$$

which implies transitivity of $\Pi \circ \Pi^{-1}$. ¤

Lemma 4.2. Let $\Pi: M \to M/S$ be a canonical mapping. Then

$$\Pi = \Pi \circ \Pi^{-1} \circ \Pi_1,$$

where Π_1 is a crisp mapping defined by

$$\Pi_1(x,[y]) = \begin{cases} 1, & \text{if } [x]=[y], \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We have

$$(\Pi \circ \Pi^{-1} \circ \Pi_1)(x,[y]) = \bigvee_{a,z} \{\Pi(x,[a]) \wedge \Pi(a,[z]) \wedge \Pi(z,[y])\}$$

$$= \bigvee_{a} \{S(a,x) \wedge S(a,y)\} = S(x,y) = \Pi(x,[y]). \qquad ¤$$

A natural decomposition $F = i \circ \Pi$ of crisp surjective mappings, generally speaking, has no place in a fuzzy case even if $N = F(M)$ is demanded. Let us consider, for instance, a fuzzy mapping F from 3-element set M into 2-element set N defined by matrix

1	0.3
1	0.5
0.7	1

It is easy to verify that there are no fuzzy isomorphism i and similarity relation S such that $F = i \circ \Pi$.

The following definition suggests a proper generalization of (crisp) surjective mappings.

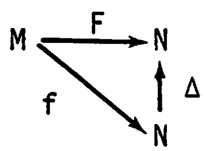
Definition 4.2. A fuzzy mapping $F: M \to N$ is said to be a fuzzy surjective mapping iff there is a similarity relation S on M such that a diagram

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ \Pi \downarrow & \nearrow i & \\ M/S & & \end{array}$$

is commutative, where Π is a (fuzzy) canoncial mapping defined by S and i is a crisp isomorphism.

Evidently, a problem of a proper generalization find the simplest solution in Definition 4.2: we employ a desirable property as a definition. Nevertheless, it is possible to characterize a class of fuzzy surjective mappings in the following way:

Theorem 4.1. A fuzzy mapping $F:M \to N$ is a fuzzy surjective mapping iff there is a crisp surjective mapping $f:M \to N$ and a fuzzy diagonal relation Δ on N such that $F = \Delta \circ f$:



Proof. 1) Let F be a fuzzy mapping from M onto N. Then $F = i \circ \Pi$, by definition. Let us define $f = i \circ \Pi_1$ (see Lemma 4.2) and $\Delta = F \circ F^{-1}$. Then

$$\Delta \circ f = i \circ \Pi \circ \Pi^{-1} \circ i^{-1} \circ i \circ \Pi_1 = i \circ \Pi \circ \Pi^{-1} \circ \Pi_1 = i \circ \Pi = F,$$

by Lemma 4.2. Further, we have

$$\Delta(u,v) = \bigvee_x \{F(x,u) \wedge F(x,v)\},$$

which implies symmetry and reflexivity of Δ. Moreover, $\Delta(u,v) = 1$ iff $u = v$, since F is a fuzzy mapping. Finally, we have

$$\begin{aligned} \Delta \circ \Delta &= i \circ \Pi \circ \Pi^{-1} \circ i^{-1} \circ i \circ \Pi \circ \Pi^{-1} \circ i^{-1} \\ &= i \circ \Pi \circ \Pi^{-1} \circ \Pi \circ \Pi^{-1} \circ i^{-1} = i \circ \Pi \circ \Pi^{-1} \circ i^{-1} = \Delta, \end{aligned}$$

since $\Pi \circ \Pi^{-1}$ is a diagonal relation, by Lemma 4.1. Hence, Δ is a fuzzy diagonal relation on N.

2) Conversely, let us have a decomposition $F = \Delta \circ f$. Let us define $S = F^{-1} \circ F$ and $i([a]) = f(a)$. Obviously, S is a reflexive and symmetrical fuzzy binary relation on M. We have

$$S \circ S = F^{-1} \circ F \circ F^{-1} \circ F = f^{-1} \circ \Delta^{-1} \circ \Delta \circ f \circ f^{-1} \circ \Delta^{-1} \circ \Delta \circ f$$
$$= f^{-1} \circ \Delta^{-1} \circ \Delta \circ f = S,$$

since f is a crisp mapping onto. Hence, S is a similarity relation. Evidently, i is a surjective mapping. Let us prove that it is an injective mapping too. We have $S = f^{-1} \circ \Delta \circ f$, since Δ is a diagonal relation. Hence,

$$S(x,y) = \Delta(f(x),f(y))$$

and we have

$$[x] = [y] \text{ iff } S(x,y) = 1 \text{ iff } \Delta(f(x),f(y)) = 1 \text{ iff } f(x) = f(y).$$

Therefore, i is an isomorphism. Finally, we have

$$(i \circ \Pi)(x,u) = \Pi(x,[f^{-1}(u)]) = S(x,f^{-1}(u))$$
$$= \Delta(f(x),u) = (\Delta \circ f)(x,u) = F(x,y).$$ ¤

It is easy to verify that

$$S = F^{-1} \circ F = f^{-1} \circ \Delta \circ f$$

if F is a fuzzy surjective mapping. This formula provides dual descriptions of similarity relations: as inverse images of a crisp diagonal under fuzzy mappings and as inverse images of fuzzy diagonals under crisp mappings.

5. FUZZY WEAK ORDERINGS

Concepts of a weak ordering and a linear ordering play basic role in the theory of ordinal scales. Remind the reader that a weak ordering R is a reflexive, complete and trasitive binary relation, i.e.,

1) xRx for all x,
2) xRy or yRx for all $x \neq y$,
3) xRy and yRz imply xRz for all x,y,z.

The following definition gives a proper extension of this notion on fuzzy set theory.

Definition 5.1. A fuzzy weak ordering is a fuzzy binary relation R fulfilling

1) $R(x,y) \vee R(y,x) = 1$ for all x,y, (completeness)

2) $R(x,y) \wedge R(y,z) \leq R(x,z)$ for all x,y,z (transitivity)

Remark. 1) implies $R(x,x) = 1$ for all x (reflexivity).

Crisp weak orderings have a very simple structure. Namely, $I = R^{-1} \cap R$ is an equivalence relation and R can be regarded as a linear ordering over classes of F. In this section we investigate analogous properties of fuzzy weak orderings.

Theorem 5.1. Let R be a fuzzy weak ordering. Then $S = R^{-1} \cap R$ is a similarlity relation.

Proof. Obviously, S is a reflexive and symmetric fuzzy binary relation. Further, we have

$$S(x,y) \wedge S(y,z) = R(x,y) \wedge R(y,x) \wedge R(y,z) \wedge R(z,y)$$
$$\leq R(x,z) \wedge R(z,x) = S(x,z),$$

by transitivity of R. ¤

Hence, we may employ notions of quotient-set and canonical mapping in a study of fuzzy weak orderings.

Let us define a crisp relation $\geq$ on M by

$$x \geq y \quad \text{iff} \quad R(x,y) = 1. \tag{5.1}$$

Obviously, $\geq$ is a 1-level set of R and, therefore, a weak ordering on M.

Lemma 5.1. Let $x \geq y$ and $[x'] = [x]$, $[y'] = [y]$. Then $x' \geq y'$.

Proof. We have $S(x',x) = S(y',y) = 1$ which implies

$$R(x',x) = R(y,y') = 1.$$

Further,

$$R(x',y') \geq R(x',x) \wedge R(x,y') = R(x,y') \geq R(x,y) \wedge R(y,y') = 1,$$

since $R(x,y) = 1$. ¤

The lemma proven provides correctness of the following definition

$$[x] \geq [y] \quad \text{iff} \quad x \geq y. \tag{5.2}$$

Lemma 5.2. $\geq$ defined by (5.2) is a linear ordering on M/S.

Proof. It suffices to prove that $[x] \geq [y]$ and $[y] \geq [x]$ imply $[x] = [y]$. We have $R(x,y) = R(y,x) = 1$, or $S(x,y) = 1$, which impliy $[x] = [y]$. ¤

An important property of a similarity relation associated with a fuzzy weak ordering is established in the following

Lemma 5.3. $x \geq y \geq z$ implies $S(x,z) = S(x,y) \wedge S(y,z)$.

Proof. We have $R(x,y) = R(y,z) = R(x,z) = 1$. Hence, $S(x,y) = R(y,x)$, $S(y,z) = R(z,y)$ and $S(x,z) = R(z,x)$. By transitivity of R

$$R(y,x) \geq R(y,z) \wedge R(z,x), \text{ or } S(x,y) \geq S(x,z)$$

and

$$R(z,y) \geq R(y,x) \wedge R(x,y), \text{ or } S(y,z) \geq S(x,z).$$

Hence, $S(x,y) \wedge S(y,z) \geq S(x,z)$ which imply $S(x,z) = S(x,y) \wedge S(y,z)$, by transitivity of S. ¤

If I is an equivalence relation, then any linear ordering over its classes generates a weak ordering R such that $I = R^{-1} \cap R$. The previous lemma shows that it is not, generally speaking, true in a fuzzy case.

Nevertheless, as the following theorem shows, for any similarity relation S there is a fuzzy weak ordering R such that $S = R^{-1} \cap R$.

Theorem 5.2. For any similarity relation S there is a fuzzy weak ordering R such that $S = R^{-1} \cap R$.

Proof. Let us consider a resolution (see [5])

$$S = \alpha_1 \cdot S_{\alpha_1} + \alpha_2 S_{\alpha_2} + \ldots + \alpha_n S_{\alpha_n},$$

where $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n = 1$, $S_{\alpha_1} \supset S_{\alpha_2} \supset \ldots \supset S_{\alpha_n}$ and all S_{α_i} are equivalence relations. Then there is a nested sequence $R_{\alpha_1} \supset R_{\alpha_2} \supset \ldots \supset R_{\alpha_n}$ of weak orderings such that $S_{\alpha_i} = R^{-1}_{\alpha_i} \cap R_{\alpha_i}$ for all i. A fuzzy binary relation R defined by

$$R = \alpha_1 R_{\alpha_1} + \alpha_2 R_{\alpha_2} + \ldots + \alpha_n R_{\alpha_n}$$

is a fuzzy weak ordering such that $S = R^{-1} \cap R$. ¤

Remark. Note that R_{α_n} is, actually, an ordering defined by (5.1) which induces a linear ordering over classes of S.

Now we will study relationships between fuzzy weak orderings and canonical mappings.

Theorem 5.3. Let R be a fuzzy weak ordering on M and $S = R^{-1} \cap R$. Then R is an inverse image of a linear ordering $\geq$ on M/S with respect to a canonical mapping $\Pi : M \to M/S$.

Proof. We need to prove that

$$R(x,y) = \bigvee_{[z] \geq [t]} \{\Pi(x,[z]) \wedge \Pi(y,[t])\} = \bigvee_{z \geq t} \{S(x,z) \wedge S(y,t)\} \quad (5.3)$$

If $x \geq y$, then both sides of (5.3) are equal to 1. Let $x < y$. Then

$R(x,y) = S(x,y)$. It is obvious that right side of (5.3) is greater than $S(x,y)$. Hence, it suffices to prove that

$$S(x,z) \wedge S(y,t) \leq S(x,y) \quad \text{for all} \quad z \geq t. \tag{5.4}$$

Let $z \geq y > x$. Then $S(x,z) \leq S(x,y)$, by Lemma 5.3, which implies (5.4). Let $y > x \geq t$. Then $S(y,t) \leq S(x,y)$, by Lemma 5.3, which implies (5.4). Let now $y > z \geq t > x$. Then, by Lemma 5.3,

$$\begin{aligned} S(x,z) \wedge S(y,t) &= S(x,t) \wedge S(t,z) \wedge S(y,t) \\ &= S(x,y) \wedge S(z,t) \leq S(x,y), \end{aligned}$$

which implies (5.4). ⌑

It follows from Theorem 5.3 that fuzzy weak orderings are inverse images of crisp linear orderings induced on quotient-sets. They can also be described as quasi-inverse images. We define $u = f(x)$ if $F(x,u) = 1$.

<u>Definition 5.2</u>. Let $F: M \to N$ be a fuzzy mapping, and $\geq$ be a crisp linear ordering on N. A fuzzy binary relation R on M with membership function

$$R(x,y) = \begin{cases} 1, \text{ if } f(x) \geq f(xy) \\ \\ F(x,f(y)), \text{ otherwise} \end{cases}$$

is said to be a <u>quasi-inverse image</u> of $\geq$ with respect to F.

It is easy to verify that any fuzzy weak ordering R on M is a quasi-inverse image of $\geq$ defined by (5.2) with respect to Π defined by $S = R^{-1} \cap R$.

Generally speaking, it is false that an inverse image or a quasi-inverse image of a crisp linear ordering on N with respect to a fuzzy surjective mapping F is a fuzzy weak ordering. For we have the following

Theorem 5.4. Let $F = \Delta \circ f$ be a fuzzy mapping from M onto N and $\geq$ be a crisp linear ordering on N. A quasi-inverse image of $\geq$ with respect to F is a fuzzy weak ordering iff

$$\Delta(u,v) = \Delta(u,w) \wedge \Delta(w,v) \quad \text{for any} \quad u \geq w \geq v \tag{5.5}$$

Proof. We have

$$R(x,y) = \begin{cases} 1, \text{ if } f(x) \geq f(y) \\ (f(x),f(y)), \text{ if } f(x) \leq f(y) \end{cases}$$

by the definition of a quasi-inverse image.

1) Let R be a fuzzy weak ordering. Then $S(x,y) = \Delta(f(x),f(y))$. Let $u \geq w \geq v$ and $u = f(x)$, $w = f(y)$, $v = f(z)$. We have

$$R(x,y) = R(y,z) = R(x,z) = 1,$$

which imply $S(x,z) = S(x,y) \wedge S(y,z)$, by Lemma 5.3. It means that $\Delta(u,v) = \Delta(u,w) \wedge \Delta(w,v)$.

2) Let (5.5) holds. Obviously, R is a complete fuzzy binary relation. Let us prove transitivity, i.e.,

$$R(x,y) \wedge R(y,z) \leq R(x,z). \tag{5.6}$$

It is true if $f(x) \geq f(z)$. Let $f(x) < f(z)$. If $f(y) \geq f(z) > f(x)$, then $\Delta(f(y),f(x)) \leq \Delta(f(z),f(x))$, by (5.5), which implies (5.6). Let $f(z) \geq f(y) \geq f(x)$. Then

$$\Delta(f(z),f(y)) \wedge \Delta(f(y),f(x)) \leq \Delta(f(x),f(z)),$$

by (5.5), which implies (5.6). If $f(z) > f(x) \geq f(y)$, then $\Delta(f(z)f(y)) \leq \Delta(f(x),f(z))$, by (5.5), which implies (5.6). ¤

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