STATE SPACE THEORY OF NONLINEAR TWO-TERMINAL HIGHER-ORDER ELEMENTS

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Abstract

Higher-order elements have been introduced to provide a logically complete formulation for nonlinear circuit theory. A distinctive feature of higher-order elements is that they possess internal dynamics that are more complicated than those of conventional circuit elements (namely, the resistor, inductor, capacitor and memristor). In this paper, we shall provide a state space formulation for studying two-terminal higher-order elements. State-space properties such as local controllability, input-observability, passivity and losslessness will be investigated in great detail.

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1. Introduction

Higher-order elements[1] have been introduced in [1] to provide a logically complete formulation for nonlinear circuit theory. In [2], it has been shown that these elements can be synthesized using only linear reactances, linear controlled sources and nonlinear two-terminal resistors. The synthesis method indicates that a distinctive feature of higher-order elements is that they possess internal dynamics that are more complicated than those of conventional circuit elements. In this paper, we shall provide an analysis of the state-space properties of two-terminal higher-order elements.

It is our intention to treat each section in the paper as independent. In section 2, we shall show that each two-terminal higher-order element can be described by a state representation which satisfies the usual state-space axioms (see, for example, [3]). Section 3 treats the problems of local-controllability and input-observability. It is advisable to skim through sections 2 and 3 before moving onto sections 4 and 5, which deal with passivity and losslessness, respectively, of two-terminal higher-order elements. At the time of writing of this paper, we feel that it would require a major effort to extend our present results to the case of n-port elements.

2. State Representations of Two-Terminal Higher-Order Elements

Our interest in the state (dynamical) representations of higher-order elements was originally motivated by studying the passivity and losslessness properties of these elements in the setting of [3] and [4]. It turns out that because of the complicated internal dynamics of higher-order elements (abbr., h.o.e.'s), their state representations are of a highly specific form which enables us to draw interesting conclusions concerning their state space properties. Before stating our results, it is necessary to introduce the following concepts:

Definition 2.1 [3]

A state representation S for an n-port is a quintuplet $\{U, \mathcal{U}, \Sigma, E, R\}$ where:

(i) $U \subset \mathbb{R}^n$ is the set of admissible input values,

(ii) $\mathcal{U} = \{u | u : \mathbb{R}_+ \to U\}$ is the set of admissible input waveforms,

(iii) $\Sigma \subset \mathbb{R}^m$ is the state space,

(iv) E is the state equation[2]

$$\dot{x} = f(x,u)$$

where $f(\cdot,\cdot) : \Sigma \times U \to \mathbb{R}^m$, and

[1]Throughout this paper, we shall use the term "higher-order elements" to include both higher- and mixed-order elements as introduced in [1].

[2]The only difference between our definition and that in [3] is that we have excluded the "output equation" from E.

(v) R is a pair of readout maps:

$V : \Sigma \times U \to \mathbb{R}^n$ is the port voltage readout map,

$I : \Sigma \times U \to \mathbb{R}^n$ is the port current readout map. ¤

Definition 2.2 [3]

The power input function $p : \Sigma \times U \to \mathbb{R}^n$ is defined by

$$p(x,u) = \sum_{j=1}^{n} V_j(x,y)\ I_j(x,y).$$ ¤

The state representation S is assumed to satisfy all the state space axioms as stated in [3]. In particular, we would like to remind the reader that the following have to be satisfied before we are able to apply the theory of [3,4] for passivity and losslessness in the later sections:

Standing Assumptions:

(A1) For every $x_0 \in \Sigma$ and every $u(\cdot) \in \mathcal{U}$, there exists a unique solution[3] $x(\cdot) : \mathbb{R}^+ \to \Sigma$ of the differential equation $\dot{x} = f(x,u)$ such that $x(0) = x_0$.

(A2) For every $\{u(\cdot), x(\cdot)\}$ described in (A1), the port voltage and port current of the n-port are, respectively,
$v(t) = V(x(t),u(t))$ and
$i(t) = I(x(t),u(t))$.

(A3) For every pair $\{u(\cdot), x(\cdot)\}$ as described in (A2), the function $t \to p(x(t), u(t))$ is locally L^1 [3].

(A4) The set of admissible input waveforms $\mathcal{U}$ is translation invariant and closed under concatenation, and all functions in $\mathcal{U}$ are measurable [5].

Definition 2.3

$\mathcal{P}(\mathbb{R}_+)$ is defined to be the set of all piecewise continuous functions $g : [t_0,t_1] \to \mathbb{R}$ (where $[t_0,t_1]$ denotes any finite interval in $\mathbb{R}_+$) with a finite number of discontinuities. ¤

In the following, we shall concern ourselves only with a 2-terminal h.o.e. described explicitly by

$$v^{(\alpha)} = f(i^{(\beta)}) \tag{2.1}$$

[3] x is a solution to the differential equation $\dot{x} = f(x,u)$ if $x(t) : \mathbb{R}_+ \to \Sigma$ is absolutely continuous on every bounded interval $[0,T]$ with $T \geq 0$, and satisfies $\dot{x} = f(x,u)$ for almost all t.

Dual results can be derived for the representation

$$i^{(\beta)} = g(v^{(\alpha)}) \tag{2.2}$$

by interchanging the roles of $v^{(\alpha)}$ and $i^{(\beta)}$. Depending on the integral values of α and β, the h.o.e. of equation (2.1) can have different state representations. The state representations $S = \{U, \mathcal{U}, \Sigma, E, R\}$ for the case $\alpha \geq 0$ are listed in tables 1a and 1b; and those for $\alpha < 0$ can be found in table 1c.

For the case $\alpha < 0$, we need to consider the following equivalent representation for the h.o.e. of equation (2.1):

$$\begin{aligned} v &= \frac{d^{|\alpha|}}{dt^{|\alpha|}} f(z)\Big|_{z=i^{(\beta)}} \\ &\triangleq \tilde{f}(i^{(\beta)}, i^{(\beta+1)}, \cdots, i^{(\beta-\alpha)}) \; . \end{aligned} \tag{2.3}$$

Note that the function f in equation (2.1) has to be at least $|\alpha|$ times differentiable for the existence of the representation in equation (2.3). We say that these two representations are "equivalent" in the sense that every $(i^{(\beta)}, v^{(\alpha)})$ satisfying the constitutive relation of (2.1) gives rise to $(i^{(\beta)}, i^{(\beta+1)}, \cdots, i^{(\beta-\alpha)}, v)$ which satisfies (2.3). Conversely, we can pass from (2.3) to (2.1) by integrating the former equation $|\alpha|$ times, and taking the initial conditions $v^{(-1)}(0), \cdots, v^{(\alpha)}(0)$ into account. We wish to bring up this point here because the role of these initial conditions is of particular importance when we consider passivity and losslessness in the later sections of this paper. It may be of interest to note that representation (2.3) is precisely what is needed in considering the state representation of a charge controlled memristor in [6].

From Table 1, we see that there are certain cases in which the "input space" $\mathcal{U}$ consists of unusual, or rather, unconventional "inputs," namely $i^{(\lambda)}$, for $\lambda \geq 1$. It is to be noted that such a choice is chosen purely for mathematical convenience, and that it does not affect the validity of our state representations. The different choice of $\mathcal{U}$ for each case enables us to prove that the state representations listed in Tables 1 satisfy assumption (A1), i.e. existence and uniqueness of solutions.

Theorem 2.1.

Consider the 2-terminal higher-order element described by equation (2.1) and the corresponding state representations listed in Table 1. Assuming that $\mathcal{U}$, Σ and f satisfy the conditions given in the tables, then the standing assumptions (A1)-(A4) (stated previously) are satisfied. ¤

For the case $\alpha \geq 0$, all that is required for existence and uniqueness of solutions is for the function f to take on finite values almost everywhere in $\mathbb{R}$. This is a much weaker condition than the usual continuity assumptions needed in proving existence and uniqueness of solutions to state equations [7]. We shall see shortly that the continuity restrictions on f can be relaxed because of the highly specific form of the state equations for h.o.e.'s. The result of this theorem enables us to consider passivity and losslessness in the (state-space) framework of [3,4] in our later sections.

Before proving the theorem, we need the following result:

Lemma 2.1

Consider the set of equations

$$\left\{ \begin{aligned} \dot{x}_1(t) &= \rho(t) \\ \dot{x}_2(t) &= x_1(t) \\ &\vdots \\ \dot{x}_m(t) &= x_{m-1}(t) \end{aligned} \right\} \tag{2.4}$$

where $\rho(t)$ is integrable[4] over any finite interval in $\mathbb{R}_+$. There exists a unique solution $x(t)$ to equation (2.4). ¤

Proof of Lemma 2.1

Since $\rho(t)$ is integrable over any finite interval in $\mathbb{R}_+$,

$$x_1(t) = \int_{t_0}^{t} \rho(\tau)d\tau + x_1(t_0) \tag{2.5}$$

is absolutely continuous on the interval $[t_0,t]$ and $\dot{x}_1(\tau) = \rho(\tau)$ almost everywhere (a.e.) on $[t_0,t]$ [5]. Since $x_1(t)$ is absolutely continuous, it is also integrable on $[t_0,t]$ [5], so we have

$$x_2(t) = \int_{t_0}^{t} x_1(\tau)d\tau + x_2(t_0) \ ,$$

where $x_2(t)$ is absolutely continuous and $\dot{x}_2(\tau) = x_1(\tau)$ a.e. on $[t_0,t]$.

Using similar arguments, we can show that for $k = 2,3, \cdots,m$,

$$x_k(t) = \int_{t_0}^{t} x_{k-1}(\tau)d\tau + x_k(t_0) \ ,$$

[4] By "integrable," we mean Lebesgue-integrable [5]

where $x_k(\tau)$ is absolutely continuous on $[t_0,t]$ and $\dot{x}_k(\tau) = x_{k-1}(\tau)$ a.e. on $[t_0,t]$. Therefore there exists a solution (cf footnote 3) to equation (2.4).

For the uniqueness part of the proof, we assume that the solution obtained by the above process is not unique, i.e. there exist $x(t)$ and $\hat{x}(t)$ satisfying equation (2.4), with

$$x_k(t_0) = \hat{x}_k(t_0), \quad \text{for all} \quad k = 1,2, \cdots,m.$$

Define $e_k(t) = x_k(t) - \hat{x}_k(t)$ for each k. From equation (2.5),

$$\dot{e}_1(t) = 0 \quad \text{a.e. on } [t_0,t],$$

which means that

$$|\dot{e}_1(t)| = 0 \; . \tag{2.6}$$

Since $x_1(t)$ is integrable over any finite interval $[t_0,t]$ in $\mathbb{R}_+$, so is $e_1(t)$; and since $e_1(t)$ is absolutely continuous on $[t_0,t_1]$ (because $x_1(t)$ and $\hat{x}_1(t)$ are both absolutely continuous), we get [8]

$$\left|\int_{e_1(t_0)}^{e_1(t)} de_1\right| = \left|\int_{t_0}^{t} \dot{e}_1(\tau)d\tau\right| \leq \int_{t_0}^{t} |\dot{e}_1(\tau)|d\tau = 0 \; .$$

Therefore

$$E_1(\tau) \triangleq \int_{e_1(t_0)}^{e_1(\tau)} de_1 = e_1(\tau) - e_1(t_0) = 0$$

for all $\tau \in [t_0,t]$, i.e.,

$$e_1(\tau) = e_1(t_0) = 0, \text{ or}$$

$$x_1(\tau) = \hat{x}_1(\tau) = 0 \quad \text{for all } \tau \in [t_0,t] \; . \tag{2.7a}$$

A similar argument can be used to show that

$$x_k(\tau) = \hat{x}_k(\tau) \quad \text{for all } \tau \in [t_0,t] \tag{2.7b}$$

It follows from (2.7a) and (2.7b) that $x = \hat{x}$ on every finite interval $[t_0,t]$ and therefore the solution is unique. ¤

<u>Partial Proof of Theorem 2.1</u>

We shall show that under the conditions listed in Tables 1a-c, each assumption for the state representation is satisfied.

(A2) It is obvious from the tables that R consists of the port voltage and current readout maps.

(A4) Since all L^p and locally L^p functions from $\mathbb{R}_+ \to \mathbb{R}$ are translation invariant and closed under concatenation [3], the result follows immediately from Tables 1a-c.

(A1) and (A3) The proof for these parts involves a case by case study of the various values of α and β as given in Table 1. We shall consider only two of these cases here, which should be illustrative of the general strategy used in proving the theorem. The proof for the other cases can be found in Appendix A.

<u>$\alpha = 0,\ \beta \leq 1$</u> (i.e., case ① (i) in Table 1a)

- Existence and uniqueness of a solution to the state equation E follows from Lemma 2.1.
- From the existence part of the proof of Lemma 2.1, $x_{|\beta|}(t)$ is continuous, and hence Lebesgue-measurable on any finite interval I in $\mathbb{R}_+$ [5]. By hypothesis, f is Borel measurable, which implies that $f(x_{|\beta|}(t))$ is Lebesgue measurable on I [12]. Since $|f|$ is integrable on I by hypothesis, $v(t) = f(x_{|\beta|}(t))$ is also integrable on I, and is therefore $L^1_{loc}(\mathbb{R}_+ \to \mathbb{R})$ [8]. Since $i(t) = u(t) \in L_{loc}(\mathbb{R}_+ \to \mathbb{R})$ in this case, we have, for the power input function (cf Definition 2.2)

$$p(t) = i(t)\ v(t) \in L^1_{loc}(\mathbb{R}_+ \to \mathbb{R})$$

by Hölder's Inequality [5]. So (A3) is satisfied.

<u>$\beta = \alpha \leq -1$</u> (i.e., case ③ in Table 1c)

- Existence and uniqueness of a solution to E follows from Lemma 2.1.
- Since $v^{(\alpha)} = f(i^{(\beta)})$, and f is at least $|\alpha|$ times differentiable, we can write

$$v = \tilde{f}(i^{(\beta)}, i^{(\beta+1)}, \cdots, i^{(\beta-\alpha)}) \ , \tag{2.8}$$

where $\tilde{f} = \sum_{j=1}^{|\alpha|} f_j(i^{(\beta)})\ h_j(i^{(\beta+1)}, i^{(\beta+2)}, \cdots, i^{(\beta-\alpha-j+1)})$,

$$f_j(i^{(\beta)}) = \frac{d^j}{dz^j} f(z)\Big|_{z=i^{(\beta)}} \quad \text{for } j = 1,2, \cdots, |\alpha| \ ,$$

and $h_j = \sum_{\sigma}\{K(\sigma) \times \prod_{i=1}^{j} (i^{(\beta+\theta_i)})\}$

with $K(\sigma)$ a constant dependent on σ and σ denoting the set of all permutations such that $\sum_{i=1}^{j} \theta_i = |\alpha|$.

We can rewrite equation (2.8) as

$$v = f_1(i^{(\beta)})\; i^{(\beta-\alpha)} + \sum_{j=2}^{|\alpha|} f_j(i^{(\beta)})\; h_j(i^{(\beta+1)}, \cdots, i^{(\beta-\alpha-j+1)}) \tag{2.9}$$

By hypothesis, $f \in C^{|\alpha|-1}$, therefore f_j is continuous for $j = 1,2, \cdots, |\alpha|-1$ and $f_{|\alpha|}$ is piecewise continuous. Since we are considering the case $\alpha = \beta \leq -1$, we can further decompose equation (2.9) as follows:

$$\begin{aligned} v(t) &= \underbrace{f_1(i^{(\beta)})[i]}_{} + \underbrace{f_{|\beta|}(i^{(\beta)})[i^{(\beta+1)}]^{|\beta|}}_{} + \underbrace{\sum_{j=2}^{|\beta|-1} f_j(i^{(\beta)})\; h_j(i^{(\beta+2)}, \cdots, i^{(-j+1)})}_{} \, . \\ &= \quad A(t) \quad + \quad B(t) \quad + \quad C(t) \end{aligned} \tag{2.10}$$

From the proof of Lemma 2.1 and the state equations for this case, $i^{(-1)}(t)$, $i^{(-2)}(t)$, $\cdots$, $i^{(\beta)}(t)$ are all continuous on any finite interval I in $\mathbb{R}+$. Therefore $h_j(t)$ is also continuous in that interval for each $j = 2,3, \cdots, |\beta|$. Hence $C(t)$ in equation (2.9) is continuous in I and

$$C(t) \in L^{\infty}_{loc}(\mathbb{R}_+ \to \mathbb{R}) \, . \tag{2.11}$$

Similarly, it can be shown that

$$A(t) \in L^{\infty}_{loc}(\mathbb{R}_+ \to \mathbb{R}) \tag{2.12}$$

Since $i^{(\beta)}(t)$ is continuous in I and $f_{|\beta|} \in P(\mathbb{R}_+)$ (cf definition 2.3), $f_{|\beta|}(i^{(\beta)}(t)) \in P(\mathbb{R}_+)$. This, and the continuity of $i^{(\beta+1)}(t)$ in I implies that the term $B(t)$ in (2.10) $\in P(\mathbb{R}_+)$ and therefore,

$$B(t) \in L^{2}_{loc}(\mathbb{R}_+) \, . \tag{2.13}$$

Since $u \in L^2_{loc}(\mathbb{R}_+)$, equations (2.11)-(2.13) give that $p(t) = i(t)\; v(t) \in L^1_{loc}(\mathbb{R}_+ \to \mathbb{R})$.
¤

3. Input-Observability and Local-Controllability

(I) INPUT-OBSERVABILITY

Definition 3.1 [3]

Given a state representation S, an input-trajectory pair is a pair of functions $u(\cdot) \in U$ and $x : \mathbb{R}^+ \to \Sigma$ such that $x(\cdot)$ is a solution of $\dot{x} = f(x,u)$. ¤

Definition 3.2 [4]

A state representation S is input-observable if the following condition holds for any two input-trajectory pairs $\{u_1(\cdot), x_1(\cdot)\}$, $\{u_2(\cdot), x_2(\cdot)\}$ with a common initial state $x_1(0) = x_2(0)$:

For all $t' \geq 0$,

If $\{V(x_1(t), u_1(t)), I(x_1(t), u_1(t))\} = \{V(x_2(t), u_2(t)), I(x_2(t), u_2(t))\}$

for all $t \in [0,t')$,

then $u_1(t) = u_2(t)$ for all $t \in [0,t')$. ¤

Input-observability means that every admissible pair (v,i) [3] with a given initial state is associated with a unique input waveform $u(\cdot)$. Assuming the solutions to the state equation are unique, the state representation S is input-observable if every admissible pair $\{v(\cdot), i(\cdot)\}$ with a given initial state x_0 is associated with a unique input-trajectory pair $\{u(\cdot), x(\cdot)\}$. This concept was introduced in reference [4] to formulate a complete theory for losslessness of nonlinear networks in a state space setting. Before we can apply that theory to study losslessness for h.o.e.'s, we need to check if all the state representations for h.o.e.'s are input-observable. The following result provides an answer:

Theorem 3.1

The state representations listed in Tables 1a-c for the 2-terminal h.o.e. described by equation (2.1) are all input observable. ¤

Proof

(i) For the cases ($\alpha \geq 0$, $\beta \leq 0$) and ($\beta \leq \alpha \leq -1$), the input u is the current i, so the condition for input-observability given in definition 3.2 is trivially satisfied.

(ii) For ($\beta \geq 0$) and ($\alpha < \beta \leq -1$), the state representation is of the form

$$\left\{\begin{array}{l} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \quad\vdots \\ \dot{x}_\lambda = x_{\lambda-1} \\ \quad\vdots \end{array}\right\} \tag{3.1}$$

where $u = i^{(\lambda)}$ for some $\lambda \geq 1$, and $u \in L^1_{loc}(\mathbb{R}_+ \to \mathbb{R})$. Consider two input-trajectory pairs $\{i_1^{(\lambda)}(\cdot), x_1(\cdot)\}$ and $\{i_2^{(\lambda)}(\cdot), x_2(\cdot)\}$. For j = 1 or 2, we have shown in the proof of Theorem 2.1 that $i_j^{(\lambda-1)}$, $i_j^{(\lambda-2)}$, $\cdots$, $i_j^{(0)}(t)$ are absolutely continuous on every finite interval in $\mathbb{R}_+$. For all $t' \geq 0$, suppose

$$i_1(t) = i_2(t) \quad \text{for all } t \in [0,t') \tag{3.2}$$

For any $t \in [0,t')$, by definition of the derivative [8]

$$i_j^{(1)}(t) = \lim_{s\to 0}\left\{\sup_{0<h<s} \frac{i_j(t+h)-i_j(t)}{h}\right\}$$
$$= \lim_{s\to 0}\left\{\inf_{0<h<s} \frac{i_j(t+h)-i_j(t)}{h}\right\}$$

Since the interval $[0,t')$ can be expressed as a finite union of closed intervals

$$[0,t') = \bigcup_{k=1}^{\infty} [0,\ t' - \frac{t'}{k}] \ ,$$

for any $t \in [0,\ t')$ and for $0 < h < \frac{(t'-t)}{2}$ there exists $N_t < \infty$ such that

$$t + h \in \bigcup_{k=1}^{N_t} [0,\ t' - \frac{t'}{k}] \triangleq J_t \subset [0,\ t') \tag{3.3}$$

Naturally, we also have

$$t \in J_t \subset [0,\ t') \tag{3.4}$$

Equations (3.2)-(3.4) imply that for all $t \in [0,\ t')$,

$$\lim_{s\to 0}\left\{\sup_{0<h<s} \frac{i_1(t+h)-i_1(t)}{h}\right\} = \lim_{s\to 0}\left\{\sup_{0<h<s} \frac{i_2(t+h)-i_2(t)}{h}\right\}$$

(similarly, equality holds for the lim inf)

and therefore,

$$i_1^{(1)}(t) = i_2^{(1)}(t) \quad \text{for all } t \in [0,\ t') \tag{3.5}$$

If $\lambda = 1$, we are done because equations (3.2) and (3.5) give the input-observability of the state representation (3.1). If $\lambda \geq 2$, we can repeat the above argument to get the following: for all $t \in [0,\ t')$,

$$i_1^{(1)}(t) = i_2^{(1)}(t) \Rightarrow i_1^{(2)}(t) = i_2^{(2)}(t) \Rightarrow \cdots \Rightarrow \underbrace{i_1^{(\lambda)}(t) = i_2^{(\lambda)}(t)}_{\text{i.e. } u_1(t)=u_2(t)}$$

Therefore the state representation is input-observable by Definition 3.2. ¤

(II) LOCAL CONTROLLABILITY

Given a circuit with state representation S, (global) controlability means that it is possible to go from any initial state in the state space Σ to any other state in Σ via some appropriate input applied over some finite time interval. A formal definition of this concept can be found in [3,9].

For linear systems, there exist well-known criteria for controllability [9,10]. Unfortunately, for nonlinear systems, it is difficult to derive the corresponding criteria for this property, which is global in nature. Most of the results in the current literature deals with a local version of this concept [9]. Roughly speaking, the system is locally controllable about a point x_0 in Σ if it is possible to travel a "short" distance along the state trajectory for a considerably "short" period of time to reach points that are "close to" x_0. The following definitions are adapted from [9]:

Definition 3.3

Given a state representation S, let $x_0, x_1 \in \Sigma$. The state x_1 is reachable from x_0 if there exists a finite $T \geq 0$ and an input-trajectory pair (c.f. Definition 3.1) $\{u(\cdot), x(\cdot)\}|[0,T]$ from x_0 to x_1. ¤

Definition 3.4[5]

The state representation S is locally controllable at $x_0 \in \Sigma$ if for any state x in a neighborhood $\Omega(x_0)$ of x_0, x is reachable from x_0, and x_0 is reachable from x. S is locally controllable if it is so at every $x_0 \in \Sigma$. ¤

The above definition deals only with the existence of controls or inputs that can provide for state transitions in a local neighborhood about some point in the state space. It does not account for the amount of energy needed for the transitions. The following definition, adapted from [11], imposes the condition that the amount of energy required for each local transition be "sufficiently small."

Definition 3.5[6]

S is locally continuously controllable at x_0 if it is locally controllable at x_0 (in the sense of Definition 3.3) with the additional assumption that

$$\left|\int_{t_0}^{t} p(t)dt\right| \leq \rho\|x-x_0\| \tag{3.6}$$

[5]More precisely, this type of local controllability is referred to in [9] as "weak local controllability." For the sake of brevity, we have chosen to adopt the present terminology. Note that for linear systems, controllability is equivalent to local controllability.

[6]Again, this notion is being introduced as "local controllability" in reference [11].

where p(t) is just the power input function introduced in Definition 2.2, $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function satisfying $\rho(0) = 0$, t_0 denotes the initial time for the transition from x_0 to x (or from x to x_0), and t denotes the final time. The state representation S is <u>locally continuously controllable</u> if it is so at every state $x_0 \in \Sigma$. ¤

We are interested in the local (continuous) controllability properties of h.o.e.'s because this concept will prove useful when we explore the subject of the existence of storage functions for these elements [12].

<u>Theorem 3.2</u>

Assume that for the h.o.e. $v^{(\alpha)} = f(i^{(\beta)})$, the function f satisfies the conditions given in Tables 2a-c for different values of α and β. Under these conditions, all the state representations for this h.o.e. as given in Tables 1a-c are locally controllable in the sense of Definition 3.4. ¤

We have imposed the C^∞ restriction on f in the cases mentioned because of technical details, which would become obvious in the following proof. We suspect that such a smoothness condition can be relaxed, especially for the case of h.o.e.'s, but not without a major effort in re-formulating the existing theory of local controllability in [9].

<u>Partial Proof of Theorem 3.2</u>

Referring to Tables 2:

- For all cases, except for ① (ii), (iv) and ② (ii), the state equations are of the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & & & & \bigcirc \\ 1 & 0 & & & \\ 0 & 1 & \ddots & & \\ & 0 & \ddots & \ddots & \\ \bigcirc & & \ddots 0 & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{B} u$$

or $\dot{x} = Ax + Bu$

It is easy to verify that the matrix

$$Q = [B \vdots AB \vdots A^2B \vdots \cdots \vdots A^{n-1}B]$$

has rank n. Therefore, the state representation S satisfies the controllability rank condition given in reference [9], and is therefore locally controllable. In fact, it is also controllable because the two concepts are equivalent in the case of linear systems.

- Consider case ① (iii) for ($\alpha \geq 1$, $\beta = 0$), where the state representation is given by

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_\alpha \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & & & & 0 \\ 1 & 0 & & & \\ 0 & 1 & \ddots & & \\ & 0 & \ddots & \ddots & \\ 0 & & \ddots & 1 & 0 \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ \vdots \\ x_\alpha \end{bmatrix} + \overbrace{\begin{bmatrix} f(u) \\ 0 \\ \vdots \\ 0 \end{bmatrix}}^{\hat{f}(u)} \tag{3.7a}$$

$$\text{for } \dot{x} = Ax + \hat{f}(u) \tag{3.7b}$$

Using the terminology in reference [9], the subset of the vector fields generated by each constant control (or input) u is given by

$$F^0 = \{Ax + \hat{f}(u),\ u = \text{constant} \in \mathcal{U}\}$$

So the Lie algebra containing F^0 is generated by the vector fields

$$\{Ax,\ \hat{f}(u)\}$$

Let F denote the smallest subalgebra which contains F^0. By computing the Jacobi brackets, we get

$$[\hat{f}(u)\ \hat{f}(u)] = 0$$

$$[Ax,\ \hat{f}(u)] = \begin{bmatrix} 0 \\ -f(u) \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$[Ax,[Ax,\ f_u]] = \begin{bmatrix} 0 \\ 0 \\ f(u) \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and so on. So the Lie algebra F is spanned by the linear vector fields Ax and the (constant) vectors

$$\begin{bmatrix} f(u) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ f(u) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(u) \end{bmatrix}.$$

Provided that $f(u) \neq 0$ for every constant value of admissible u, F has dimension α, and therefore satisfies the controllability rank condition given in reference [9]. By Theorem 2.2 in [9], the state representation of equations (3.7) is locally controllable.

- For cases ① (iv) and ② (ii), corresponding to ($\alpha \geq 1$, $\beta \leq -1$) and ($\alpha \geq 1$, $\beta \geq 1$), respectively, the proof is very similar to the one above, and is given in Appendix B. ¤

Unfortunately, we have not yet devised a means of testing for local continuous controllability of these h.o.e.'s. We conjecture that under sufficient smoothness assumptions, every state representation for h.o.e.'s is locally continuously controllable if the associated linearized representation has this property. It is also possible that under very weak conditions, the state representations of h.o.e.'s will always be locally continuously controllable, as is illustrated in the following example:

<u>Example 3.1</u>

Consider the h.o.e. described by

$$v^{(-1)} = f(i^{(-2)}),$$

where <u>$f \in C^1$</u>.

This element has a state representation

$$\dot{x}_1 = u$$
$$\dot{x}_2 = x_1$$
$$i = u$$
$$v = f_1(x_2)x_1, \text{ where } f_1(x_2) \triangleq \frac{d}{dx_2} f(x_2)$$

Consider the transition from the state $x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$ to $x_1 = \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}$, where x_1

lies in a small neighborhood $\Omega(x_0)$ of x_0. Suppose we apply a constant input over a small time interval:

$$u(t) = a, \quad t \in [0,t_1]$$

The desired boundary conditions at both ends of the transition can be met if we had chosen

$$t_1 = 2\,\frac{x_{21}-x_{20}}{x_{11}+x_{10}}$$

and

$$a = \left(\frac{x_{11}-x_{10}}{2}\right)\left(\frac{x_{21}-x_{20}}{x_{11}+x_{10}}\right) \tag{3.8}$$

(Note that we can always make $\Omega(x_0)$ small enough to ensure that $x_{11} + x_{10} \neq 0$)
Now as t_1 and a both tend to zero, x_1 tends to x_0.

By definition of the power input function,

$$\int_0^{t_1} p(t)dt = \int_0^{t_1} i(t)\, v(t)dt = \int_0^{t_1} a\, f_1(x_2(t))(\overbrace{x_{10}+at}^{x_1(t)})dt \tag{3.9}$$

where $x_2(t) = x_{20} + x_{10}(t) + \frac{1}{2}\,at^2$

By assumption, f is C^1, therefore f_1 is continuous. The integrand in equation (3.9) has constant sign on $t \in [0, t_1]$ if $\|x_0-x_1\|$ is small enough. Also by continuity of f_1, there exists a constant $M \geq 0$ such that

$$|f_1(x_2(t))| \leq M \quad \text{for all } t \in [0, t_1]$$

This gives

$$\left|\int_0^{t_1} p(t)dt\right| \leq |a|M\; |x_{10}t_1 + \frac{1}{2}\,at_1^2| = |a|M\; |x_{21}-x_{20}|$$

Substituting the expression for a from equation (3.8) we get

$$\left|\int_0^{t_1} p(t)dt\right| \leq \overbrace{\frac{M}{2|x_{11}+x_{10}|}\; |x_{11}-x_{10}|\,|x_{21}-x_{20}|^2}^{E}$$

Since the expression E on the right side of the above inequality satisfies E = 0 for $x_0 = x_1$, and E > 0 otherwise, and it varies continuously with $\|x_0-x_1\|$, we can conclude that this h.o.e. satisfies condition (3.6) in Definition 3.5.

Therefore, the h.o.e. in this example is locally continuously controllable.

A similar result can probably be obtained for other h.o.e.'s, using the same technique. It is still unclear as to whether a general analysis can be obtained; meanwhile, we can only satisfy ourselves with a case-by-case analysis.

4. Passivity of 2-Terminal H.O.E.'s

Traditionally, an electrical network is called "passive" if it absorbs energy, i.e., it never delivers any energy to the outside world. From a synthesis viewpoint, this means that a passive network can be built without any energy, except possibly for energy losses during the fabrication process. With this notion of passivity, an interconnection of passive circuit elements always results in a passive network, which is also stable in the sense of Lyapunov [13].

It has been shown in [3] that this traditional definition can lead to an anomalous classification of passive nonlinear n-ports. There, a state space theory of passivity has been introduced, whereby, a passive n-port is one that is capable of delivering only a finite amount of energy to the external world. One can therefore interpret a passive n-port as one that can be built using only a finite amount of energy. Using this definition, an interconnection of passive elements results in a passive n-port which is not necessarily stable (in the sense of Lyapunov).

Other publications (see for example, [14]) have adopted an input-output approach, retaining the traditional definition and stability properties of a passive network and referring to the passivity of [3] as "weak passivity." Such an approach considers a fixed initial state and treats passivity as a property of one instance of the network, instead of the overall network.

Despite all the work that has been done on the subject of passivity, it still remains unclear as to which definition is most suitable for the general class of electrical networks. Recently, an attempt has been made to unify all existing definitions. In [15] an electrical circuit (or a "device") is modelled via a singal space and a parameter space. A definition has been proposed for a "frame" of the circuit, i.e., the circuit operating under a specific set of "parameter settings." Where a state description exists for the circuit, the parameter space can include the set of all initial states. The frame of the circuit is passive if only a finite amount of energy can be extracted from it; and the overall circuit is passive if all its frames of interest are passive. When a state representation is possible, this notion of passivity is identical to that in [3].

In the following, we have chosen the definition in [3] as the basis for our study of the passivity of h.o.e.'s. As in Section 2, we shall concern ourselves only with 2-terminal h.o.e.'s having an explicit representation $v^{(\alpha)} = f(i^{(\beta)})$. It is important to note that our choice of a definition for passivity is simply a matter of convenience -- in Theorem 2.1, we have shown that under very mild conditions on the function f, the state representations of h.o.e.'s always satisfy the state space axioms in [3]; and we have also adopted the same set of notations for our state space descriptions as in [3]. Our subsequent results would still remain valid, had we chosen another definition. The reason for this is simple: we are going to show that a large class of 2-terminal h.o.e.'s is active by the definition in [3], because they can deliver an infinite amount of energy to the outside world. Since the amount of energy delivered is dependent solely on the admissible voltage and current waveforms, necessarily, the same class of h.o.e.'s is active by all other definitions. Whenever relevant, we shall discuss the impact of [15] on the classification of passive or active h.o.e.'s.

We now owe it to the reader to state our definition of passivity, which is adapted from [3]:

Definition 4.1

Let S denote the state representation of a 2-terminal h.o.e. and let Σ denote the state space of such a representation. We define the available energy $E_A : \Sigma \to \mathbb{R}^+ \cup \{\infty\}$ by

$$E_A(x) = \sup_{\substack{x\to \\ T\geq 0}} \{-\int_0^T i(t)\, v(t)dt\}$$

where the notation $\sup\limits_{\substack{x\to \\ T\geq 0}}$ indicates that the supremum is taken over all $T \geq 0$ and all admissible pairs $\{v(\cdot), i(\cdot)\}$ with the fixed initial state x. ¤

Definition 4.2

The h.o.e. is passive iff

$$E_A(x) < +\infty \quad \text{for all } x \in \Sigma.$$

Otherwise, the element is said to be active. ¤

(I) MAIN RESULTS

We are now ready to state the results in this section. There are basically three main theorems: Theorem 4.1 takes the well-known result that a negative resistor, inductor or capacitor is active, and extends it to a much larger class of elements. Theorem 4.2 shows, in essence, that almost independently of the properties of $f(\cdot)$, the condition $|\beta-\alpha| \geq 2$ implies activity. (Note, though, that some of the cases where $|\beta-\alpha| < 2$ are also covered by Theorem 4.2). Finally, we discuss the linear case in great detail, since in that case, it is possible to state explicit necessary and sufficient conditions for passivity.

Most of the results in this section are statements about when a h.o.e. is <u>not</u> passive. Equivalently, we are establishing a set of necessary conditions for passivity. The problem of finding conditions which are both necessary and sufficient (and also explicit in the sense that they involve the element constitutive relations directly) remains open, except of course, in some special cases.

<u>Theorem 4.1</u>

Assume:

(i) There exists $a \in \mathbb{R}$ such that $af(a) < 0$ and

(ii) ($\alpha = 0$, $\beta \geq 0$) or ($\alpha \geq 0$, $\beta = 0$).

The element described by $v^{(\alpha)} = f(i^{(\beta)})$ is active under these assumptions.

$\square$

For this result to hold, the function f has to take on values in the 2nd or 4th quadrant. We know that a current-controlled resistor (with $(\alpha,\beta) = (0,0)$ is passive if, and only if its v-i characteristic passes through the origin and lies only in the 1st and 3rd quadrants. This agrees with the result of the theorem.

<u>Proof</u>

(i) We refer to Table 1b for the state representation for the case $\alpha = 0$, $\beta \geq 1$.

By hypothesis, there exists $a \in \mathbb{R}$ such that if we denote $b \triangleq f(a)$, then $\mathrm{sgn}(b) = -\mathrm{sgn}(a)$ with $a \neq 0$, $b \neq 0$.

Choose initial state $x_0 = 0$.

Let $u(t) = i^{(\beta)}(t) = a \quad \forall t \geq 0$

Then $i(t) = x_\beta(t) = \frac{at}{\beta!} \quad ,t \geq 0$

$v(t) = f(a) = b \quad ,t \geq 0$

$p(t) = i(t)\,v(t) = ab\,\frac{t^\beta}{\beta!} \quad ,t \geq 0$

Therefore,

$$E_A(x_0) = \sup_{\substack{\text{admissible } u \\ T \geq 0}} \left\{-\int_0^T p(t)dt\right\} \geq \sup_{T \geq 0} \left\{-\int_0^T ab\,\frac{t^\beta}{\beta!}\,dt\right\} = \sup_{T>0}\left\{-ab\,\frac{T^{\beta+1}}{(\beta+1)!}\right\} = +\infty$$

So the element is active.

(ii) For $\alpha \geq 1$, $\beta = 0$, the state representation is given in Table 1a.

Let $i(t) = a \;\; \forall t \geq 0$, where a is in assumption (1) in the theorem and $x_0 = 0$. The proof proceeds as in (i) above.

(iii) For $\alpha = 0$, $\beta = 0$, the state representation is given in Table 1a.

Pick $i(t) = a$, $t \geq 0$, where a is as given in assumption (1), and some arbitrary x_0. It is easy to verify that for any $x_0 \in \Sigma$, $E_A(x_0) = +\infty$, and so the element is active. ¤

Theorem 4.2

Assume:

(i) The state representation $S = \{U, \mathcal{U}, \Sigma, E, R\}$ and the function f satisfy the respective conditions given in Tables 1a-c, and

(ii) furthermore, f satisfies the conditions given in Table 3 for different values of α and β.

Under these assumptions, the 2-terminal h.o.e. described by $v^{(\alpha)} = f(i^{(\beta)})$ is active. ¤

Remarks

The essential content of this theorem is that $|\beta-\alpha| \geq 2$ implies activity, regardless of f, which is not too surprising when one considers the linear case. The remaining assumptions of the theorem may look complicated, but they actually boil down to excluding pathological cases (except perhaps in cases 3 and 9, where the assumptions needed to validate the proof are a bit stronger than one would have expected). Note, however, that some parts of the theorem statement -- specifically cases 1, 2 and 10, allow the possibility of $|\beta-\alpha| \leq 1$, so the results are not simply an extension of "intuitively obvious" linear circuit properties. In particular, the following needs to be pointed out (cf. Table 3):

Case 1: For ($\alpha \geq 1$, $\beta \geq 1$), so long as the function f is Borel-measurable and $|f|$ is integrable over every finite interval in $\mathbb{R}$, the h.o.e. described by $v^{(\alpha)} = f(i^{(\beta)})$ is always active. An obvious corollary to this is that the linear h.o.e. $v^{(\alpha)} = ki^{(\beta)}$ with $\alpha,\beta \geq 1$ can never be passive, no matter what value k takes on.

Cases 2 and 10: The results here for ($\alpha \geq 1$, $\beta = 0$) and ($\alpha = 0$, $\beta \geq 1$), only work for functions that are not linear. We shall see later, when we consider the linear case that passivity implies linearity in these cases.

This is a truly interesting result because it is the first known instance (in state space theory) of elements which are passive when linear, and which for any deviation, no matter how small, from linearity, become active.

Case ③: Applied to the linear case, this result for ($\alpha \geq 1$, $\beta = -1$) states that, the element $v^{(\alpha)} = ki^{(-1)}$ with $\alpha \geq 1$ can never be passive. Note that $|\beta-\alpha| \geq 2$ in this case and the result here will appear as a consequence of Theorem 4.3, to be presented later.

Case ④: In this case ($\alpha \geq 0$, $\beta \leq -2$), almost all h.o.e.'s of practical interests are active. Examples of f where the theorem is inapplicable are:

a) $f(z) \equiv 0 \quad \forall z \in \mathbb{R}$, which is a trivial case, and

b) $f : \mathbb{R} \to \mathbb{R}$ has a discontinuity at every point, e.g.

$$f(z) = \begin{cases} 1, & z \text{ is rational} \\ 0, & z \text{ is irrational} \end{cases} .$$

To prove activity in Theorem 4.2, we work in the state space Σ for the particular element: we find an input waveform $u \in \mathcal{U}$ and an initial condition $x_0 \in \Sigma$ such that

$$E_A(x_0) = \sup_{\substack{x_0 \to \\ T \geq 0}} \{-\int_0^T ivdt\} \geq \sup_{T \geq 0} \{-\int_0^T i_u v_u dt\} = +\infty ,$$

where (i_u, v_u) denote the admissible current-voltage pair due to the particular input u that we have picked.

We have proved the theorem using the three input-types of Figure 1: a pulse, a piecewise-constant cyclic waveform with period T and a step. Unfortunately, it is impossible to combine our proofs according to the type of input used because of the variety in the state representations of these h.o.e.'s. We shall next give detailed proofs of two cases which involve a pulse input and a cyclic input respectively. (A step input has already been introduced in the proof of Theorem 4.1). The rest of the proof follows the same line of argument, and can be found in Appendix C. Table 4 includes a summary of the input types and the corresponding current waveforms, as well as the restrictions on f, α and β that will guarantee the activity of h.o.e.'s.

Proof of part ④ ($\alpha \geq 0$, $\beta \leq -2$) of Theorem 4.2 (pulse input)

(I) For $\alpha \geq 1$, $\beta \leq -2$, the state representation is given in Table 1a. By assumption (i) in the theorem, there exists $a \in \mathbb{R}$ such that $b = f(a) \neq 0$. Let

$$i(t) = \begin{cases} K/\varepsilon^{1.5}, & t \in [0, \varepsilon] \triangleq I(\varepsilon) \\ 0 & , \end{cases}$$

where $K \triangleq -\text{sgn}(b)$ and $\varepsilon > 0$ is small.

Choose the initial state

$$x_0 = (0, 0, \cdots, 0, a, 0, \cdots, 0, b)$$

with $x_{0|\beta|} = a$ and $x_{0(\alpha+|\beta|)} = b$.

It follows that for $t \in I(\varepsilon)$,

$$x_{|\beta|}(t) \in \underbrace{\left[a - \frac{\varepsilon^{|\beta|-1.5}}{|\beta|!}, \; a + \frac{\varepsilon^{|\beta|-1.5}}{|\beta|!} \right]}_{\Delta(\varepsilon)}$$

By hypothesis, f is continuous in a small neighborhood of a. Supposing $|\beta| \geq 2$, this means that there exists $\varepsilon^* > 0$ such that

$$\Delta(\varepsilon) \subseteq \Delta(\varepsilon^*), \text{ for all } \varepsilon \in (0, \varepsilon^*).$$

Since $\Delta(\varepsilon^*)$ is obviously compact, f must attain a maximum and a minimum value in that interval. Denote them by $f_2(\varepsilon^*)$ and $f_1(\varepsilon^*)$ respectively, i.e., we now have

$$f(x_{|\beta|}(t)) \in [f_1(\varepsilon^*), f_2(\varepsilon^*)] \text{ for all } t \in I(\varepsilon).$$

This implies that

$$v(t) = x_{|\beta|+\alpha}(t) \in \left[\frac{f_1(\varepsilon^*)}{\alpha!} t^\alpha + b, \; \frac{f_2(\varepsilon^*)}{\alpha!} t^\alpha + b\right]$$

for all $t \in I(\varepsilon)$, and $\varepsilon \in (0, \varepsilon^*]$

There are two cases to consider now:

(i) $b > 0$, in which case $i(t) = -1/\varepsilon^{1.5}$, for $t \in I(\varepsilon)$. Then considering $p(t) = i(t)\, v(t)$, we obtain, for $t \in I(\varepsilon)$ and $\varepsilon \in (0, \varepsilon^*)$

$$0 < \frac{f_1(\varepsilon^*)}{\alpha!} \varepsilon^{\alpha-0.5} + \frac{b}{\sqrt{\varepsilon}} \leq \int_{I(\varepsilon)} -p(t)dt \leq \frac{f_2(\varepsilon^*)}{\alpha!} \varepsilon^{\alpha-0.5} + \frac{b}{\sqrt{\varepsilon}}$$

For any $\alpha \geq 1$, as ε tends to 0,

$$\sup_{\varepsilon\in(0,\varepsilon^*]} \int_{I(\varepsilon)} -p(t)dt = +\infty .$$

Therefore

$$E_A(x_0) \geq \sup_{\varepsilon\ (0,\varepsilon^*]} \{-p(t)dt\} = +\infty,$$

and so the element is active for $|\beta| \geq 2$ and $\alpha \geq 1$.

(ii) For $b > 0$, a similar argument shows that for $\varepsilon \in (0,\varepsilon^*]$ and $t \in I(\varepsilon)$,

$$0 < \frac{-f_2(\varepsilon^*)}{\alpha!} \varepsilon^{\alpha-0.5} - \frac{b}{\sqrt{\varepsilon}} \leq -\int_{I(\varepsilon)} p(t)dt \leq \frac{-f_2(\varepsilon^*)}{\alpha!} \varepsilon^{\alpha-0.5} - \frac{b}{\sqrt{\varepsilon}} .$$

Activity then follows as in (i) above.

(II) For $\alpha = 0$, $\beta \leq -2$, the state representation is also given in Table 1a. The current here is the same as in part (i), except that we choose an initial state

$$x_0 = (0, 0, \cdots, 0, a).$$

By continuity arguments, we can show that for $|\beta| \geq 2$,

$$v(t) = f\left(x_{|\beta|}(t)\right) \in [f_1(\varepsilon^*), f_2(\varepsilon^*)], \text{ for } t \in I(\varepsilon),$$

where $\varepsilon^* > 0$ is chosen such that

$$\operatorname{sgn} f_1(\varepsilon) = \operatorname{sgn} f_2(\varepsilon) = \operatorname{sgn} f(a), \text{ for all } \varepsilon \in (0, \varepsilon^*].$$

Therefore, for $\operatorname{sgn} f(a) = +1$,

$$\frac{f_1(\varepsilon^*)}{\sqrt{\varepsilon}} \leq - \int_{I(\varepsilon)} p(t)dt \leq \frac{f_2(\varepsilon^*)}{\sqrt{\varepsilon}}$$

and $E_A(x_0) \geq \sup_{\varepsilon\in(0,\varepsilon^*]} \{-\int_{I(\varepsilon)} p(t)dt\} = +\infty$, and the element is active.
For $\operatorname{sgn} f(a) = -1$, a similar conclusion follows. ¤

Before we proceed with the proof of case 10 of Theorem 4.2, we need the following result:

<u>Lemma 4.1</u>

Consider the following state equation:

$$
\begin{cases}
\dot{x}_1 = u \\
\dot{x}_2 = x_1 \\
\quad\vdots \\
\dot{x}_n = x_{n-1}\,, \quad \text{where } n \geq 1
\end{cases}
\tag{4.1}
$$

If the input satisfies

$$\int_0^T u(\tau)d\tau = 0 \quad \text{for some } T \geq 0, \tag{4.2}$$

then there exist $x_{01}, x_{02}, \cdots, x_{0(n-1)}$ such that

$$x(T) = x(0) \triangleq x_0 = (x_{01}, \cdots, x_{0n}) \tag{4.3}$$

for <u>any</u> value of the last component x_{0n}. ¤

We shall include the proof of this lemma here because this result is essential in the next section, when we consider losslessness.

<u>Proof of Lemma 4.1</u>

Rewrite equation (4.1) as

$$\dot{x} = Ax + Bu \tag{4.1a}$$

where

$$A = \begin{bmatrix} 0 & & & & \bigcirc \\ 1 & 0 & & & \\ 0 & 1 & \ddots & & \\ & 0 & \ddots & \ddots & \\ \bigcirc & & \ddots & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The solution to this equation can be given explicitly by

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \ .$$

In order that condition (4.2)in the lemma is satisfied, we need

$$\int_0^T e^{A(T-\tau)}Bu(\tau)d\tau = [I - e^{At}]x_0 \tag{4.4}$$

For the particular A for equation (4.1a),

$$e^{At} = \begin{bmatrix} 1 & 0 & & & \\ t & 1 & \ddots & \underset{\sim}{0} & \\ t^2/2 & t & \ddots & \ddots & \\ & t^2/2 & \ddots & \ddots & 0 \\ \vdots & \cdots & & t & 1 \end{bmatrix} \quad \text{and } e^{A(T-\tau)}\, B = \begin{bmatrix} 1 \\ (T-\tau) \\ \frac{1}{2}(T-\tau)^2 \\ \frac{1}{6}(T-\tau)^3 \\ \vdots \end{bmatrix},$$

so equation (4.4) becomes

$$\int_0^T \begin{bmatrix} 0 \\ (T-\tau) \\ \frac{1}{2}(T-\tau)^2 \\ \vdots \end{bmatrix} u(\tau)d\tau = \begin{bmatrix} 0 & & & & \underset{\sim}{0} \\ -T & 0 & & & \\ -T^2/2 & -T & \ddots & & \\ \vdots & -T^2/2 & \ddots & \ddots & \\ \vdots & \cdots & \cdots & -T & 0 \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \\ \vdots \\ x \end{bmatrix} \tag{4.5}$$

For (4.5) to have a solution, we need

$$\int_0^T u(\tau)d\tau = 0$$

Then we have to solve

$$\int_0^T \begin{bmatrix} (T-\tau) \\ \frac{1}{2}(T-\tau)^2 \\ \vdots \end{bmatrix} \rho(\tau)d\tau = - \begin{bmatrix} T & 0 & & \underset{\sim}{0} \\ T\ /2 & T & \ddots & \\ \vdots & T^2/2 & \ddots & 0 \\ \cdots & \vdots & \ddots & T \end{bmatrix} \begin{bmatrix} x_{01} \\ \vdots \\ x_{0(n-1)} \end{bmatrix} \tag{4.5a}$$

Since the matrix on the right of the above equation is nonsingular for all $T > 0$, we can always find a solution $(x_{01}, x_{02}, \cdots x_{0(n-1)})$. This implies that there always exists a solution $x_0 = (x_{01}, \cdots, x_{0n})$ such that $x(T) = x(0) \triangleq x_0$ for <u>any</u> value of the last component x_{0n}, provided that $T > 0$ and u satisfies condition (4.2). ⌑

Proof of case 10 ($\alpha = 0$, $\beta \geq 1$) of Theorem 4.2 (cyclic input)

From Table 1b, the state representation for this case is of the form of equation (4.1), with $n = \beta$, and the readout maps are

$$\begin{cases} i = \underbrace{[0 \;\; 0 \;\; \cdots \;\; 0 \;\; 1]}_{C} x \\ v = f(u) \end{cases} \tag{4.6}$$

Let $u(t)$ be periodic with period T given by

$$u(t) = \begin{cases} a & t \in [0, t_1], \;\; t_1 > 0 \\ b & t \in [t_1, T] \end{cases}$$

where $T = (1 - \frac{a}{b})t_1$, and the values a, b satisfy assumption (i) in the statement of the theorem.

It is easy to verify that the above input satisfies equation (4.2) and so, by Lemma 4.1, we can always find an initial condition such that

$$x(T) = x(0) = (x_{01}, x_{02}, \cdots, x_{0\beta})$$

for any value of $x_{0\beta}$.

With this choice of input, $v(t)$ is given by

$$v(t) = \begin{cases} f(a) \; , & t \in [0, t_1) \\ f(b) \; , & t \in [t_1, T) \end{cases} \tag{4.7}$$

Since

$$\int_0^T f(u(t))dt = f(a) - \frac{af(b)}{b} \; ,$$

assumption (ii) in the theorem guarantees that

$$\int_0^T f(u(t))dt \neq 0 \; .$$

Using the matrices A, B from equation (4.1a) and C from equation (4.6), we can write the current $i(t)$ explicitly as

$$i(t) = C \, e^{At}x_0 + \int_0^t C \, e^{A(t-\tau)}Bu(\tau)d\tau$$

which can in turn be evaluated as

$$i(t) = \sum_{j=1}^{\beta} \frac{1}{(\beta-j)!} t^{\beta-j} x_{0j} + \int_0^t \frac{1}{(\beta-1)!} (t-\tau)^{\beta-1} u(\tau)d\tau$$

$$= x_{0\beta} + g(t) + \int_0^t h(t-\tau)\ u(\tau)d\tau\ , \tag{4.8}$$

where $g(t)$ is a polynomial whose coefficients depend on the solution to equation (4.5a): $(x_{01}, x_{02}, \cdots, x_{0(\beta-1)})$ and $h(t-\tau) \triangleq \frac{1}{(\beta-1)!} (t-\tau)^{\beta-1}$ is a polynomial independent of the initial condition x_0.

From equations (4.7) and (4.8),

$$-\int_0^T p(t)dt = -\int_0^T i(t)\ v(t)dt$$

$$= -x_{0\beta} \int_0^T f\big(u(t)\big)dt - \int_0^T g(t)\ f\big(u(t)\big)dt - \int_0^T\int_0^t f\big(u(t)\big)h(t-\tau)u(\tau)d\tau dt \tag{4.9}$$

The last two integrals on the right side of equation (4.9) give only a finite contribution, independent of $x_{0\beta}$, since $T > 0$ is finite. Therefore since $f\big(u(t)\big) \neq 0$, we can always choose $x_{0\beta}$ such that

$$-\int_0^T p(t)dt = c > 0, \text{ where } c \text{ is a constant.}$$

For this choice of $x_{0\beta}$, and the solution $(x_{01}, x_{02}, \cdots, x_{0(\beta-1)})$ to equation (4.5a), we can repeat the above for N cycles to get

$$E_A(x_0) \geq \sup_N\{-\int_0^{NT} p(t)dt\} = \sup_N\{Nc\} = +\infty\ .$$

Hence the element is active in this case. ¤

In Theorem 4.2, we have derived the sufficient conditions for activity for the general class of 2-terminal h.o.e.'s. Since most of these conditions are not at all restrictive, a large class of h.o.e.'s can never be passive. This situation is depicted in Figure 2. The shaded portions on the circuit element array [3] contain those elements which are always active, so long as f satisfies certain mild conditions. The remaining elements, in the unshaded portion (with $\alpha \leq 0$, $\beta \leq 0$ and $|\beta-\alpha| \leq 1$) are thus the only possible candidates for passivity. This class includes the four known basic circuit elements: the resistor, inductor, capacitor and memristor. It is to be noted that cases 2 and 10 in Figure 2 applies only to nonlinear (i.e., strictly not linear h.o.e. 's). We shall show later in Theorem 4.4 that for

$(\alpha,\beta) = (1,0)$ or $(0,1)$, the linear h.o.e. $v^{(\alpha)} = Ki^{(\beta)}$ can be passive, provided that $K > 0$. We have not yet been able to identify a passive nonlinear (i.e., strictly not linear) element, other than the four basic circuit elements. Therefore, we conjecture that the only passive nonlinear elements are the four basic circuit elements. We shall now proceed to show that in the linear case, there exist certain h.o.e.'s which can be passive.

(II LINEAR CASE

In this subsection, we concentrate on the linear 2-terminal h.o.e. described by

$$v^{(\alpha)} = Ki^{(\beta)},\ K \in \mathbb{R}\,. \tag{4.10}$$

We are going to show that for a subclass of these linear elements, it is possible to derive a necessary and sufficient condition for passivity.

Let P_L denote the class of linear h.o.e.'s of equation (4.10) satisfying

(i) $\alpha \leq 0$, or
(ii) $\alpha \geq 1$ $\beta \geq 1$ and $v^{(\alpha-j)}(0) = Ki^{(\beta-j)}(0)$, $j = 2, 3, \cdots, \alpha$.

Theorem 4.3

Any h.o.e. belonging to the class P_L is passive if, and only if
(i) $|\beta-\alpha| \leq 1$ and
(ii) $K \geq 0$ (with $K > 0$ only when $\alpha = 1$). ¤

Remark

Although Theorem 4.3 considers a relatively small subclass of the linear h.o.e.'s, it is nevertheless all that we really need to consider, since most of the h.o.e.'s not belonging in P_L have been shown to be active in Theorem 4.2. In fact, the only important linear h.o.e.'s which are covered by neither Theorem 4.2 nor Theorem 4.3 are those in cases 2 and 10 of Figure 2. These cases will be considered later, in Theorem 4.4.

We would also like to stress that in Theorem 4.3, for the case $\alpha \geq 1$, we are only considering those h.o.e.'s with constrained initial conditions. In the terminology of [15], if we consider the linear h.o.e. of equation (4.10) to be a "device," then each of h.o.e.'s included in the class P_L is only a different "frame" of some device -- the overall device may be active, but it may possess certain "frames" which are passive. From the synthesis viewpoint, this appears to be a most reasonable classification of passive or active linear h.o.e.'s. Consider, for example, the h.o.e. $v^{(1)} = Kv^{(1)}$ constrained to satisfy $v(0) = Ki(0)$.

This is equivalent to a linear resistor $v = Ki$, which is passive for $K \geq 0$. The very same element with unconstrained initial conditions is very different from a linear resistor -- it has to be built using linear reactive elements and controlled sources, as shown in Figure 3, (where the linear 1F capacitors can have arbitrary initial conditions).

As the following proof will indicate, the necessary and sufficient conditions for the passivity of those h.o.e.'s in P_L form in fact the "positive real criterion," so commonly encountered in classical network synthesis. This should not be too surprising, since the basic linear circuit elements (the resistor, inductor and capacitor) are all members of P_L.

<u>Proof of Theorem 4.3</u>

a) We first note that <u>except for $\alpha = 1$</u>, every h.o.e. in P_L has an equivalent representation:[7]

$$v = Ki^{(\beta-\alpha)} \tag{4.11}$$

Next, we show that equation (4.11) has a completely controllable [10] state representation whenever $\beta-\alpha \neq 0$:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{|\beta-\alpha|} \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & & & \mathbf{O} \\ 1 & \ddots & & \\ 0 & \ddots & \ddots & \\ \mathbf{O} & \cdots 0 & 1 & 0 \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{|\beta-\alpha|} \end{bmatrix} + \overbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}^{B} u \tag{4.12}$$

$$\begin{bmatrix} i \\ v \end{bmatrix} = \begin{cases} \overbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & K \end{bmatrix}}^{C} x + \overbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}^{D} u, & (\beta-\alpha) < 0 \\ \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} x + \begin{bmatrix} K \\ 0 \end{bmatrix} u, & (\beta-\alpha) > 0 \end{cases}$$

Since $[B \,\vdots\, AB \,\vdots\, A^2B \,\vdots\, \cdots \,\vdots\, A^{|\beta-\alpha|-1}B]$ has rank $|\beta-\alpha|$, (4.12) is completely controllable [15]. By Theorem 8 in [8], the state representation (4.12) is passive iff the transfer function matrix

---

[7] Equation (4.11) is equivalent to the original representation (4.10) in the sense that both representations posses the same set of admissible voltage-current pairs.

$$H(s) = C(sI-A)^{-1}B + D$$

is positive real.

A simple calculation shows that for (4.12),

$$H(s) = \begin{cases} \begin{bmatrix} 1 \\ K/s^{\beta-\alpha} \end{bmatrix}, & \beta-\alpha < 0 \\ \begin{bmatrix} 1/s^{\beta-\alpha} \\ K \end{bmatrix}, & \beta-\alpha > 0 \end{cases}$$

In either case, $H(s)$ is positive real if, and only if $K \geq 0$ and $|\beta-\alpha| \leq 1$.

For $\beta-\alpha = 0$, representation (4.11) just describes a linear resistor, which is known to be passive iff $K \geq 0$.

b) For $\alpha = 1$, the only h.o.e. that is not active by Theorem 4.2 is when $\beta = 0$. It is easy to show that the transfer function for the h.o.e. $v^{(1)} = Ki$ is $H(s) = K/s$.
For $K = 0$, the element is just a constant voltage source, which is active. $H(s)$ is positive real only if $K > 1$. Therefore using Theorem 8 in [3], this h.o.e. is passive iff $K > 1$. ¤

The following result applies to a small subclass of unconstrained h.o.e.'s $v^{(\alpha)} = f(i^{(\beta)})$ and shows that linearity (with nonnegative slope) of the constitutive relation is necessary for passivity.

Theorem 4.4

The only passive elements of the form $v = f(i^{(\beta)})$ with $\beta \geq 1$ or $v^{(\alpha)} = f(i)$ with $\alpha \geq 1$ are the linear elements:

$$v = Ki^{(1)}, \quad K \geq 0$$

and

$$v^{(1)} = Ki \quad , \quad K > 0$$

¤

The proof of this theorem is quite complicated and can be found in Appendix D. A summary of the above results can be found in Figure 5. The linear h.o.e.'s in region P_{L1} are passive provided that $K \geq 0$. Those in P_{L2} are passive whenever $K \geq 0$ only if they are constrained to zero initial conditions. The two crosses in the circuit element array indicate those elements which are passive if, and only if $K > 0$.

(III) <u>CONCLUSIONS</u>

So far, we have derived the sufficient conditions for activity for a nonlinear 2-terminal h.o.e. described explicitly by $v^{(\alpha)} = f(i^{(\beta)})$. We have also obtained the necessary and sufficient conditions for passivity for a subclass of h.o.e.'s. We feel, at this point, that we should be able to generalize some of our present results to the case of a 2-terminal h.o.e. with an implicit representation

$$h(v^{(\alpha)}, i^{(\beta)}) = 0 .$$

In this case, we might run into certain difficulties in applying the theory in [3] because of the following:

(i) Using the Implicit Function Theorem [16], we might be able to find a local state representation for the h.o.e. However, the theory in [3] requires the existence of a global state representation. If the local state representation can be extended to a global one, i.e., if the implicit function in $v^{(\alpha)}$ and $i^{(\beta)}$ can be transformed into an explicit function, then we would have no problems. Unfortunately, this may not be possible in general.

(ii) A global state representation may exist, which does not necessarily agree with the local representation given by the Implicit Function Theorem. In this case, we may find ourselves dealing with an element which is passive (or active) with respect to some choices of the state space, but not with respect to other choices. The problem here is that the definition in [3] is coordinate-dependent.

(iii) We can use the Global Implicit Function Theorem [17] to transform the implicit representation to an explicit one. However, any form of global implicit function Theorem would impose too stringent a set of conditions on the original (implicit) function, which is undesirable.

We suspect that the above problems may be circumvented if we were to adopt the definition of passivity given in [15]. In fact, the work in [15] was motivated by studying h.o.e.'s. But we need to point out that it would be by no means a trivial extension of our present work to the context of passivity as given in [15].

5. <u>Losslessness of H.O.E.'s</u>

Unlike passivity, the current literature does not provide an adequate definition for losslessness of nonlinear circuits and systems that unifies the input-output and state-space viewponts. Intuitively, an electrical system or circuit is "lossless" if it is incapable of delivering energy to, or absorbing energy from the external world.

Throughout this section, we shall adopt the state space approach to study the losslessness of h.o.e.'s. Basically, we treat losslessness as the path-independence of the energy consumed while traversing any two points in the state space. The following definitions are adapted from [4]:

Definition 5.1

The energy consumed by the input-trajectory pair $\{u(\cdot), x(\cdot)\}|[t_1, t_2]$ is the quantity

$$\int_{t_1}^{t_2} p(x(t), u(t))\, dt,$$

where $p(\cdot,\cdot)$ is just the power input function in Definition 2.2. ¤

Definition 5.2

A state representation S is defined to be lossless if the following condition holds for every pair of states x_a, x_b in the state space Σ: For any two input-trajectory pairs $\{u_1(\cdot), x_1(\cdot)\}|[0, T_1]$, $\{u_2(\cdot), x_2(\cdot)\}|[0, T_2]$ from x_a to x_b, the energy consumed by $\{u_1(\cdot), x_1(\cdot)\}|[0, T_1]$ equals the energy consumed by $\{u_2(\cdot), x_2(\cdot)\}|[0, T_2]$. ¤

Definition 5.3

A 2-terminal h.o.e. is lossless if there exists for this h.o.e. a totally observable [9] state representation S which is lossless by Definition 5.2. ¤

The concept of "total observability" introduced in the above definition is equivalent to that of "complete observability" [10] plus "input-observability" (as introduced in Definition 3.2). In order to prove that a h.o.e. is lossless by directly applying Definition 5.3, we need to find a state representation for the h.o.e. that is both completely observable and input-observable. However, it is simpler to prove that a h.o.e. is not lossless. According to Lemma 3.3 in [4], if a 2-terminal h.o.e. is lossless, then every input-observable state representation for the element is lossless. Hence, to show that a h.o.e. is not lossless, it suffices to find only one input-observable state representation that is not lossless. This is the approach that we shall take in arriving at the first theorem of this section.

(I) MAIN RESULTS

Just as in the case of passivity, we shall introduce three main theorems in this section. Theorem 5.1 is analogous to Theorem 4.1 in that we shall show that a large class of h.o.e.'s can never be lossless. But unlike its counterpart in passivity, Theorem 5.1 does not cover all the cases in the circuit-element-array. The reason for this is because it is extremely hard to find input waveforms that will drive the state-space trajectory from one specific point to another. (This is actually a problem in global nonlinear controllability, and to the best of our knowledge, such a concept has not been well-formulated in the current literature.) Theorem 5.2 states a sufficient condition for losslessness of the state representations for a very specific class of h.o.e.'s -- namely, those lying on the -45° line $\alpha+\beta = -1$. There is no analogy to this result in our passivity theory. We then proceed to study the linear case in detail and state, in Theorem 5.3, a necessary and sufficient condition for losslessness for a subclass of linear h.o.e.'s.

Theorem 5.1

Assume:

(i) The state representations $S = \{U, \mathcal{U}, \Sigma, E, R\}$ and the function f satisfy the respective conditions given in Tables 1a-c and

(ii) furthermore, f satisfies the conditions given in Table 5 for different values of α and β.

Under these assumptions, the 2-terminal h.o.e. described by $v^{(\alpha)} = f(i^{(\beta)})$ is not lossless. ¤

Remarks

For case ①, i.e., $\alpha \geq 1$ and $\beta \geq 1$, almost any function satisfying the conditions in Table 3 are active by Theorem 4.2. Note also that this includes the class of all linear elements described by $v^{(\alpha)} = Ki^{(\beta)}$, and all nonlinear elements $v^{(\alpha)} = f(i^{(\beta)})$ for which f is an odd function.

For case ②, i.e., $\alpha \geq 1$ and $\beta = 0$, the criteria for non-losslessness and activity are identical; hence we can conclude that all active h.o.e.'s falling in this category are not lossless, and vice-versa. We also note that in this case, for almost all functions f that are not linear, the h.o.e. can never be lossless.

The restrictions on f in case ③, i.e., $\alpha = 0$, $\beta \leq -2$ can be satisfied by a large class of functions. In particular, the linear h.o.e. $v = Ki^{(\beta)}$, for any even, negative value of β is not lossless. Also, in comparison with Theorem 4.2, almost all h.o.e.'s that are not lossless in this category are active.

The above observations may lead us to believe that only passive h.o.e.'s are lossless. However, as will be apparent in Theorems 5.2 and 5.3, this is not always the case. In fact, it is worthwhile to point out here that traditionally, losslessness has been considered only for the case of passive elements. One novel feature in the definition proposed in [4] is that it allows for the consideration of losslessness even for active circuit elements or circuits in general.

Proof of Theorem 5.1

By Theorem 3.2, all the state representations in Tables 1a-c are input-observable. So by Lemma 3.3 in [4], to prove that a certain h.o.e. is not lossless, we need only to prove that its corresponding state representation (as given in Table 1) is not lossless. We shall now proceed to prove only cases ② and ③ in Theorem 5.1. The proof for case ① dwells on similar ideas and can be found in Appendix E.

② For $\alpha \geq 1$, $\beta = 0$, the state representation can be found in Table 1a. Following the same arguments as in case ② of Theorem 4.2, and using Lemma 4.1, we can always find a T-periodic input $u(t)$ and initial condition x_0 such that

(i) $x_0 = x(0) = x(T), = x(2T),$

(ii) $\int_0^T p(x(t), u(t))dt = c \neq 0$, and

(iii) $\int_0^{2T} p(x(t), u(t))dt = 2c$

These conditions ensure that the h.o.e. is not lossless since the energy consumed in one period of the input is different from those consumed in two periods, even though the initial state and final states are the same in both cases. ¤

Before proving case ③, we need the following result:

Lemma 5.1

Consider the state equations

$$\left\{\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= x_1 \\ &\vdots \\ \dot{x}_n &= x_{n-1} \end{aligned}\right\} \qquad (5.1)$$

Let $u(\cdot) \in \mathcal{U}$ be an odd T-periodic function,

i.e. $u(T-t) = u(t)$ for all $t \in [0,T]$.

Then the following hold for all $t \in [0, T]$:

a) For $j = 1,2, \cdots, n-1$,

$$x_j(T-t) = x_j(t) \, , \qquad \text{for } j \text{ odd}$$

and $$x_j(T-t) = -x_j(t) \text{ and } x_{j0} = 0 \, , \qquad \text{for } j \text{ even.}$$

b) $x_n(T-t) = x_n(t)$, for j odd

and $x_n(T-t) - x_{n0} - x_n(t)$, for j even. ¤

The proof of this lemma can be found in Appendix E. What we need to complete the proof of Theorem 5.1 is the following:

<u>Corollary to Lemma 5.1</u>

Consider the state equation (5.1) of Lemma 5.1. Let the input be given by

$$u(t) = \begin{cases} -K \, , & t \in [0,1) \\ K \, , & t \in [1, 2) \end{cases} \tag{5.2}$$

with $K > 0$.

Then it is possible to find initial conditions x_{0j} for $j = 1,2, \cdots, n$, such that for <u>even</u> values of n, $x_n(t)$ is odd-symmetric about the $t = 1$ axis and $x_n(t) = 0$ only at $t = 0$, 1 and 2. ¤

This result follows directly from Lemma 5.1.

③ <u>For $\alpha = 0$, $\beta \le -2$</u>, the state representation is given in Table 1a.

Choose an input as given in the Corollary to Lemma 5.1 (with K to be determined later) and fix the initial condition $x_{0|\beta|} = 0$. By the corollary, we can choose initial conditions x_{0j}, $j = 1, \cdots, |\beta|-1$ such that

a) $x(0) = x(2)$ and

b) $x_n(t)$ is odd symmetric about the axis $t = 1$ and is zero only at $t = 0, 1, 2$, for $t \in [0, 2)$

From the proof of Lemma 4.1, x is a linear function of the input, and with our choice of x_{0n}, $x_n(t)$ takes on the form

$$x_n(t) = \begin{cases} K \sum\limits_{j=1}^{|\beta|} c_j t^j & , \; t \in [0, 1) \\ K \sum\limits_{j=0}^{|\beta|} \hat{c}_j (t-1)^j & , \; t \in 1, 2 \end{cases}$$

where C_j and C_j are constants dependent only on the initial conditions x_{0j} for $j = 1, 2, \cdots, |\beta|-1$. By the continuity assumption, there exists δ', $0 < \delta' \leq \delta$ such that

$$\operatorname{sgn} f(z) = \operatorname{sgn} f(0) \quad \forall z \in \underbrace{[-\delta'/2, \delta'/2]}_{\triangleq \Delta'} \tag{5.3}$$

We can always choose $K = K^* \neq 0$ such that

$$x_n(t) \in \Delta' \quad \forall t \in [0, 2) \tag{5.4}$$

Denote the intervals [0, 1] and [1, 2] by I_1 and I_2, respectively. Equations (5.3) and (5.4) imply that

$$\operatorname{sgn}\{f(x_n(t)) = \operatorname{sgn}\{f(0)\}, \quad \forall t \in I_1 \text{ and } I_2 \tag{5.5}$$

Condition (b) and the injectiveness of f within Δ' give that for every $t_1 \in I_1$ and $t_2 \in I_2$,

$$\operatorname{sgn}\{f(x_n(t_1)) - f(0)\} = -\operatorname{sgn}\{f(x_n(t_2))-f(0)\} \tag{5.6}$$

Equation (5.5) implies that

$$\operatorname{sgn}\{\int_{I_1} f(x_n(t))dt\} = \operatorname{sgn}\{\int_{I_2} f(x_n(t))dt\} \tag{5.7}$$

Equations (5.6), (5.7) and assumption (ii) in the theorem results in

$$\int_{I_1} f(x_n(t))dt \neq \int_{I_2} f(x_n(t))dt \tag{5.8}$$

Noting that $f(x_n(t)) = v(t)$, we finally obtain

$$\int_0^2 p(t)dt = -K^* \int_{I_1} v(t)dt + K^* \int_{I_2} v(t)dt = K^*[\int_{I_2} v(t)dt - \int_{I_1} v(t)dt]$$

$$= C \neq 0 \text{ by equation (5.8)}$$

By condition (b) and repeating the input for one more cycle, we also get

$$\int_0^4 p(t)dt = 2c.$$

Therefore, different amounts of energy are consumed along two different admissible paths in the state space with the same endpoints. Hence the h.o.e. cannot be lossless. ¤

Theorem 5.2

Assume

(i) $\alpha+\beta = -1$

(ii) For α, β satisfying (i) above, the function f satisfies the corresponding restrictions in Table 1a or Table 1c.

Under these assumptions, the corresponding state representations for the 2-terminal h.o.e. $v^{(\alpha)} = f(i^{(\beta)})$ (as given in Table 1a or c) is lossless. ¤

Remarks

Notice that the charge-controlled capacitor [with $(\alpha,\beta) = (0, -1)$] and the current-controlled inductor [with $(\alpha,\beta) = (-1,0)$] both fall under the considerations of this theorem. In particular, a (positive or negative) linear capacitor or inductor belongs to this category. It is important to point out that the result of the theorem does not imply that the h.o.e.'s satisfying assumptions (i) and (ii) are lossless. According to Definition 5.3, to show losslessness of the h.o.e., we have to find a totally-observable lossless state representation for the element. In section 3, we have only shown that the state representations listed in Tables 1a-c are input-observable. Whether or not they are totally observable (i.e., input-observable and completely obserable) is still an open question because to the best of our knowledge, there does not exist any means of testing for complete-observability of nonlinear circuits and systems. All we can say at this point is that whether or not the state representation is completely observable depends largely on the behavior of the function f. In the case where f is linear, we shall see shortly that the complete-observability issue does not pose any problems.

Proof of Theorem 5.2

The basic idea behind the proof is to show that the energy consumed during the time interval $[t_1, t_2]$ is dependent only on the initial state $x(t_1)$ and the final state $x(t_2)$.

a) For $\alpha = 0$, $\beta = -1$, the state representation can be found in case ① (i) of Table 1a:

$$\dot{x}_1 = i$$

$$v = f(x_1)$$

The energy consumed during the interval $[t_1, t_2]$ is

$$\int_{t_1}^{t_2} p(t)dt = \int_{t_1}^{t_2} i(t)\ v(t)dt = \int_{t_1}^{t_2} f(x_1)\dot{x}_1\ dt = \int_{x_1(t_1)}^{x_1(t_2)} f(x_1)dx_1 \ .$$

Hence $\int_{t_1}^{t_2} p(t)dt$ is dependent only on $x_1(t_1)$ and $x_1(t_2)$ and therefore this state representation is lossless.

b) <u>For $\alpha \geq 1$, $\beta = -\alpha-1$,</u> the state representation for $v^{(\alpha)} = f(i^{(-\alpha-1)})$ is given in case 1 (iv) in Table 1a.

Consider the integral

$$\int_{t_1}^{t_2} ivdt = \int_{i^{(-1)}(t_1)}^{i^{(-1)}(t_2)} vdi^{(-1)}$$

Using integration by parts recursively on the right integral, we get

$$\int_{t_1}^{t_2} i(t)\ v(t)dt = \sum_{p=1}^{\alpha} \Big[(-1)^{p-1}v^{(p-1)}(t)i^{(-p)}(t)\Big]_{t_1}^{t_2} + \int_{i^{(-\alpha-1)}(t_1)}^{i^{(-\alpha-1)}(t_2)} v^{(\alpha)}di^{(-\alpha-1)}$$

Each term in the summation on the right side of the above equation is just a product of the components of the state vectors at the initial and final times. The integral on the right side can be rewritten as

$$\int_{i^{(-\alpha-1)}(t_1)}^{i^{(-\alpha-1)}(t_2)} f(i^{(-\alpha-1)})di^{(-\alpha-1)}$$

which is a function only of the $(\alpha+1)$-th component of the state vector at the initial and final times. Hence the energy consumed as the state trajectory traverses from initial state $x(t_1)$ to final state $x(t_2)$ is dependent only on the endpoints. Therefore we can conclude that the state representation for this case is lossless.

c) <u>For $\alpha \leq -1$, $\beta = -\alpha-1$</u>

The state representation is given in case 6 in Table 1c. The energy consumed within time interval $[t_1, t_2]$ is $\int_{t_1}^{t_2} p(t)dt$ and using integration by parts recursively, it is found to be equal to

$$\sum_{p=1}^{|\alpha|} \Big[(-1)^{p-1} i^{(p-1)}(t) v^{(-p)}(t)\Big]_{t_1}^{t_2} + \int_{i^{(-\alpha-1)}(t_1)}^{i^{(-\alpha-1)}(t_2)} f(i^{(-\alpha-1)}) di^{(-\alpha-1)}$$

By comparison with the corresponding state representation, this amount of energy can be found to be dependent only on the initial state $x(t_1)$ and the final state $x(t_2)$. Hence the representation is lossless. ¤

(II) LINEAR CASE

The linear h.o.e. $v^{(\alpha)} = Ki^{(\beta)}$, $K \in \mathbb{R}$, is being considered in this subcase. Theorem 5.3 below parallels Theorem 4.3 in the sense that the same subclass, $\mathcal{P}_L$, of linear elements are being considered; but unlike the case for passivity, a much weaker condition than the "positive real criterion" is used in deriving the necessary and sufficient conditions for losslessness. This results in a larger subclass of lossless elements in P_L than the corresponding subclass of passive elements.

Theorem 5.3

Any h.o.e. belonging to class P_L is lossless if, and only if

(i) $K = 0$ or

(ii) $|\beta-\alpha|$ is odd. ¤

Remarks

The conclusion of this theorem is in agreement with case 2 of Theorem 5.2. However, case 1 in Theorem 5.2 states that all linear h.o.e.'s with $\alpha \geq 1$ and $\beta \geq 1$ can never be lossless. We must, once again, draw the distinction between unconstrained h.o.e.'s and constrained h.o.e.'s. The present result for losslessness holds only for those constrained h.o.e.'s satisfying $v^{(\alpha-j)}(0) = Ki^{(\beta-j)}(0)$ for $j = 2, 3, \cdots, |\alpha|$. This is not surprising because, as pointed out in [3], h.o.e.'s in this category whose initial conditions are such constrained, and satisfying condition (ii) in Theorem 5.3 are precisely those which behave like inductors or capacitors, and should therefore be lossless. Another distinguishing feature of this result is that even negative linear capacitors and inductors are classified as lossless elements. Since these element are <u>active</u>, they have not even been considered in classical circuit theory in the context of losslessness.

Figure 5 shows the linear lossless elements. Except for those h.o.e.'s in the first quadrant which have to satisfy constrained initial conditions all other h.o.e.'s are unconstrained.

Proof of Theorem 5.3

a) For $K = 0$, we have, for elements belonging to class P_L, either $i = 0$ or $v = 0$. The h.o.e. is obviously lossless in this case.

b) Suppose $K \neq 0$.

Then using the same techniques as in the proof of Theorem 4.3, we can show that h.o.e.'s belonging to class P_L have a completely controllable state representation with transfer function

$$H(s) = \begin{bmatrix} 1 \\ K/S^{\beta-\alpha} \end{bmatrix}, \begin{bmatrix} 1/S^{\beta-\alpha} \\ K \end{bmatrix}, \text{ or } K/S .$$

Applying Theorem (5.1) in [9], it is quite clear that $H(j\omega) = - H(-j\omega)$ if, and only if $|\beta-\alpha|$ is odd. ¤

(III) Conclusions

Even though losslessness in the linear case has been covered quite thoroughly in this section, the nonlinear case is still incomplete. This is inevitable, because unless the subject of nonlinear controllability and observability has been further investigated, we have no means of testing the lossless properties of the rest of these h.o.e.'s . One way to circumvent the problem is to find a definition for losslessness that is based solely on the behavior of the admissible (v,i) pairs, thus rendering losslessness as a property that is independent of state representations.

By comparing the results in this section and the previous one, we can see that, at least in the linear case, the number of lossless h.o.e.'s far exceeds that of passive h.o.e.'s (cf. Figures 4 and 5). Thus, even though the present definition does have its limitations, it nevertheless enables us to consider a much broader class of lossless elements that is allowed by previous definitions.

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APPENDICES

(A) Completion of Proof of Theorem 2.1

For $\alpha \geq 1$, $\beta = 0$ (i.e., case ① (ii) in Table 1a)

- Since $u \in P(\mathbb{R}_+)$, $u(t)$ is measurable on any finite interval I in $\mathbb{R}_+$. Since f is Borel measurable, it follows that $f(u(t))$ is integrable in I [5]. Existence and uniqueness of a solution follows from Lemma 2.1.
- From the proof of Lemma 2.1, $v(\tau) = x_\alpha(\tau)$ is absolutely continuous in I and so $v(t) \in L^\infty_{loc}(\mathbb{R}_+ \to \mathbb{R})$. It follows from $u \in L^1_{loc}(\mathbb{R}_+ \to \mathbb{R})$ and Holder's inequality that $p(t) = i(t)\ v(t) = u(t)\ x_\alpha(t) \in L^1_{loc}(\mathbb{R}_+ \to \mathbb{R})$.

For $\alpha = 0$, $\beta = 0$ (i.e., ① (iii) in Table 1a)

- existence and uniqueness of a solution to the state equation is redundant in this case.
- By reasonings similar to case ① (ii) above,

$$v(t) = f(u(t)) \in L^1_{loc}(\mathbb{R}_+ \to \mathbb{R}) .$$

$i(t) = u(t) \in P(\mathbb{R}_+)$ implies that $i(t) \in L_{loc}(\mathbb{R}_+ \to \mathbb{R})$. Therefore, $p(t) \in L^1_{loc}(\mathbb{R}_+ \to \mathbb{R})$ by Hölder's Inequality.

For $\alpha \geq 1$, $\beta \leq -1$ (① (iv) in Table 1a)

- Consider the first $|\beta|$ equations. Existence and uniqueness of a solution $(x_1,\ldots,x_{|\beta|})$ follows from Lemma 2.1.
  Since $x_{|\beta|}$ is continuous, $f(x_{|\beta|}(t)) \in L^1_{loc}(\mathbb{R}_+ \to \mathbb{R})$ and is therefore integrable. By lemma 3.1 again, there exists a unique solution $(x_{|\beta|+1},\ldots,x_{|\beta|+\alpha})$ to the last α equations. So we have a unique solution $x = (x_1,x_2,\ldots,x_{|\beta|+\alpha})$ to the state equation E.
- From the proof of Lemma 2.1, $v(t) = x_{|\beta|+\alpha}(t)$ is bounded and measurable over any finite interval I. Since $u(t) = i(t)$ is integrable,

$$p(t) = v(t)\ i(t) \in L^1_{loc}(\mathbb{R}_+ \to \mathbb{R}) \text{ [8]}.$$

For $\alpha = 0$, $\beta \geq 1$ (i.e., case ② (i) in Table 1b)

- Existence and uniqueness of a solution to E follows from Lemma 3.1.
- $v(t) = f(u(t)) \in L^1_{loc}(\mathbb{R}_+ \to \mathbb{R})$ as in case ① (ii) above. $i(t) = x_{|\beta|}(t)$ is bounded and integrable in I by the proof of Lemma 2.1. It follows, therefore, that $p(t) \in L^1_{loc}(\mathbb{R}_+ \to \mathbb{R})$ [8].

For $\alpha \geq 1$, $\beta \geq 1$ (i.e., case (2) (ii) in Table 1b)

- Existence and uniqueness of a solution is similar to case (1) (iv) above
- Since x_β and $x_{\beta+\alpha}$ are both continuous by the proof of Lemma 2.1, $p(t) = x_\beta(t)x_{\beta+\alpha}(t)$ follows trivially.

For $\beta < \alpha \leq -1$ (i.e., case (4) in Table 1c)

- Existence and uniqueness follows from Lemma 2.1.
- Rewrite $v(t)$ as

$$v(t) = \underbrace{\sum_{j=1}^{|\alpha|-1} \cdot f_j(i^{(\beta)})h_j(i^{(\beta+2)},\ldots,i^{(\beta-\alpha-j+1)})}_{} + \underbrace{f_{|\alpha|}(i^{(\beta)})[i^{(\beta+1)}]^{|\alpha|}}_{}$$

$$= \qquad A(t) \qquad + \qquad B(t) \tag{A.1}$$

The term $A(t)$ in equation (A.1) is continuous over any finite interval I since (i) $i^{(\beta)}(t)$, ..., $i^{(\beta-\alpha)}(t)$ are continuous in I and (ii) f_j is continuous for $j = 1,2, \ldots, |\alpha|-1$, by hypothesis.
As in the proof for case (3) (see main text), the second term $B(t)$ in equation (A.1) belongs in $P(\mathbb{R}_+)$ and is therefore $L^\infty_{loc}(\mathbb{R}_+\to\mathbb{R})$. So $v(t) = A(t) + B(t)$ is bounded and measurable over any finite interval in $\mathbb{R}_+$. Since $i(t) = u(t) \in L^1_{loc}(\mathbb{R}_+\to\mathbb{R})$ is integrable, $p(t) = i(t)\, v(t) \in L^1_{loc}(\mathbb{R}_+\to\mathbb{R})$ [8].

For $\alpha < \beta \leq -1$ (i.e., case (5) in Table 1c)

- Existence and uniqueness of a solution follows from Lemma 2.1.
- Rewrite $v(t)$ as

$$v(t) = \underbrace{f_1(i^{(\beta)})i^{(\beta-\alpha)}}_{} + \underbrace{f_{|\alpha|}(i^{(\beta)})[i^{(\beta+1)}]^{|\beta|}}_{} + \underbrace{\sum_{j=2}^{|\alpha|-1} f_j(i^{(\beta)})h_j(i^{(\beta+2)},\ldots,i^{(-j+1)})}_{}$$

$$= \qquad A(t) \qquad + \qquad B(t) \qquad + \qquad C(t) \tag{A.2}$$

By reasonings similar to previous cases, we can show that $A(t)$ and $C(t)$ are continuous over any finite interval I in $\mathbb{R}_+$. Also, $B(t) \in P(\mathbb{R}_+)$. Since $i(t) = x_{|\alpha|-|\beta|}$ is continuous in I, $i(t)\, B(t) \in P(\mathbb{R}_+)$ and therefore belongs in $L^1_{loc}(\mathbb{R}_+\to\mathbb{R})$. So $p(t) = i(t)\, v(t) = i(t)[A(t) + B(t) + C(t)] \in L^1_{loc}(\mathbb{R}_+\to\mathbb{R})$.

For $\alpha \leq -1$, $\beta \geq 0$ (i.e., case (6) in Table 1c)

The proof of this case is almost identical to that of case (5) and is therefore omitted. ¤

(B). Completion of Proof of Theorem 3.2

For $\alpha \geq 1$, $\beta \leq -1$ (i.e., case ① (iv) in Table 2a), the state representation can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{|\beta|} \\ \hdashline \dot{x}_{|\beta|+1} \\ \dot{x}_{|\beta|+2} \\ \vdots \\ \dot{x}_{|\beta|+\alpha} \end{bmatrix} = \begin{bmatrix} 0 \\ x_1 \\ \vdots \\ x_{|\beta|-1} \\ \hdashline f(x_{|\beta|}) \\ x_{|\beta|+1} \\ \vdots \\ x_{|\beta|+\alpha-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \hdashline 0 \\ \vdots \\ 0 \end{bmatrix} u \tag{B.1}$$

Adopting the same notations as in the main text, by computing the Jacobi brackets, we can verify that the subalgebra F is spanned by the vectors:

$$\begin{array}{c} |\beta| \left\{ \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \right. \\ \\ \alpha \left\{ \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \right. \end{array} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \hdashline 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ \hdashline 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ \hdashline 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hdashline f_1(x_{|\beta|}) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hdashline 0 \\ f_1(x_{|\beta|}) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hdashline 0 \\ \vdots \\ 0 \\ f_1(x_{|\beta|}) \end{bmatrix}$$

where $f_1(x_{|\beta|}) = \frac{d}{dz} f(z)\Big|_{z=x_{|\beta|}}$.

Assuming that $f_1(x_{|\beta|}) \neq 0$ for all admissible values of $x_{|\beta|}$, F has dimension $|\alpha| + \beta$ and therefore satisfies the controllability rank condition in [9]. Hence the state representation is locally controllable in this case.

For $\alpha \geq 1$, $\beta \geq 1$ (i.e., case ② (ii) in Table 2b),

The state equation is

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_\beta \\ \hline \dot{x}_{\beta+1} \\ \dot{x}_{\beta+2} \\ \vdots \\ \dot{x}_{\beta+\alpha} \end{bmatrix}
=
\begin{bmatrix} 0 & & & & & \\ 1 & 0 & & & \tilde{\bigcirc} & \\ 0 & 1 & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & \tilde{\bigcirc} & & \ddots & \ddots & \\ & & & 0 & 1 & 0 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_\beta \\ \hline x_{\beta+1} \\ x_{\beta+2} \\ \vdots \\ x_{\beta+\alpha} \end{bmatrix}
+
\begin{bmatrix} u \\ 0 \\ \vdots \\ 0 \\ \hline f(u) \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\tag{B.2}
$$

Computing Jacobi brackets, we can see that the smallest subalgebra F is spanned by the vectors

$$
\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \hline 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix},
\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ \hline 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix},\ldots,
\begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \\ \hline 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix},
\begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \hline f(u) \\ 0 \\ \vdots \\ 0 \end{bmatrix},
\begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \hline 0 \\ f(u) \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ldots,
\begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \hline 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ f(u) \end{bmatrix}
$$

Assuming $f(u) \neq 0$ for all constant values of u, F has dimension $\alpha+\beta$ and therefore satisfies the controllability rank condition in [9]. So the state representation (B.2) in this case is locally controllable. ¤

(C) Completion of proof of Theorem 4.2

① $\alpha \geq 1,\ \beta \geq 1$

Referring to the state representation in Table 1b, we pick $u(t) = 0$ and choose the initial state

$$x_0 = (0, 0, \cdots 0, i_0, 0, 0, \cdots, 0, v_\alpha),$$

where $x_{0\beta} = i_0 = \begin{cases} 1 & , \text{ if } f(0) = 0 \\ -\text{sgn } f(0), & \text{otherwise,} \end{cases}$

and $x_{0(\beta+\alpha)} = v_\alpha = -i_0$.

Under these choices of input and initial condition,

$$-\int_0^T p(t)dt = -i_0\, f(0)\, \frac{T^{\alpha+1}}{(\alpha+1)!} + i_0^2\, T$$

By our choice of i_0,

$$E_A(x_0) \sup_{\substack{\text{admissible } u \\ T \geq 0}} \{-\int_0^T p(t)dt\} \geq \sup_{T \geq 0} \{-i_0\, f(0)\, \frac{T^{\alpha+1}}{(\alpha+1)!} + i_0^2\, T\} = +\infty$$

Hence activity follows. ¤

② $\alpha \geq 1,\ \beta = 0$

The state representation for this case is given in Table 1a.

Let a,b satisfy the assumptions in the theorem.

Pick $i(t) = u(t) = \begin{cases} a & t \in [0, t_1) \triangleq I_1 \\ b & t \in [t_1, T) \triangleq I_2 \end{cases}$

where $T = (1 - \frac{f(a)}{f(b)})\, t_1$.

Then

$$f(u(t)) = \begin{cases} f(a)\ , & t \in I_1 \\ f(b)\ , & t \in I_2 \end{cases}$$

Note that $\int_0^T f(u(t))dt = 0$.

By Lemma 5.1, it is possible to choose the first $(\alpha-1)$ components of an initial state x_0 such that

$$x(T) = x_0$$

for <u>any</u> value of the last component $x_{0\alpha}$.

By writing out explicitly what v(t) and i(t) are, it can be shown that

$$-\int_0^T p(t)dt = -x_{0\alpha}\, t_1[a - \frac{bf(a)}{f(b)}] + M ,$$

where $M \in \mathbb{R}$ gives only a finite contribution to the above integral. By assumption (ii) in the Theorem, $af(b) \neq bf(a)$, so it is always possible to choose $x_{0\alpha}$ such that

$$-\int_0^T p(t)dt = c > 0.$$

Repeating this for N cycles, we get

$$E_A(x_0) \geq \sup_N\{ -\int_0^{NT} p(t)dt\} = \sup_N\{NC\} = +\infty,$$

so the element is active. ¤

③ <u>$\alpha \geq 1,\ \beta = -1$</u>

The state representation is given in Table 1a.

(i) Pick

$$i(t) = \begin{cases} \varepsilon^{-p} , & t \in (0, \varepsilon] \triangleq I(\varepsilon) \\ 0 , & t \notin I(\varepsilon) \end{cases}$$

with $\varepsilon > 0$ and $1 < p < (1 + \frac{\alpha}{k+1})$, where k is the integer as stated in the theorem. Choose an initial condition

$$x_0 = (0, \cdots, 0, v_0), \text{ with } v_0 < 0.$$

Then it is possible to deduce by assumption (i) that for $t \in I(\varepsilon)$,

$$v(t) \leq \sum_{j=0}^{k} \frac{A_j}{(j+\alpha)!}\, \varepsilon^{-jp} t^{j+\alpha} + v_0,$$

and hence

$$\int_{I(\varepsilon)} -p(t)dt \geq \sum_{j=0}^{k} \frac{A_j}{(j+\alpha+1)!}\, \varepsilon^{j+\alpha+1-(j+1)p} - v_0 \varepsilon^{1-p} \qquad \text{(C.1)}$$

Since $\alpha \geq 1$ and $(1-p) < 0$, the first term in (C.1) tends to zero as $\varepsilon \to 0$ and the second term tends to infinity, we have

$$E_A(x_0) \geq \sup_{\varepsilon>0}\{ -\int_{I(\varepsilon)} ivdt\} = +\infty ,$$

and the element is active.

(ii) To prove activity using assumption (ii), we simply repeat the above arguments, choosing a negative pulse as inputs i.e.

$$i(t) = \begin{cases} -\varepsilon^{-p} , & t \in I(\varepsilon), \varepsilon > 0 \\ 0 , & t \notin I(\varepsilon) . \end{cases}$$ ¤

⑤ $\alpha \leq -1, \beta \leq \alpha-2$

The state representation for this case can be found in Table 1c.

Let $a = (a_0, a_1, \cdots, a_{|\alpha|})$ and b satisfy the assumptions in the theorem.

Pick

$$i(t) = \begin{cases} K/\varepsilon^{1.5}, & t \in (0, \varepsilon] \triangleq I(\varepsilon) \\ 0 , & t \notin I(\varepsilon) \end{cases}$$

where $K \triangleq -\mathrm{sgn}(b)$ and $\varepsilon > 0$.

Choose the initial condition

$$x_0 = (0, \cdots, 0, a_0, a_1, \cdots, a_{|\alpha|})$$

Under these condtions, for $t \in I(\varepsilon)$ and $q = 0, 1, \cdots, |\alpha|$,

$$i^{(\beta-\alpha-q)}(t) = \frac{K}{\varepsilon^{1.5}} \frac{t^{|\beta-\alpha-q|}}{|\beta-\alpha-q|!} + h_q(t) + a_q,$$

where $h_q(t)$ is a polynomial with coefficients dependent on $a_0, a_1, \cdots, a_{q-1}$ and degree less than or equal to q.

By the compactness of $I(\varepsilon)$, we can show that $(i^{(\beta-\alpha)}, \cdots, i^{(\beta)})$ lies in some compact interval, say, $\Delta(\varepsilon)$ in $\mathbb{R}^{|\alpha|}$. Then, using the locally continuity of $\tilde{f}$ about a and the fact that $|\beta-\alpha| \geq 2$, and bearing in mind that $v(t) = \tilde{f}(i^{(\beta-\alpha)}(t), \cdots, i^{(\beta)}(t))$, we can repeat the same argument as in case ④ (see main text) to show that $E_A(x_0) = \infty$, for our particular choice of x_0. ¤

Before proceeding with the proof of cases ⑥-⑨, we need the following result:

Lemma C.1

Assume:

(1) $v^{(\alpha)} = f(i^{(\beta)})$, where $\alpha \leq -1$ and $\beta > \alpha$

(2) $f \in C^{|\alpha|-1}$

(3) $\exists a \in \mathbb{R} \ni F : \mathbb{R} \to \mathbb{R}$ is locally $C^{|\alpha|}$ about a.

Under these assumptions, for $1 \leq k \leq |\alpha|$, we have

$$v^{(\alpha+k)} = f_1(i^{(\beta)})i^{(\beta+k)} + (k-1)\, f_2(i^{(\beta)})\, i^{(\beta+1)}\, i^{(\beta+k-1)} + g_k(i^{(\beta+k-2)}, \cdots, i^{(\beta+1)}, i^{(\beta)})$$

where $f_j(z) \triangleq \left.\dfrac{d^j f(z)}{dz^j}\right|_{z=i^{(\beta)}}$,

and $g_k : \mathbb{R}^{k-1} \to \mathbb{R}$ satisfies

(i) $g_k \in C^{|\alpha|-k-1}$ and

(ii) g is locally $C^{|\alpha|-k}$ in all its arguments. ¤

This lemma can be proved by induction on k. The proof is algebraically complicated but straightforward and is therefore omitted. The following corollary will be useful in our subsequent proofs:

Corollary

Under the same assumptions as in the lemma,

$$v = f_1(i^{(\beta)})\, i^{(\beta-\alpha)} + (|\alpha|-1)\, f_2(i^{(\beta)})\, i^{(\beta+1)}\, i^{(\beta-\alpha-1)} + g(i^{(\beta-\alpha-2)}, \ldots, i^{(\beta+1)}, i^{(\beta)}),$$

where g is locally C^0 and therefore takes on finite values provided its arguments are confined to a small enough region in $\mathbb{R}^{k-1}$. ¤

⑥ $\alpha \leq \beta-2$, $\beta \leq -1$

The state representation for this case can be found in Table 1c.

Since $f \in C^{|\alpha|-1}$, we can write

$$v = f_1(i^{(\beta)})i^{(\beta-\alpha)} + (|\alpha|-1)f_2(i^{(\beta)})i^{(\beta+1)}i^{(\beta-\alpha-1)} + g(i^{(\beta-\alpha-2)}, \cdots, i^{(\beta+1)}, i^{(\beta)}) \tag{C.2}$$

Let $u(t) = \varepsilon^{-1.5}$, $t \in [0, \varepsilon] \triangleq I(\varepsilon)$, with $\varepsilon > 0$. Choose the initial condition

$$x_0 = (0, 0, \cdots, 0, a_0, 0, \cdots, 0, a_1),$$

i.e., $x_{0(\beta-\alpha)} = a$ and $x_{0|\alpha|} = a_1$, where $a_0 \triangleq -\text{sgn}\{f_1(a_1)\}$ and a_1 is as given in the assumption of the theorem. Under these conditions, for $t \in I(\varepsilon)$,

$$i(t) = x_{\beta-\alpha}(t) = \frac{1}{(\beta-\alpha)!}\,\varepsilon^{-1.5}\,t^{\beta-\alpha} + a_0,$$

$$i^{(\beta)}(t) = \frac{1}{|\alpha|!}\,\varepsilon^{-1.5}\,t^{|\alpha|} + \frac{1}{|\beta|!}\,a_0\,t^{|\beta|} + a_1,$$

and

$$i^{(\beta+1)}i^{(\beta-\alpha-1)} = \frac{1}{(|\alpha|-1)!}\,\varepsilon^{-3}\,t^{|\alpha|} + \frac{1}{(|\beta|-1)!}\,a_0\,\varepsilon^{-1.5}\,t^{|\beta|} \tag{C.3}$$

Combining (C.2) and (C.3), for $t \in I(\varepsilon)$,

$$p(t) = f_1\varepsilon^{-1.5}[\frac{1}{(\beta-\alpha)!}\,\varepsilon^{-1.5}\,t^{\beta-\alpha} + a_0] + (|\alpha|-1)\,f_2(\frac{\varepsilon^{-3}}{(|\alpha|-1)!}t^{|\alpha|} + \frac{a_0\varepsilon^{-1.5}t^{|\beta|}}{(|\beta|-1)!})$$

$$(\frac{1}{(\beta-\alpha)!}\,\varepsilon^{-1.5}\,t^{\beta-\alpha} + a_0) + i\,g(\cdot,\cdots,\cdot), \tag{C.4}$$

where $f_1 \triangleq f_1(i^{(\beta)}(t))$ and $f_2 \triangleq f_2(i^{(\beta)}(t))$. Since f is locally $C^{|\alpha|}$ about a_1, for $j = 1, 2, \cdots, |\alpha|$, f_j is locally C^0 about a_1, and therefore has a maximum $f_{j\,max}$ and a minimum $f_{j\,min}$ about some ε_0-neighborhood of a_1. So we have

$$f_j \le \max\{|f_{j\,max}|,\ |f_{j\,min}|\} \triangleq \bar{f}_j$$

For $t \in I(\varepsilon)$, it can be deduced that

$$i^{(\beta+p)} = K_p + \lambda_p(\varepsilon,t)$$

where

$$K_p = \begin{cases} a_1 & , p = 0 \text{ or } p = |\beta| \\ 0 & , \text{otherwise} \end{cases}$$

and $\lambda_p(\varepsilon,t) \le \hat{\lambda}\varepsilon^q$ for some constant $\hat{\lambda} \ge 0$ and $q > 0$.

Therefore, as $\varepsilon \to 0$,

$\forall t \in I(\varepsilon)$, $t \to 0$, $\lambda(\varepsilon) \to 0$ and $i^{(\beta)} \to a_1$.

Using the corollary to Lemma C.1, for small enough ε_0, $\exists M(\varepsilon)$ such that

$$|g(i^{(\beta-\alpha-2)},\ \cdots,\ i^{(\beta+1)},\ i^{(\beta)})| \le M(\varepsilon), \quad \forall\varepsilon \in (0,\ \varepsilon_0] \tag{C.5}$$

Combining equations (C.3)-(C.5) we have

$$p(t) \leq \bar{f}_1[K_1\varepsilon^{\beta-\alpha-2} + a_0\varepsilon^{-0.5}] + \bar{f}_2[K_2\varepsilon^{\beta-2\alpha-3.5} + K_3a_0\varepsilon^{|\alpha|-2}]$$

$$+ \bar{f}_2a_0[K_4\varepsilon^{|\alpha|-2} + K_5a_0^2\varepsilon^{|\beta|-0.5}] + M[K_1\varepsilon^{\beta-\alpha-0.5} + a_0\varepsilon], \quad \text{(C.6)}$$

where $M \triangleq \max_{0<\varepsilon\leq\varepsilon_0} \{M(\varepsilon)\}$ and $K_j (j = 1, \cdots, 5)$ are constants dependent on the values of α and β. Assuming $(\beta-\alpha) \geq 2$ and $|\alpha| \geq 3$, as $\varepsilon \to 0$, $i^{(\beta)}(t) \to a_1$ and all the terms on the right side of inequality (C.6) tend to zero, except for

$$f_1(a_1)\ (a_0\varepsilon^{-0.5} + K_1\varepsilon^{(\beta-\alpha)-2}) \quad \text{(C.7)}$$

Since we have chosen $a_0 = -\text{sgn}\{f_1(a_1)\}$, we can combine (C.6) and (C.7) to get

$E_A(x_0) = +\infty$, and hence the element is active. ¤

⑦ $\beta = 0, \alpha \leq -2$

The state representation for this case can be found in Table 1c. Pick the input

$$u(t) = K\varepsilon^{-1.5},\ t \in (0, \varepsilon] \triangleq I(\varepsilon),\ \varepsilon > 0$$

where $K \triangleq a\ f_1(a)$ and a satisfies the assumption in the theorem. Choose the initial condition

$$x_0 = (0, 0, \cdots, 0, a),$$

i.e., $x_{0|\alpha|} = a$.

Using arguments similar to the previous case, it can be shown that $E_A(x_0) = +\infty$. ¤

⑧ $\beta \geq 1, \alpha \leq -2$

The state representation for this case is in Table 1c. The input to use is

$$u(t) = \varepsilon^{-1.5},\ t \in I(\varepsilon),\ \varepsilon > 0;$$

and the initial condition is

$$x_0 = (0, \cdots, 0, a_1, 0, \cdots, 0, a_0),$$

i.e., $x_{0\alpha} = a_1$ and $x_{0(\beta-\alpha)} = a_0$, where $a_0 \triangleq -\text{sgn}\{f_1(a_1)\}$ and a_1 is as given in the assumption of the theorem. The proof for this case is similar to that of case ⑥ and is omitted. ¤

⑨ $\alpha = -1,\ \beta \geq 1$

The state representation for this case is given in Table 1c. Pick an initial condition

$$x_0 = (b,\ 0,\ \cdots,\ 0,\ a)$$

where b is as given in the assumption of the theorem, and $a < 0$. Let the input be

$$u(t) = \varepsilon^{-1.5} \text{ for } t \in (0,\ \varepsilon] \triangleq I(\varepsilon),\ \varepsilon > 0$$

Then

$$x_1(t) = \varepsilon^{-1.5}t + b$$

and $v(t) = f_1(\varepsilon^{-1.5}t + b)\varepsilon^{-1.5}$ (C.7)

Since $a < 0$, it is always possible to choose an $\varepsilon_0 > 0$ such that

$$i(t) = \frac{1}{(\beta+1)}\varepsilon_0^{\beta-0.5} + \frac{1}{\beta!}\, b\, t^{\beta} + a \leq 0 \tag{C.8}$$

Then, for any $\varepsilon \in (0,\ \varepsilon_0]$,

$$i(t) \leq 0 \quad \forall t \in I(\varepsilon) \tag{C.9}$$

and $v(t) \geq M\varepsilon^{-1.5}$ (by assumption in the theorem).

Equations (C.7)-(C.9) imply that for all $t \in I(\varepsilon)$,

$$v(t)\ i(t) \leq \frac{1}{(\beta+1)!} M\varepsilon^{-3}t^{\beta+1} + \frac{1}{\beta!}\, b\, M\varepsilon^{-1.5}t^{\beta} + aM\varepsilon^{-1.5}$$

Therefore,

$$\int_{I(\varepsilon)} p(t)dt \leq \frac{M}{(\beta+2)!}\, \varepsilon^{\beta-1} + \frac{bM}{(\beta+1)!}\, \varepsilon^{\beta-0.5} + aM\varepsilon^{-0.5}$$

As $\varepsilon \to 0$, the above expression $\to +\infty$, since $aM < 0$.

So $E_A(x_0) \geq \sup_{\varepsilon \in (0,\varepsilon_0)} \{-\int_{I(\varepsilon)} p(t)dt\} = +\infty$. ¤

(D) Proof of Theorem 4.4

The following two lemmas are required for the proof.

Lemma D.1

Assume:

(i) $\alpha = 0$ and $\beta \geq 1$

(ii) $f(0) \neq 0$

Under these assumptions, the h.o.e. $v = f(i^{(\beta)})$ is active. ¤

Proof

Let $i(t) = -\mathrm{sgn}\, f(0)$

Then $v(t) = f(0)$

So $i(t)\, v(t) = -f(0)\, \mathrm{sgn}\{f(0)\} < 0$, and activity follows.

Lemma D.2

Assume:

(i) $\alpha \geq 1$ and $\beta = 0$

(ii) $\exists a \neq 0 \ni a\, f(a) \leq 0$

The h.o.e. $v^{(\alpha)} = f(i)$ is active under these assumptions. ¤

Proof

If $\exists a \ni af(a) < 0$, activity follows by Theorem 4.1

If $a \neq 0 \ni af(a) = 0$, then, necessarily, $f(a) = 0$.

Let $i(t) = a$

Then $v^{(\alpha)}(t) = f(a) = 0$.

By an appropriate choice of initial conditions, we get

$v(t) = -\mathrm{sgn}\, a$,

so $i(t)\, v(t) = -a\, \mathrm{sgn}\, a < 0$, and activity follows.

Proof of Theorem 4.4

(I) Consider the case $\alpha = 0$, $\beta \geq 1$:

By part 10 of Theorem 4.2, this element is active

if $\exists a,b \in \mathbb{R}$ such that: (i) $ab < 0$ and

(ii) $bf(a) \neq af(b)$

Assume (i) and (ii) are not both satisfied. This implies that for all $a,b \in \mathbb{R}$

$$ab \geq 0 \quad \text{or} \quad bf(a) = af(b) \tag{D.1}$$

By considering the values $b = \pm 1$ in (D.1) and applying Lemma D.1 to rule out the case where $f(0) \neq 0$, we can deduce that

$$f(a) = Ka, \tag{D.2}$$

for some value $K \in \mathbb{R}$ and for all $a \in \mathbb{R}$. Hence any function satisfying condition (D.2) cannot be active. This implies that the only passive h.o.e.'s with $\alpha = 0$ and $\beta \geq 1$ are the linear h.o.e.'s of the form

$$v = Ki^{(\beta)}, \ \beta \geq 1 \text{ and } K \in \mathbb{R} \tag{D.3}$$

By Theorem 4.3, the only passive h.o.e. described by equation (D.3) is the element

$$v = Ki^{(1)}, \text{ with } K \geq 0.$$

(II) Consider the case $\alpha \geq 1$ and $\beta = 0$:

By part ② of Theorem 4.2, the h.o.e. is active if $\exists a,b \in \mathbb{R}$ such that

(i) $af(b) \neq bf(a)$ and

(ii) $f(a)\,f(b) < 0$.

Passivity would imply that the above two conditions are not satisfied simultaneously. Also, by Theorem 4.1 and Lemma D.2, passivity further implies that for all $a \neq 0 \in \mathbb{R}$, $af(a) > 0$.

From the above, we can see that for the h.o.e. to be passive, it must satisfy the following:

$$\forall a \neq 0 \text{ and } b \in \mathbb{R}, \quad \begin{aligned} & af(b) = bf(a) \quad \text{or} \\ & f(a)\,f(b) \geq 0 \ . \end{aligned} \tag{D.4}$$

By considering the values of $b = 0$ and ± 1 in (D.4), it can be deduced that in this case, passivity implies linearity. Therefore, the only possible candidate for passivity for the case $\alpha \geq 1$ and $\beta = 0$ is the linear element

$$v^{(\alpha)} = Ki, \quad K \in \mathbb{R} \text{ and } \alpha \geq 1 \tag{D.5}$$

By Theorem 4.3, the only passive h.o.e. described by (D.5) is the following:

$$v^{(1)} = Ki, \text{ with } K > 0 \ . \qquad \square$$

(E) I. Proof of case ① of Theorem 5.1

For $\alpha \geq 1$, $\beta \geq 1$, the state representation can be found in Table 1b.

We choose the input

$$u(t) = i^{(\beta)}(t) = \begin{cases} a \ , & t \in [0, t_1) \\ b \ , & t \in [t_1, T) \end{cases}$$

with a,b as given in the hypothesis of the theorem and $T = (1 - \frac{a}{b})\, t_1$.

By Lemma 4.1, it is possible to find initial conditions x_{j0}, $j = 1, \cdots, (\beta-1)$ such that

$$x_{0j} = x_j(nT), \ n = 1, 2, \cdots$$

and $x_{\beta 0}$ can be chosen arbitrarily.

Under these conditions,

$$x_\beta(t) = i(t) = x_{0\beta} + g(t) + \int_0^t h(t-\tau)\ (\tau)d\tau \tag{E.1}$$

where $g(t)$ is a polynomial whose coefficients depend only on the first $(\beta-1)$ components of x_0 and $h(t-\tau) \triangleq \frac{1}{(\beta-1)!}(t-\tau)^{\beta-1}$ is a polynomial independent of the components of x_0.

Then

$$\dot{x}_{\beta+1}(t) = f(u(t)) = \begin{cases} f(a), & t \in [0, t_1) \\ f(b), & t \in [t_1, T) \end{cases}$$

It can easily be verified that

$$\int_0^T f(u(t))dt = 0$$

using the definition of T and assumption (ii) in the theorem. We can therefore apply Lemma 4.1 again to show that we can solve for x_{0j}, $j = (\beta+1), \cdots, (\beta+\alpha-1)$ and choose $x_{0(\beta+\alpha)}$ arbitrarily such that

$$x_{j0} = x_j(nT) \quad \text{for } n = 1, 2, \text{ and } j = (\beta+1), \cdots, (\beta+\alpha-1)$$

Now

$$v(t) = x_{\alpha+\beta}(t) = x_{0(\beta+\alpha)} + \tilde{g}(t) + \int_0^t h(t-\tau)\ (\tau)d\tau \tag{E.2}$$

where $\tilde{g}(t)$ is a polynomial whose coefficients depend only on x_{0j} for $j = (\beta+1), \cdots, (\beta+\alpha-1)$, $h(t-\tau)$ is as defined in equation (E.1) and $\tilde{u}(\tau) \triangleq f(u(\tau))$.

From equations (E.1) and (E.2), we can deduce that

$$\int_0^T p(t)dt = x_{0\beta}x_{0,(\alpha+\beta)}T + M_1x_{0\beta} + M_2x_{0(\beta+\alpha)} + M_3 \tag{E.3}$$

where M_1, M_2 and M_3 are just some finite constants. From (E.3), it is always possible to pick $x_{0\beta}$ and $x_{0(\alpha+\beta)}$ such that $\int_0^T p(t)dt = c \neq 0$.

This gives that

$$\int_0^{2T} p(t)dt = 2c$$

and hence the element is lossless by previous arguments. ¤

II. <u>Proof of Lemma 5.1</u>

a) By induction on $j = 1, 2, \cdots, n-1$: Integrating the j-th state equations, we get

$$x_{j+1}(T-t) = x_{0,j+1} - \int_0^t x_j(T-\tau)d\tau .$$

(i) If $x_j(t)$ has odd symmetry, i.e., $x_j(T-\tau) = -x_j(\tau)$ $\forall\tau$, then $x_{j+1}(T-t) = x_{j+1}(t)$, i.e., x_{j+1} has even symmetry.

(ii) If $x_j(t)$ has even symmetry, then

$$x_{j+1}(T-t) = x_{0,j+1} - \int_0^t x_j(\tau)d\tau, \text{ and}$$

$$x_{j+1}(t) = x_{0,j+1} + \int_0^T x_j(\tau)d\tau.$$

This implies that

$$\int_0^T x_{j+1}(T-t) = 2x_{0,j+1} - x_{j+1}(t).$$

Therefore

$$\int_0^T x_{j+1}(T-t)dt = 2Tx_{0,j+1} - \int_0^T x_{j+1}(t)dt .$$

Since both integrals in the above equation are zero, we have: $x_{j+1,0} = 0$.

b) The proof for odd n is the same as in part a). For even n, $x_{n-1}(t)$ is an even function and we have essentially the same proof, except that we cannot conclude that $x_{0n} = 0$. ¤

III. Proof of Corollary to Lemma 5.1

First note that the following facts are true:

(i) $x_{0j} = 0$ for even values of $j = 2, 4, \cdots, (n-2)$ (by part (a) of Lemma 5.1)

(ii) x_{0n} can be chosen arbitrarily such that for fixed values of x_{j0}, $j = 1, \cdots, (n-1)$, we have

$$x_0 \triangleq x(0) = x(2m),\ m = 1, 2, \cdots\cdot\ . \quad \text{(by Lemma 4.1).}$$

Fact (ii) implies that without loss of generality, x_{0n} can be chosen to be zero, so we get from part b) of Lemma 5.1

$$x_n(2-t) = -x_n(t) \quad \text{for } t \in [0, 2)$$

Setting t = 0 and 1 in the above equation, we can arrive at the conclusion of the corollary. ¤

①	(i) $\alpha = 0,\ \beta \le -1$	(ii) $\alpha \ge 1,\ \beta = 0$	(iii) $\alpha = 0,\ \beta = 0$	(iv) $\alpha \ge 1,\ \beta \le -1$
U	$= \mathbb{R}$ in all cases			
$\mathcal{U}$	$= L^1_{loc}(\mathbb{R}_+ \to \mathbb{R})$	$\mathcal{P}(\mathbb{R}_+)$	$\mathcal{P}(\mathbb{R}_+)$	$= L^1_{loc}(\mathbb{R}_+ \to \mathbb{R})$
Σ	$\subseteq \mathbb{R}^{\lvert\beta\rvert}$	$\subseteq \mathbb{R}^{\alpha}$	$\subseteq \mathbb{R}$	$\subseteq \mathbb{R}^{\lvert\beta\rvert+\alpha}$
E	$\dot{x}_1 = u$ $\dot{x}_2 = x_1$ $\vdots$ $\dot{x}_{\lvert\beta\rvert} = x_{\lvert\beta\rvert-1}$	$\dot{x}_1 = f(u)$ $\dot{x}_2 = x_1$ $\vdots$ $\dot{x}_{\alpha} = x_{\alpha-1}$	$\dot{x}_1 = 0$	$\dot{x}_1 = u$ $\dot{x}_2 = x_1$ $\vdots$ $\dot{x}_{\lvert\beta\rvert} = x_{\lvert\beta\rvert-1}$ $\dot{x}_{\lvert\beta\rvert+1} = f(x_{\lvert\beta\rvert})$ $\dot{x}_{\lvert\beta\rvert+2} = x_{\lvert\beta\rvert+1}$ $\vdots$ $\dot{x}_{\lvert\beta\rvert+\alpha} = x_{\lvert\beta\rvert+\alpha-1}$
R	$i = u$ $v = f(x_{\lvert\beta\rvert})$	$i = u$ $v = x_{\alpha}$	$i = u$ $v = f(u)$	$i = u$ $v = x_{\lvert\beta\rvert+\alpha}$
Assumptions on f	f: $\mathbb{R} \to \mathbb{R}$ is Borel measurable [5] and $\lvert f\rvert$ is integrable [5] over any finite interval in $\mathbb{R}$.			

Table 1a: $\alpha \ge 0,\ \beta \le 0$

②	(i) $\alpha = 0,\ \beta \geq 1$	(ii) $\alpha \geq 1,\ \beta \geq 1$
U	$= \mathbb{R}$	
$\mathcal{U}$	$= \mathcal{P}(\mathbb{R}_+)$	$= L^1_{loc}(\mathbb{R}_+ \to \mathbb{R})$
Σ	$\subseteq \mathbb{R}^{\beta}$	$\subseteq \mathbb{R}^{\beta+\alpha}$
E	$\dot{x}_1 = u \quad (= i^{(\beta)})$ $\dot{x}_2 = x_1$ $\vdots$ $\dot{x}_\beta = x_{\beta-1}$	$\dot{x}_1 = u \quad (= i^{(\beta)})$ $\dot{x}_2 = x_1$ $\vdots$ $\dot{x}_\beta = x_{\beta-1}$ $\dot{x}_{\beta+1} = f(u)$ $\dot{x}_{\beta+2} = x_{\beta+1}$ $\vdots$ $\dot{x}_{\beta+\alpha} = x_{\beta+\alpha-1}$
R	$i = x_\beta$ $v = f(u)$	$i = x_\beta$ $v = x_{\beta+\alpha}$
Ass. on f	f: $\mathbb{R} \to \mathbb{R}$ is Borel measurable and $\|f\|$ is integrable over any finite interval in $\mathbb{R}$.	

Table 1b: $\alpha \geq 0,\ \beta \geq 1$

	③ $\beta = \alpha \leq -1$	④ $\beta < \alpha \leq -1$	⑤ $\alpha < \beta \leq -1$	⑥ $\alpha \leq -1,\ \beta \geq 0$
U	$= \mathbb{R}$			
$\mathcal{U}$	$= L^2_{loc}(\mathbb{R}_+ \to \mathbb{R})$	$= L^p_{loc}(\mathbb{R}_+ \to \mathbb{R})$, $p \in [1, \infty]$	$= L^1_{loc}(\mathbb{R}_+ \to \mathbb{R})$	$= L^1_{loc}(\mathbb{R}_+ \to \mathbb{R})$
Σ	$\subseteq \mathbb{R}^{\lvert\beta\rvert}$	$\subseteq \mathbb{R}^{\lvert\beta\rvert}$	$\subseteq \mathbb{R}^{\lvert\alpha\rvert}$	$\subseteq \mathbb{R}^{\beta-\alpha}$
E	$\dot{x}_1 = u$ $\dot{x}_2 = x_1$ $\vdots$ $\dot{x}_{\lvert\beta\rvert} = x_{\lvert\beta\rvert-1}$	$\dot{x}_1 = u$ $\dot{x}_2 = x_1$ $\vdots$ $\dot{x}_{\lvert\beta\rvert} = x_{\lvert\beta\rvert-1}$	$\dot{x}_1 = u \quad (=i^{(\beta-\alpha)})$ $\dot{x}_2 = x_1$ $\vdots$ $x_{\lvert\alpha\rvert-\lvert\beta\rvert} = x_{\lvert\alpha\rvert-\lvert\beta\rvert-1}$ $\vdots$ $\dot{x}_{\lvert\alpha\rvert} = x_{\lvert\alpha\rvert-1}$	$\dot{x}_1 = u(=i^{(\beta-\alpha)})$ $\dot{x}_2 = x_1$ $\vdots$ $\dot{x}_{\beta-\alpha} = x_{\beta-\alpha-1}$
R	$i = u$ $v = \tilde{f}(\underset{\sim}{x},u)$	$i = u$ $v = \tilde{f}(x_{\lvert\beta\rvert-\lvert\alpha\rvert},$ $\cdots, x_{\lvert\beta\rvert-1},$ $x_{\lvert\beta\rvert})$	$i = x_{\lvert\alpha\rvert-\lvert\beta\rvert}$ $v = \tilde{f}(x,u)$	$i = x_{\beta-\alpha}$ $v = \tilde{f}(u,x_1,\cdots,x_{\lvert\alpha\rvert})$
Ass. on f	$f \in C^{\lvert\alpha\rvert-1}$ throughout			

Table 1c: $\alpha \leq -1$

① (i) $\alpha = 0, \beta \leq -1$		(ii) $\alpha \geq 1, \beta = 0$	(iii) $\alpha = 0, \beta = 0$	(iv) $\alpha \geq 1, \beta \leq -1$
Assumptions on f	$f \in C^\infty$	$f \in C^\infty$ and $f(u) \neq 0$ $\forall u \in U$	$f \in C^\infty$	$f \in C^\infty$, and $\frac{d}{dz} f(z)\Big\vert_{z=x_{\vert\beta\vert}} \neq 0$ for any admissible value of $x_{\vert\beta\vert}$.

Table 2a. $\alpha \geq 0, \beta \leq 0$

② (i) $\alpha = 0, \beta \geq 1$		(ii) $\alpha \geq 1, \beta \geq 1$
Assumptions on f	$f \in C^\infty$	$f \in C^\infty$ and $f(u) \neq 0$ $\forall u \in U$

Table 2b. $\alpha \geq 0, \beta \geq 1$

③ $\beta = \alpha \leq -1$	④ $\beta < \alpha \leq -1$ ⑤ $\alpha < \beta \leq -1$ ⑥ $\alpha \leq -1, \beta \geq 0$
Assumptions on f	$f \in C^\infty$ throughout

Table 2c. $\alpha \leq -1$

	α	β	f
①	≥ 1	≥ 1	$f: \mathbb{R} \to \mathbb{R}$
②	≥ 1	$= 0$	$\exists a,b \in \mathbb{R}$ such that (i) $af(b) \neq bf(a)$, and (ii) $f(a)\, f(b) < 0$
③	≥ 1	$= -1$	$\exists$ integer k and $A_j \in \mathbb{R}$ for $j = 0, 1, \cdots, k$, such that $f(z) \leq \sum_{j=0}^{k} A_j z^j$ (i) $\forall z \in [0,\infty)$ or (ii) $\forall z \in (-\infty, 0]$
④	≥ 0	≤ -2	(i) $\exists a \in \mathbb{R} \ni b \triangleq f(a) \neq 0$, and (ii) $f: \mathbb{R} \to \mathbb{R}$ is continuous in an ε-neighborhood of a
⑤	≤ -1	$\leq \alpha-2$	(i) $\tilde{f}: \mathbb{R}^{\lvert\alpha\rvert+1} \to \mathbb{R}$ and $\exists a \in \mathbb{R}^{\lvert\alpha\rvert+1} \ni b \triangleq \tilde{f}(a) \neq 0$, and (ii) $\tilde{f}$ is locally C^0 about a
⑥	$\leq \beta-2$	≤ -1	$\exists a_1 \in \mathbb{R} \ni f$ is locally $C^{\lvert\alpha\rvert}$ about a_1 and $f_1(a_1) \neq 0$
⑦	≤ -2	$= 0$	$\exists a \in \mathbb{R} \ni f$ is locally $C^{\lvert\alpha\rvert}$ about a and $af_1(a) \neq 0$
⑧	≤ -2	≥ 1	$\exists a_1 \in \mathbb{R} \ni f$ is locally $C^{\lvert\alpha\rvert}$ about a_1 and $f_1(a_1) \neq 0$
⑨	$= -1$	≥ 1	$\exists b > 0,\ M > 0 \ni f_1(z) \geq M,\ \forall z \in [b,\infty)$
⑩	$= 0$	≥ 1	$\exists a,b \in \mathbb{R} \ni$ (i) $ab < 0$ and (ii) $bf(a) \neq af(b)$

Table 3

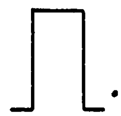
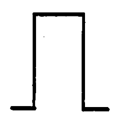
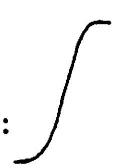
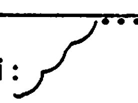
Theorem 5.2	input waveform u	current waveform i	class of functions f (satisfying the assumptions in Tables 1) to which theorem can be applied
1 $\alpha \geq 1,\ \beta \geq 1$	$u \equiv 0$	i: 	any $f:\ \mathbb{R} \to \mathbb{R}$
2 $\alpha \geq 1,\ \beta = 0$	u = i: 		nonlinear functions only
3 $\alpha \geq 1,\ \beta = -1$	u = i: 		f satisfies a polynomial bound
4 $\alpha \geq 0,\ \beta \leq -2$			almost all functions, except for pathological cases
5 $\alpha \leq -1,\ \beta \leq \alpha-2$			f is locally $C^{\lvert\alpha\rvert}$ about a point
6 $\alpha \leq -2,\ \beta \leq -1$ 7 $\alpha \leq -2,\ \beta = 0$ 8 $\alpha \leq -2,\ \beta \leq 1$	u: 	i: 	f is locally $C^{\lvert\alpha\rvert}$ about some point and has nonzero first-derivative at that point
9 $\alpha = -1,\ \beta \geq 1$			$f_1(z) \leq M \quad \forall z \in [b, \infty)$
10 $\alpha = 0,\ \beta \geq 1$	u: 	i: 	nonlinear functions only

Table 4

	α	β	f
①	≥ 1	≥ 1	$\exists a,b \in \mathbb{R}$ such that (i) $ab < 0$, and (ii) $af(b) = bf(a) \neq 0$
②	≥ 1	$= 0$	$\exists a,b \in \mathbb{R}$ such that (i) $af(b) \neq bf(a)$, and (ii) $f(a)\, f(b) < 0$
③	$= 0$	≤ -2 and takes on only <u>even</u> values	$\exists \delta > 0$ such that (i) f is injective and continuous in the interval $[-\delta/2, \delta/2]$, and (ii) $f(x) \neq f(0)$ on a set of nonzero measure $\forall x \in [-\delta/2, \delta/2]$.

Table 5

Figure Captions

Fig. 1. Input waveforms used in proving Theorems 4.1 and 4.2.

Fig. 2. Illustration of the results of Theorem 4.2

Fig. 3. Synthesis of the (unconstrained) linear h.o.e. $v^{(1)} = Ki^{(1)}$.

Fig. 4. Linear h.o.e.'s in regions P_{L1} and P_{L2} have to satisfy $K \geq 0$ for passivity. Those marked with crosses have to satisfy $K = 0$.

Fig. 5. Lossless linear h.o.e.'s. (Those in the first quadrant and the positive half-axes are constrained h.o.e.'s).

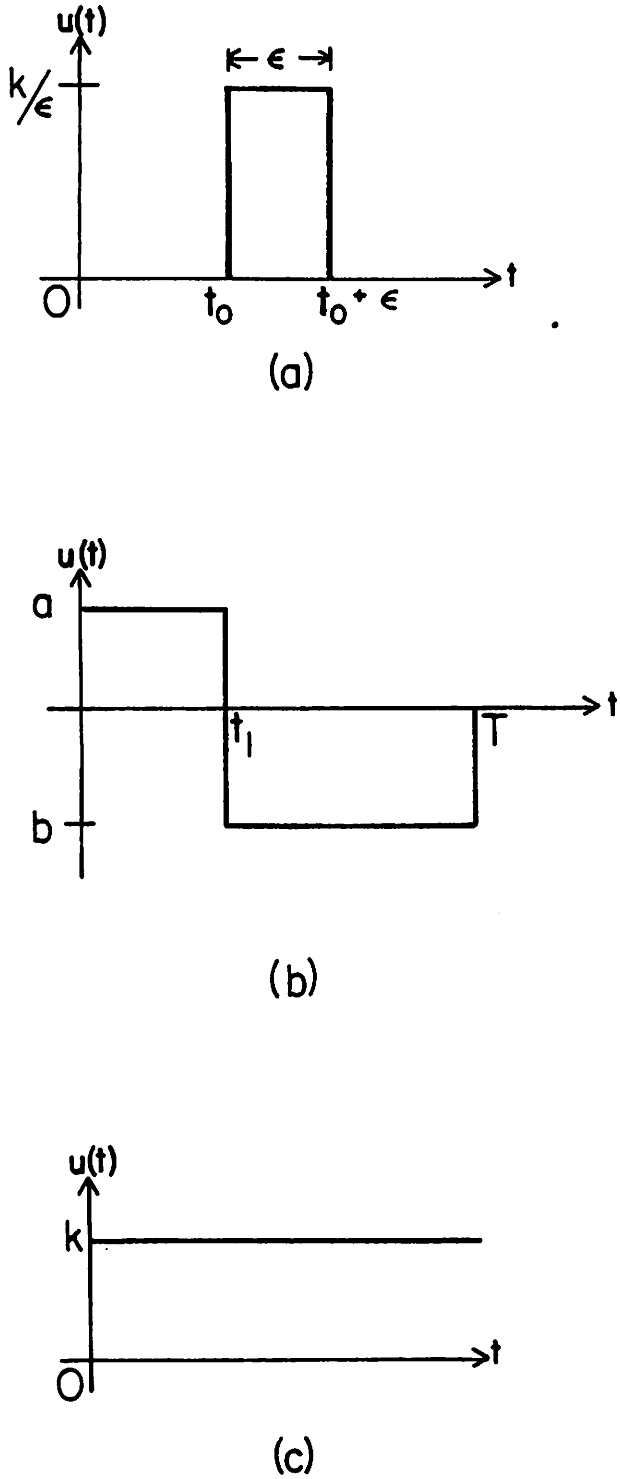


Fig. 1

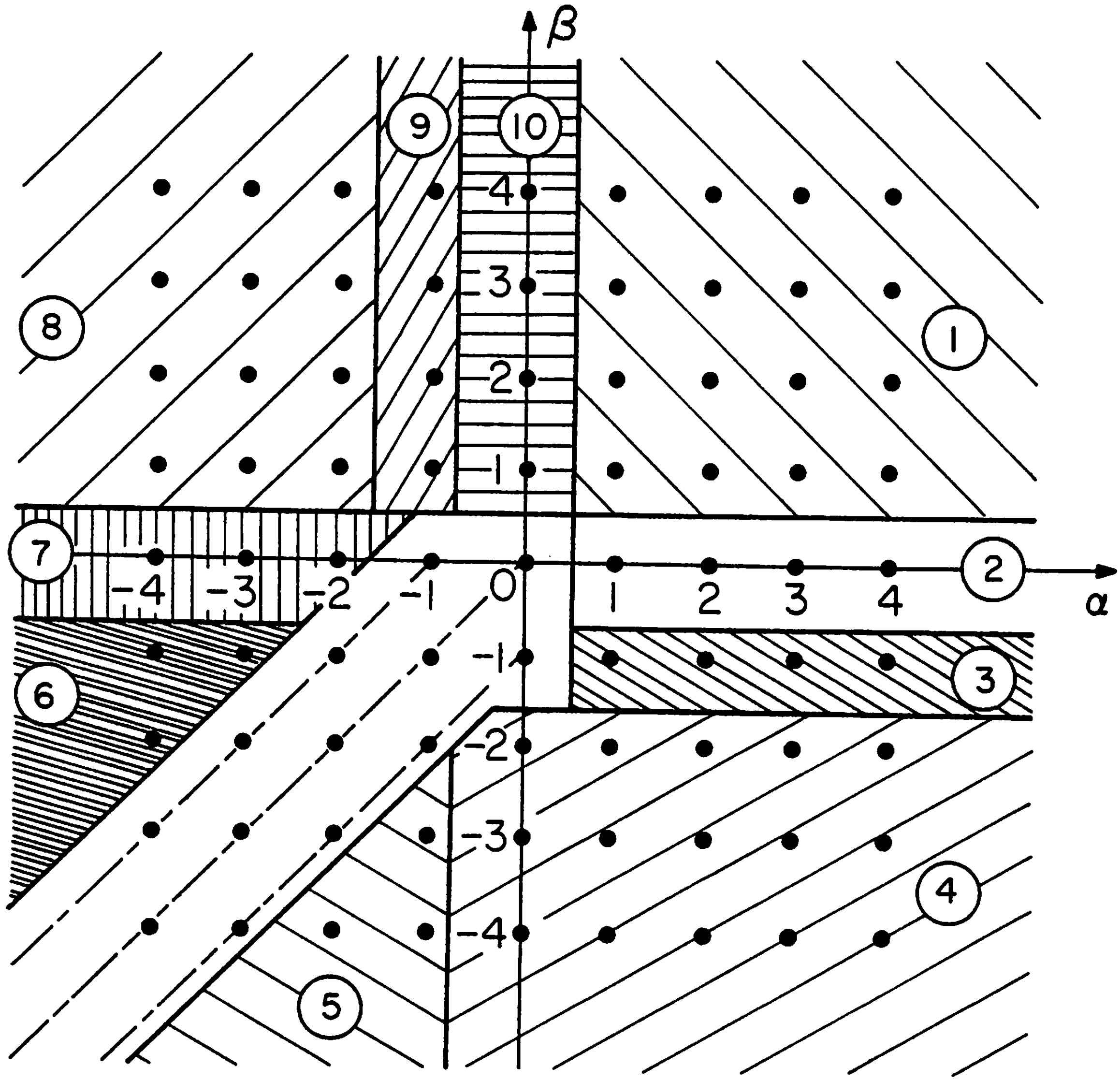


Fig. 2

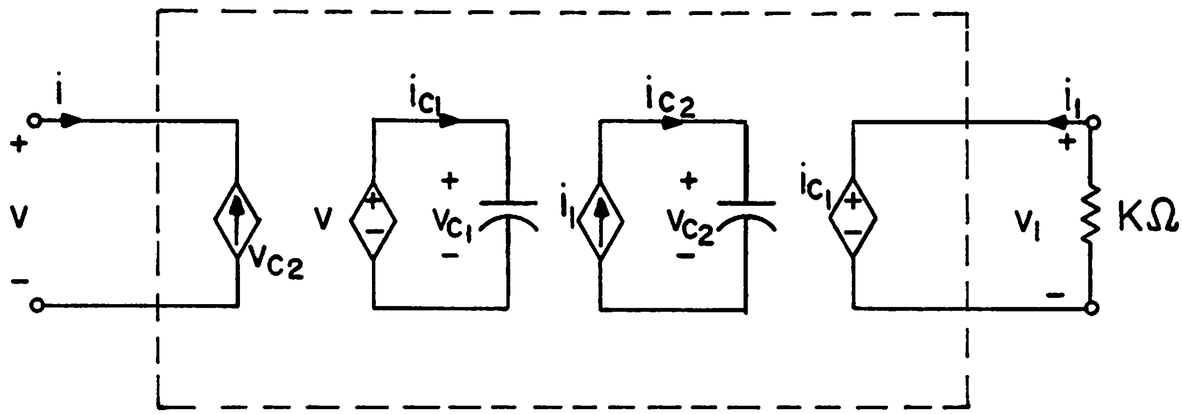


Fig. 3

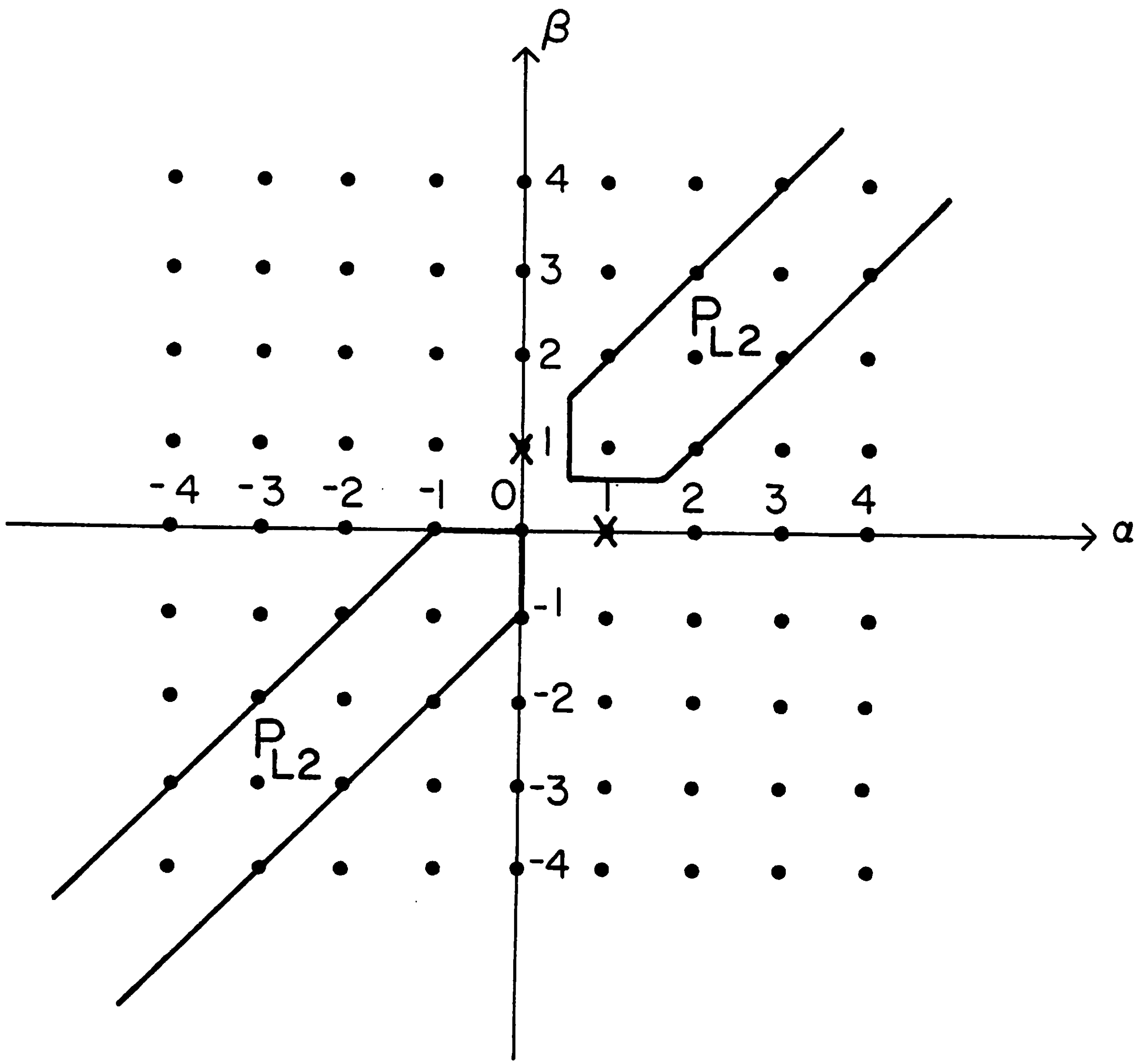


Fig. 4

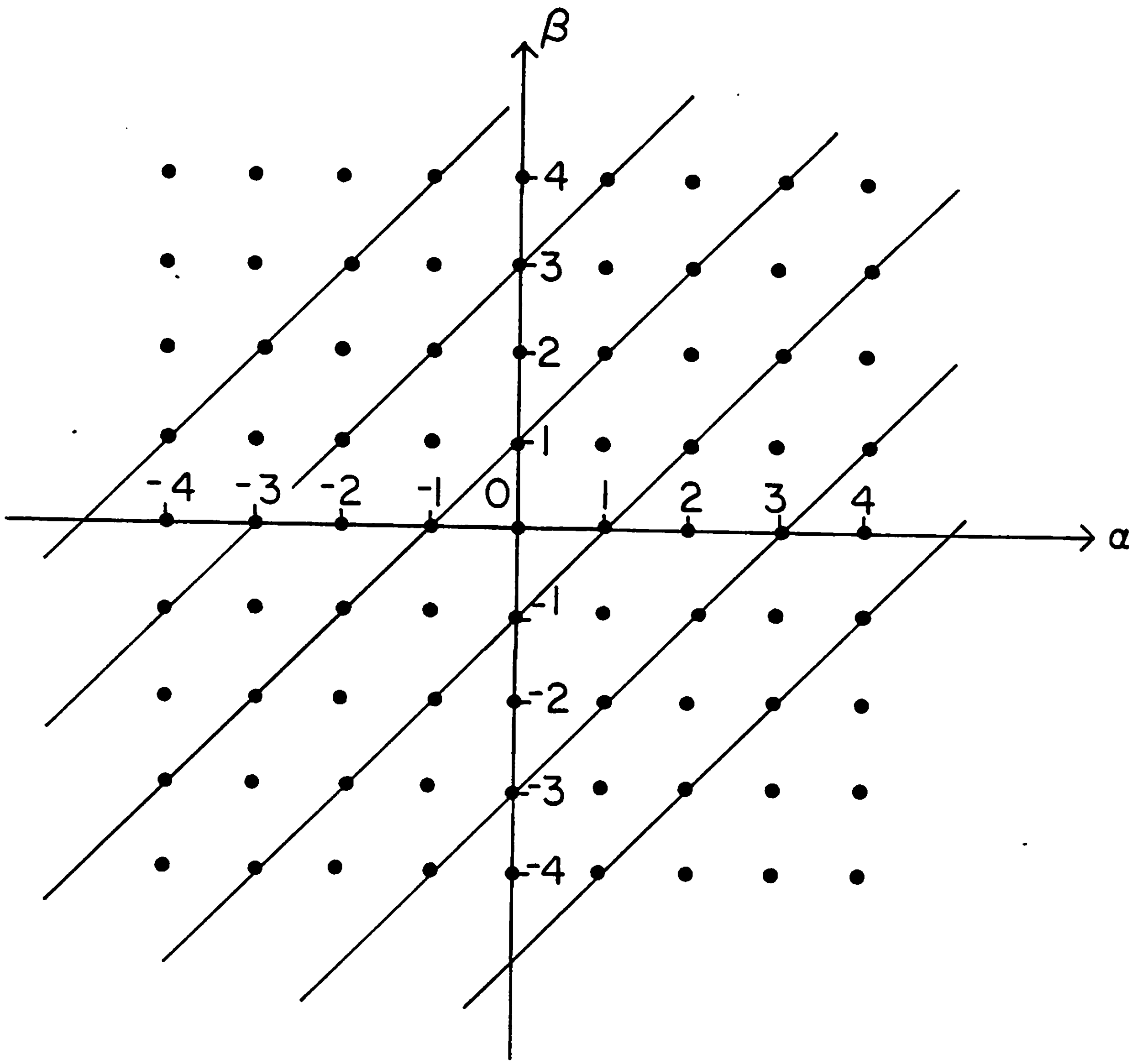


Fig. 5