DYNAMIC SECURITY REGIONS OF POWER SYSTEMS

by

R. John Kaye and Felix F. Wu

Memorandum No. UCB/ERL M81/85

16 November 1981

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

DYNAMIC SECURITY REGIONS OF POWER SYSTEMS

R. John Kaye and Felix F. Wu

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

ABSTRACT

A method for deriving dynamic security regions of power systems is developed. A power system operating state is defined to be dynamically secure with respect to a given disturbance if the system, starting in that state maintains transient stability after experiencing the disturbance. Specifically, these are regions of pre-fault angles such that the post-fault system is asymptotically stable. The proposed approach is to construct affine approximations to the nonlinearities in the transient stability model and then derive quadratic bounds on the errors between the nonlinearities and their approximation. These are then used to derive sufficient conditions for a polytope of operating states to be dynamically secure.

1. INTRODUCTION

Security of a power system refers to its robustness relative to a set of imminent disturbances during operation [1]. In [2], a framework for probabilistic dynamic security assessment was established using both steady state and dynamic security regions. The problem of finding steady state security regions, being sets of power system operating points which satisfy the load flow and associated constraints, is dealt with in [3]. In this paper, a method of finding dynamic security regions is derived.

A power system operating state is defined to be dynamically secure with respect to a given fault or disturbance if the system, starting in that state and then undergoing that disturbance, is transiently stable. A set of such states is called a dynamic security region. Loss of transient stability corresponds to the system failing to maintain sychronism in operation which is, of course, a severe breach of security.

The concept of power system security was introduced by Dy Liacco [4]. He also established a framework for deterministic security assessment, in which the robustness of the system is tested with respect to a set of selected disturbances or contingencies. For each of these contingencies, a digital simulation is performed to obtain the system response [5,6,7]. This "scenario-study" approach is also traditionally used in transmission system planning [8]. The salient feature of this type of approach is that, because each scenario defines system configurations and a trajectory, it is possible to perform the analysis using load flow and transient stability programmes. One major difficulty with this approach to dynamic security assessment is the on-line computational burden. In our framework for probabilistic dynamic security assessment [2], the use of regions of security operation in the state space of present operations is advocated. It is intended that these regions be computed off-line. Other proposed techniques for dynamic security assessment include the use pattern recognition [9], Liapunov functions [9,10], transient energy functions [11], transient energy margins [12] and transient security indices [13,14].

In transient stability analysis, a power system is generally considered as undergoing two changes in configuration: from pre-fault to fault-on and thence to post-fault. In the pre-fault configuration, the power system is in a steady state condition, θ. The fault occurs and the system is then in the fault-on condition for a fixed time period, during which the state of the system changes dynamically. The fault is then cleared by protective relay systems operation, moving the system to its post-fault configuration. The state of the system then changes according to different dynamics, the initial condition of which is the value of the fault-on state at the instance of fault clearing. If this post-fault system is asymptotically stable to a post-fault equilibrium operating point, θ_p, then the system is transiently stable. The fault-on dynamics can be considered as a map D from the pre-fault operating condition, θ, to the dynamic state value at the instance of fault clearing. The post fault equilibrium, θ_p, depends on the pre-fault operating point, θ, via assumptions about the way in which the steady state value of the power injections change with configuration (e.g. they remain constant). A map $\tilde{\theta}_p$ can be defined so that $\tilde{\theta}_p(\theta) = \theta_p$. We assume that the post-fault dynamics have been analysed using either Liapunov or transient energy techniques to yield a region of attraction to θ_p. In other words, if $D(\theta)$ is in this region, then the post-fault system is asymptotically stable to θ_p. Thus, we shall define a region L such that if $(\theta_p, D(\theta))$

is in L, the post-fault system is asymptotically stable. In fact, if θ is a pre-fault steady state such that $(\tilde{\theta}_p(\theta), D(\theta))$ lies in L then θ is dynamically secure. The problem of finding a dynamic security region is then to find a subset of the inverse image of L under the map defined by $(\tilde{\theta}_p(\theta), D(\theta))$.

Evaluating D at a particular θ involves solving a set of nonlinear differential equations representing the generator swing equations and the power flow equations of the fault-on transmission network. In the classical "scenario study" approach, this is achieved using numerical integration [15]. Various approximations for D have been proposed [10,11], with the aim of reducing computation time. Analytic studies of these approximations have not been reported. Evaluating $\tilde{\theta}_p(\theta)$, on the other hand, requires solving a power flow, which is a set of nonlinear algebraic equations [16]. The DC load flow [17] is a commonly used approximation to the power flow.

Our approach in this paper is to construct affine approximations to D and $\tilde{\theta}_p$, using the derivative values obtained from one numerical simulation of the model. Quadratic bounds on the errors between the nonlinear maps and their approximations are then found. The affine nature of the approximations, and the quadratic form of the bounds are used to derive sufficient conditions for a polytope of pre-fault operating states to be dynamically secure.

The paper is organized as follows. In section 2, details of the model are given. This model is a generalization of a large variety of transient stability models to which Liapunov and transient energy techniques have been applied. Further details of the approximation and bound philosophy are given in section 3 as well as the fundamental circuit theoretic lemma upon which most of the analysis is based. In sections 4 and 5, the affine approximations and bounds for D and θ_p, respectively, are derived. The sufficient conditions for a polytope to be a dynamic security region are presented in section 6. In section 7, we apply the preceding analysis to a slightly different transient event. Here, we are considering the case of instantaneous changes in power injection or of a single switching action on a line.

The notation used in this paper is standard. $\mathbb{R}_+$ is the set $\{z \in \mathbb{R} : z > 0\}$. If $x, y \in \mathbb{R}^n$, then the vector inequality $x \leqq y$ implies the partial ordering relation $x_k \leq y_k$ for all $k = 1,\ldots,n$, where x_k refers to the k^{th} component of x. Similarly, if $A \in \mathbb{R}^{m\times n}$, then $[A]_{ij}$ is the i,j^{th} entry of A. For $x \in \mathbb{R}^n$, $\|x\|$ denotes any norm on $\mathbb{R}^n$, while $\|x\|_1$ and $\|x\|_\infty$ denote $\sum_{k=1}^{n} |x_k|$ and $\max\{|x_k| : k = 1, \cdots, n\}$ respectively. $\bar{B}(x,\lambda)$ is the set $\{z \in \mathbb{R}^n : \|z-x\| \leq \lambda\}$ and $\bar{B}_\infty(x,\lambda)$ is $\{z \in \mathbb{R}^n : \|z-x\|_\infty \leq \lambda\}$. The statement "$x := E$" implies that x is defined by the expression E.

2. TRANSIENT STABILITY MODEL

2.1. General Features

The sequence of events considered for transient stability analysis of power systems are the following. The power system is in steady state prior to a balanced three phase fault occurring on the transmission network. The fault is then cleared by the protective relay system. The model for transient stability analysis therefore has three components: pre-fault, fault-on and post-fault. The pre-fault system is in steady state and is represented by power flow equations. The fault-on system is represented by a differential equation corresponding to the generator swing equations and the power flow equations of the transmission network with the faulted line [15]. The post-fault system is again a dynamic system. The steady state condition of the post-fault system is represented by power flow equations. The stability of this system is represented by a Liapunov region [18] or by a region established using transient energy functions [11,19].

These three components are coupled together in the following fashion. Suppose a pre-fault equilibrium angle is specified. Then the vector of steady state power injections is determined by the pre-fault load flow and is assumed to remain constant throughout the fault-on and post-fault periods. Thus

(i) The fault-on dynamics contain a term in the pre-fault angle, representing the dependency of the swing equations on the injections.

(ii) The initial condition of the fault-on dynamics is a linear function of the pre-fault angle.

(iii) The post-fault network is asymptotically stable if the value of the fault-on state at the time of fault clearing is in the Liapunov stable region.

(iv) The post-fault equilibrium angle depends on the pre-fault equilibrium angle via the constancy of steady state power.

(v) Finally, the Liapunov stable region depends on the post-fault equilibrium angle [18].

The model is given in detail below. It can easily be seen that it is a generalization of the models in [11,18,19] and thus represents a wide range of transient stability formulations to which Liapunov or transient energy techniques have been applied.

2.2 Prefault System

We assume the network has n busses, excluding the slack bus. The steady state power injections, $p \in \mathbb{R}^n$, and the pre-fault equilibrium angle, $\theta \in \mathbb{R}^n$, are related by the decoupled real power flow equations [11,18,19,20].

$$f_o(\theta) = p \tag{1}$$

where $f_o : \mathbb{R}^n \to \mathbb{R}^n$ is the load flow function and is assumed to have the form

$$f_o(\phi) = A_o g_o(A_o^T \phi) \tag{2}$$

Hence, $A_o \in \mathbb{R}^{n \times \ell_o}$ and $g_o : \mathbb{R}^{\ell_o} \to \mathbb{R}^{\ell_o}$ where

$$g_o(\sigma) = (g_{o1}(\sigma_1), g_{o2}(\sigma_2), \ldots, g_{o\ell_o}(\sigma_{\ell_o})) \tag{3}$$

Further, g_o is assumed to be twice continuously differentiable. The pre-fault angle, θ, is said to be a stable equilibrium if the Jacobian, $f_o'(\theta)$, is positive definite. A justification of this definition is given in [18].

Remarks. 1. Although it is not assumed for the work in this paper, A_o is generally a reduced node incidence matrix [21,p. 417]. In this case, ℓ_o is the number of branches in the network and the kth component of g_o, written $g_{ok} : \mathbb{R} \to \mathbb{R}$, gives the power flow in the kth branch as a function of the angle difference across that branch. The decoupled nature of g_o as seen in equation 3, is equivalent to assumming no two branches of the network are coupled. It has been shown [18,22] that equation 1 can then be interpreted as the node equations of a nonlinear resistive network. We will have occasion to refer to this analogy.

2. Under the assumption of lossless transmission lines, $g_{ok}(\sigma_k) = Y_k \sin(\sigma_k)$ where Y_k is a positive real constant [8,ch 7]. ◻

2.3 Fault-on System

The fault occurs at time t = 0 and is cleared at time $t = t_F$. Suppose that the pre-fault equilibrium angle is θ. Then it is assumed that the fault-on system at time $t \in [0, t_F]$ can be represented by a state vector $x(t) \in \mathbb{R}^m$ given by the solution of

$$\dot{x}(t) = E\, x(t) + f_d(x(t)) + Ff_0(\theta) \qquad x(0) = G\theta \tag{4}$$

where $E \in \mathbb{R}^{m\times m}$, $F \in \mathbb{R}^{m\times n}$, $G \in \mathbb{R}^{m\times n}$ and $f_d : \mathbb{R}^m \to \mathbb{R}^m$ is Lipschitz continuous and is given by

$$f_d(\hat{x}) = B_d g_d(A_d^T \hat{x}) \tag{5}$$

Hence, B_d and $A_d \in \mathbb{R}^{m\times \ell_d}$ and $g_d : \mathbb{R}^{\ell_d} \to \mathbb{R}^{\ell_d}$ such that

$$g_d(\sigma) = (g_{d1}(\sigma_1), g_{d2}(\sigma_2), \ldots, g_{d\ell_d}(\sigma_{\ell_d})) \tag{6}$$

g_d is also assumed to be twice continuously differentiable.

Examining equation 4, it can be seen that the state at time $t = t_F$, $x(t_F)$, is uniquely determined by θ. Thus, a map $D : \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$D(\theta) := x(t_F) \tag{7}$$

That is, D maps a pre-fault equilibrium angle into the corresponding fault-on dynamic state at the instance of fault clearing.

Remarks: 1. Equation 4 can be seen to be a generalization of the transient stability models used in [11,18 and 19]. Specifically for the case of the model of Bergen and Hill [18], the matrix A_d is composed of the reduced node incidence matrix of the faulted network and some zero blocks to remove the frequency terms in the state vector. Thus the term

$g_d(A_d^T x(t))$ is the vector of real power flows in the branches at time t.
$f_o(\theta)$ is the steady state value of the injected powers.

2. The generality of the model presented in this section thus allows for loads which vary dynamically in time [eg. 18]. However we shall still require that the steady state values of the power injections remain constant throughout the event. ⌑

2.4 Post-Fault System

2.4.1 Equilibrium: It is assumed that the post-fault network also has n busses, excluding the slack bus. In a fashion similar to the pre-fault case, if $\theta_p \in \mathbb{R}^n$ is the post-fault equilibrium angle and $p \in \mathbb{R}^n$ the steady state injections then

$$f_p(\theta_p) = p \tag{8}$$

where $f_p : \mathbb{R}^n \to \mathbb{R}^n$ is the post-fault load flow function. It is assumed to have the form

$$f_p(\phi) = A_p g_p(A_p^T \phi) \tag{9}$$

where $A_p \in \mathbb{R}^{n \times \ell_p}$ and $g_p : \mathbb{R}^{\ell_p} \to \mathbb{R}^{\ell_p}$ is twice continuously differentiable with the form

$$g_p(\sigma) = (g_{p1}(\sigma_1), g_{p2}(\sigma_2), \ldots, g_{p\ell_p}(\sigma_{\ell_p})) \tag{10}$$

The post-fault equilibrium angle, θ_p, is said to be stable if the Jacobian, $f_p'(\theta_p)$, is positive definite.

2.4.2 Stability Set: We represent the post-fault dynamics by defining a stability set $L \subset \mathbb{R}^n \times \mathbb{R}^m$ by the following property. Suppose $\theta_p \in \mathbb{R}^n$ is a stable post-fault equilibrium and $\hat{x} \in \mathbb{R}^m$ is the initial condition of the post-fault dynamical system (i.e. the post-fault dynamical state at time $t = t_F$). If

$$(\theta_p, \hat{x}) \in L \tag{11}$$

then the post-fault system is asymptotically stable.

Remarks: 1. The fault-on dynamic state at time $t = t_F$, $x(t_F)$, is, in general, the initial condition for the post-fault dynamical system, $\hat{x}$.

2. It is more usual, when refering to Liapunov or transient energy stability techniques for power systems, to calculate a set $V(p) \subset \mathbb{R}^m$ with the property that if $p \in \mathbb{R}^n$ is the steady state post-fault injections and $\hat{x} \in \mathbb{R}^m$ the post-fault initial dynamic state and if

$$\hat{x} \in V(p) \tag{12}$$

then the post-fault system is asymptotically stable. That is, $V(p)$ is a domain of attraction [23, p.8] of the post-fault dynamical system to a stable equilibrium when the injection is p. The stable equilibrium in question will be the dynamic state corresponding to θ_p where θ_p is a solution of the following nonlinear problem.

$$f_p(\theta_p) = p \text{ and } f_p'(\theta_p) \text{ is positive definite} \tag{13}$$

That there might be more than one such θ_p and that $V(p)$ might consequently be disconnected is immaterial to this discussion.

A region L can then be found using the following procedure. A set $I \subset \mathbb{R}^n$ is selected such that for each $\theta_p \in I$ the Jacobian $f_p'(\theta_p)$ is positive definite. Then L can be the union of the cartesian products of θ_p and $V(p)$ where $p = f_p(\theta_p)$. That is,

$$L = \bigcup_{\theta_p \in I} \{\theta_p\} \times V(f_p(\theta_p)) \tag{14}$$

In this fashion the state space of the problem is augmented to obtain a single stability set. ¤

2.5 Problem Statement

We seek a set of dynamically secure pre-fault equilibrium angles. A stable pre-fault equilibrium angle, $\theta \in \mathbb{R}^n$, is said to be dynamically secure if the resulting dynamic state at fault clearing, $D(\theta) \in \mathbb{R}^m$, is an asymptotically stable initial condition for the post-fault network. Since the steady state power injection is assumed to remain constant in time, it is sufficient that

$$D(\theta) \in V(f_o(\theta)) \tag{15}$$

Thus, we define a dynamic security region in the pre-fault state space to be

$$\begin{aligned} \bar{\Omega}_d \triangleq \{\theta \in \mathbb{R}^n : \; & f_o'(\theta) \text{ is positive definite} \\ & \text{and } \exists \theta_p \in \mathbb{R}^n \text{ such that} \\ \text{(i)} \quad & f_p(\theta_p) = f_o(\theta) \\ \text{(ii)} \quad & f_p'(\theta_p) \text{ is positive definite} \\ \text{(iii)} \quad & (\theta_p, D(\theta)) \in L\} \end{aligned} \tag{16}$$

In [2], a dynamic security set, Ω_d, was defined in the space of injections. This is the image of $\bar{\Omega}_d$ under the map, f_o, (i.e. $f_o(\bar{\Omega}_d)$) and is thus a set of injectious such that for each $p \in \Omega_d$, there exists a dynamically secure pre-fault stable equilibrium, $\theta \in \mathbb{R}^n$ such that p is the associated injection (i.e. $p = f_o(\theta)$.)

In equation 16, the constraint "$f_o(\theta) = f_p(\theta_p)$" expresses the assumption of constant steady state power. It will be shown below (section 5) that it locally defines an implicit function $\tilde{\theta}_p : \tilde{U}_o \to \mathbb{R}^n$, where $\tilde{U}_o \subset \mathbb{R}^n$. That is, $\forall \theta \in \tilde{U}_o$

$$f_p(\tilde{\theta}_p(\theta)) = f_o(\theta) \tag{17}$$

so that $\tilde{\theta}_p$ maps a pre-fault equilibrium into a post-fault equilibrium with the same injection.

A set of dynamically secure states would be $(\tilde{\theta}_p x D)^{-1}(L)$, the inverse image of L under the map $\tilde{\theta}_p \times D : \tilde{U}_o \to \mathbb{R}^n \times \mathbb{R}^m$ where

$$(\tilde{\theta}_p x D)(\theta) = (\tilde{\theta}_p(\theta), D(\theta)) \tag{18}$$

Both $\tilde{\theta}_p$ and D are nonlinear and no closed form expression for the inverse operator is known. Evaluating $\tilde{\theta}_p$ and D at a particular point θ involves solving a nonlinear algebraic equation and a nonlinear differential equation, respectively. In this paper, we develop a viable method for finding a subset of the dynamic security region, $(\tilde{\theta}_p x D)^{-1}(L)$ in the space of pre-fault equilibrium angles.

3. EXACT TAYLOR SERIES ANALYSIS

3.1 Solution Approach

The method developed in this paper for finding a dynamic security region in the prefault angle space is based on expanding the nonlinear maps D and $\tilde{\theta}_p$ in the above model about an operating point θ to exact three term Taylor series. The first two terms correspond to the linearization of the maps D and $\tilde{\theta}_p$. We then obtain bounds on the third terms and use them to derive quadratic bounds on the difference between D and its linearization, and between $\tilde{\theta}_p$ and its linearization. In the final part of our analysis we derive a condition under which a polytope in the pre-fault equilibrium angle space contains only dynamically secure angles.

That is, the image of this polytope under $\tilde{\theta}_p \times D$ lies in L. Verification of this condition requires only evaluation of the linearized maps and the quadratic bounds. Only one solution of the full nonlinear model, to obtain the derivative values and the operation point, is required for verification of the security of an entire polytope.

3.2 Linearization

The process of Taylor series expansion of the model is most easily seen in the treatment of the map D. Suppose that $\theta \in \mathbb{R}^n$ is a stable pre-fault equilibrium and $x(\cdot)$ is the corresponding fault-on trajectory i.e.

$$\dot{x} = Ex + f_d(x) + Ff_o(\theta) \qquad x(0) = G\theta \tag{19}$$

Suppose, in addition, that for $\gamma \in \mathbb{R}^n$, $x_\gamma(\cdot)$ is the fault-on trajectory when the pre-fault angle is $\theta + \gamma$, so that

$$\dot{x}_\gamma = Ex_\gamma + f_d(x_\gamma) + Ff_o(\theta+\gamma) \qquad x_\gamma(0) = G(\theta+\gamma) \tag{20}$$

Thus

$$D(\theta) = x(t_F) \quad \text{and} \quad D(\theta+\gamma) = x_\gamma(t_F) \tag{21}$$

Defining

$$\varepsilon_\gamma(t) := x_\gamma(t) - x(t) \tag{22}$$

from equations 19 and 20 we obtain

$$\dot{\varepsilon}_\gamma = E\varepsilon_\gamma + f_d(x+\varepsilon_\gamma) - f_d(x) + F[f_o(\theta+\gamma) - f_o(\theta)]$$

$$\varepsilon_\gamma(0) = G\gamma \tag{23}$$

This can be re-written

$$\dot{\varepsilon}_\gamma = [E + f_d'(x)]\varepsilon_\gamma + Ff_o'(\theta)\gamma + r_d(t,\gamma) + r_o(\gamma) \qquad \varepsilon_\gamma(0) = G\gamma \tag{24}$$

where

$$r_d(t,\gamma) := f_d(x(t)+\varepsilon_\gamma(t)) - f_d(x(t)) - f_d'(x(t))\varepsilon_\gamma(t) \tag{25}$$

and

$$r_0(\gamma) := F[f_0(\theta+\gamma) - f_0(\theta) - f_0'(\theta)\gamma] \tag{26}$$

The terms $r_d(t,\gamma)$ and $r_0(\gamma)$, which are both functions of γ and θ, are the third terms in a three term Taylor series expansion of $f_d(x+\varepsilon_\gamma)$ and $Ff_0(\theta+\gamma)$, respectively. They are thus given by [24, p.190]

$$r_d(t,\gamma) = \int_0^1 (1-\lambda)f_d''(x(t)+\lambda\varepsilon_\gamma(t))\cdot(\varepsilon_\gamma(t),\varepsilon_\gamma(t))d\lambda \tag{27}$$

$$r_0(\gamma) = F\int_0^1 (1-\lambda)f_0''(\theta+\lambda\gamma)\cdot(\gamma,\gamma)d\lambda \tag{28}$$

where the notation $f''(\alpha)\cdot(\beta,\beta)$ implies the second derivative of f at α, evaluated as a bilinear operator at (β,β) [24, p.179]. If $r_d(t,\gamma)$ and $r_0(\gamma)$ are small, then the linear part equation 24 forms an approximation. That is, $\hat{\varepsilon}_\gamma(t) \approx \varepsilon_\gamma(t)$ where

$$\dot{\hat{\varepsilon}}_\gamma = [E + f_d'(x)]\hat{\varepsilon}_\gamma + Ff_0'(\theta)\gamma \qquad \hat{\varepsilon}_\gamma(0) = G\gamma \tag{29}$$

Thus $\hat{\varepsilon}_\gamma(t)$ is a linear function of γ. In the language of small signal analysis of nonlinear circuits [21, pp.91-95], $\hat{\varepsilon}_\gamma(t)$ is the linearized output perturbation when the input is γ away from its quiescent value, θ. (The input, in this case, is the initial condition).

Our approach is to bound $r_d(t,\gamma)$ and $r_0(\gamma)$ using their specific form as derived from equations 2, 3, 5 and 6 and then to use the Bellman-Gronwell Lemma [25, p.38] to thus obtain an upper bound on $\|\hat{\varepsilon}_\gamma(t_F) - \varepsilon_\gamma(t_F)\|$. Since $D(\theta) + \hat{\varepsilon}_\gamma(t_F)$ is affine in γ and

$$\|D(\theta+\gamma) - [D(\theta)+\varepsilon_\gamma(t_F)]\| = \|\hat{\varepsilon}_\gamma(t_F) - \varepsilon_\gamma(t_F)\| \tag{30}$$

we thus have an affine approximation and error bound. A similar but more sophisticated approach is used to deal with the variation of the post-fault equilibrium angle with the pre-fault equilibrium angle.

3.3 Bounding The Third Terms

$r_d(t,\gamma)$ and $r_0(\gamma)$ depend on $\varepsilon_\gamma(t)$ and γ respectively in a very complicated fashion. However, if, in equations 27 and 28, the points of evaluation, $x(t) + \lambda\varepsilon_\gamma(t)$ and $\theta + \lambda\gamma$, are ignored, these expressions become quadratic in $\varepsilon_\gamma(t)$ and γ. Thus, it makes sense to seek quadratic bounds.

In order to consider both these terms simultaneously, as well as a similar term in f_p, a function $f : \mathbb{R}^{n_i} \to \mathbb{R}^{n_0}$ is defined by

$$f(z) = Bg(A^Tz) \tag{31}$$

where $A \in \mathbb{R}^{n_i \times \ell}$, $B \in \mathbb{R}^{n_0 \times \ell}$ and $g : \mathbb{R}^\ell \to \mathbb{R}^\ell$ is given by

$$g(\sigma) = (g_1(\sigma_1), g_2(\sigma_2), \ldots, g_\ell(\sigma_\ell)) \tag{32}$$

g is assumed to be twice continuously differentiable. It can be seen that, by appropriate choice of n_i, n_0, ℓ, A, B and g that f generalizes any matrix multiplied by f_0, f_d of f_p. The following lemma gives an upper bound on

$$\|f(z+\rho) - f(z) - f'(z)\rho\| = \|\int_0^1 (1-\lambda)f''(z+\lambda\rho)\cdot(\rho,\rho)d\lambda\| \tag{33}$$

Lemma 1:

Let $\xi \in \mathbb{R}_+^\ell$ and define

$$S(\xi) := \{\rho \in \mathbb{R}^{n_i} : -\xi \leqq A^T\rho \leqq \xi\} \tag{34}$$

Let $z \in \mathbb{R}^{n_i}$, $\sigma = A^Tz$ and for each $i \in \{1,\ldots,\ell\}$ let $a_i \in \mathbb{R}_+$ such that

$$a_i \geq \max\{|g_i''(\sigma_i+\delta_i)| : -\xi_i \leq \delta_i \leq \xi_i\}$$

For each $j \in \{1,\ldots,n_o\}$, let Q_j be the diagonal $\ell \times \ell$ matrix with i,i^{th} entry

$$\frac{1}{2}\,|[B]_{ji}|\,a_i$$

Under these conditions, $\forall \rho \in S(\xi)$

(1) $|$jth component of $f(z+\rho) - f(z) - f'(z)\rho| \leq \rho^T A Q_j A^T \rho$ (35)

(2) If $k_n \in \mathbb{R}_+$ such that

$$\|w\| \leq k_n \|w\|_1 \qquad \forall w \in \mathbb{R}^{n_o} \tag{36}$$

then

$$\|f(z+\rho) - f(z) - f'(z)\rho\| \leq \rho^T A\tilde{Q}A^T \rho \tag{37}$$

where $\tilde{Q} = k_n \sum_{j=1}^{n_o} Q_j \in \mathbb{R}^{n_i \times n_i}$.

(3) $A\tilde{Q}A^T$ and AQ_jA^T are positive semidefinite symmetric matrices. ¤

Remarks: 1. The bounding technique in Lemma 1 can be interpreted as follows, in the case where $B = A$ is the reduced node incidence matrix and f is the node admittance function of a nonlinear resistive circuit. By bounding the voltage perturbations in each branch (equation 34), limits can be found for the branch current swings and hence for the node current deviation (equation 35).

2. If f is a load flow function such as f_o or f_p, then $S(\xi)$ can be considered as a set of constraints on line angle deviations. These naturally arise from line power flow constraints in the study of steady state security [3].

3. $\tilde{Q}$ and Q_j depend on z and ξ, but are independent of ρ. In the

case where g_i is a constant multiplying a sinusoid, the calculation of a_i is trivial.

4. Values of k_n for different norms on $\mathbb{R}^{n_0}$ are available in [26, p.170]. ¤

Proof

Consider $\rho \in S(\xi)$ and let $\delta = A^T\rho$. Thus

$$-\xi \leqq \delta \leqq \xi$$

and $f(z+\rho) - f(z) - f'(z)\rho$

$$= B[g(\sigma+\delta) - g(\sigma) - g'(\sigma)\delta]$$

$$= B\ \mathrm{vec}[g_i(\sigma_i+\delta_i) - g_i(\sigma_i) - g_i'(\sigma_i)\delta_i]$$

$$= B\ \mathrm{vec}[\delta_i^2 \int_0^1 (1-\lambda_i)g_i''(\sigma_i+\lambda_i\delta_i)d\lambda_i] \tag{38}$$

where the terminology $\mathrm{vec}(x_i)$ means x, a vector in $\mathbb{R}^\ell$ with ith component x_i. It follows that

$$|j^{th} \text{ component of } f(z+\rho) - f(z) - f'(z)\rho|$$

$$\leq \sum_{i=1}^{\ell} |[B]_{ji}| \int_0^1 (1-\lambda_i)|g_i''(\sigma_i+\lambda_i\delta_i)|d\lambda_i\ \delta_i^2$$

$$\leq \sum_{i=1}^{\ell} |[B]_{ji}| \cdot \frac{1}{2} a_i \cdot \delta_i^2$$

$$= \delta^T Q_j \delta = \rho^T A Q_j A^T \rho$$

which proves part (1) of the lemma. Part (2) follows immediately and part (3) is true because $\tilde{Q}$ and Q_j are diagonal with non-negative entries.

q.e.d.

4. THE FAULT-ON DYNAMICS MAP

The analysis of the fault-on dynamics is herein completed. Our method does not require that a solution to the exact fault-on dynamics (equation 19) be found. Instead, an approximate quiescent trajectory can be found by replacing the nonlinear part of the dynamics, f_d, with a simpler term, $\hat{f}_d$. This latter might be, for example, piecewise linear [27] or any other function which approximates f_d but renders the differential equation easier to solve.

In section 3 (equations 19 and 20), $x(\cdot)$ and $x_\gamma(\cdot)$ were defined to be the trajectories of the fault-on dynamics corresponding to pre-fault equilibria θ and $\theta + \gamma$, respectively. Now suppose $\hat{f}_d : \mathbb{R}^m \to \mathbb{R}^m$ is Lipschitz continuous and that there exists $e_\infty \in \mathbb{R}_+$ such that

$$\sup\{\|f_d(z) - \hat{f}_d(z)\| : z \in \mathbb{R}^m\} \leq e_\infty \tag{39}$$

Thus, equation 19 remains unsolved and instead $y(\cdot)$, our approximation for $x(\cdot)$, is found from

$$\dot{y}(t) = Ey(t) + \hat{f}_d(y(t)) + Ff_o(\theta) \qquad y(0) = G\theta \tag{40}$$

In the spirit of equation 29, for each $\gamma \in \mathbb{R}^n$, define $\tilde{\varepsilon}_\gamma(t) \in \mathbb{R}^m$ for $t \in [0,t_F]$ by

$$\dot{\tilde{\varepsilon}}_\gamma(t) = [E+f_d'(y(t))]\tilde{\varepsilon}_\gamma(t) + Ff_o'(\theta)\gamma \qquad \tilde{\varepsilon}_\gamma(0) = G\gamma \tag{41}$$

Note that the exact fault-on dynamics derivative, f_d', is used in this linearization but that it is evaluated at the approximate quiescent trajectory, $y(t)$ and not at the exact value, $x(t)$. Thus, we place no restrictions on $\hat{f}_d$ other than equation 39 and those necessary to ensure the existence and uniqueness of a solution to equation 40.

Equation 41 has a closed form solution in terms of $\Phi(\cdot,\cdot)$, the state transition matrix [25, p.63] associated with the matrix function $t \to E + f_d'(y(t))$,

It is

$$\tilde{\varepsilon}_\gamma(t) = \Psi(t)\,\gamma \tag{42}$$

where $\Psi(t) \in \mathbb{R}^{mxn}$ is

$$\Psi(t) := \Phi(t,0)G + \int_0^t \Phi(t,\tau)d\tau\; Ff_0'(\theta) \tag{43}$$

Note that $\Psi(\cdot)$ depends only on $y(\cdot)$ and not on γ. Our approximation for $D(\theta+\gamma)$ is $y(t_F) + \Psi(t_F)\gamma$ and a bound on $\|x_\gamma(t) - [y(t)+\Psi(t)\gamma]\|$ is thus sought.

To this end, we apply the Bellman-Gronwell lemma to obtain lemma 2 below. First, however, it is useful to make the following definitions. For $\gamma \in \mathbb{R}^n$, $t \in [0,t_F]$, let

$$\hat{r}_d(t,\gamma) := f_d(y(t)+\psi(t)\gamma) - f_d(y(t)) - f_d'(y(t))\psi(t)\gamma \tag{44}$$

Comparing this to equation 25, we see that $x(t)$ has been replaced by $y(t)$, and $\varepsilon_\gamma(t)$ by a linear approximation $\tilde{\varepsilon}_\gamma(t) = \psi(t)\gamma$. Also, let

$$k_d := \|E\| + k_f \tag{45}$$

where k_f is the Lipschitz constant of f_d.

<u>Lemma 2</u>:

For all $t \in [0,t_F]$,

$$\|x_\gamma(t) - [y(t) + \Psi(t)\gamma]\| \leq k_d^{-1}(e^{k_d t}-1)(e_\infty+\|r_0(\gamma)\|) + \int_0^t \|\hat{r}_d(\tau,\gamma)\| e^{k_d(t-\tau)}d\tau \qquad ¤$$

Proof

From equations 19, 40 and 41, it is easily seen that

$$\dot{x}_\gamma - (\dot{y}+\dot{\tilde{\varepsilon}}_\gamma) = E[x_\gamma - (y+\tilde{\varepsilon}_\gamma)] + f_d(x_\gamma) - f_d(y+\tilde{\varepsilon}_\gamma)$$
$$+ f_d(y) - \hat{f}_d(y) + r_0(\gamma) + \hat{r}_d(t,\gamma)$$

Integrating and taking norms yields

$$\|x_\gamma(t) - [y(t)+\tilde{\varepsilon}_\gamma(t)]\| \leq k_d \int_0^t \|x_\gamma(\tau) - [y(\tau)+\tilde{\varepsilon}_\gamma(\tau)]\| d\tau$$
$$+ (e_\infty + \|r_0(\gamma)\|)t + \int_0^t \|\hat{r}_d(\tau,\gamma)\| d\tau \qquad (46)$$

Applying the extended Bellman-Gronwell lemma [25, p.38] to equation 46 and integrating by parts, we get

$$\|x_\gamma(t) - [y(t)+\tilde{\varepsilon}_\gamma(t)]\| \leq \int_0^t [e_\infty + \|r_0(\gamma)\| + \|\hat{r}_d(\tau,\gamma)\|]\, e^{k_d(t-\tau)} d\tau$$

whence the lemma follows

q.e.d.

Lemma 1 is now applied to convert the terms $\|r_0(\gamma)\|$ and $\|\hat{r}_d(\tau,\gamma)\|$ in Lemma 2 into quadratics in γ. To achieve this, we need the following definitions and assumptions.

Let $\xi \in \mathbb{R}_+^{\ell_0}$. Define

$$S_0(\xi) := \{\gamma \in \mathbb{R}^n : -\xi \leqq A_0^T\gamma \leqq \xi\} \qquad (47)$$

$$R(\xi,t) := \{\Psi(t)\gamma \in \mathbb{R}^m : -\xi \leqq A_0^T\gamma \leqq \xi\} = \Psi(t)S_0(\xi) \qquad (48)$$

For each $\xi_d \in \mathbb{R}_+^{\ell_d}$, let

$$\overline{S}_d(\xi_d) := \{z \in \mathbb{R}^m : -\xi_d \leqq A_d^T z \leqq \xi_d\} \qquad (49)$$

Assumption A1

For each $t \in [0,t_F]$, there exists a $\eta(t) \in \mathbb{R}_+^{\ell_d}$ such that

$$\mathcal{R}(\xi,t) \subset \overline{\mathcal{S}}_d(\eta(t)) \tag{50}$$

That is, $\forall t \in [0,t_F]$, if $\gamma \in \mathcal{S}_0(\xi)$ then $\tilde{\varepsilon}_\gamma(t) \in \overline{\mathcal{S}}_d(\eta(t))$ ¤

Remark: If A_0^T is of rank n, then assumption A1 is satisfied. This is true because, if A_0^T is of rank n, $\mathcal{S}_0(\xi)$ is compact and its image under the continuous map $\gamma \mapsto \Psi(t)\gamma$, which is $\mathcal{R}(\xi,t)$, is also compact. Thus, this latter set is bounded in $\mathbb{R}^m$ and a $\eta(t) \in \mathbb{R}^{\ell_d}$ can thence be found so that it is included in $\overline{\mathcal{S}}_d(\eta(t))$. In the case where A_0 is the reduced node incidence of the pre-fault power system, it is sufficient for this network to be connected [21, p.417]. ¤

For each $t \in [0,t_F]$, we define $\rho(t) \in \mathbb{R}^{\ell_d}$ by

$$\rho(t) := A_d^T y(t) \tag{51}$$

and, for each $i \in \{1,\ldots,\ell_d\}$, let $b_i(t) \in \mathbb{R}_+$ such that

$$b_i(t) \geq \max\{|g''_{di}(\rho_i(t)+z_i)| : -\eta_i(t) \leq z_i \leq \eta_i(t)\} \tag{52}$$

Let $Q_d(t)$ be the $\ell_d \times \ell_d$ diagonal matrix with i,ith entry $\frac{1}{2}\sum_{j=1}^{m} |[B_d]_{ji}|b_i(t)$.

Similarly, let $\sigma := A_0^T\theta \in \mathbb{R}^{\ell_0}$ and $\tilde{B} := FA_0 \in \mathbb{R}^{m \times \ell_0}$. For each i in $\{1,\ldots,\ell_0\}$, let $c_i \in \mathbb{R}_+$ such that

$$c_i \geq \max\{|g''_{0i}(\sigma_i+z_i)|: -\xi_i \leq z_i \leq \xi_i\} \tag{53}$$

Let Q_0 be the $\ell_0 \times \ell_0$ diagonal matrix with i,ith entry $\frac{1}{2}\sum_{j=1}^{m} |[\tilde{B}]_{ji}|c_i$.

Theorem 1

Under assumption A1 and if k_n is a norm bound described in the hypothesis of lemma 1, then $\forall \gamma \in \mathcal{S}_0(\xi)$, $\forall t \in [0,t_F]$

$$\|x_\gamma(t) - [y(t)+\Psi(t)\gamma]\| \leq \gamma^T N_d(t)\gamma + e_d(t)$$

where

$$N_d(t) = k_n \int_0^t \Psi(\tau)^T A_d Q_d(\tau) A_d^T \Psi(\tau) e^{k_d(t-\tau)} d\tau + k_n k_d^{-1}(e^{k_d t}-1)A_0 Q_0 A_0^T$$

and

$$e_d(t) = k_d^{-1}(e^{k_d t}-1)e_\infty$$

Further, $N_d(t)$ is positive semidefinite symmetric in $\mathbb{R}^{n\times n}$. ¤

Remarks: 1. The proof is immediate from lemmas 1 and 2.

2. A consequence of this theorem is that, under these conditions, $\forall \gamma \in \mathcal{S}_0(\xi)$

$$\|D(\theta+\gamma) - [y(t_F) + \Psi(t_F)\gamma]\| \leq \gamma^T N_d(t_F)\gamma + e_d(t_F) \tag{54}$$

That is, the error between the approximate state at the fault switching time and the exact state is less than $\gamma^T N_d(t_F)\gamma + e_d(t_F)$.

3. The approximation $y(\cdot)$ can also be chosen as an approximate solution to equation 19 rather than as the solution of a differential equation. We only insist that $y(\cdot)$ be continuously differentiable and that $y(0) = G\theta$. It might be obtained using approximate numerical integration or as some convenient form, such as a sinusoidal waveform [11]. In this case, $\hat{f}_d$ is defined on the set $\{y(t) : t \in [0,t_F]\}$ by

$$\hat{f}_d(y(t)) := \dot{y}(t) - Ey(t) - Ff_o(\theta) \tag{55}$$

and e_∞ by

$$\sup\{\|f(y(t)) - \hat{f}(y(t))\| : t \in [0,t_F]\} \leq e_\infty \tag{56}$$

The results of this section would then remain unchanged. ¤

5. PRE-FAULT TO POST-FAULT EQUILIBRIUM MAP

In this section, the dependency of the post-fault equilibrium angle on the pre-fault equilibrium angle is studied. The objective is to reproduce the above approach, dealing now with the equilibrium map. That is, we aim to find a linear approximation for the perturbation in the post-fault equilibrium when the pre-fault equilibrium is disturbed γ from its quiescent value θ and a bound on the error involved in this approximation. Recall that the pre-fault and post-fault equilibrium angles are coupled by the constancy of power assumption.

Specifically, suppose θ and $\theta_p^* \in \mathbb{R}^n$ are respectively stable pre-fault and post-fault equilibria with the same steady state power injections. That is

$$f_p(\theta_p^*) = f_o(\theta) \tag{57}$$

We seek a $\phi_\gamma \in \mathbb{R}^n$ such that $\theta + \gamma$ and $\theta_p^* + \phi_\gamma$ are pre- and post-fault equilibria with the same injections. That is

$$f_p(\theta_p^*+\phi_\gamma) = f_o(\theta+\gamma) \tag{58}$$

We first find a local map $\tilde{\phi} : \gamma \to \phi_\gamma$ using the implicit function theorem [24, p.270]. Define $\Delta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Delta(\gamma,\phi) := f_p(\theta_p^*+\phi) - f_o(\theta+\gamma) \tag{59}$$

Thus,

$$\Delta(0,0) = 0 \tag{60}$$

and

$$\frac{\partial}{\partial\phi}\Delta(0,0) = f_p'(\theta_p^*) \tag{61}$$

Since θ_p^* is a stable equilibrium, $f_p'(\theta_p^*)$ is symmetric positive definite and thus nonsingular. Hence, there exists $\mathcal{U}_o$ and $\mathcal{U}_p$, neighborhoods of zero in $\mathbb{R}^n$, and a twice continuously differentiable map $\tilde{\phi} : \mathcal{U}_o \to \mathcal{U}_p$ with the properties that

$$\text{graph}(\tilde{\phi}) = \{(\gamma,\phi) \in \mathcal{U}_o \times \mathcal{U}_p : \Delta(\gamma,\phi) = 0\} \tag{62}$$

and

$$\tilde{\phi}'(0) = [f_p'(\theta_p)]^{-1} f_o'(\theta) \tag{63}$$

Thus ϕ can be expressed as a function of γ in a neighborhood of zero and $\gamma \to [f_p'(\theta_p^*)]^{-1} f_o'(\theta)\gamma$ is a best linear approximation to that function in that neighborhood.

Note the equation 62 also establishes the existence of the map $\tilde{\theta}_p : \tilde{\mathcal{U}}_o \to \mathbb{R}^n$ referred to in section 2 (see equation 17). $\tilde{\theta}_p$ and $\tilde{\phi}$ are related by

$$\tilde{\theta}_p(\theta+\gamma) = \theta_p^* + \tilde{\phi}(\gamma) \qquad \forall\gamma \in \mathcal{U}_o \tag{64}$$

Further,

$$\tilde{\mathcal{U}}_o = \{\theta+\gamma : \gamma \in \mathcal{U}_o\} \tag{65}$$

Our method does not require an exact solution of equation 57 for the post-fault equilibrium angle θ_p^*. Suppose that θ_p^* is unknown and, instead, $\theta_p \in \mathbb{R}^n$ has been calculated such that

$$\|f_p(\theta_p) - f_o(\theta)\|_\infty \leq e_m \tag{66}$$

where e_m is some positive parameter. In practice, this corresponds to terminating some iterative solution scheme for equation 57 when the power mismatch, $\|f_p(\theta_p) - f_o(\theta)\|_\infty$, is sufficiently small. Defining

$$J_o := f_o'(\theta) \text{ and } J_p := f_p'(\theta_p) \tag{67}$$

our approximation for $\theta_p^* + \phi_\gamma$ is now $\theta_p + J_p^{-1} J_o \gamma$, so that a bound on $\|(\theta_p^* + \phi_\gamma) - (\theta_p + J_p^{-1} J_o \gamma)\|_\infty$ is now sought. The technique used is to pose the above as an existence problem. To achieve this, some definitions and assumptions are required.

<u>Assumption A2</u>

A_p is of rank n

<u>Remarks</u>: In the case where A_p is the reduced node incidence matrix of the post-fault network and this network is connected, then part (i) of the assumption is valid [21, p.417].

¤

Let $\xi \in \mathbb{R}_+^{\ell_o}$, $\xi_p \in \mathbb{R}_+^{\ell_p}$ and $\hat{\beta} \in \mathbb{R}_+$. Define

$$S_o(\xi) := \{\gamma \in \mathbb{R}^n : -\xi \leqq A_o^T \gamma \leqq \xi\} \tag{68}$$

$$S_p(\xi) := \{\phi \in \mathbb{R}^n : -\xi_p \leqq A_p^T \phi \leqq \xi_p\} \tag{69}$$

From Lemma 1, there exists positive semi-definite symmetric matrices N_o and N_p in $\mathbb{R}^{n\text{x}n}$ such that

$$\|f_o(\theta+\gamma) - f_o(\theta) - f_o'(\theta)\gamma\|_1 \leq \gamma^T N_o \gamma \qquad \forall_\gamma \in S_o(\xi) \qquad (70)$$

$$\|f_p(\theta_p+\phi) - f_p(\theta_p) - f_p'(\theta_p)\phi\|_1 \leq \phi^T N_p \phi \qquad \forall\phi \in S_p(\xi_p) \qquad (71)$$

Note use of the 1-norm here.

For each $j \in \{1,\ldots,\ell_p\}$, let $A^p_{\cdot j} \in \mathbb{R}^n$ be the jth column of A_p. Let $\sigma := A_p^T \theta_p \in \mathbb{R}^{\ell_p}$

<u>Assumption A3</u>

$\forall j \in \{1,\ldots,\ell_p\}$

$$\min\{g_{pj}'(\sigma_j+z_j) : |z_j| \leq \xi_{pj} + \hat{\beta}\|A^p_{\cdot j}\| \qquad (72)$$

¤

<u>Remark</u>: If $g_{pj}'(\sigma_j) > 0$ for all $j \in \{1,\ldots,\ell_p\}$, then, by the continuously differentiable nature of $g_{pj}(\cdot)$, there exists a $\xi_p \in \mathbb{R}^{\ell_p}$ and a $\hat{\beta} \in \mathbb{R}_+$ such that assumption A3 is valid. $g_{pj}'(\sigma_j) > 0$ is a sufficient condition, under assumption A2, for θ_p to be a stable equilibrium. This follows from

$$f_p'(\theta_p) = A_p g_p'(\sigma) A_p^T \qquad (73)$$

In the case where g_{pj} is a positive constant multiplying the sin function [18], it is sufficient that the magnitude of the angle difference across each transmission line is less than $\pi/2$. This latter condition usually arises in steady state security studies from thermal limits on power flow.

¤

For each $j \in \{1,\ldots,\ell_p\}$, we define $d_j \in \mathbb{R}_+$ by

$$0 < d_j \leq \min\{g'_{pj}(\sigma_j+z_j) : |z_j| \leq \xi_{pj} + \hat{\beta}\|A^p_{.j}\|_1\} \tag{74}$$

Let M be the $\ell_p \times \ell_p$ diagonal matrix with j,jth entry $[M]_{jj} = d_j$. Let $\alpha \in \mathbb{R}_+$ be such that

$$0 < \alpha \leq \min\{z^T A_p M A_p^T z : \|z\|_\infty = 1\} \tag{75}$$

Remarks 1. Such an α exist since, under assumption A2, $A_p M A_p^T$ is positive definite.

2. In the case where A_p is the reduced node incidence matrix of the post-fault network, there exists a simple technique for calculating α. Consider an electrical network with graph described by A_p and with j^{th} branch conductance $d_j > 0$. For $k = 1, \cdots, n$, let

$$\alpha_k = (\underline{1}^k)^T (A_p M A_p^T)^{-1} \underline{1}^k \tag{76}$$

where $\underline{1}^k$ is the k^{th} standard unit basis vector in $\mathbb{R}^n$. α_k is then the voltage at node k when a unit element source is connected to node k and all other nodes are left unconnected. Then

$$\min\{z^T A_p M A_p^T z : \|z\| = 1\} = \min\{\frac{1}{\alpha_k} : k = 1, \cdots, n\} \tag{77}$$

This result can be derived from simple linear circuit theory considerations. Equations (76) and (77) remain valid when $A_p M A_p^T$ is a Stieltjes matrix [28, p. 54] with a diagonally dominant inverse. ¤

Let N_s be the positive semidefinite symmetric n x n matrix

$$N_s := \frac{1}{\alpha} (J_o J_p^{-1} N_p J_p^{-1} J_o + N_o) \tag{78}$$

and $e_s \in \mathbb{R}_+$ be given by

$$e_s := \frac{1}{\alpha} e_m \tag{79}$$

Further let $S_s \subset \mathbb{R}^n$ be defined by

$$S_s := J_o^{-1} J_p S_p(\xi_p) \cap S_o(\xi) \cap \{z \in \mathbb{R}^n : z^T N_s z \leq \hat{\beta} - e_s\} \tag{80}$$

Here, it is implicitly assumed that $\hat{\beta} > e_s$.

<u>Theorem 2</u>

Under assumptions A2 and A3, for each $\gamma \in S_s$, there exists a <u>unique</u> $\phi \in \mathbb{R}^n$ satisfying

$$(1) \quad f_p(\theta_p^* + \phi) = f_o(\theta + \gamma) \tag{81}$$

and

$$(2) \quad \|(\theta_p^* + \phi) - (\theta + J_p^{-1} J_o \gamma)\|_\infty \leq \gamma^T N_s \gamma + e_s \tag{82}$$

Also, $\forall \overline{\theta}_p \in \overline{B}_\infty(\theta_p + J_p^{-1} J_o \gamma, \gamma^T N_s \gamma + e_s)$, $f_p'(\overline{\theta}_p)$ is positive definite and thus $f_p'(\theta_p^* + \phi)$ is positive definite. That is, $\overline{\theta}_p$ and $\theta_p^* + \phi$ are stable post-fault equilibria. ¤

<u>Proof</u>

Fix γ in S_s, and define $h : \mathbb{R}^n \to \mathbb{R}^n$ by

$$h(\mu) := f_p(\theta_p + J_p^{-1} J_o \gamma + \mu) - f_o(\theta + \gamma) \tag{83}$$

and let

$$\beta := \gamma^T N_s \gamma + e_s \tag{84}$$

We seek to prove that h has a zero inside the closed ball $\overline{B}_\infty(0,\beta)$. For suppose there exists a $\hat{\mu} \in \overline{B}_\infty(0,\beta)$ such that $h(\hat{\mu}) = 0$. Then

$$f_p(\theta_p + J_p^{-1} J_o \gamma + \hat{\mu}) = f_o(\theta + \gamma) \tag{85}$$

Defining

$$\hat{\phi} = \theta_p + J_p^{-1} J_o \gamma - \theta_p^* + \hat{\mu} \tag{86}$$

and substituting into equation 85, we get

$$f_p(\theta_p^* + \hat{\phi}) = f_o(\theta + \gamma) \tag{87}$$

and, from equation 86,

$$\|(\theta_p^* + \hat{\phi}) - (\theta_p + J_p^{-1} J_o \gamma)\|_\infty = \|\hat{\mu}\|_\infty \leq \beta \tag{88}$$

Equations 87 and 88 are the existence part of the theorem. Thus, we need to establish the existence of a zero of h in $\overline{B}_\infty(0,\beta)$. This is achieved using the following variant of the Leray-Schauder fixed point theorem.

Existence Theorem [28, p.163]

Let C be an open and bounded subset of $\mathbb{R}^n$, and $\tilde{h} : \overline{C} \to \mathbb{R}^n$ be continuous. If there exists a $\tilde{\mu} \in \overline{C}$ such that

$$(\mu - \tilde{\mu})^T \tilde{h}(\mu) \geq 0 \quad \forall \mu \in \partial C \tag{89}$$

then $\tilde{h}(\hat{\mu}) = 0$ for some $\hat{\mu} \in \overline{C}$. ¤

It is thus sufficient to establish that

$$\mu^T h(\mu) \geq 0 \quad \forall \mu \ni \|\mu\|_\infty = \beta \tag{90}$$

From equation 83, it can be seen that

$$\mu^T h(\mu) = v(\mu) + w(\mu) \tag{91}$$

where

$$v(\mu) := \mu^T [f_p(\theta_p + J_p^{-1} J_o \gamma + \mu) - f_p(\theta_p + J_p^{-1} J_o \gamma)] \tag{92}$$

$$w(\mu) := \mu^T[f_p(\theta_p+J_p^{-1}J_o\gamma) - f_p(\theta_p) - f_p'(\theta_p)J_p^{-1}J_o\gamma]$$

$$- \mu^T[f_o(\theta+\gamma) - f_o(\theta) - f_o'(\theta)\gamma] + \mu^T[f_p(\theta_p) - f_o(\theta)] \qquad (93)$$

Our strategy is to obtain an upper bound on $|w(\mu)|$ and a lower bound on $v(\mu)$ for all μ such that $\|\mu\|_\infty = \beta$, and thus show that the sum of these two terms is positive.

First, we address the v term. Consider $\mu \in \mathbb{R}^n$ such that $\|\mu\|_\infty = \beta$ and define $\check{\sigma}$ and $\delta \in \mathbb{R}^{\ell_p}$ by

$$\delta := A_p^T\mu \qquad (94)$$

$$\check{\sigma} := A_p^T(\theta_p+J_p^{-1}J_o\gamma) \qquad (95)$$

From equation 80, $J_p^{-1}J_o\gamma \in S_p(\xi_p)$ so that $\forall j \in \{1,\ldots,\ell_p\}$,

$$|\sigma_j - \check{\sigma}_j| \leq \xi_{pj} \text{ and } |\delta_j| \leq \beta\|A^p_{.j}\|_1 \qquad (96)$$

Further, since $\gamma^T N_s \gamma \leq \hat{\beta} - e_s$, we get that

$$0 \leq \beta \leq \hat{\beta} \qquad (97)$$

We conclude from equations 96 and 97 that $\forall\lambda \in [0,1]$

$$\check{\sigma}_j + \lambda\delta_j \in [\sigma_j - \xi_{pj} - \hat{\beta}\|A^p_{.j}\|_1, \sigma_j + \xi_{pj} + \hat{\beta}\|A^p_{.j}\|_1] \qquad (98)$$

Thus,

$$v(\mu) = \delta^T[g_p(\check{\sigma}+\delta) - g_p(\check{\sigma})] \qquad (99a)$$

$$= \sum_{j=1}^{\ell_p} \delta_j^2 \int_0^1 g_{pj}'(\check{\sigma}_j+\lambda_j\delta_j)d\lambda_j \qquad (99b)$$

$$\geq \sum_{j=1}^{\ell_p} \delta_j^2 \, d_j \qquad (99c)$$

$$= \delta^T M \delta \tag{99d}$$

$$= \mu^T A_p M A_p^T \mu \tag{99e}$$

$$\geq \alpha \|\mu\|_\infty^2 \tag{99f}$$

$$= \alpha\beta^2 \tag{99g}$$

Equation 99b follows from the decoupled nature of g_p(see equation 10) and an exact two term Taylor series expansion [24, p.190] for $g_{pj}(\gamma_j+\delta_j)$. The validity of equation 99c can be seen from equations 74 and 98. The inequality in 99f follows from equation (75).

We now obtain a lower bound on the w term, using Holder's inequality. Let $\mu \in \mathbb{R}^n \ni \|\mu\|_\infty = \beta$. Then

$$|w(\mu)| \leq \|\mu\|_\infty \|f_p(\theta_p+J_p^{-1}J_o\gamma) - f_p(\theta_p) - f_p'(\theta_p)J_p^{-1}J_o\gamma\|_1$$

$$+ \|\mu\|_\infty \|f_o(\theta+\gamma) - f_o(\theta) - f_o'(\theta)\gamma\|_1$$

$$+ \|\mu\|_\infty \|f_p(\theta_p) - f_o(\theta)\|_1 \tag{100}$$

Since $\gamma \in J_o^{-1}J_p S_p(\xi_p) \cap S_o(\xi)$, from equations (70) and (71) and from equation (66)

$$|w(\mu)| \leq \beta[\gamma^T J_o J_p^{-1} N_p J_p^{-1} J_o \gamma + \gamma^T N_o \gamma + e_m] \tag{101}$$

Thus, from equations (78), (79), (84) and (99g), $\forall\mu \in \mathbb{R}^n \ni \|\mu\|_\infty = \beta$

$$|w(\mu)| \leq \alpha\beta^2 \leq v(\mu) \tag{102}$$

and it follows that $\mu^T h(\mu) \geq 0$, and the existence part of the theorem is thereby proved.

We establish uniqueness by showing that there exists a $\bar{\nu} > 0$ such that $\forall\bar{\theta}_p \in B_\infty(\theta_p+J_p^{-1}J_o\gamma, \beta+\bar{\nu})$, $f_p'(\bar{\theta}_p)$ is positive definite. For then it follows that f_p is injective on $\overline{B}_\infty(\theta_p+J_p^{-1}J_0\gamma,\beta)$ [28, p.143] and thence that ϕ in the theorem is unique.

Consider $\overline{\theta}_p := \theta_p + J_p^{-1}J_0\gamma + \hat{z} + \tilde{z}$ where $\hat{z}, \tilde{z} \in \mathbb{R}^n$ such that $\|\hat{z}\|_\infty \leq \beta$. As used in equation 99c, we have $\forall j \in \{1,\ldots,\ell_p\}$

$$g'_{pj}(\check{\delta}_j+(A^p_{.j})^T\hat{z}) \geq d_j > 0 \tag{103}$$

Since each g'_{pj} is continuous, there exists a $\nu_j > 0$ such that whenever $\|\tilde{z}\|_\infty < \nu_j$

$$g'_{pj}(\check{\delta}_j+(A^p_{.j})^T\hat{z} + (A^p_{.j})^T\tilde{z}) > 0 \tag{104}$$

Defining $\overline{\nu} := \min\{\nu_1,\ldots,\nu_{\ell_p}\} > 0$, we get that whenever $\overline{\theta}_p \in B_\infty(\theta_p+J_p^{-1}J_0\gamma, \beta+\overline{\nu})$, the diagonal matrix $g'_p(A_p^T\overline{\theta}_p)$ has strictly positive diagonal entries. Since A_p is full rank, the Jacobian

$$f'_p(\overline{\theta}_p) = A_p\, g'_p(A_p^T\overline{\theta}_p)A_p^T \tag{105}$$

is positive definite [21, p.768]

q.e.d.

6. POLYTOPES OF DYNAMIC SECURITY: SHORT-CIRCUIT FAULT

In this section, the results of sections 4 and 5 are used to derive a sufficient condition for a polytope in $\mathbb{R}^n$ to contain only dynamically secure pre-fault equilibrium angles. These results are first presented in summary form.

From theorem 1, we have a convex set $S_d \subset \mathbb{R}^n$, $e_d \in \mathbb{R}_+$ and a positive semidefinite symmetric matrix $N_d \in \mathbb{R}^{n\times n}$ such that $\forall\gamma \in S_d$,

$$\|D(\theta+\gamma) - (y+\Psi\gamma)\| \leq \gamma^T N_d \gamma + e_d \tag{106}$$

Note that for simplicity we have not shown explicitly the dependency on t and it is understood that the appropriate functions are evaluated at $t = t_F$ (eg. $N_d = N_d(t_F), \Psi = \Psi(t_F)$). Further, we have renamed $S_0(\xi)$ as S_d.

From theorem 2, there exists a convex set $S_s \subset \mathbb{R}^n$, $e_s \in \mathbb{R}_+$ and a positive semidefinite symmetric matrix $N_s \in \mathbb{R}^{n\times n}$ such that $\forall \gamma \in S_s$

$$\|\tilde{\theta}_p(\theta+\gamma) - (\theta_p + K\gamma)\|_\infty \leq \gamma^T N_s \gamma + e_s \tag{107}$$

Note that here

$$K := J_p^{-1} J_o \in \mathbb{R}^{n\times n} \tag{108}$$

Recall from section 2 that a stability set $L \subset \mathbb{R}^n \times \mathbb{R}^m$ was defined, in terms of the above notation, by the following relation.

If $\gamma \in S_d \cap S_s$ and $(\tilde{\theta}_p(\theta+\gamma), D(\theta+\gamma)) \in L$ then $\theta + \gamma$ is a dynamically secure pre-fault equilibrium. The following theorem gives a sufficient condition for a polytope of pre-fault angles to be mapped by $\tilde{\theta}_p \times D$ into L.

<u>Theorem 3</u>

Assume L is convex. Suppose Ω is a polytope in $\mathbb{R}^n$ such that for each vertex $\hat{\gamma}$ of Ω,

$$\overline{B}_\infty(\theta_p + K\hat{\gamma}, \hat{\gamma}^T N_s \hat{\gamma} + e_s) \times \overline{B}(y + \Psi\hat{\gamma}, \hat{\gamma}^T N_d \hat{\gamma} + e_d) \subset L \tag{109}$$

Then, $\forall \gamma \in \Omega \cap S_d \cap S_s$,

$$(\tilde{\theta}_p(\theta+\gamma),\ D(\theta+\gamma)) \in L \tag{110}$$

and hence $\theta + \gamma$ is a dynamically secure pre-fault angle. In other words, if we define the set addition $\Omega + \{\theta\}$ by $\{\theta + \gamma : \gamma \in \Omega\}$, then

$$(S_d \cap S_s \cap \Omega) + \{\theta\} \subset \overline{\Omega}_d \tag{111}$$

¤

Remark: Here, $\overline{B}$ refers to the closed ball using the arbitrary norm of sections 3 and 4 while $\overline{B}_\infty$ is defined using the infinity norm as in section 5. ¤

Proof

Let $\gamma \in \Omega$ and $w_s \in \overline{B}(0,\gamma^T N_s \gamma + e_s)$ and $w_d \in \overline{B}_\infty(0,\gamma^T N_d \gamma + e_d)$. It is then sufficient to show that $(\theta_p + K\gamma + w_s, y + \Psi\gamma + w_d) \in L$. Let $\{\hat{\gamma}^j\}_{j=1}^{\tilde{m}}$ be the $\tilde{m}$ vertices of Ω. Then there exists $\lambda_j \geq 0$, $j = 1,\ldots,\tilde{m}$ such that

$$\sum_{j=1}^{\tilde{m}} \lambda_j = 1 \quad \text{and} \quad \gamma = \sum_{j=1}^{m} \lambda_j \hat{\gamma}^j \tag{112}$$

For $j = 1,\ldots,\tilde{m}$, let

$$q_{sj} := (\hat{\gamma}^j)^T N_s \hat{\gamma}^j + e_s \quad \text{and} \quad q_{dj} := (\hat{\gamma}^j)^T N_d \hat{\gamma}^j + e_d \tag{113}$$

$$\tau_s := \frac{\|w_s\|_\infty}{\sum_{j=1}^{\tilde{m}} \lambda_j q_{sj}} \quad \text{and} \quad \tau_d := \frac{\|w_d\|}{\sum_{j=1}^{\tilde{m}} \lambda_j q_{dj}} \tag{114}$$

$$\hat{w}_s := \frac{1}{\|w_s\|_\infty} w_s \quad \text{and} \quad \hat{w}_d = \frac{1}{\|w_d\|} w_d \tag{115}$$

Since the matrices N_s and N_d are positive semidefinite symmetric, the maps $z \to z^T N_s z + e_s$ and $z \to z^T N_d z + e_d$ of $\mathbb{R}^n \to \mathbb{R}$ are convex [29, p.27]. Thus, by Jensen's inequality [29, p.25],

$$\begin{aligned} 0 &\leq \gamma^T N_s \gamma + e_s \leq \sum_{j=1}^{m} \lambda_j q_{sj} \\ 0 &\leq \gamma^T N_d \gamma + e_d \leq \sum_{j=1}^{m} \lambda_j q_{dj} \end{aligned} \tag{116}$$

so that τ_s and $\tau_d \in [0,1]$. Thus for $j = 1,\ldots,\tilde{m}$

$$(\theta_p+K\hat{\gamma}^j+\tau_s q_{sj}\hat{w}_s, y+\Psi\hat{\gamma}^j+\tau_d q_{dj}\hat{w}_d)$$

$$\in \bar{B}_\infty(\theta_p+K\hat{\gamma}^j, q_{sj}) \times \bar{B}(y+\Psi\hat{\gamma}^j, q_{dj}) \subset L \qquad (117)$$

It then follows from the convexity of L that

$$\sum_{j=1}^{\tilde{m}} \lambda_j(\theta_p+K\hat{\gamma}^j+\tau_s q_{sj}\hat{w}_s, y+\Psi\hat{\gamma}^j+\tau_d q_{dj}\hat{w}_d) \in L \qquad (118)$$

i.e.

$$(\theta_p+K\gamma+w_s, y+\Psi\gamma+w_d) \in L \qquad (119)$$

q.e.d.

7. POLYTOPES OF DYNAMIC SECURITY: STEP CHANGES IN INJECTION AND LINE SWITCHING

7.1 Introduction

In this section, the above analysis is modified to handle the second major type of transient stability event, a step change in the injections. In this case, there is no fault-on period. Instead, the steady-state injections change instantaneously at time t = 0 from their constant pre-fault value to their constant post-fault value. This represents the situations of an instantaneous generator outage or repair, and a switching in or out of a block of load. Also, since we do not insist that the pre-fault and post-fault load flow functions be the same, this can also represent a line switching action, in which case the difference between pre-fault and post-fault steady state injections is zero.

The model for this type of events is a special case of the previous model, given in section 2 (i.e. with t_F=0) with one minor modification (i.e. the steady state injections are no longer the same for pre-fault and post-fault.) The analysis required to derive a sufficient condition for dynamic security of a polytope of pre-fault angles for this situation differs from the above in a minor fashion. Thus, after establishing the model, we state the two theorems which are the analogues of theorems 2 and 3. No proofs are given, as these can be done by inspection of the proofs of the latter two theorems.

7.2 The Model

The pre-fault and post-fault models are precisely as they are stated in sections 2.2 and 2.4. That is, there is f_o, the pre-fault load flow function and f_p, the post-fault load flow function given by equations 1,2,3 and by equations 8,9,10 respectively. Also, we have a region L given by equation 11, representing the post-fault dynamics. The difference is in the way that these two models are coupled together. We do not assume constancy of steady state injected power, nor do the fault-on dynamics couple these two, as there are no fault-on dynamics. Instead, they are coupled as follows.

Suppose $\theta \in \mathbb{R}^n$ is the pre-fault equilibrium angle, so that the pre-fault injection is $f_o(\theta)$. Then the steady-state value of the post-fault power injections, $p_p \in \mathbb{R}^n$ is given by

$$p_p = Wf_o(\theta) + \hat{p} \tag{120}$$

where $W \in \mathbb{R}^{nxn}$ and $\hat{p} \in \mathbb{R}^n$ are fixed by the nature of the event. Thus, the corresponding post-fault equilibrium angle, $\theta_p \in \mathbb{R}^n$, is given by a solution (if one exists) of

$$\begin{aligned} &f_p(\theta_p) = Wf_o(\theta) + \hat{p} \\ &\text{and } f_p'(\theta_p) \text{ is positive definite} \end{aligned} \tag{121}$$

The post-fault steady state injection is an affine function of the pre-fault injection.

The way in which the choice of W and $\hat{p}$ can represent the different types of transient stability events in as follows.

(i) Partial generator outage or repair and switching a block of load of fixed size. This is the event where pre-fault and post-fault injections differ by a constant amount, $\hat{p}$, independently of θ. Thus, W is set equal to the $n \times n$ identity matrix. If, for example, a generator connected to node j were to loose 50 MW of its capacity at time $t = 0$ then $\hat{p}$ is the vector of all zeros except at the jth entry which is -50 and, then,

$$p_p = f_o(\theta) + \hat{p} \tag{122}$$

Note that the range of θ is restricted in order that the jth entry of p_p does not become negative thereby representing a load.

(ii) Total outage of a generating station or load at a node. This is the event where the post-fault injection at the node at which the fault occurs is fixed independently of the pre-fault injections. Thus, supposing the fault occurs at the jth node, W is the diagonal $n \times n$ matrix with zero in the j,jth position and unity in all other diagonal entries. All entries of $\hat{p}$ are zero except, possibly the jth, which contains the post-fault value of the injection at the jth node.

(iii) Line switching. Here, W is the $n \times n$ identity matrix and $\hat{p}$ is zero, so that

$$p_p = f_o(\theta) \tag{123}$$

The significant feature of this model is the difference between A_p and A_o. ¤

The second way in which the pre-fault and post-fault models are coupled is that the initial condition for the post-fault dynamical system in uniquely determined by the pre-fault equilibrium. As before, we use the function $D : \mathbb{R}^n \to \mathbb{R}^m$ but, now, because $t_F = 0$ we assume that D is the linear function

$$D(\theta) = G\theta \tag{124}$$

where $G \in \mathbb{R}^{m \times n}$ is the "selector" matrix.

The problem of finding regions of the pre-fault equilibrium space which are dynamically secure can then be stated as seeking subsets of

$$\begin{aligned} \hat{\Omega}_d = \{\theta \in \mathbb{R}^n : \ & f_o'(\theta) \text{ is positive definite and} \\ & \exists\ \theta_p \in \mathbb{R}^n \text{ such that} \\ \text{(i)}\quad & f_p(\theta_p) = Wf_o(\theta) + \hat{p} \\ \text{(ii)}\quad & f_p'(\theta_p) \text{ is positive definite} \\ \text{(iii)}\quad & (\theta_p, D(\theta)) \in L\} \end{aligned} \tag{125}$$

This differs from $\overline{\Omega}_d$ in equation 16 only in condition (i) and in that D is a linear function. Thus, the philosophy of the the analysis remains unchanged. However, since D is linear, the work of section 4 is no longer required, and we need only concentrate on the pre-fault to post-fault equilibrium map. It can easily be seen using the same process with which the existence of $\tilde{\theta}_p$ was established (equation 59 et. seq.) that there exists a $\hat{\theta}_p : \hat{U}_o \to \mathbb{R}^n$ where $\hat{U}_o$ is an open set of $\mathbb{R}^n$ such that $\forall \theta \in \hat{U}_o$

$$f_p(\hat{\theta}_p(\theta)) = Wf_o(\theta) + \hat{p} \tag{126}$$

Defining $\hat{\theta}_p \times D$ in the same fashion as equation 18, the problem is to find subsets of $(\hat{\theta}_p \times D)^{-1}(L)$. The process of linearizing $\hat{\theta}_p$ about a quiescent θ, finding quadratic error bounds and then using these to give sufficient conditions for a polytope to be mapped by $\theta_p \times D$ into L is now derived from the above work.

7.3 Results

First, the analysis of section 5 is modified for these types of events. We maintain almost all the definitions of section 5 and make only the following alterations to θ_p^* and θ_p. Let $\theta \in \mathbb{R}^n$. We assume there exists a $\theta_p^* \in \mathbb{R}^n$ such that

$$f_p(\theta_p^*) = Wf_0(\theta) + \hat{p} \tag{127}$$

but, as before, we do not expect to know it exactly. Instead, we assume that we have a $\theta_p \in \mathbb{R}^n$ such that

$$\| f_p(\theta_p) - [W f_0(\theta)+\hat{p}] \|_\infty \leq \hat{e}_m \tag{128}$$

where $\hat{e}_m$ is some positive parameter. It is assumed that these values of θ_p^* and θ_p are used in all other subsequent definitions. The following additional definitions are made. Let $\hat{J}_o \in \mathbb{R}^{n \times n}$ be

$$\hat{J}_o := Wf_o'(\theta) \tag{129}$$

and $\hat{N}_o$ be the positive semidefinite n x n matrix, derived from Lemma 1, such that

$$\| Wf_o(\theta+\gamma) - Wf_o(\theta) - Wf_o'(\theta)\gamma \|_1 \leq \gamma^T \hat{N}_o \gamma \qquad \forall \gamma \in S_o(\xi) \tag{130}$$

Remark: In lemma 1, we let $B = WA_o$ and $A = A_o$ to obtain $\hat{N}_o = A\tilde{Q}A^T$.

¤

Let $\hat{N}_s$ be the positive semidefinite n x n matrix

$$\hat{N}_s := \frac{1}{\alpha}(\hat{J}_o J_p^{-1} N_p J_p^{-1} \hat{J}_o + \hat{N}_o) \tag{131}$$

and $\hat{e}_s \in \mathbb{R}_+$ be

$$\hat{e}_s := \frac{1}{\alpha}\hat{e}_m \tag{132}$$

Further, let $\hat{S}_s \subset \mathbb{R}^n$ be defined by

$$\hat{S}_s := \hat{J}_o^{-1} J_p S_p(\xi_p) \cap S_o(\xi) \cap \{z \in \mathbb{R}^n : z^T \hat{N}_s z \leq \hat{\beta} - \hat{e}_s\} \tag{133}$$

Theorem 3

Under assumptions A2 and A3, for each $\gamma \in \hat{S}_s$ there exists a unique $\phi \in \mathbb{R}^n$ such that

(1) $f_p(\theta_p^* + \phi) = Wf_o(\theta+\gamma) + \hat{p}$

and

(2) $\|(\phi_p^* + \phi) - (\theta_p + J_p^{-1}\hat{J}_o\gamma)\|_\infty \leq \gamma^T N_s \gamma + \hat{e}_s$

Also, $\forall \overline{\theta}_p \in \overline{B}_\infty(\theta_p + J_p^{-1}\hat{J}_o\gamma, \gamma^T\hat{N}_s\gamma + \hat{e}_s)$, $f_p'(\overline{\theta}_p)$ is positive definite and thus $f_p'(\theta_p^* + \phi)$ is positive definite. That is, $\overline{\theta}_p$ and $\theta_p^* + \phi$ are stable post-fault equilibria. ¤

Remarks: If we define $\hat{f}_o : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\hat{f}_o(\overline{\theta}) := Wf_o(\overline{\theta}) + \hat{p} \tag{134}$$

then this theorem can be proved in precisely the same fashion as theorem 2 by changing f_o, J_o, N_o, N_s, e_m and e_s to $\hat{f}_o$, $\hat{J}_o$, $\hat{N}_o$, $\hat{N}_s$, $\hat{e}_m$ and $\hat{e}_s$, respectively whenever they occur. Note that eqution 83 becomes

$$h(\mu) := f_p(\theta_p + J_p^{-1}\hat{J}_o\gamma + \mu) - \hat{f}_o(\theta+\gamma)$$

$$= f_p(\theta_p + J_p^{-1}\hat{J}_o\gamma + \mu) - Wf_o(\theta+\gamma) - \hat{p} \qquad (135)$$

and, thus equation 93 becomes

$$w(\mu) := \mu^T[f_p(\theta_p + J_p^{-1}\hat{J}_o\gamma) - f_p(\theta_p) - f_p'(\theta_p)J_p^{-1}\hat{J}_o\gamma]$$

$$- \mu^T[Wf_o(\theta+\gamma) - Wf_o(\theta) - Wf_o'(\theta)\gamma]$$

$$+ \mu^T[f_p(\theta_p) - Wf_o(\theta) - \hat{p}] \qquad (136)$$

¤

The analysis is completed by setting N_d and e_d to zero in theorem 3. The following definition is required in addition to those already made in this section. Let

$$\hat{K} := J_p^{-1}\hat{J}_o \qquad (137)$$

Theorem 5

Assume L is convex. Suppose Ω is a convex polytope in $\mathbb{R}^n$ such that for each vertex $\hat{\gamma}$ of Ω

$$\overline{B}_\infty(\theta_p + \hat{K}\hat{\gamma}, \hat{\gamma}^T\hat{N}_s\hat{\gamma}^T + \hat{e}_s) \times \{G\theta + G\hat{\gamma}\} \subset L$$

Then, $\forall\gamma \in \Omega \cap \hat{S}_d$

$$(\hat{\theta}_p(\theta+\gamma), D(\theta+\gamma)) \in V$$

and hence $\theta + \gamma$ is a dynamically secure pre-fault angle (i.e. $(S_d \cap \Omega) + \{\theta\} \subset \hat{\Omega}_d$) ¤

8. CONCLUSION

Two classes of transient stability events were considered in this paper. One is short circuit faults on a transmission lines and the other is sudden changes in injection or line switching actions. We have given the analysis required to design an algorithm for finding dynamic security regions in the pre-fault state space. The proposed approach was to linearize the maps representing the fault-on dynamics (in the former class) and the dependency of the post-fault equilibrium on the pre-fault equilibrium (both classes) about a quiescent pre-fault equilibrium angle. Bounds on the difference between the nonlinear maps and their affine approximations were then found. A sufficient condition for a polytope of pre-fault angles to be a dynamic security region was then derived, using the convexity of L, the affine nature of the approximations and the quadratic plus constant nature of the bounds. The proposed approach is fairly general and represents the first attempt to treat the problem rigorously.

The computational scheme herein suggested requires that the full nonlinear model be approximately solved once only after the quiescent values. Calculation of the bounds and checking sufficient conditions are straightforward numerical procedures. The actual design of an algorithm for computing dynamic security regions is outside the scope of this paper.

Acknowledgments

This research was sponsored by the Department of Energy, Division of Electric Energy Systems, under contract DE-AC01-79-ET29364. The support of the Stella Mary Langford Scholarship fund of the University of Melbourne, Victoria, Australia is also gratefully acknowledged.

References

[1] L.H. Fink, K. Carlsen, "Operating under stress and strain," IEEE Spectrum, Vol. 15, March 1978, pp. 48-53.

[2] F.F. Wu, Y.-K. Tsai, "Probabilistic dynamic security assessment of power systems part I: basic model," to appear in IEEE Trans. on Circuits and Systems.

[3] F.F. Wu, S. Kumagai, "Steady state security regions of power systems," submitted for publication.

[4] T.E. Dy Liacco, "The adaptive reliability control system," IEEE Trans. Power App. and Sys., Vol. PAS-86, May 1967, pp. 517-531.

[5] T.E. Dy Liacco, "System security: the computer's role," IEEE Spectrum, Vol. 15, June 1978, pp. 43-50.

[6] T.E. Dy Liacco, "Real-time computer control of power systems," Proc. IEEE, Vol. 62, July 1974, pp. 884-891.

[7] S.N. Talukdar, F.F. Wu, "Computer-aided dispatch for electric power systems," Proc. IEEE, Vol. 69, Oct. 1981, pp. 1212-1231.

[8] O.I. Elgerd, Electric Energy Systems Theory: an Introduction, McGraw-Hill,1971.

[9] O. Saito, K. Koizumi, M. Vdo, M. Sato, H. Mukae, T. Tsuji, "Security monitoring systems including fast transient stability studies," IEEE Trans. Power App. and Sys., Vol. PAS-94, Sept/Oct 1975, pp. 1789-1805.

[10] M. Ribbens-Pavella, B. Lemal, W. Pinard, "On-line operation of Liapunov criterion for transient stability studies," Proc. IFAC Symposium, Melbourne, February, 1977.

[11] T. Athay, R. Podmore, S. Virmani, "A practical method for direct analysis of transient stability," IEEE Trans. Power App. and Sys., Vol. PAS-98, March/April 1979, pp. 573-580.

[12] A.A. Fouad, et. al., "Transient stability margin as a tool for dynamic security assessment," Final Report, EPRI EL-1755, March 1981.

[13] T.A. Hughes, et.al., "Application of security indices and restraints to a real power system," Final Report, EPRI RP90-3, March 1975.

[14] M. Ribbens-Pavella, A. Calvaer, J. Gheury, "Transient stability index for on-line evaluation," paper A 80 013-3, presented at IEEE PES Winter Meet., New York, N.Y., Feb. 1980.

[15] B. Stott, "Power systems dynamic response calculations," Proc. IEEE, Vol. 67, February 1979, pp. 219-241.

[16] B. Stott, "Review of load flow calculation methods," Proc. IEEE, Vol. 62, July 1974, pp. 916-929.

[17] B. Stott, O. Alsac, "Fast decoupled load flow," IEEE Trans. Power App. and Sys., Vol. PAS-93, May/June 1974, pp. 859-867.

[18] A.R. Bergen, D.J. Hill, "A structure preserving model for power systems stability analysis," IEEE Trans. Power App. and Sys., Vol. PAS-100, January 1981, pp. 25-35.

[19] F.S. Prabhakara, A.H. El-Abiad, "A simplified determination of transient stability," IEEE Trans. Power App. and Sys., Vol. PAS-94, March/April 1975, pp. 672-688.

[20] A. Araposthatis, S. Sastry, and P. Varaiya, "Analysis of power-flow equation," Int. J. Electrical Power Energy Systems, Vol. 3, July 1981, pp. 115-126.

[21] C.A. Desoer, E.S. Kuh, Basic Circuit Theory, McGraw-Hill, 1969.

[22] F.F. Wu, R.L. Sullivan, "Nonlinear resistive circuit models for power system steady-state analysis," Proc. 1976 Allenton Conference.

[23] W. Hahn, Theory and Application of Liapunov's Direct Method, Prentice Hall, 1963.

[24] J. Dieudonne, Foundations of Modern Analysis, Academic Press, 1960.

[25] C.A. Desoer, Notes for a second course on linear systems, D. Van Nostrand, 1970.

[26] G.W. Stewart, Introduction to matrix computations, Academic Press, 1973.

[27] I. Hajj, K. Singhal, J. Vlach, "Efficient analysis of nonlinear networks by piecewise linear approximations," Proc. 11th Allerton Conf. on Circuit Theory, Allerton House, IL., October 1973, pp. 635-642.

[28] J.M. Ortega, W.C. Rheinboldt, Iterative Solutions of Nonlinear Equations in Several Variables, Academic Press, 1970.

[29] R.T. Rockafellar, Convex Analysis, Princeton University Press, 1970.