MODELING NONLINEAR DEVICES EXHIBITING FREQUENCY - DEPENDENT CAPACITANCES AND INDUCTANCES

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ABSTRACT

The incremental capacitance and inductance of many electronic devices are frequency and bias dependent. This paper presents a systematic method for modeling such devices using only positive and/or negative linear time-invariant capacitances and inductances. In the nonlinear case, these capicitances and inductances are functions of either the port voltage or the port current. Using the theory developed in tis paper, any (2k + 2) order linear higher-order capacitor or inductor is shown realizable by a one-port containing (2k + 2) positive and/or negative linear inductors and capacitors.

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I. INTRODUCTION

It is well known that the incremental resistance, capacitance and inductance parameters in the circuit models of many devices are found to vary with frequency. Indeed, device manufacturers sometimes publish experimental curves giving the measured incremental resistance $R(\omega)$, capacitance $C(\omega)$, or inductance $L(\omega)$ as a function of frequency ω [1] - [4]. In [6] and [7] the capacitance of an MOS system is examined and it has been shown that the small-signal MOS capacitance depends upon the operating frequency. In [4] the frequency-dependent capacitance curve of a varactor diode measured by a network analyzer is given. It is also well known that the surface inductance of a conductor is frequency dependent as a result of skin effect [9].

The incremental capacitance of MOS systems and varactor diodes also vary with bias voltages [4, 6, 10]. Therefore, in some cases the incremental resistance $R(\omega,X)$, incremental capacitance $C(\omega,X)$ and incremental inductance $L(\omega,X)$ are also nonlinear functions of another parameter X, such as bias voltage, temperature, light intensity, etc.

In the following, modeling of $C(\omega)$ and $L(\omega)$ will be considered first. Modeling of $C(\omega,X)$ and $L(\omega,X)$ will be considered in Section III.F.

For the sinusoidal steady-state analysis of a linear time-invariant (LTI) circuit, frequency-dependent capacitors and inductors can be modeled by an LC one-port. In fact, for a passive LC one-port N_{LC},

$$Z_{LC}(j\omega) \triangleq jX(\omega) \triangleq j\omega A(\omega), \quad \text{where} \quad A(\omega) = X(\omega)/\omega$$

indicates that N_{LC} can be considered as a frequency-dependent inductance when $A(\omega) > 0$, and a frequency-dependent capacitance when $A(\omega) < 0$. However, since $X(\omega)$ is an increasing function of ω for passive LC one-ports, the type of $L(\omega)$ and $C(\omega)$ characteristics that can be modeled using only positive inductors and capacitors is rather restrictive. If negative inductors and capacitors are also used, then a much wider class of frequency-dependent capacitors and inductors can be modeled. For example, negative inductors and capacitors are used in [5] to model solid state cavity masers.

In this paper, modeling $C(\omega)$ and $L(\omega)$ by rational functions will be considered. It will be shown that any odd rational function $F(s)$ can be realized as the driving-point function of a one-port which contains $\pm L$, $\pm C$ elements only. Consequently, a LTI "generalized" capacitor or inductor of any order [1] can be realized by a $\pm L, \pm C$ one-port. On the other hand, if the incremental capacitance $C(\omega)$ or incremental inductance $L(\omega)$ is given as a continuous function in a frequency band $\omega_1 \leq \omega \leq \omega_2$ then it can be approximated to any desired accuracy by an odd rational function F(s) such that

$$|C(\omega) - \frac{1}{\omega} I_m[F(j\omega)| \leq \varepsilon \text{ or } |L(\omega) - \frac{1}{\omega} I_m[F(j\omega)]| \leq \varepsilon$$

where ε is an arbitrarily small positive number. Hence any continuous $C(\omega)$ or $L(\omega)$ can be modeled by a ±L, ±C one-port having F(s) as its driving-point function.

As it is known, the driving-point function F(s) of a +L, +C (LC) one-port is the ratio of two polynomials, one being odd and the other even, such that there is no missing term in F(s), i.e., there is no missing even (resp., odd) term in the even (resp., odd) polynomial. On the other hand, all but the highest (resp., the lowest) degree term in the odd (resp., even) polynomial are missing if F(s) is the driving-point function of a LTI "higher-order" capacitor or inductor as defined in [1]. Hence, in this respect, a higher-order capacitor or inductor is an extreme type of element and a realization of an m-th order capacitor or inductor by a ±L, ±C one-port requires at least m+1 elements for m > 1.

Properties of -L, +C one-ports(henceforth abbreviated as -LC one-ports) and +L, -C one-ports (abbreviated as -CL one-ports) are quite similar to that of LC one-ports. For example, in all cases the driving-point functions have no missing terms.

In Section II, modeling frequency-dependent capacitances and inductances by -LC and -CL one-ports will be considered and necessary and sufficient conditions for an odd rational function to be realized as the driving-poing function of a -LC or -CL one-port will be given.

In Section III, modeling frequency-dependent capacitances and inductances by ±L,±C one-ports will be considered. First, it will be shown that an odd polynomial with no missing term, whose order is 2k + 1 can be realized as the driving-point function of a ±L, ±C one-port which contains 2k + 1 elements. Then it will be shown that a minimal ±L, ±C realization of a LTI generalized capacitor or inductor of order 2k + 1 contains at least 2k + 2 elements if k > 0.

II. MODELING FREQUENCY-DEPENDENT CAPACITANCES AND INDUCTANCES BY -LC AND -CL ONE-PORTS.

A one-port made of negative inductors and positive capacitors as shown in Figure 1 realizes a frequency-dependent capacitor; namely, if the input admittance is denoted by $Y(s)$, then $C(\omega) \triangleq Y(j\omega)/j\omega \geq 0$ for all real ω values.

The analysis and synthesis of -LC one-ports can be reduced to the analysis and synthesis of LC one-ports which is a well established topic. However, only some restricted class of odd rational functions can be realized by -LC one-ports. For example, the following section shows that a fifth-order capacitor whose input admittance is $Y(s) = Ds^5$, where D is a real number, can not be realized by a -LC one-port.

A. Modeling frequency-dependent capacitance by -LC one-port

Choosing node $1'$ as the datum node and applying node equations, the input impedance of the one-port shown in Fig. 1 can be expressed as,

$$Z(s) = \frac{\Delta_{11}(s)}{\Delta(s)} \tag{2.1}$$

where $\Delta(s)$ and $\Delta_{11}(s)$ are the determinant and (1,1) cofactor of the node admittance matrix, respectively. Note that $\Delta(s)$ and $\Delta_{11}(s)$ can be expressed as the summations of tree-admittance products [13]:

$$\Delta(s) = \sum_{\text{all trees of } N} (\text{tree-admittance product of tree } t_i \text{ of } N) \tag{2.2a}$$

$$\Delta_{11}(s) = \sum_{\text{all trees of } N'} (\text{tree-admittance product of tree } t_i \text{ of } N') \tag{2.2b}$$

where N is the one-port circuit shown in Fig. 1 and N' is the circuit obtained from N by joining nodes 1 and 1'. A term in (2.2) has the following form:

$$\prod_{i=1}^{q} \gamma_i \prod_{i=1}^{p} C_i s^{p-q} \tag{2.3}$$

where $\gamma_i \triangleq 1/L_i$ and, p and q are the numbers of capacitors and inductors in the associated tree, respectively. If q is odd then the coefficient of s^{p-q} is negative; otherwise positive. If the number of nodes in Fig. 1 is denoted by $n + 1$, then, $p + q = n$ when (2.3) is a term in (2.2a) and $p + q = n - 1$ when (2.3) is a term in (2.2b). Let $p + q = n$ then $s^{p-q} = s^{n-2q}$. Hence if q is increased (resp., decreased) by 1 then n-2q is decreased (resp., increased) by 2 and the sign of the coefficient of s^{p-q} in (2.3) is changed. Clearly, the number of terms of the degree s^{n-2q} is equal to the number of different trees whose q branches are inductors and n-q branches are capacitors, and the coefficients are positive (resp., negative) when q is even (resp., odd). The results can be stated as a theorem.

Theorem 1. The input impedance of a -LC one-port can be expressed as,

$$Z(s) = \frac{a_m s^m + a_{m-2}\, s^{m-2} + \ldots}{b_n s^n + b_{n-2}\, s^{n-2} + \ldots} \tag{2.4}$$

where

i) a_i(resp., b_i) and a_{i-2}(resp., b_{i-2}) have opposite signs,

ii) If the negative coefficients in (2.4) are multiplied by -1, then the resulting impedance function is the input impedance of a one-port which is obtained from the given one-port by replacing each $-L_i$ by L_i. ¤

It is well-known that an odd rational function can be realized as the input impedance of an LC one-port if and only if it can be written in the form of [14].

$$Z(s) = k_\infty s + \frac{k_o}{s} + \frac{k_1 s}{s^2+\omega_1^2} + \ldots + \frac{k_p s}{s^2+\omega_p^2} \tag{2.5}$$

where $k_\infty \geq 0$ and $k_i \geq 0$, $i = 0,1,2,\ldots,p$, and $\omega_i \in \mathbb{R}$, $i = 1,2,\ldots,p$. $Z(s)$ in (2.5) can also be written as,

$$Z(s) = L_\infty s + \frac{1}{C_o s} + \frac{1}{C_1 s + \frac{1}{L_1 s}} + \ldots + \frac{1}{C_p s + \frac{1}{L_p s}} \tag{2.6}$$

where $L_\infty = k_\infty$, $C_o = 1/k_o$, $C_i = 1/k_i$ and $L_i = k_i/\omega_i^2$, $i = 1,2,\ldots,p$.

Equation (2.6) leads to the Foster 1 type realization shown in Fig. 2.

In the case of a -LC one-port, (2.5) and (2.6) take the following forms, respectively,

$$Z(s) = -k_\infty s + \frac{k_o}{s} + \frac{k_1 s}{s^2-\omega_1^2} + \ldots + \frac{k_p s}{s^2-\omega_p^2} \tag{2.7}$$

$$Z(s) = -L_\infty s + \frac{1}{C_o s} + \frac{1}{C_1 s - \frac{1}{L_1 s}} + \ldots + \frac{1}{C_p s - \frac{1}{L_p s}} \tag{2.8}$$

where $L_\infty = k_\infty$, $C_o = 1/k_o$, $C_i = 1/k_i$ and $L_i = k_i/\omega_i^2$, $i = 1,2,\ldots,p$. On the other hand, if the right hand side of Eqn. (2.5) represents an LC input admittance then partial fraction expansion of a -LC input admittance takes the following form,

$$Y(s) = k_\infty s - \frac{k_o}{s} + \frac{-k_1 s}{s^2-\omega_1^2} + \ldots + \frac{-k_p s}{s^2-\omega_p^2} \tag{2.9}$$

where $k_\infty \geq 0$ and $k_i \geq 0$, $i = 0,1,2,\ldots,p$. Hence the poles and zeros of an LC input impedance $Z_{LC}(s)$ and -LC input impedance $Z_{-LC}(s)$ are in one-to-one correspondence, respectively; namely if $j\omega_i$ is a pole (resp., zero) of $Z_{LC}(s)$ then $\omega_i = \mathrm{Im}[j\omega_i]$ is a pole (resp., zero) of $Z_{-LC}(s)$. The results can be stated as a theorem:

Theorem 2. A rational function Z(s) can be realized as the input impedance of a -LC one-port if and only if it can be expressed in one of the following two forms:

$$Z(s) = \frac{k(s^2-\omega_1^2)(s^2-\omega_3^2)\dots(s^2-\omega_r^2)}{s(s^2-\omega_2^2)(s^2-\omega_4^2)\dots(s-\omega_q^2)} \tag{2.10a}$$

$$Z(s) = \frac{ks(s^2-\omega_2^2)(s^2-\omega_4^2)\dots(s^2-\omega_q^2)}{(s^2-\omega_1^2)(s^2-\omega_3^2)\dots(s^2-\omega_r^2)} \tag{2.10b}$$

where $\omega_i \in \mathbb{R}$, $\omega_i > \omega_j$ if $i > j$; r (resp., q) is odd (resp., even) integer and either $q = r + 1$ or $r = q + 1$; $k < 0$ if $Z(\infty) \neq 0$ and $k > 0$ if $Z(\infty) = 0$. ¤

The susceptance $B(\omega)$ and frequency-dependent capacitance $C(\omega)$ associated with Y(s) in (2.9) are given by,

$$B(\omega) = Y(j\omega)/j = k_\infty\omega + \frac{k_0}{\omega} + \sum_{i=1}^{p} \frac{k_i\omega}{\omega^2+\omega_i^2} \tag{2.11}$$

and

$$C(\omega) = B(\omega)/\omega = k_\infty + \frac{k_0}{\omega^2} + \sum_{i=1}^{p} \frac{k_i}{\omega^2+\omega_i^2} \tag{2.12}$$

It follows from Eqn. (2.12) that the frequency-dependent capacitance $C(\omega)$ which is realized by a -LC one-port is non-negative for all real ω values. However, it is too restrictive to realize an arbitrary frequency-dependent $C(\omega)$. For example, $C(\omega) = D\omega^4$, which corresponds to a fifth-order capacitor, can not be obtained from Eqn. (2.12), i.e., $C(\omega) = D\omega^4$ can not be realized by a -LC one-port.

Example 1. Assume that the odd rational function which realizes a given freuqency-dependent capacitance is,

$$Z(s) = \frac{-4}{3} \cdot \frac{(s^2-1)(s^2-9)}{s(s^2-4)} \tag{2.13}$$

Since Z(s) is in the form (2.10a), it can be realized by a -LC one-port. In fact expanding Z(s) as,

$$Z(s) = \frac{-4}{3}s + \frac{3}{s} + \frac{5s}{s^2-4} = \frac{-4}{3} + \frac{3}{s} + \frac{1}{\frac{s}{5} - \frac{4}{5s}}$$

the one-port shown in Fig. 3 is obtained.

B. Modeling frequency-dependent inductance by -CL one-port

Replacing C_i by $-C_i$ in Eqn. (2.6) a general expression of the impedance of a -CL one-port is obtained as follows:

$$Z(s) = L_\infty s + \frac{1}{-C_o s} + \sum_{i=1}^{p} \frac{1}{-C_i s + \frac{1}{L_i s}} \tag{2.14}$$

Similarly a general expression of admittance becomes,

$$Y(s) = -C_\infty s + \frac{1}{L_o s} + \sum_{i=1}^{p} \frac{1}{L_i s - \frac{1}{C_i s}} \tag{2.15}$$

The frequency-dependent inductance associated with Z(s) given in (2.14) is,

$$L(\omega) = Z(j\omega)/j\omega = L_\infty + \frac{1}{C_o \omega^2} + \sum_{i=1}^{p} \frac{1}{C_i \omega + \frac{1}{L_i \omega}} \tag{2.16}$$

Note that $L(\omega)$ is non-negative for all ω values. However,Eqn. (2.16) is too restrictive to realize an arbitrary frequency-dependent inductance. For example $L(\omega) = \omega^4$ which corresponds to the fifth-order inductor $v(t) = d^5 i(t)/dt^5$ can not be realized by a -CL one-port.

Note that Eqns. (2.8) and (2.15) have the same form. Hence the following theorem can be stated.

Theorem 3. An odd rational function H(s) can be realized as the input admittance of a -CL one-port if and only if H(s) can be realized as the input impedance of a -LC one-port.

III. REALIZATION OF ARBITRARY ODD RATIONAL FUNCTIONS

Consider a higher-order linear time-invariant inductor defined in [1] as,

$$v(t) = m \frac{d^{2k+1} i(t)}{dt^{2k+1}} \tag{3.1}$$

and a higher-order linear time-invariant capacitor defined as,

$$i(t) = m \frac{d^{2k+1} v(t)}{dt^{2k+1}} \tag{3.2}$$

where $m \in \mathbb{R}$ and k is a non-negative integer. Hence, the input impedance (resp., admittance) of a higher-order inductor (resp., capacitor) is given by $F(s) = ms^{2k+1}$. Its associated frequency-dependent incremental inductance (resp., capacitance) is given by $F(j\omega)/j\omega = (-1)^k m\omega^{2k}$. It follows from Theorems 2 and 3 that a higher-order inductor or capacitor with $k \geq 1$ can not be realized by a -CL or a -LC one port. In the next section we will show that any odd rational function, consequently any higher-order inductor or capacitor can be realized as the input impedance or admittance of a ±L, ±C one-port.

A. Realization of a special type of odd polynomial

First, assume that our frequency-dependent inductance or capacitance is modeled in a frequency band $\omega_1 \leq \omega \leq \omega_2$ by a polynomial of the following form,

$$p(s) = \sum_{i=0}^{k} a_{2i+1}\, s^{2i+1} \tag{3.3}$$

where $a_{2i+1} \neq 0$ for $i = 0,1,2,\ldots,k$, i.e., $p(s)$ is an odd polynomial with no missing terms. In the following we will give a method to realize $p(s)$ as the input impedance of a ±L, ±C one-port. However, using the same method $p(s)$ can also be realized as the input admittance of a ±L, ±C one-port.

Consider $Y(s) = 1/p(s)$; without loss of generality, let $a_{2k+1} = 1$ and remove the pole of $Y(s)$ at $s = 0$. Then $Y(s)$ can be expressed as,

$$Y(s) = \frac{1}{a_1 s} + Y_1(s) \tag{3.4}$$

where

$$Y_1(s) = Y(s) - \frac{1}{a_1 s} = \frac{-1}{a_1} \cdot \frac{s^{2k-1} + a_{2k-1}\, s^{2k-3} + \ldots + a_5 s^3 + a_3 s}{s^{2k} + a_{2k-1}\, s^{2k-2} + \ldots + a_3 s^2 + a_1} \tag{3.5}$$

From Eqn. (3.4) it is seen that realization of $Y(s)$ is reduced to the realization of $Y_1(s)$. Note that if $\hat{Y}_1(s) = -a_1 Y_1(s)$ is realized then $Y_1(s)$ can be realized by impedance scaling.

Removing the pole of $1/\hat{Y}_1(s)$ at ∞ $\hat{Y}_1(s)$ can be written as,

$$Y_1(s) = \frac{1}{s+Z_2(s)} = \cfrac{1}{s + \cfrac{a_1}{s^{2k-1} + a_{2k-1}\, s^{2k-3} + \ldots + a_5 s^3 + a_3 s}} \tag{3.6}$$

From Eqn. (3.6) it is seen that $\hat{Y}_2(s) = a_1/Z_2(s)$ is an odd polynomial with no missing terms, whose degree differs by 2 from that of $Y(s)$. Iterating the above reduction process, we obtain the following expression for $Z(s)$:

$$Z(s) = \cfrac{1}{\cfrac{q_1}{s} + \cfrac{1}{q_2 s + \cfrac{q_3}{s} + \cfrac{1}{q_4 s + \cfrac{q_5}{s} + \cfrac{1}{\ddots + q_{2k}s + \cfrac{q_{2k+1}}{s}}}}} \tag{3.7}$$

The one-port realizing Z(s) is shown in Fig. 4. The above result can be stated as a theorem.

<u>Theorem 4</u>. An <u>odd polynomial</u> $p(s) = \sum_{i=0}^{k} a_{2i+1}s^{2i+1}$ where $a_i \neq 0$, $i = 0,1,2,\ldots,k$, can be realized as the input impedance or admittance of a ±L, ±C one-port which contains 2k+1 elements.

<u>Example 2</u>. Consider the polynomial

$$Z(s) = s^7 + s^5 + s^3 + s. \tag{3.8}$$

The expression which corresponds to (3.7) is,

$$Z(s) = \cfrac{1}{\cfrac{1}{s} + \cfrac{1}{-s + \cfrac{1}{-s} + \cfrac{1}{s + \cfrac{1}{s} + \cfrac{1}{-s + \cfrac{1}{-s}}}}} \tag{3.9}$$

and the one-port which realizes (3.9) is given in Fig. 5.

B. <u>Realization of higher-order capacitances and inductances</u>

As it is indicated in <u>Theorem 4</u>, an odd polynomial of order 2k+1 can be realized using 2k+1 elements if there are no missing terms. Assume that all the terms but $a_{2k+1}s^{2k+1}$ are missing. Then $p(s) = a_{2k+1}s^{2k+1}$ can be considered as the input impedance (resp., admittance) of an inductor (resp., capacitor) of order 2k+1. Without loss of generality, assume that $a_{2k+1} = 1$ and $p(s) = s^{2k+1}$ is the input impedance of an inductor. Assume that a ±L, ±C one-port realizes $Z(s) = s^{2k+1}$. Then, write the input impedance of the one-port such that element values are not assigned; the input impedance can be written as,

$$Z(s) = \frac{a_{2k+1}s^{2k+1} + a_{2k-1}s^{2k-1} + \ldots + a_1 s}{b_{2k}s^{2k} + b_{2k-2}s^{2k-2} + \ldots + b_0} \tag{3.10}$$

where the coefficients a_i and b_i are functions of circuit parameters L_i and C_i. Since the one-port realizes $Z(s) = s^{2k+1}$ there exists a set of elements values such that $a_{2k+1} = b_o = 1$ and all other coefficients are equal to zero. Assume that an LC one-port realizing Eqn. (3.10) is the one shown in Fig. 6. $Z(0) = 0$ implies that there exists at least one L-path between the terminals 1 and 1'. On the other hand $\lim_{s\to\infty} Z(s) = \infty$ implies that there can not be a C-path between 1 and 1'. As a result of these properties there can be only two different realizations of the impedance

$$Z(s) = \frac{a_3s^3+a_1s}{b_2s^2+b_o} \tag{3.11}$$

if (3.11) is to be realized using only three elements. The configurations are shown in Fig. 7. The input impedance corresponding to the one-port shown in Fig. 7a is

$$Z(s) = \frac{L_1L_2Cs^3+(L_1+L_2)s}{L_2Cs^2+1} \tag{3.12}$$

In order to realize $Z(s) = s^3$ the element values must satisfy the following equations,

$$L_1L_2C_2 = 1 \tag{3.13a}$$

$$L_1 + L_2 = 0 \tag{3.13b}$$

$$L_2C = 0 \tag{3.13c}$$

But there exists no solution of Eqns. (3.13).

The input impedance of the one-port shown in Fig. 7b is

$$Z(s) = \frac{L_1L_2Cs^3+L_1s}{(L_1C+L_2C)s^2+1} \tag{3.14}$$

Again it can be shown that there does not exist a set of element values realizing $Z(s) = s^3$. Hence, the total number of elements in an $\pm L$, $\pm C$ one-port realizing $Z = s^3$ must be greater than 3. In the following it is shown that the total number of elements in a $\pm L$, $\pm C$ one-port realizing $Z(s) = s^{2k+1}$ is greater than $2k+1$ for $k \geq 2$.

Assume that there is a realization of $Z(s) = s^{2k+1}$ with a $\pm L$, $\pm C$ one-port which contains $2k+1$ elements. Then the general form of $Z(s)$ is of the form of Eqn. (3.10). Considering the partial fraction expansion, it is seen that the one-port must contain $k+1$ inductors and k capacitors. Furthermore, there is no loops of inductors only, otherwise $s = 0$ would be a pole of $Z(s)$.

Consequently, a tree of Eqn. (2.2a) contains k+1 elements and a tree of Eqn. (2.2b) contains k elements, $k \geq 2$.

It follows from the above observations that if Z(s) is expressed in the form of Eqn. (2.1) and all coefficients but those corresponding to a_{2k+1} and b_0 in (3.10) are set equal to zero, then 2k equations are obtained with 2k+1 unknowns such that each term in an equation is the multiplication either of k or k+1 unknowns, i.e., each equation is a multi-variable polynomial of degree k of k+1.

Let a solution of these equations be $\underset{\sim}{x}^* = [x_1^*, x_2^*, \ldots, x_{2k+1}^*]^T$. Then, obviously, $k\underset{\sim}{x}^*$ is also a solution where k is any real number. Now $x_i^* \neq 0$, $i = 1,2,\ldots,2k+1$ by assumption. Hence, if any unknown say x_{2k+1} is set equal to 1, then the resulting set of 2k equations which contain 2k unknowns has also a solution $\bar{\underset{\sim}{x}}^* = [\bar{x}_1^*, \bar{x}_2^*, \ldots, \bar{x}_{2k}^*]^T$ such that $\bar{x}_i \neq 0$, $i = 1,2,\ldots,2k$. However, $\bar{x}_1 = \bar{x}_2 = \ldots = \bar{x}_{2k} = 0$ is a solution of the 2k equations. Hence, a solution $\bar{x}^*$ with $\bar{x}_i \neq 0$, $i = 1,2,\ldots,2k$ can not exist.

The preceding analysis shows that any ±L, ±C one-port which realizes $Z(s) = s^{2k+1}$ must have <u>at least</u> 2k+2 elements. In the following a realization of $Z(s) = s^{2k+1}$ using only the minimal number of 2k+2 elements will be given.

Let

$$Z_1(s) \triangleq s^{2k+1} - s. \tag{3.15}$$

Then

$$Z(s) = Z_1(s) + s. \tag{3.16}$$

Therefore realization of Z(s) is reduced to the realization of $Z_1(s)$. Notice that the degree of $Z_1(s)$ is also 2k+1. If $Z_1(s)$ is realized by a ±L, ±C one-port with 2k+1 elements then Z(s) can be realized with a ±L, ±C one-port with 2k+2 elements.

Consider,

$$Y_1(s) = \frac{1}{Z_1(s)} = \frac{1}{s(s^{2k}-1)} \tag{3.17}$$

Removing the pole of $Y_1(s)$ at zero we obtain,

$$Y_1(s) = \frac{-1}{s} + Y_2(s) \tag{3.18}$$

where

$$Y_2(s) = Y_1(s) + \frac{1}{s} = \frac{s^{2k-1}}{s^{2k}-1} \tag{3.19}$$

Now, remove the poles of $Y_2(s)$ at $s = 1$ and $s = -1$:

$$Y_2(s) = \frac{\frac{1}{k}s}{s^2-1} + Y_3(s) \tag{3.20}$$

where

$$Y_3(s) = Y_2(s) - \frac{\frac{1}{k}s}{s^2-1} = \frac{s^{2k-1} - \frac{1}{k}(s^{2k-1}+s^{2k-3}+\ldots+s^3+s)}{(s^2-1)(s^{2k-2}+s^{2k-4}+\ldots+s^2+1)} \tag{3.21}$$

Factoring out (s^2-1) in the numerator and cancelling the common factor (s^2-1), $Y_3(s)$ takes the following form,

$$Y_3(s) = \frac{1}{k} \cdot \frac{(k-1)s^{2k-3}+(k-2)s^{2k-5}+(k-3)s^{2k-7}+\ldots+3s^5+2s^3+s}{s^{2k-2}+s^{2k-4}+\ldots+s^2+1} \tag{3.22}$$

Using continued-fraction expansion we obtain,

$$Z_3(s) = \frac{1}{Y_3(s)} = \frac{k}{k-1}s + Z_4(s) \tag{3.23}$$

where

$$Z_4(s) = \frac{k}{k-1} \cdot \frac{s^{2k-4}+2s^{2k-6}+\ldots+(k-2)s^2+k-1}{(k-1)s^{2k-3}+(k-2)s^{2k-5}+\ldots+2s^3+s} = \frac{\sum_{j=2}^{k}(j-1)s^{2(k-j)}}{s\sum_{j=1}^{k-1} js^{2(j-1)}} \tag{3.24}$$

Then

$$Y_4(s) = \frac{(k-1)^2}{k}s + Y_5(s) \tag{3.25}$$

where

$$Y_5(s) = (1-k) \cdot \frac{s^{2k-5}+2s^{2k-7}+\ldots+(k-3)s^3+(k-2)s}{s^{2k-4}+2s^{2k-6}+\ldots+(k-2)s^2+k-1}$$

in short,

$$Y_5(s) = (1-k)\frac{s\sum_{j=3}^{j=k}(j-2)s^{2(k-j)}}{\sum_{j=2}^{j=k}(j-1)s^{2(k-j)}} \tag{3.26}$$

Finally,

$$Z_5(s) = \frac{s}{1-k} + Z_6(s) \tag{3.27}$$

where

$$Z_6(s) = \frac{-1}{s^{2k-5}+2s^{2k-6}+\ldots+(k-2)s} = \frac{-1}{s\sum_{j=3}^{j=k}(j-2)s^{2(k-j)}} \tag{3.28}$$

Now, using (3.15), (3.18), (3.20), (3.23), (3.25), (3.27) and (3.28) $Z(s) = s^{2k+1}$ can be expanded as,

$$Z(s) = s + \cfrac{1}{\frac{-1}{s} + \frac{\frac{1}{k}s}{s^2-1} + \cfrac{1}{\frac{k}{k-1}s + \cfrac{1}{\frac{(k-1)^2}{k}s + \cfrac{1}{\frac{s}{1-k} + \cfrac{-1}{s^{2k-5}+2s^{2k-7}+\ldots+(k-2)s}}}}} \tag{3.29}$$

and (3.29) can be realized by the one-port shown in Fig. 8.

By Theorem 4. $Z_6(s)$ can be realized using 2k-5 elements. Therefore, Z(s) can be realized using 7+2k-5 = 2k+2 elements. Hence we can state the following theorem.

Theorem 5. A generalized inductor or capacitor of order 2k+1 can be realized by a ±L, ±C one-port which contains 2k+2 elements.

Example 3. Consider,

$$Z(s) = s^7 \tag{3.30}$$

Then expansion (3.29) takes the following form,

$$Z(s) = s + \cfrac{1}{\frac{-1}{s} + \frac{\frac{1}{3}s}{s^2-1} + \cfrac{1}{\frac{3}{2}s + \cfrac{1}{\frac{4}{3}s + \cfrac{1}{-\frac{1}{2}s + \frac{-1}{s}}}}} \tag{3.31}$$

The ±L, ±C one-port which realizes (3.31) is given in Fig. 9.

C. Realization of any Odd Polynomial

It has been shown that the polynomial given by (3.3) and the polynomial,

$$p(s) = s^{2k+1} - s \tag{3.32}$$

can be realized using 2k+1 elements. However, a minimal realization of the polynomial

$$p(s) = s^{2k+1} \tag{3.33}$$

requires 2k+2 elements.

Obviously any other odd polynomial can be expressed as a summation of the polynomials given by (3.3), (3.32) and (3.33); hence it can be realized as a driving-point function of a $\pm L$, $\pm C$ one-port.

Example 4. Consider

$$Z(s) = s^7 + s^5 + s \tag{3.34}$$

If Z(s) is written as,

$$Z(s) = Z_1(s) + Z_2(s) + Z_3(s)$$

where $Z_1(s) = s^7$, $Z_2(s) = s^5$ and $Z_3(s) = s$ then a minimal realization requires 8+6+1 = 15 elements. If Z(s) is written as,

$$Z(s) = Z_1(s) + Z_2(s)$$

where $Z_1(s) = s^7+s^5+s^3+s$ and $Z_2(s) = -s^3$ then a minimal realization requires 7+4 = 11 elements. On the other hand, consider the following decomposition of $Y(s) = 1/Z(s)$,

$$Y(s) = \frac{1}{Z(s)} = \frac{1}{s^7+s^5+s} = \frac{1}{s} + \frac{-s^5-s^3}{s^6+s^4+1} = \frac{1}{s} + \cfrac{1}{-s + \cfrac{-1}{s^5+s^3}}$$

Using

$$\frac{-1}{s^5+s^3} = \frac{1}{s} + \frac{-1}{s^3} + \frac{-s}{s^2+1} = \cfrac{1}{s + \cfrac{s}{s^2-1}} + \frac{-s}{s^2+1}$$

the following decomposition is obtained,

$$Y(s) = \frac{1}{s} + \cfrac{1}{-s + \cfrac{1}{s + \cfrac{s}{s^2-1}} + \cfrac{-s}{s^2+1}}$$

which requires 7 elements. Since the degree of Z(s) is seven, a realization with fewer elements is not possible. Minimal realization of an arbitrary odd polynomial will not be discussed here.

D. Realization of an odd rational function

Given an odd rational function Z(s), if Z(s) or Y(s) is not a polynomial then using continued-fraction expansion, realization of Z(s) can always be reduced to the realization of polynomials. However, a decomposition of Z(s) may not lead to a minimal realization. On the other hand, removal of a simple pole of Z(s) or Y(s) simplifies the realization and leads to a minimal realization. This is because after removal of a simple pole (including a pole at ∞) if the remaining impedance (resp., admittance) $Z_r(s)$ (resp., $Y_r(s)$) has a minimal realization then Z(s) (resp., Y(s)) has a minimal realization. Hence, if all the poles of Z(s) (resp., Y(s)) are simple and if the polynomial part of the partial fraction expansion of Z(s) (resp., Y(s)) has a minimal realization, then Z(s) (resp., Y(s)) has a minimal realization. Pole removal from an odd rational function is discussed in the Appendix.

Example 5. Consider,

$$Z(s) = \frac{s^5}{(s^2+1)(s^4+s^2+1)} \tag{3.35}$$

The poles are : $p_1 = j$, $p_2 = -j$, $p_3 = \frac{-1}{2} + j\frac{-\sqrt{3}}{2}$ $p_4 = \bar{p}_3$, $p_5 = -p_3$ and $p_6 = -\bar{p}_3$.
Partial fraction expansion of Z(s) is given by,

$$Z(s) = \frac{s}{s^2+1} + \frac{-s}{s^4+s^2+1} = \frac{1}{s+\frac{1}{s}} + \cfrac{1}{\frac{-1}{s}+\cfrac{1}{\frac{-1}{s}+\cfrac{1}{s+\frac{1}{s}}}}$$

The corresponding realization is given in Fig. 10.

E. A method to approximate a given $L(\omega)$ or $C(\omega)$ curve by an odd rational function

In the following approximation of a $C(\omega)$ curve will be discussed. However, the method can also be applied to the approximation of an $L(\omega)$ curve.

Given $C(\omega)$ a rational function Y(s) can be found such that $C(\omega)$ and $\tilde{C}(\omega) \triangleq \frac{1}{j\omega} Y(j\omega)$ are equal at ω_i, $i = 1,2,\ldots,n$. In general Y(s) can be chosen as any odd rational function,but to simplify the discussion the following general form will be considered,

$$Y(s) = \frac{s^{m+1}+a_{m-1}s^{m-1}+\ldots+a_3s^3+a_1s}{a_ms^m+a_{m-2}s^{m-2}+\ldots+a_2s^2+a_0} \tag{3.36}$$

Then,

$$\tilde{C}(\omega_i) = C(\omega_i) \quad i = 1,2,\ldots,n \tag{3.37}$$

give n linear algebraic relations among the coefficients a_i. Hence choosing m appropriately the coefficients a_i $i = 1,2,\ldots,n$ are determined.

Example 6. Consider the $C(\omega)$ curves shown in Fig. 11. It is given in [4] as the frequency-dependent capacitance of a varactor diode biased at 10V and 15V, respectively.

The 10V bias voltage curve will be approximated using the points corresponding to $\omega_1 = 2\pi \times 460 \times 10^6$ r/s, $\omega_2 = 2\pi \times 640 \times 10^6$ r/s and $\omega_3 = 2\pi \times 840 \times 10^6$ r/s. Choose,

$$Y(s) = \frac{s^3+a_1 s}{a_2 s^2+a_0} \tag{3.38}$$

Then $\tilde{C}(\omega)$ is,

$$\tilde{C}(\omega) = \frac{a_1-\omega^2}{a_0-a_2\omega^2} \tag{3.39}$$

From Fig. 11 it is seen that $C(\omega_1) = 6.5$ pF

$C(\omega_2) = 7.5$ pF and $C(\omega_3) = 10$ pF.

Scaling frequency by 10^{10}, $\tilde{C}(\omega_i) = C(\omega_i)$, $i = 1,2,3$, give the following equation.

$$\begin{bmatrix} -0.065 & 1 & 0.0054298 \\ -0.075 & 1 & 0.0121277 \\ -0.1 & 1 & 0.0278559 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0.083536 \\ 0.1617 \\ 0.27855 \end{bmatrix} \tag{3.40}$$

Substituting the solution from (3.40) into (3.38) and then using continued-fraction expansion, we obtain the following approximating admittance:

$$Y(s) = \frac{s^3+2.5199s}{77.27s^2+43.9387} = 0.01294s + \cfrac{1}{39.598s + \cfrac{1}{0.04441s}} \tag{3.41}$$

The denormalized circuit corresponding to (3.41) is shown in Fig. 12. Although we have matched $C(\omega)$ and $\tilde{C}(\omega)$ only at 3 points it can be seen that $\frac{|C(\omega)-\tilde{C}(\omega)|}{C(\omega)} \leq 5\%$ in the entire band of $100 \text{ MHz} \leq f \leq 1000 \text{ MHz}$.

Observe that for this example, which represents a real physical device, all capacitances and inductance in the circuit model of Fig. 12 are positive. Note that at low frequencies, L_s is negligible and the model reduces to a 5.735 pF capacitance. It is remarkable that although this circuit model was derived strictly from a black box approach, the elements in the model can actually be given physical interpretations. Indeed, as is typical in microwave applications, a "lead inductance" L_s is generally inserted in series with the "junction capacitance" C_j, and the entire model is connected across a "case capacitance" C_C [10,11].†

F. Modeling frequency and X-dependent capacitances and inductances

Besides being a function of frequency the incremental capacitance or inductance which is used in the model of a nonlinear device may also be a nonlinear function of other parameters, such as bias voltage, light intensity, temperature etc. In such a case the incremental capacitance or inductance can be modeled by a $\pm L$, $\pm C$ one-port whose elements are X-dependent, i.e., instead of $\pm L$, $\pm C$ one-port, a $\pm L(X)$, $\pm C(X)$ one-port is used. For example, the capacitance of a varactor diode is a function of both frequency and bias voltage. Then, one way to realize $C(\omega,V)$ approximately is to realize $C(\omega,V)$ for a certain V and repeat the process for different V values. Another way is to use optimization [12]; once $C(\omega,V)$ is realized by a $\pm L$, $\pm C$ one-port for $V = V_i$, changing the circuit parameters in an optimum way would model $C(\omega,V)$ for different V values.

The circuit for realizing an arbitrary $C(\omega,X)$ or $L(\omega,X)$ can be found by repeating the preceding method for each value of the parameter X. Note that the circuit remains the same. Only the elements are now (generally nonlinear) functions of the parameter X. In the usual case where X denotes either the driving-point voltage or current, the resulting circuit model is no longer linear. Rather, it is a time-invariant nonlinear circuit. In this case, it is important to note that the X-dependent capacitances (resp., inductances) in this model are not nonlinear capacitors. Rather, they are algebraic 2-ports as defined in [1].

For example, let $C_j(v_1)$ denotes a capacitance in the circuit which depends on the driving-point voltage v_1. In order for the circuit model containing this capacitance to correctly simulate on a computer the measured small-signal capacitance $C_j(V_1)$ at each dc bias voltage $v_1 = V_1$, it is necessary that the computer be instructed to describe this element by

$$i_j = C_j(v_1)\frac{dv_j}{dt} \tag{3.42}$$

To describe this element as a v_1-controlled nonlinear capacitor,

$$q_j = \int_0^{v_j} C_j(v_1)dv_j \triangleq f(v_1,v_j) \tag{3.43}$$

would be incorrect because this would give

†The microwave varactor circuit model would also normally include the small series resistance of the wafer. Since our measured data in Fig. 11 consists of a frequency dependent capacitance only, the model in Fig. 12 includes only lossless elements.

$$i_j(t) = \frac{dq_j}{dt} = \frac{\partial f}{\partial v_1}\frac{dv_1}{dt} + C_j(v_1)\frac{dv_j}{dt} \tag{3.44}$$

which contains an extra term not present in (3.42). Indeed, this paradoxical element is actually an (α_1,β_1)-(α_2,β_2) algebraic 2-port (with $\alpha_1 = \beta_1 = \beta_2 = 0$ and $\alpha_2 = 1$) described by (where $j = 2$)

$$i_1 = 0 \tag{3.45a}$$

$$i_2 = C_j(v_1)\frac{dv_2}{dt} \tag{3.45b}$$

Note that the variables associated with port 1 are $v_1^{(0)} \triangleq v_1$ and $i_1^{(0)} \triangleq i$; those with port 2 are $v_2^{(1)} \triangleq dv_2/dt$ and i_2.

IV. STABILITY CONSIDERATIONS

It has been shown that any odd rational function can be realized as the driving-point function of a $\pm L$, $\pm C$ one-port. Hence frequency-dependent inductances and capacitances can be modeled by $\pm L$, $\pm C$ one-ports. However, the one-port which approximate $L(\omega)$ or $C(\omega)$ may have poles and/or zeros in the right-half plane.

Example 7. Consider the $C(\omega)$ curve given in Fig. 13(a). Assume that the frequency-dependent capacitance,

$$\hat{C}(\omega) = \left.\frac{Y(s)}{s}\right|_{s=j\omega} \tag{4.1}$$

where,

$$Y(s) = \frac{s(s^2-1)}{4s^2-2} \tag{4.2}$$

is equal to $C(\omega)$. Hence $Y(s)$ is an exact representation of $C(\omega)$ but it has a zero and a pole in the right-half plane. Therefore any one-port realizing $Y(s)$ is neither open-circuit nor short-circuit stable. However, if a linear time invariant circuit containing $C(\omega)$ is exponentially-stable, then sinusoidal steady state solution still exists and $C(\omega)$ can be represented either by the -LC one-port shown in Fig.13(b) or by the $\pm R$, C one-port shown in Fig. 13(c). In general an exponentially-stable linear time-invariant circuit may contain unstable elements. For example, a frequency-dependent negative resistor (FDNR) defined by,

$$i(t) = D\frac{d^2v(t)}{dt^2} \tag{4.3}$$

is not open-circuit stable but it is widely used in active circuit design provided that the circuit containing the FDNR is exponentially stable. The effect of some elements on the stability of a circuit can be examined using Nyquist criterion or other equivalent methods.

Over a given frequency interval, an arbitrary $C(\omega)$ or $L(\omega)$ curve may be approximated by several different odd rational functions. When Choosing a model the stability of the circuit should also be taken into consideration.

V. CONCLUSIONS

In this paper modeling of incremental capacitance $C(\omega,X)$ and incremental inductance $L(\omega,X)$ are considered.

In Section II it has been shown that a -LC (resp. -CL) one-port realizes a frequency-dependent capacitance (resp. inductance) and a necessary and sufficient condition for the realization of an odd rational function as the driving-point function of a -LC (resp. -CL) one-port is given.

In Section III it has been shown that an odd polynomial of order 2k+1 given by Equ. (3.3) can be realized by a $\pm L$, $\pm C$ one-port with 2k+1 elements. On the other hand, a minimal realization of the polynomial $p(s) = s^{2k+1}$ which corresponds to a higher-order inductor or capacitor of order 2k+1 contains 2k+2 elements. It has also been shown that any odd rational function can be realized as the driving-point function of a $\pm L,\pm C$ one-port.

In Section III.E a method for approximating a given $C(\omega)$ or $L(\omega)$ curve by an odd rational function is given. Once the rational function is realized by a $\pm L,\pm C$ one-port, then, using computer optimization [12] the element values can be adjusted for a better match of the $C(\omega)$ curve. Furthermore, the same technique can be used in the realization of $C(\omega,X)$ or $L(\omega,X)$ by a $\pm L,\pm C$ one-port whose elements are parametrized by X.

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FIGURE CAPTIONS

Fig. 1. Modeling of a frequency-dependent capacitance.

Fig. 2. Foster type 1 realization of an LC driving-point function.

Fig. 3. -LC realization of the impedance given by Equ. (2.13).

Fig. 4. Realization of an odd polynomial given by Equ. (3.3).

Fig. 5. Realization of the impedance given by Equ. (13.8).

Fig. 6. LC one-port associated with the impedance given by Equ. (3.10).

Fig. 7. Two different realizations of Z(s) given by Equ. (3.11).

Fig. 8. A realization of $Z(s) = s^{2k+1}$.

Fig. 9. A realization of $Z(s) = s^7$.

Fig.10. Realization of the impedance given by Equ. (3.35).

Fig.11. Capacitance of a varactor diode as function of frequency for 2 fixed bias voltages: V = 10V and V = 15V (Taken from [4]).

Fig.12. A realization of the $C(\omega)$ curve with 10V-bias voltage given in Fig. 11.

Fig.13. (a) A given $C(\omega)$ curve.

(b) and (c). Two different realizations of the admittance given by Equ. (4.2).

APPENDIX

Pole removal from an odd rational function

An odd rational function H(s) can be written in the following form,

$$H(s) = s^k \frac{N(s)}{D(s)} \tag{A.1}$$

where k is a non-zero integer and the polynomials N(s) and D(s) contain only even degree terms.

A. Removal of an imaginary pole

Assume that H(s) has a simple imaginary pole jp. Then -jp is also a pole and H(s) can be written as,

$$H(s) = H_1(s) \frac{1}{s^2+p^2} = \frac{k}{s-jp} + \frac{\bar{k}}{s+jp} + H_r(s) \tag{A.2}$$

where,

$$k = H_1(jp) \frac{1}{2jp} . \tag{A.3}$$

It follows from (A.1) that $H_1(jp)$ is an imaginary number. Hence, by (A.3) k is a real number. Therefore, Eqn. (A.2) takes the following form,

$$H(s) = \frac{2ks}{s^2+p^2} + H_r(s) \tag{A.4}$$

Since the first term in (A.4) is easily realized as the input impedance or admittance of a ±L, ±C one-port, the realization of H(s) is reduced to the realization of $H_r(s)$.

B. Removal of a real pole

If H(s) has a simple real pole then jp in (A.2) is replaced by p and (A.4) is replaced by

$$H(s) = \frac{2ks}{s^2-p^2} + H_r(s) \tag{A.5}$$

where $k \in \mathbb{R}$. Therefore the realization of H(s) is reduced to the realization of $H_r(s)$

C. Removal of a complex pole

Assume that H(s) has a simple complex pole $p = \alpha+j\beta$, $\alpha,\beta \neq 0$. Then, $\bar{p}$, $-p$ and $-\bar{p}$ are also poles of H(s). Therefore (A.2) is replaced by

$$H(s) = H_1(s)\,\frac{1}{s^4+2(\beta^2-\alpha^2)s^2+(\alpha^2+\beta^2)^2} \tag{A.6}$$

which can be expanded as follows:

$$H(s) = \frac{k_1}{s-\alpha-j\beta} + \frac{\overline{k_1}}{s-\alpha+j\beta} + \frac{k_2}{s+\alpha+j\beta} + \frac{\overline{k_2}}{s+\alpha-j\beta} + H_r(s). \tag{A.7}$$

Since p is simple,

$$k_1 = (s-\alpha-j\beta)\,H(s)\Big|_{s=\alpha+j\beta} = H_1(\alpha+j\beta)\,\frac{1}{2\alpha+2j\beta}\,\frac{1}{2\alpha}\,\frac{1}{2j\beta} \tag{A.8}$$

and

$$k_2 = H_1(-\alpha-j\beta)\,\frac{1}{-2\alpha-2j\beta}\;\frac{1}{-2\alpha}\,\frac{1}{-2j\beta} \tag{A.9}$$

Since $H_1(s)$ is an odd function of s,

$$H(-\alpha-j\beta) = -H(\alpha+j\beta)$$

It follows from Eqns. (A.8) and (A.9) that,

$$k_2 = k_1 \triangleq k = x+jy \tag{A.10}$$

substituting (A.10) into (A.7) we obtain

$$H(s) = \frac{4[xs^3+(x\alpha^2+x\beta^2-2x\alpha^2-2y\alpha\beta)s]}{s^4+2(\beta^2-\alpha^2)s^2+(\alpha^2+\beta^2)^2} + H_r(s) \tag{A.11}$$

The first term in (A.11) can be written as,

$$H_1(s) = \frac{a_3s^3+a_1s}{s^4+b_2s^2+b_0} \tag{A.12}$$

We will now show that $H_1(s)$ can be realized using 4 elements. Without loss of generality, assume that $H_1(s)$ is an input impedance.

Case 1. $a_3 = 0,\ a_1 \neq 0.$

Assume that $a_1 = 1$. Otherwise use impedance scaling. Then $Z(s) \triangleq H_1(s)$ can be decomposed as

$$Z(s) = \frac{1}{\frac{b_0}{s}+s^3+b_2s} = \cfrac{1}{\frac{b_0}{s}+\cfrac{1}{\frac{1}{b_2}+\cfrac{1}{-b_2s+\frac{b_2^2}{s}}}} \tag{A.13}$$

Obviosuly (A.13) can be realized using 4 elements.

Case 2. $a_3 \neq 0$, $a_1 = 0$

Assume $a_3 = 1$. Then Z(s) can be expanded as,

$$Z(s) = \cfrac{1}{s + \cfrac{b_2 s^2 + b_0}{s^3}} = \cfrac{1}{s + \cfrac{1}{\cfrac{1}{b_2} s + \cfrac{-b_0 s}{b_2^2 s^2 + b_0 b_2}}} \tag{A.14}$$

Again it is seen that (A.14) can be realized using 4 elements.

Case 3. $a_3 \neq 0$, $a_1 \neq 0$

Assume $a_3 = 1$. Applying continued-fraction expansion, Z(s) can be expressed as,

$$Z(s) = \cfrac{1}{s + \cfrac{1}{q_1 s + \cfrac{1}{a_2 s + \cfrac{1}{q_3 s}}}} \tag{A.15}$$

It is easily seen that $q_i \neq 0$, $i = 1,2,3$. Therefore (A.15) can be realized using 4 elements.

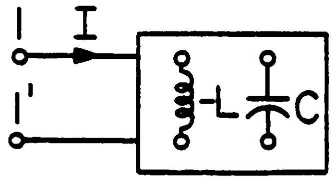

Fig. 1

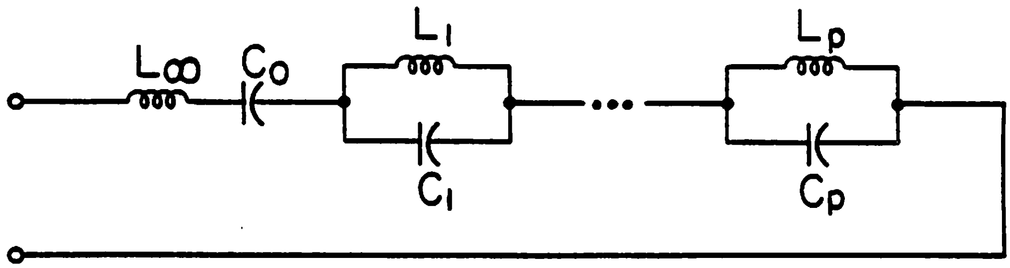

Fig. 2

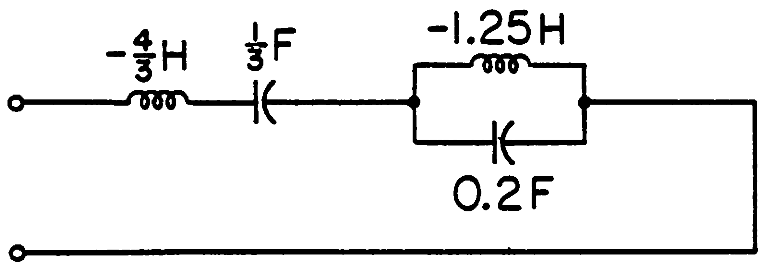

Fig. 3

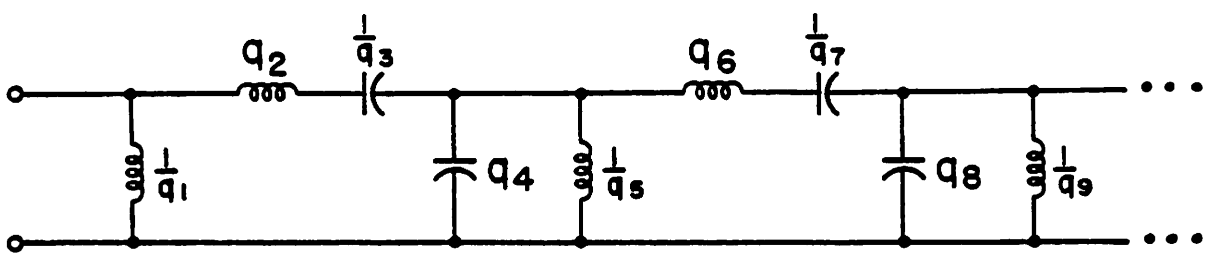

Fig. 4

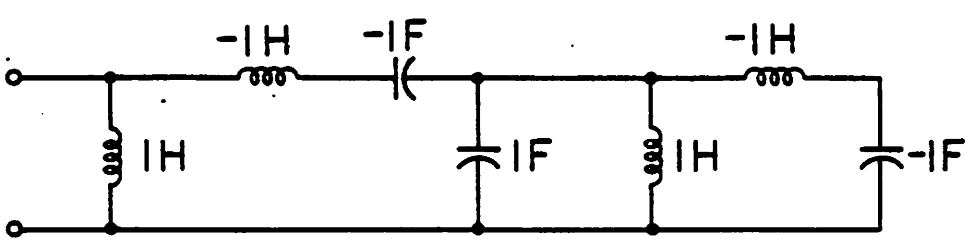

Fig. 5

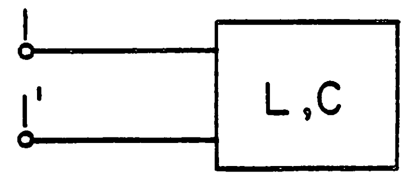

Fig. 6

Fig. 7

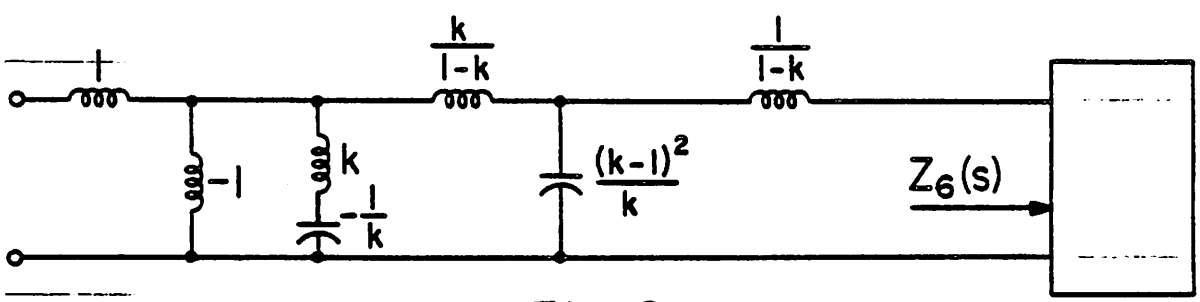

Fig. 8

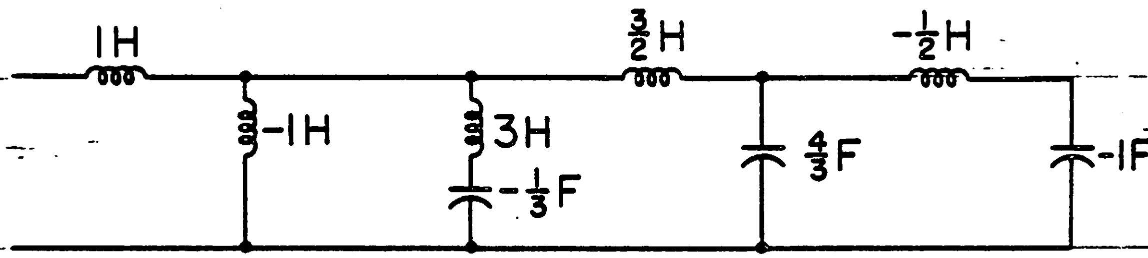

Fig. 9

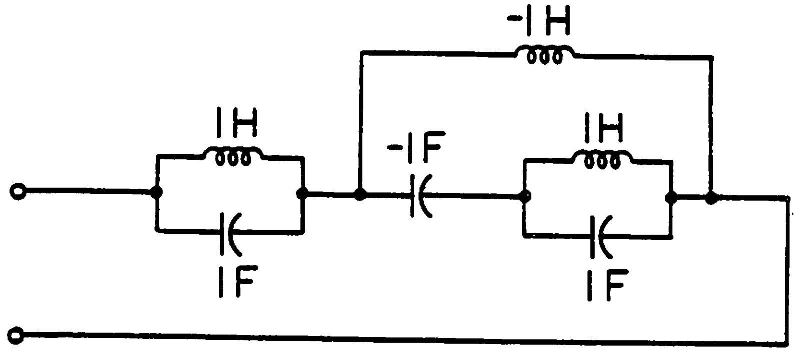

Fig. 10

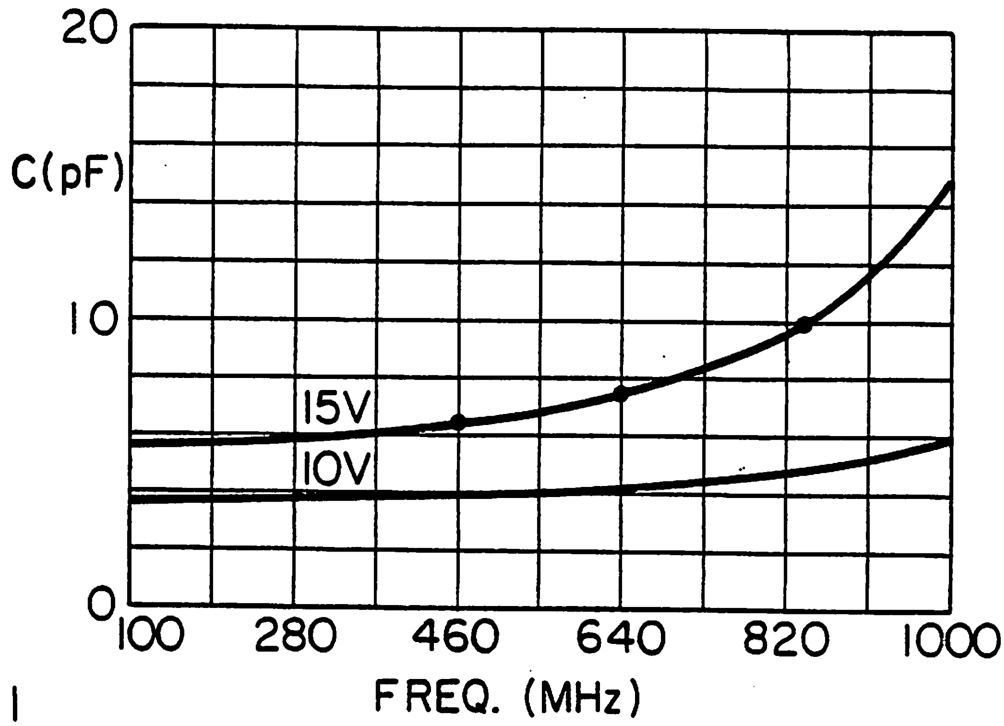

Fig. 11

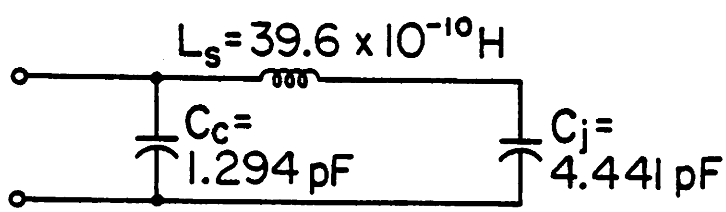

Fig. 12

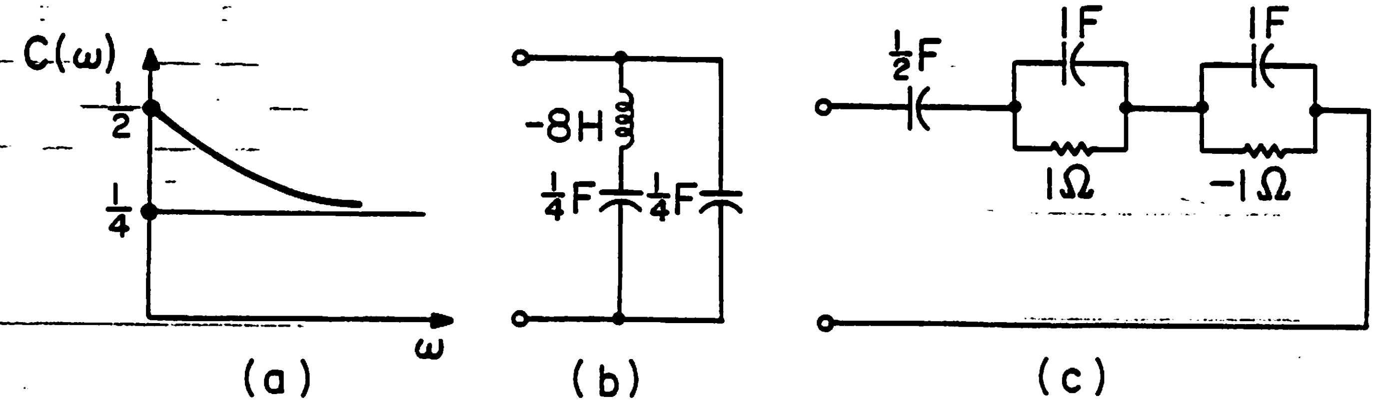

Fig. 13