SYNTHESIS OF HIGHER ORDER NONLINEAR CIRCUIT ELEMENTS

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ABSTRACT

Higher- and mixed-order n-port circuit elements were introduced recently to provide a logically complete formulation for nonlinear circuit theory. In this paper, higher-order mutators are defined and used to synthesize these elements. The class of all higher-order mutators is shown to form a group under cascade interconnections. Each mutator is realized using only linear capacitors, linear inductors and linear controlled sources. An upper bound on each type of element needed to realize a mutator is also given. Each higher- or mixed-order n-port element is realized by cascading appropriate mutators across each port of a nonlinear n-port resistor. Our main theorem shows that any higher- or mixed-order nonlinear n-port element with a constitutive relation defined on a compact set can be realized using linear capacitors, inductors and controlled sources and 2-terminal nonlinear resistors.

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I. INTRODUCTION

Higher- and mixed-order circuit elements are unconventional circuit elements introduced recently for several reasons [1]:(1) There are many nonlinear device phenomena which cannot be modelled using only conventional circuit elements. (2) Nonlinear circuit models containing only conventional circuit elements could exhibit impasse points, thereby implying that the model is non-physical and inadequate for computer simulation. (3) No nonlinear higher-order 2-terminal element can be synthesized using only conventional and/or other higher-order 2-terminal elements. Hence, each element has an independent identity. (4) A logically consistent foundation of nonlinear circuit synthesis can not be built using only conventional circuit elements.

Our objective in this paper is to introduce higher-order mutators as the building block for higher- and mixed-order circuit elements. We will present a unifying procedure for realizing any higher- or mixed-order n-port element. Moreover, we will derive an upper bound on the number of basic circuit elements needed in the synthesis.

Our study of higher- and mixed-order elements begins with two-terminal elements. Let v be the voltage across the terminals of the element and i be the current flowing through the element.[1] For $z = v$ or i and $k = 0,1,2,\ldots,$ we define

$$z^{(k)}(t) = \frac{d^k z(t)}{dt^k} \tag{1.1}$$

and

$$z^{(-k)}(t) = z^{(-k)}(0) + \int_0^t z^{(-k+1)}(t)dt \tag{1.2}$$

where $z^{(-k)}(0)$ is an arbitrary constant.

Definition 1.1 A two-terminal element E defined by a constitutive relation involving at most two dynamically independent variables [1] $v^{(\alpha)}$ and $i^{(\beta)}$ where $\alpha,\beta=0, \pm1, \pm2,\ldots)$ is called a $v^{(\alpha)}-i^{(\beta)}$ element or an algebraic higher-

[1]We shall adopt the standard associated reference direction throughout; i.e. current always flows from the positive terminal of the element to the negative terminal.

order element. It is represented by the symbol in Figure 1.

The notion of a higher- or mixed-order element introduced in the above definition can be extended to the case of (n+1)-terminal or n-port elements. In the following, we consider an n-port N whose port voltages and currents are denoted by $v=(v_1, v_2, \ldots, v_n)^T$ and $i=(i_1, i_2, \ldots, i_n)^T$, respectively.

Definition 1.2: Let N be an algebraic n-port characterized by the relation $h(\xi, \eta)=0$, where $(\xi,\eta) \in \mathbb{R}^n x \mathbb{R}^n$ denotes a pair of dynamically independent variables [1]. N is called a higher-order algebraic n-port element if the following are satisfied simultaneously:

(i) For $j = 1,2,\ldots,n$, $(\xi_j, \eta_j) = (v_j^{(\alpha_j)}, i_j^{(\beta_j)})$, and

(ii) $\alpha_1 = \alpha_2 = \ldots = \alpha_n$ and $\beta_1 = \beta_2 = \ldots = \beta_n$.

Otherwise, N is called a mixed-order algebraic n-port element.

The purpose of this paper is to study the problem of synthesizing the elements introduced in Definitions 1.1 and 1.2 above. We shall use the term "higher-order elements" loosely to include both higher- and mixed-order elements, unless otherwise stated. In sections 2 and 3, we are concerned with synthesizing a new class of linear algebraic 2-ports -- the higher-order mutators. The results presented in these two sections will be used in section 4, which deals with the general problem of synthesizing a higher- or mixed-order n-port element.

2. HIGHER-ORDER MUTATORS

Before we proceed with a general method for synthesizing any higher- or mixed-order n-port element, we need to introduce a new class of linear algebraic 2-ports, known as higher-order mutators.

Definition 2.1: A type 1 or type 2 (α_1, β_1) - (α_2, β_2) mutator, or higher-order mutator (abbr., h.o.m.) is a linear two-port with constitutive relations:

$$\text{Type 1} \quad \left\{ \begin{aligned} v_1^{(\alpha_1)} &= v_2^{(\alpha_2)} \\ i_1^{(\beta_1)} &= -i_2^{(\beta_2)} \end{aligned} \right\} \qquad \text{Type 2} \quad \left\{ \begin{aligned} v_1^{(\alpha_1)} &= -i_2^{(\beta_2)} \\ i_1^{(\beta_1)} &= v_2^{(\alpha_2)} \end{aligned} \right\} \tag{2.1}$$

It is represented by the symbol in Figure 2.

Assuming zero initial conditions, the frequency domain representation of a h.o.m. is given by:

Type 1 $\qquad\qquad\qquad\qquad$ Type 2

$$\begin{bmatrix} V_1(s) \\ I_1(s) \end{bmatrix} = \begin{bmatrix} s^{\alpha_2-\alpha_1} & 0 \\ 0 & s^{\beta_2-\beta_1} \end{bmatrix} \begin{bmatrix} V_2(s) \\ -I_2(s) \end{bmatrix}, \quad \begin{bmatrix} V_1(s) \\ I_1(s) \end{bmatrix} = \begin{bmatrix} 0 & s^{\beta_2-\alpha_1} \\ s^{\alpha_2-\beta_1} & 0 \end{bmatrix} \begin{bmatrix} V_2(s) \\ -I_2(s) \end{bmatrix} \tag{2.2}$$

where $V_j(s)$, $I_j(s)$ $(j = 1,2)$ denote the (one-sided) Laplace Transform of v_j, i_j, respectively.

From equation (2.2) we can see that the h.o.m. is just a special case of a Generalized Impedance Converter (GIC) [2], which is a common building block used in the realizations of active filters. (Of course, for nonzero initial conditions, the above frequency domain representations are invalid, just as they are for the GIC). For example, the frequency-dependent negative resistance (FDNR) is now a commercially available component which can be synthesized by an appropriate interconnection of GIC's [2]. Motivated by this observation, we shall show that just as GIC's can be used to realize the FDNR (which is a linear higher-order element), higher-order mutators can be used to realize any nonlinear[3] higher-order element.

Higher-order mutators exhibit two important properties which enable us to use them in synthesizing higher-order elements:

1. Mutation property: Just as its name suggests, the higher-order mutator can transform a 2-terminal element associated with the variables $(v^{(\alpha_2)}, i^{(\beta_2)})$ to one whose constitutive relation is between the variables $(v^{(\alpha_1)}, i^{(\beta_1)})$. More precisely, if we terminate port 2 of a type 1 h.o.m. by a higher-order element characterized by $h(v^{(\alpha_2)}, i^{(\beta_2)}) = 0$, the resulting 1-port is equivalent to a higher order element characterized by $h(v^{(\alpha_1)}, i^{(\beta_1)}) = 0$. Similarly, terminating port 2 of a type 2 h.o.m. by an element described by $h(v^{(\alpha_2)}, i^{(\beta_2)}) = 0$ results in a higher-order element at port 1 characterized by $h(i^{(\beta_1)}, v^{(\alpha_1)}) = 0$. Note that a type 2 mutator transforms (α_2, β_2) to (α_1, β_1), and the roles of

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[3]Throughout this paper, we shall use the term "nonlinear" to denote linear or strictly not linear, unless otherwise specified.

the voltages and currents are also interchanged. It may be illuminating to view this mutation property through the diagram in Figure 3.

As a result of the mutation property, any 2-terminal higher-order element characterized by $h(v^{(\alpha)}, i^{(\beta)}) = 0$ can be constructed by connecting a nonlinear resistor described by $h(v,i) = 0$ (or $h(i,v) = 0$) across port 2 of a type 1 (or type 2) (α,β) - $(0,0)$ h.o.m. Hence the problem of synthesizing any 2-terminal higher-order element is reduced to that of synthesizing a type 1 or 2 (α,β) - $(0,0)$ h.o.m., and a nonlinear 2-terminal resistor. Since there exist many well-known techniques for the synthesis of nonlinear resistors [3-5], we shall initially concern ourselves only with the problem of synthesizing a h.o.m.

2. Closure Property:

Definition 2.2: Ports j and k associated with an algebraic n-port N, or with two algebraic n-ports N_1 and N_2 are said to be compatible if the respective port variables $(v_j^{(\alpha_j)}, i_j^{(\beta_j)})$ and $(v_k^{(\alpha_k)}, i_k^{(\beta_k)})$ satisfy: $\alpha_j = \alpha_k$ and $\beta_j = \beta_k$.[4]

A straightforward application of the definition of a higher-order mutator will show that a compatible interconnection of two higher-order mutators always results in another higher-order mutator where the port variables associated with the unconnected ports remain unchanged. In particular, denoting the operation of a compatible interconnection by the symbol "+", we have [5]:

(a) [type 1 (α_1,β_1)-(α_2,β_2)] + [type 1 (α_2,β_1)-(α_3,β_3)] = type 1 (α_1,β_1)-(α_3,β_3)

(b) [type 1 (α_1,β_1)-(α_2,β_2)] + [type 2 (α_2,β_2)-(α_3,β_3)] = type 2 (α_1,β_1)-(α_3,β_3)

(c) [type 2 (α_1,β_1)-(α_2,β_2)] + [type 1 (α_2,β_2)-(α_3,β_3)] = type 2 (α_1,β_1)-(α_3,β_3)

(d) [type 2 (α_1,β_1)-(α_2,β_2)] + [type 2 (α_2,β_2)-(α_3,β_3)] = type 1 $(\alpha_1,\beta_1)(\alpha_3,\beta_3)$

As we shall see shortly, the closure property results in a considerable simplification of our synthesis problem. Although incompatible interconnections between two algebraic 2-ports would normally produce a dynamic 2-port, the following result shows a desirable property shared by all mutators:

---

[4]Note that this definition holds trivially for the case $j = k \in (1,2,\ldots,n)$.

[5]Properties (a) - (d) are reminiscent of the closure under addition of odd and even numbers. In this case, a type 1 higher-order mutator corresponds to an even number and a type 2 higher-order mutator corresponds to an odd number.

Theorem 1

The class of all higher-order mutators introduced in Definition 2.1 forms a group under cascade interconnections with zero initial conditions. ¤

Proof

Since we are considering zero initial conditions, it is more convenient to work in the frequency domain. From equation (2.2), we can see that higher-order mutators of both types have a transmission representation. Hence if M_A and M_B are h.o.m.'s with transmission matrices A and B respectively, the transmission matrix resulting from cascading M_A and M_B (denoted M_A C M_B) is simply the product of AB [6]. It now becomes easy to verify the following.

1. For j = 1, 2, let M_j' denote a type j (α_1',β_1')-(α_2',β_2') h.o.m. and M_j'' denote a type j (α_1'',β_1'')-(α_2'',β_2'') h.o.m.
   a) M_1' C M_1'' results in a Type 1 $(\alpha_1'+\alpha_1'', \beta_1'+\beta_1'')$ - $(\alpha_2'+\alpha_2'', \beta_2'+\beta_2'')$ h.o.m.
   b) M_1' C M_2'' results in a Type 2 $(\alpha_1'+\alpha_1'', \beta_1'+\beta_1'')$ - $(\alpha_2''+\beta_2', \beta_2''+\alpha_2')$ h.o.m.
   c) M_2' C M_1'' results in a Type 2 $(\alpha_1'+\beta_1'', \beta_1'+\alpha_1'')$ - $(\alpha_2'+\alpha_2'', \beta_2'+\beta_2'')$ h.o.m.
   d) M_2' C M_2'' results in a Type 1 $(\alpha_1'+\beta_1'', \beta_1'+\alpha_1'')$ - $(\alpha_2''+\beta_2', \beta_2''+\alpha_2')$ h.o.m.

   Hence the class of all mutators is closed under the operation "C."
2. Since matrix multiplication is associative, it follows that the operation of cascading mutators is also associative.
3. The inverse of a (α_1,β_1)-(α_2,β_2) h.o.m. of either type under the operation "C" is clearly the $(-\alpha_1,-\beta_1)$-$(-\alpha_2,-\beta_2)$ h.o.m. of the same type.
4. The identity element under the operation "C" is a type 1 (0,0)-(0,0) h.o.m.

It follows from facts 1-4 above that under zero initial conditions, the class of all h.o.m.'s described by equation (2.2) forms a group under the operation "C". ¤

3. SYNTHESIS OF A TYPE 1 (α,β) - (0,0) H.O.M.

There are several reasons why we are considering the problem of synthesizing a type 1 (α,β) - (0,0) h.o.m. instead of the more general one of synthesizing a (α_1, β_1) - (α_2,β_2) h.o.m:

1. Our main interest lies in the synthesis of higher-order elements. As we have shown earlier, because of the mutation property of higher-order mutators, all that is required in this synthesis is a type 1 or type 2 (α,β) - (0-0) h.o.m. and a nonlinear resistor.

2. By the closure property, any type 2 (α,β) - $(0,0)$ h.o.m. can be constructed by cascading a type 1 (α,β) - $(0,0)$ h.o.m. with a type 2 $(0,0)$ - $(0,0)$ h.o.m. The latter is simply a gyrator [2], which is a readily available circuit element. Also by the closure property, the (α_1,β_1) - (α_2,β_2) h.o.m. of either type can be constructed by cascading a (α_1,β_1) - $(0,0)$ h.o.m. and a $(0,0)$ - (α_2,β_2) h.o.m.[6] of the same type.

3. The type 1 $(\alpha_1-\alpha_2,\ \beta_1-\beta_2)$ - $(0,0)$ h.o.m. can be considered as a "minimal realization" of a type 1 (α_1,β_1) - (α_2,β_2) h.o.m. By a "minimal realization", we mean a realization in the sense that the zero initial-state responses, and therefore, the frequency responses are the same. We do not claim that the responses due to nonzero initial states are the same; but in many circuit applications, it is conventional to ignore the transient response, since it is of major relevance only in unstable circuits.

Our first result deals with the synthesis of a type 1 (α,β) - $(0,0)$ h.o.m. Starting with any type 1 (α',β') - $(0,0)$ h.o.m., it is possible to obtain a $(\alpha'\pm 1,\beta')$ - $(0,0)$ h.o.m. of the same type by appropriate interconnections with a type 1 $(0,0)$ - $(1,0)$ h.o.m. (which we shall denote by the symbol "A" for brevity). It is also possible to obtain a $(\alpha',\beta'\pm 1)$ - $(0,0)$ h.o.m. of the same type by appropriate interconnections with a type 1 $(0,0)$ - $(0,-1)$ h.o.m. (denoted by the symbol "B"). A straightforward calculation shows that the connections specified as follows will yield the above result:

$$\text{Connecting}\left\{\begin{array}{l}\text{port 2 of A}\\ \text{port 1 of A}\\ \text{port 2 of B}\\ \text{port 1 of B}\end{array}\right\}\begin{array}{l}\text{with port 1 of a type 1}\\ (\alpha',\beta')\text{-}(0,0)\text{ h.o.m.}\\ \text{will result in a}\end{array}\left\{\begin{array}{l}(\alpha'-1,\beta')\text{-}(0,0)\text{ h.o.m.}\\ (\alpha'+1,\beta')\text{-}(0,0)\text{ h.o.m.}\\ (\alpha',\beta'+1)\text{-}(0,0)\text{ h.o.m.}\\ (\alpha',\beta'-1)\text{-}(0,0)\text{ h.o.m.}\end{array}\right\}.$$

Hence, we can start with a type 1 $(0,0)$ - $(0,0)$ h.o.m. and keep increasing or decreasing the orders of v and/or i at port 1 until we end up with the desired type 1 (α,β) - $(0,0)$ h.o.m.. This leads to the following lemma:

<u>Lemma</u>

A type 1 (α,β) - $(0,0)$ h.o.m. can be synthesized with

[6] It is obvious that the synthesis of the $(0,0)$ - (α,β) h.o.m. is no different from that of the (α,β) - $(0,0)$ h.o.m..

$n_{LC} \triangleq |\alpha| + |\beta|$ linear reactances [7] and at most $3n_{LC}$ linear controlled sources.

Proof

A type 1 (0,0) - (1,0) h.o.m. can be synthesized by either of the circuits given in Figure 4a. Each circuit in Figure 4b is a realization of the type 1 (0,0) - (0,-1) h.o.m. Finally, a type 1 (0,0) - (0,0) h.o.m. simply consists of two short circuits, as shown in Figure 4c. Following the scheme discussed previously, we note that for each order of v or i that is added or subtracted, we would require one linear (1H) inductor and/or (1F) capicitor, and three linear controlled sources. Therefore a total of n_{LC} linear reactances and $3n_{LC}$ linear controlled sources are required for synthesizing the type 1 (α,β) - (0,0) h.o.m., as prescribed by the theorem. ¤

As an example, the circuit in Figure 5a realizes a type 1 (-1,-2) - (0,0) h.o.m. by cascading two type 1 (0,0) - (0,-1) h.o.m.'s in Figure 4b with a type 1 (0,0) - (1,0) h.o.m. in Figure 4a. Similarly, the circuit in Figure 5b realizes a type 1 (2,2) - (0,0) h.o.m. by appropriate interconnections of the circuits in Figures 4a and b.

A careful examination of the circuits synthesized by the above procedure shows that it is possible to reduce the number of linear controlled sources used in the synthesis. By extracting the linear reactances in the circuits of Figures 4a and b as extra ports and considering the hybrid representations [6] of the resulting 3-ports, we can use an inductive argument based on the previous lemma to deduce the following:

Theorem 2

A type 1 or type 2 (α,β) - (0,0) h.o.m. can be synthesized with

$$n_{LC} \triangleq |\alpha| + |\beta| \quad \text{linear reactances}$$

and $(n_{LC} + 2)$ linear controlled sources. ¤

As an illustration of this theorem, note that the type 1 (-1,-2) - (0,0) h.o.m. of Figure 5a which was realized using nine linear controlled sources

[7] The term "linear reactances" refers to 1F capacitors and/or 1H inductors in this context.

can now be realized using only five linear controlled sources, as shown in Figure 6.

The proof (by induction) of this theorem is tedious and complicated, and is therefore omitted. The basic idea used in the proof suggests a relatively simple and perhaps more intuitive way of synthesizing any (α,β) - (0,0) h.o.m. We shall illustrate this idea with an example:

Consider a type 1 (α,β) - (0,0) h.o.m., where $\alpha,\beta > 0$. It is easy to verify that the circuit in Figure 7 is indeed a realization of this particular h.o.m. The interesting features worth noting are:

1. Each internal section of the 2-port realizes one step of differentiation.
2. The circuit consists of two independent portions, one for realizing each equation in the definition of the h.o.m.
3. A dual synthesis can be obtained by replacing each capacitor with an inductor and each linear controlled voltage source with a linear controlled current source.
4. There exist other possible solutions by using a mixture of linear capacitors with voltage sources and linear inductors with current sources.

Since integration can be achieved by interchanging currents and voltages in an obvious way in the above, a similar reasoning will yield circuit solutions for any (α,β) - (0,0) h.o.m. (i.e. α,β not necessarily constrained to be non-negative as in the example of Figure 7.) Because such a synthesis would require one linear controlled source to be associated with each linear reactance and each of the two ports, it follows that a total of $(n_{LC} + 2)$ linear controlled sources are required.

For large values of $|\alpha|$ and $|\beta|$, this number is considerably less than that given in the lemma. We wish to point out that in Theorem 1 and its preceding lemma, we are merely trying to show the existence of a realization for the type 1 (α,β) - (0,0) h.o.m. using only <u>linear</u> circuit elements. There is no reason to believe that the proposed method gives the unique synthesis of the higher-order mutator. We can consider the results in this section as providing an <u>upper bound</u> for the number of linear circuit elements required in the synthesis of the higher-order mutator under consideration.

4. SYNTHESIS OF HIGHER AND MIXED ORDER N-PORT ELEMENTS

Recalling the mutation property of higher-order mutators (see Figure 3), the following can easily be deduced from Theorem 1:

Corollary to Theorem 2

A 2-terminal higher-order element characterized by $h(v^{(\alpha)}, i^{(\beta)}) = 0$ can be synthesized using only

$$n_{LC} \triangleq |\alpha| + |\beta| \quad \text{linear reactances}$$

$(n_{LC} + 2)$ linear controlled sources and a 2-terminal nonlinear resistor characterized by $h(v,i) = 0$. ¤

Figure 8 contains an example of this corollary. It realizes a higher-order element described by $i^{(2)} = g(v^{(2)})$ by cascading a 2-terminal resistor having the same constitutive relation with a type 1 (2,2) - (0,0) h.o.m..

We can now turn our attention to the synthesis of algebraic higher- or mixed-order n-port elements:

Theorem 3

Every higher- or mixed-order algebraic n-port element characterized by $h(\xi,\eta) = 0$ (cf definition 1.2) can be synthesized using only higher-order mutators and a nonlinear n-port resistor. ¤

Proof

Denote by R an n-port resistor characterized by $h(v,i) = 0$. Let the j^{th} component of (ξ,η) be $(\xi_j, \eta_j) = (v_j^{(\alpha_j)}, i_j^{(\beta_j)})$ or $(i_j^{(\beta_j)}, v_j^{(\alpha_j)})$. Connect the j^{th} port of R to port 2 of a (α_j, β_j) - (0,0) h.o.m. By the mutation property of h.o.m.'s the resulting n-port is characterized by $h(\xi,\eta) = 0$. ¤

In the following, let

$$\xi = (v_1^{(\alpha_1)}, v_2^{(\alpha_2)}, \ldots, v_n^{(\alpha_n)})$$

and

$$\eta = (i_1^{(\beta_1)}, i_2^{(\beta_2)}, \ldots, i_n^{(\beta_n)}).$$

Theorem 4 (Main Result)

Let A denote a compact subset of $\mathbb{R}^n$ and $g: A \to \mathbb{R}^n$ be a C^1 function. Every higher- or mixed-order n-port element characterized by $\xi = f(\eta)$ [or, dually, $\eta = f(\xi)$] can be synthesized using at most

$$n_{LC} \triangleq \sum_{j=1}^{n} |\alpha_j| + |\beta_j| \quad \text{linear reactances,}$$

$n_{LC} \triangleq 3(n_{LC} + n)$ linear controlled sources, and

$$n_R \triangleq n(2n^2 + n + 1) \quad \text{2-terminal nonlinear resistors.}$$

¤

Proof

From the proof of Theorem 3, we already know that n_{LC} linear reactances and $3n_{LC}$ linear controlled sources are required to synthesize the higher-order mutators used in the general synthesis of the n-port enelment. It remains to show that a total of n_R 2-terminal resistors and 3n linear controlled sources are needed to synthesize an n-port resistor characterized by $v = f(i)$ or $i = f(v)$. The proof of this is constructive and follows from an improvement of Kolmogorov's Theorem as introduced in [7]. We shall only consider the case of a voltage-controlled n-port resistor described by $i = f(v)$, where f satisfies the hypothesis of the theorem. A similar proof can be derived for the dual case.

From [7], it follows that each component function

$$i_k = f_k(v_1, v_2, \ldots, v_n)$$

can be expressed in the form

$$i_k = \chi_k \{ \sum_{p=1}^{n} \sum_{q=0}^{2n} (\lambda^p \psi_k(v_p + \varepsilon q) + q) \} \ , \ k = 1, 2, \ldots, n, \tag{4.1}$$

where χ_k and ψ_k are real-valued functions dependent only on f_k, and λ and ε are constants. The synthesis of the n-port resistor is based on equation (4.1), and is shown in Figure 9. For each (p,q,k), the nonlinear resistor $R^k_{p,q}$ is described by

$$i^k_{p,q} = \lambda^p \psi_k(v_p + \varepsilon q) + q \tag{4.2}$$

Note from Figure 9 that

$$I^k_p = \sum_{q=0}^{2n} i^k_{p,q} \tag{4.3}$$

The nonlinear resistor R_k is characterized by

$$V^k = \chi_k(I^k) \quad . \tag{4.4}$$

It is easy to verify that the total number of 2-terminal nonlinear resistors required is

$$n_R = n(pq) + n = n(2n^2 + n + 1)$$

and the number of linear controlled sources required is 3n. ¤

We do not claim that Theorem 4 gives the minimum number of elements used in the synthesis of a higher- or mixed-order n-port element. It merely provides an upper bound on the number of elements required. For example, the 2-port mixed-order element

$$
\begin{aligned}
i_1^{(2)} &= g_1(v_1^{(1)}, v_2^{(-1)}) \\
i_3^{(-3)} &= g_2(v_1^{(1)}, v_2^{(-1)}) \quad ,
\end{aligned}
\tag{4.5}
$$

where $g = (g_1, g_2)$ satisfies the hypothesis of Theorem 4 can be synthesized by the circuit in Figure 10. In this circuit, 22 nonlinear 2-terminal resistors and only 5 linear controlled sources are used in realizing the 2-port resistor R. The nonlinear 2-terminal resistors have the characteristics:

$$
\begin{aligned}
R_{pq}^k &: i_{pq}^k = \lambda^p \psi_k(v_p + \varepsilon q) + p \\
R_k &: V_k = \chi_k(I_k)
\end{aligned}
\tag{4.6}
$$

where ψ_k and χ_k are the functions of one variable used in the decomposition of the original functions g_k of the 2-port resistor:

$$
\begin{aligned}
i_1 &= g_1(v_1, v_2) \\
i_2 &= g_2(v_1, v_2)
\end{aligned}
\tag{4.7}
$$

5. CONCLUDING REMARKS

So far, we have shown that it is possible to synthesize any algebraic higher- or mixed-order n-port element using only linear reactances and controlled sources, and 2-terminal nonlinear resistors. Based on our above results, it would be an interesting and challenging problem to find a synthesis of these elements using only operational amplifiers (op amps) and 2-terminal elements (namely, linear reactances and nonlinear resistors) as basic building blocks. By analogy with known results for active circuit synthesis, it may be possible to obtain a bound on the number of op amps needed that is comparable to the bound for the linear controlled sources given in theorem 1.

As far as a direct op amp realization is concerned, we have successfully built two circuits which function like 2-terminal higher-order elements. The first is the circuit shown in Figure 11, which is a realization of the FDNR (cf. section 2), described by $i = mv^{(2)}$, with $m > 0$. The circuit shown in Figure 12 is a realization of the 2-terminal higher-order element characterized by $i^{(2)} = g(v^{(2)})$. From laboratory measurements, when the nonlinear resistor R is a diode, the resulting one-port mimicks (within a reasonable range of operational frequencies) the h.o.e. described by $i^{(2)} = I_s(e^{v^{(2)}/v_T} - 1)$, i.e. the $i^{(2)}$ vs $v^{(2)}$ characteristic is identical to that of the diode characteristic.

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FIGURE CAPTIONS

Figure 1. A 2-terminal higher order element.

Figure 2. A $(\alpha_1, \beta_1) - (\alpha_2, \beta_2)$ h.o.m. of type 1 or 2.

Figure 3. Mutation property of the h.o.m.

Figure 4a. 2 possible realizations for the type 1 (0,0) - (1,0) h.o.m.

Figure 4b. 2 possible realizations for the type 1 (0,0) - (0,-1) h.o.m.

Figure 4c. A type 1 (0,0) - (0,0) h.o.m.

Figure 5b. A type 1 (2,2) - (0,0) h.o.m.

Figure 6. A type (-1,-2) - (0,0) h.o.m. realized with fewer linear controlled sources.

Figure 7. A type 1 (α,β) - (0,0) h.o.m. with $\alpha,\beta > 0$.

Figure 8. A realization of the 2-terminal higher-order-element: $i^{(2)} = g(v^{(2)})$.

Figure 9. Synthesis of the n-port resistor $i = g(v)$ based on equation (4.1).

Figure 10. A realization of the mixed order 2-port element of equation (4.5).

Figure 11. A realization of the FDNR.

Figure 12. A realization of the 2-terminal higher-order element $i^{(2)} = f(v^{(2)})$.

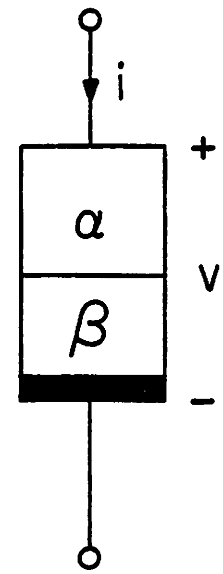


Fig. 1

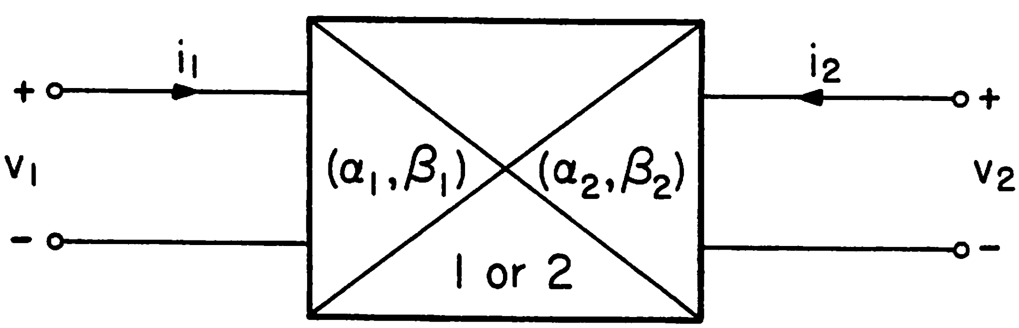
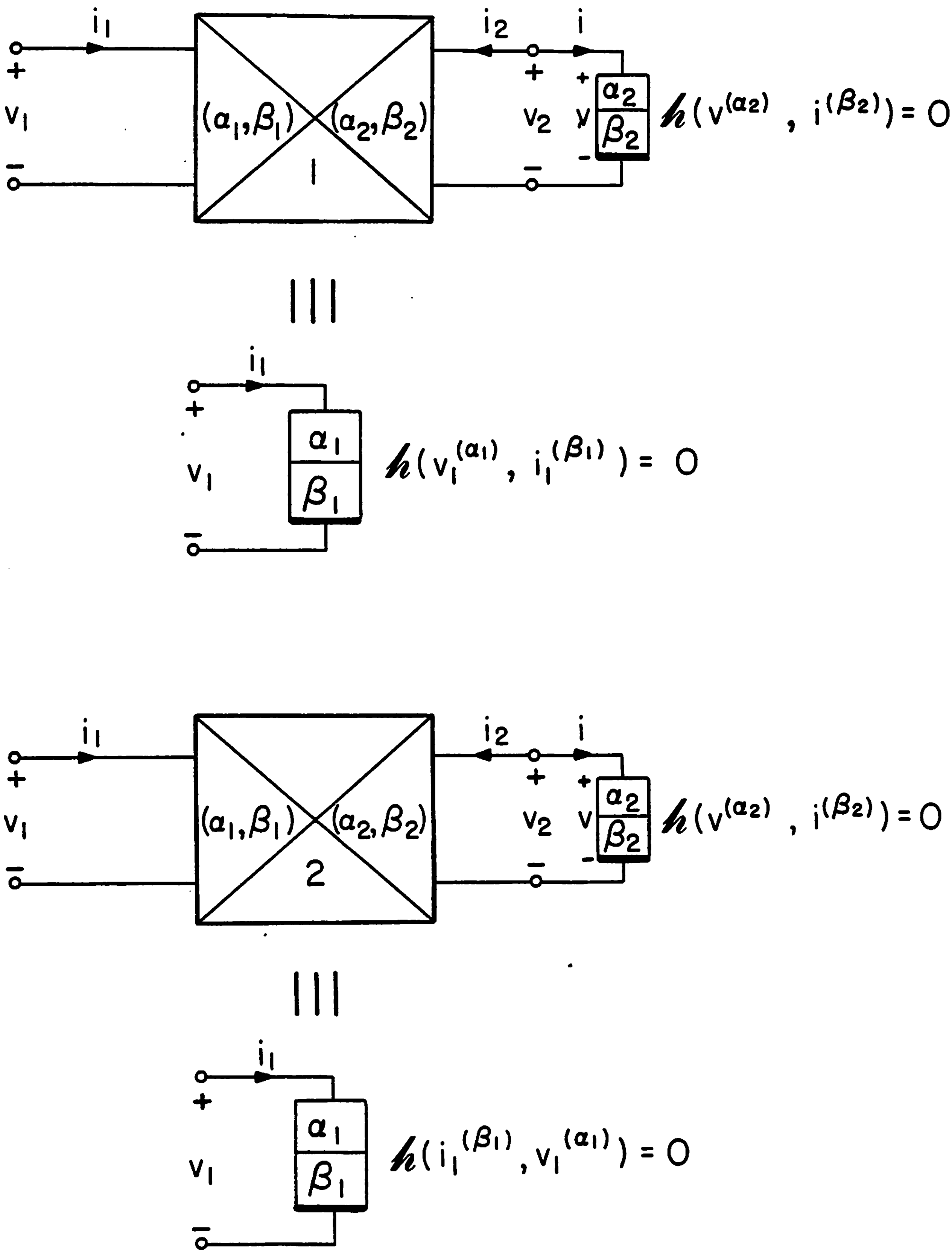
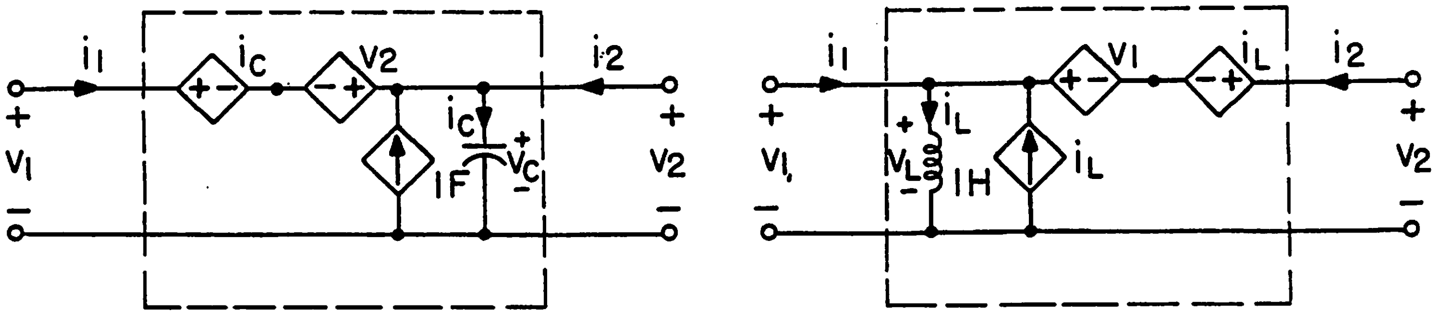


Fig. 2



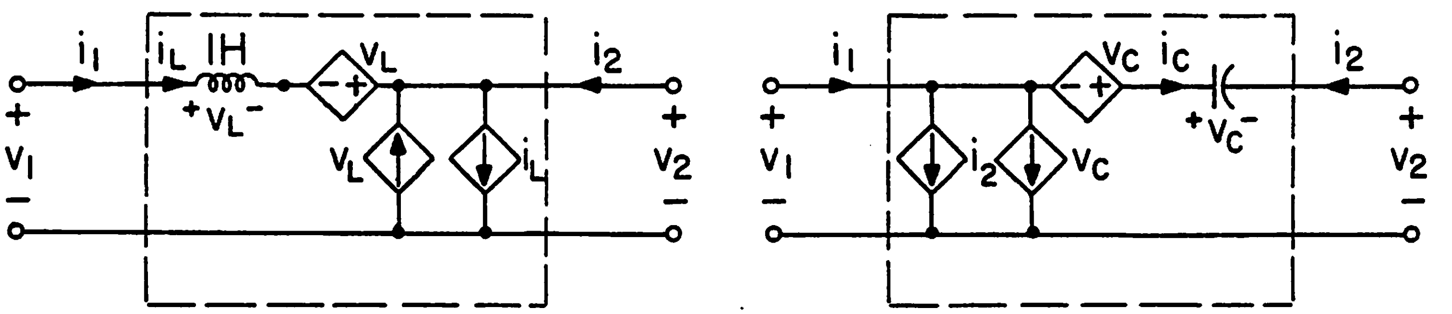
i1
i2
i
v1
v2
v
(α1,β1)
(α2,β2)
1
2
α2
β2
α1
β1
h(v(α2), i(β2)) = 0
h(v1(α1), i1(β1)) = 0
h(i1(β1), v1(α1)) = 0

Fig. 3



(i) (ii)

Fig. 4a



(i) (ii)

Fig. 4b

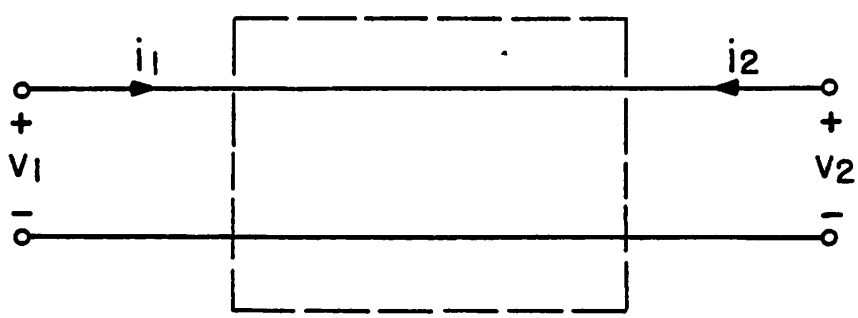
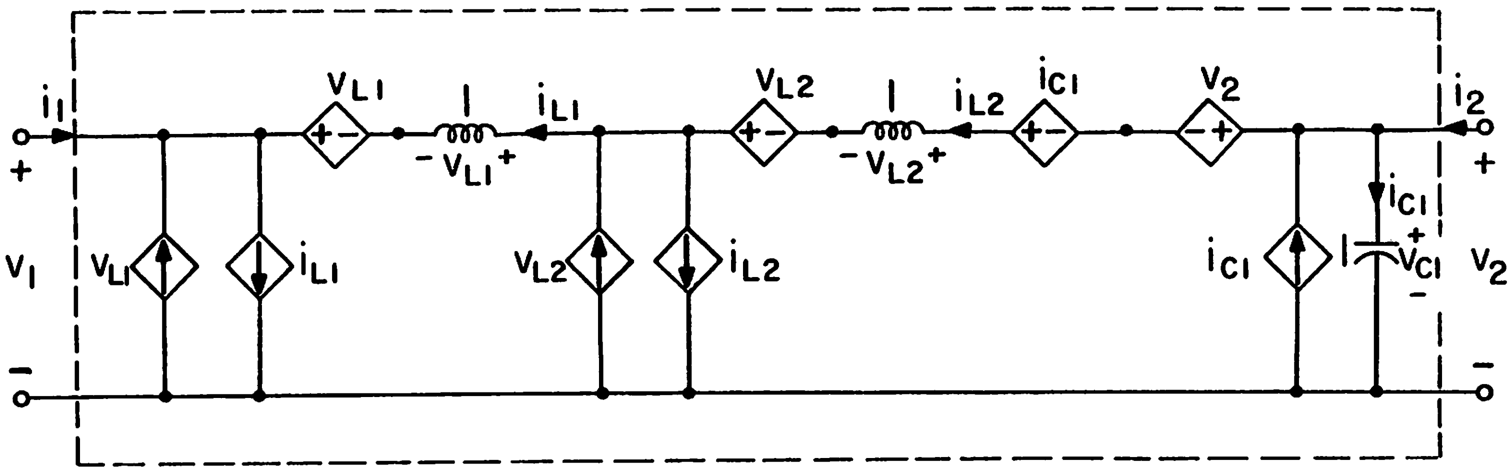
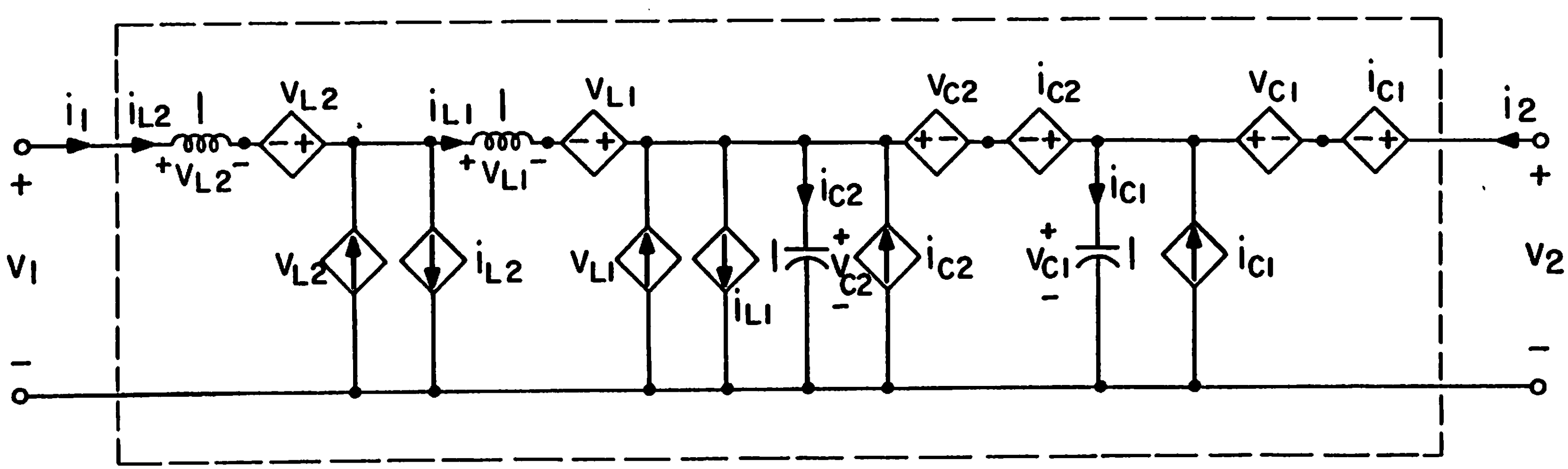


Fig. 4c



i_1
v_{L1}
1
i_{L1}
$-v_{L1}+$
v_{L2}
1
i_{L2}
$-v_{L2}+$
i_{C1}
v_2
i_2
+
v_1
v_{L1}
i_{L1}
v_{L2}
i_{L2}
i_{C1}
i_{C1}
1
v_{C1}
v_2

Fig. 5a



i_1
i_{L2}
1
v_{L2}
$+v_{L2}-$
i_{L1}
1
v_{L1}
$+v_{L1}-$
v_{C2}
i_{C2}
v_{C1}
i_{C1}
i_2
+
v_1
v_{L2}
i_{L2}
v_{L1}
i_{L1}
i_{C2}
1
v_{C2}
i_{C2}
i_{C1}
v_{C1}
1
i_{C1}
v_2

Fig. 5b

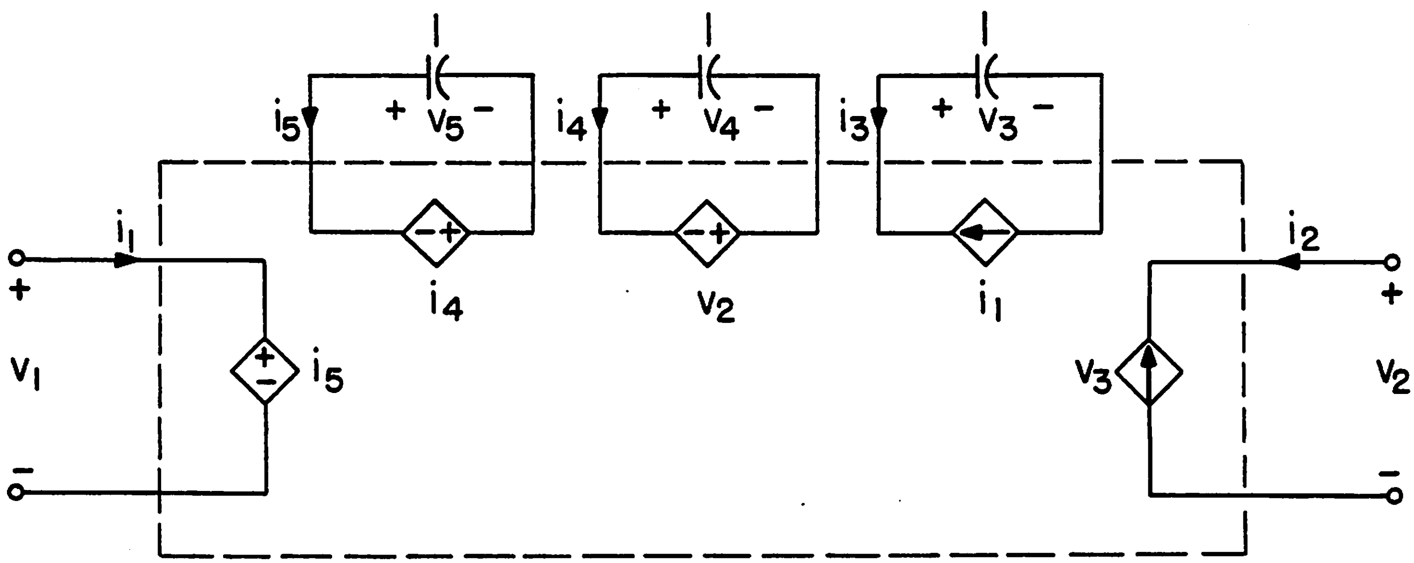
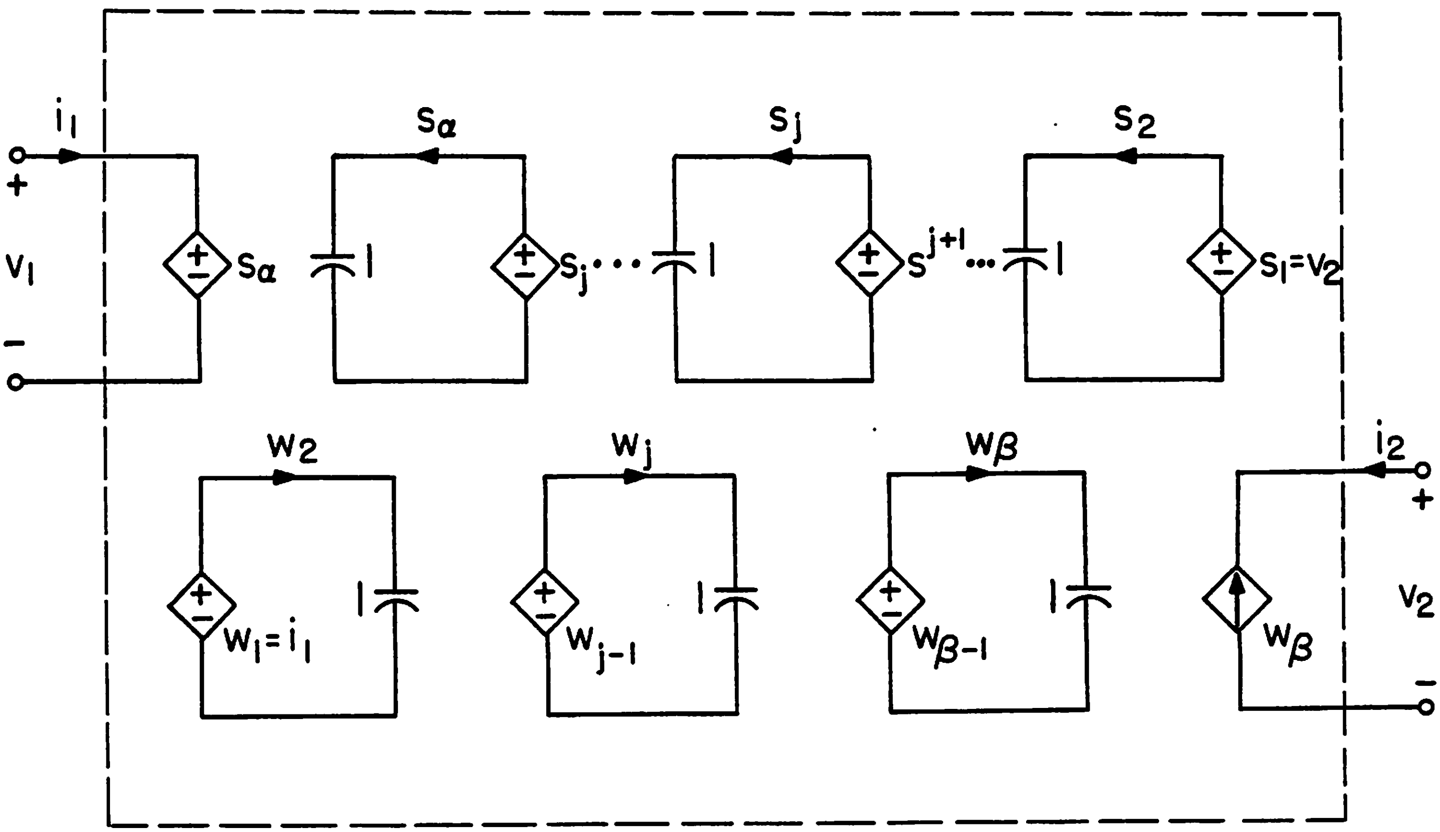


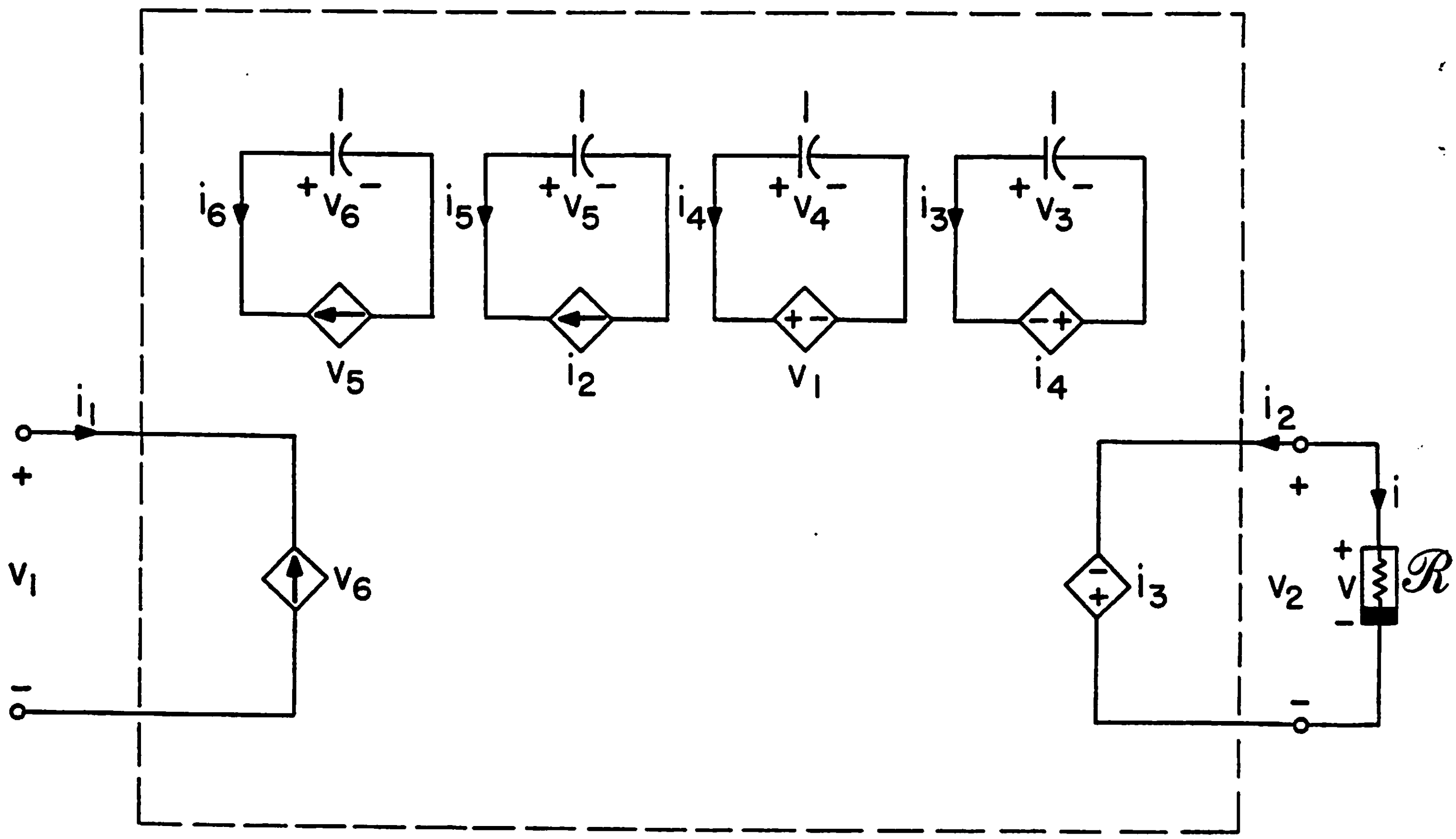
Fig. 6



$$s_j = s_{j+1}^{(1)} \qquad \forall j = 1, 2, \ldots, \alpha$$

$$\text{and } w_k = w_{k-1}^{(1)} \qquad \forall k = 1, 2, \ldots, \beta$$

Fig. 7



$\mathscr{R}$: $i = g(v)$

Fig. 8

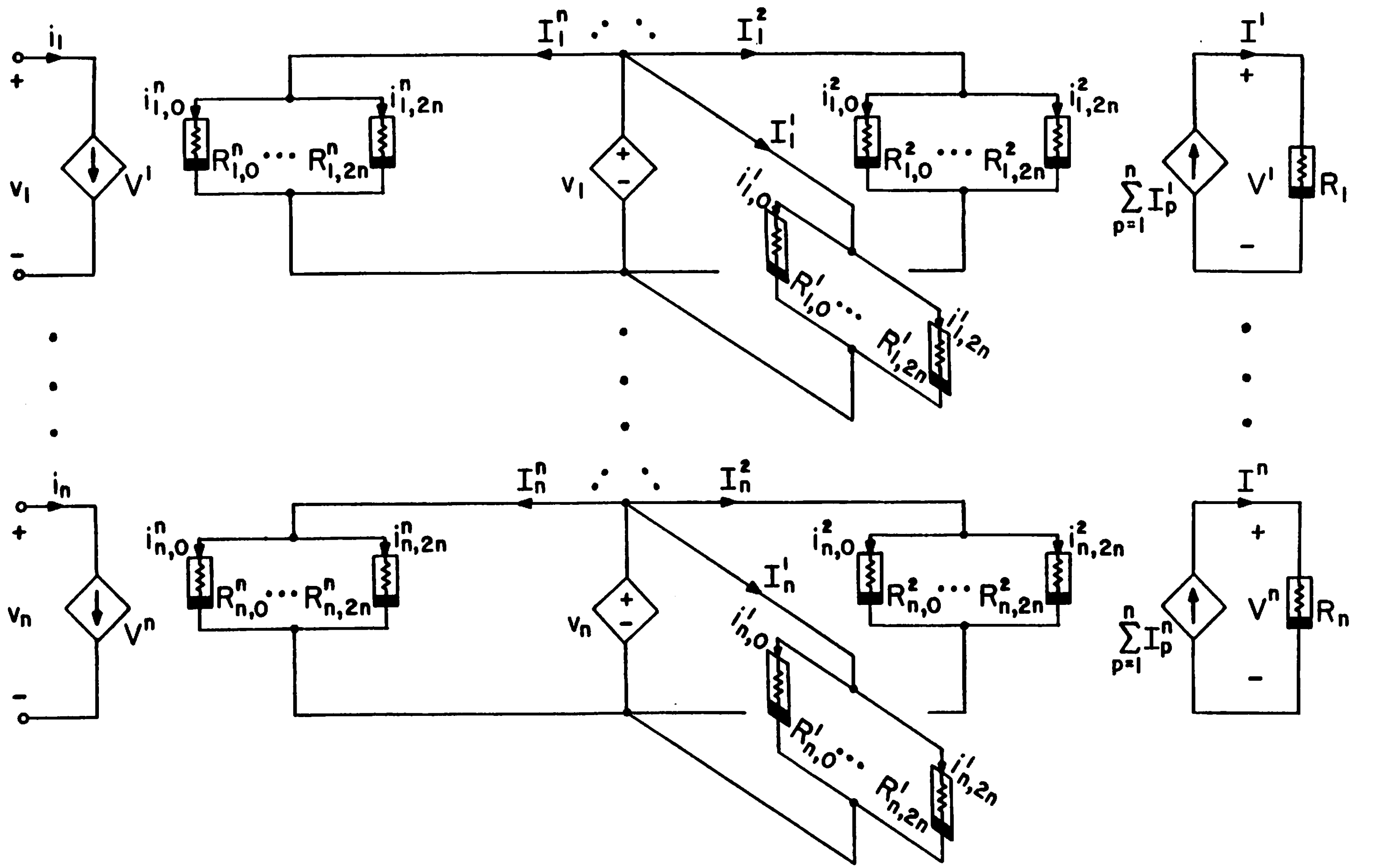


Fig. 9

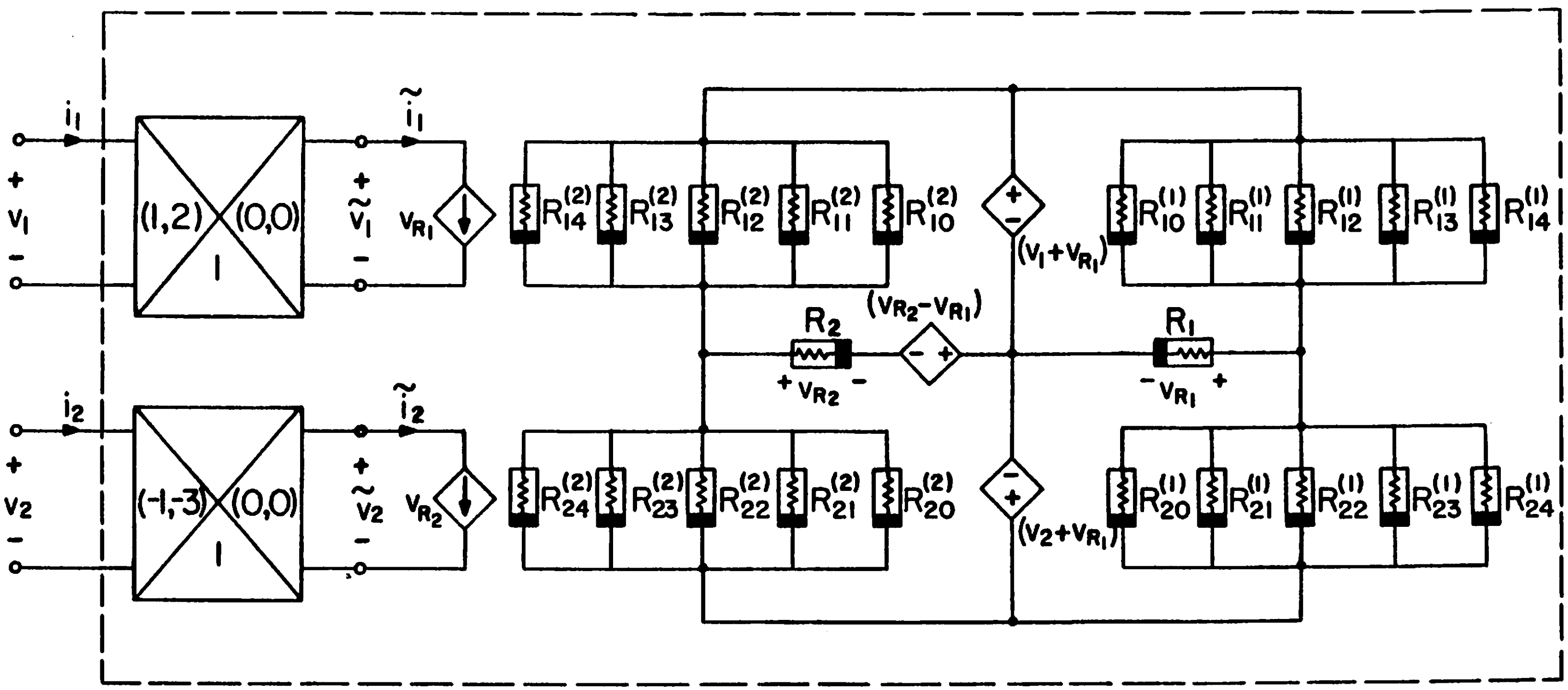
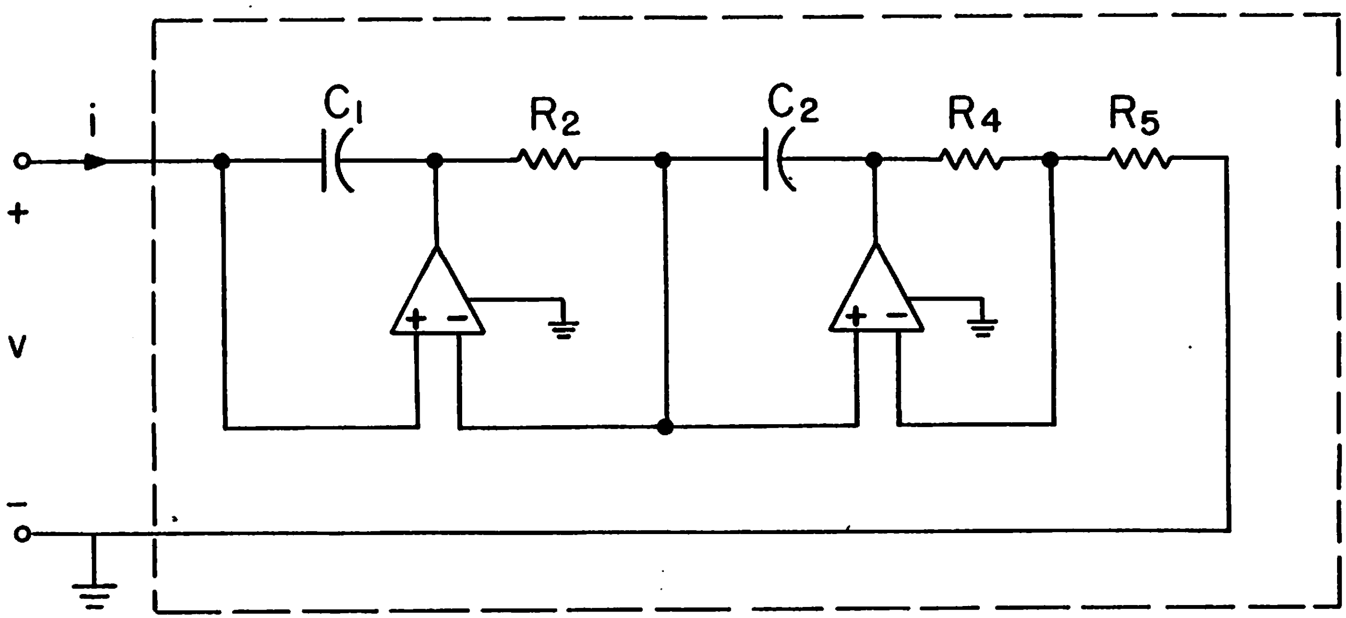


Fig. 10



$$m = \frac{(C_1 C_2 R_2 R_4)}{R_5}$$

Fig. 11

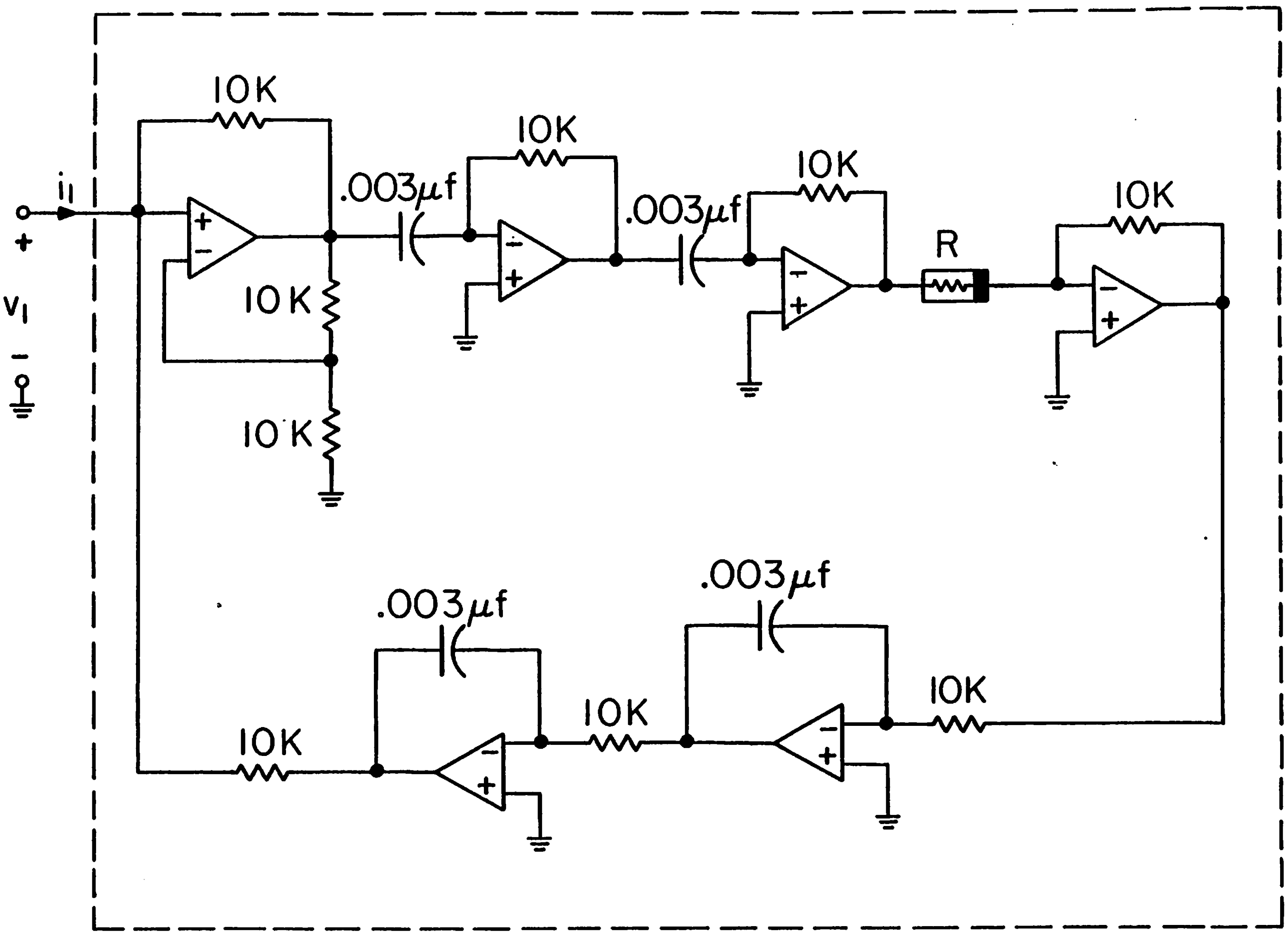

10K
10K
10K
10K
i_1
+
v_1
−
.003μf
.003μf
R
10 K
10 K
.003μf
.003μf
10K
10K
10K

Fig. 12