SYNCHRONIZATION AND CHAOS

by

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Abstract

We believe that synchronization and chaos are closely related. Common intuition suggests that when a circuit is off sychronization the observed output, although not periodic, will be a sum of periodic (intermodulation) components. In fact, at least for a large class of systems we have studied, the output does not have this relatively simple form but is actually chaotic. This paper studies a simple but realistic model for a large class of triggered oscillators. Theory and experiments both confirm that the output shows the properties of sensitivity to initial conditions, non-periodicity, broad spectrum, and complicated recurrence, that characterize chaotic motion.

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1. Loss of synchronization may imply chaos

The phenomenon of synchronization is not well-understood theoretically, yet it is very widely used in practical electronics (e.g., television, radar, oscilloscopes, etc.). In general, a small periodic synchronizing signal with an accurate period is used to drive a system which can produce a larger signal having a period not far from the driving signal, in such a way that the larger signal 'locks on' to the synchronizing signal's frequency (or to some multiple or submultiple of it). There are several possible mechanisms for this, but the most common in practical circuits, and, fortunately, the easiest to understand, is the threshold mechanism, in which the driven system produces an output that depends on when a certain threshold level is reached. In this case, the synchronization signal works by causing the threshold to be reached at a controlled time.

As we shall see, successful synchronization depends on appropriate relationships between the signal levels and other parameters, and in a system designed to synchronize, some means is always provided to adjust the parameters to make it work correctly. When the parameters are outside some range, the system is not synchronized and the output is not periodic. For example, the vertical rolling seen in TV sets with improperly adjusted vertical sweeping controls is a direct manifestation of such loss of synchronization. It would be natural to assume that in this case it is merely a sum of incommensurable periodic components due to the driving signal and the driven system. If this were so, the output would be almost periodic, and of no particular interest.

We will show that the time state of affairs is quite different: for a large class of systems that we can analyze in detail - and probably for many other systems - the output shows characteristics that are normally described as chaotic. That is, although it is quasi-recurrent, there are no stable periodic solutions; there are no stable almost periodic solutions; the signal exhibits a broad spectrum; and the system is sensitive to initial conditions, in the sense that arbitrarily close initial states eventually become mapped into different "cycles" of the recurrence. (This is chaos in rather a strong sense, since many systems described as chaotic actually have stable periodic solutions. [1]).

We want to emphasize that we have chosen a class of systems that are easy to understand mathematically and for which the circuit elements are all behaving in the way they were designed to behave†. Other chaotic circuits exist but depend on exotic behavior of components operating outside their normal design parameters. We should also remark that although it is clear in retrospect that one of the seminal papers in the study of chaos was about synchronization [2], there seems to have been no follow-up. In fact, the literature of synchronization is minimal and that on the relationship between synchronization and chaos appears to be non-existent.

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† For circuit engineers who are skeptical about chaos in common circuits, we will discuss later why artefacts of measuring equipment may often conceal such behaviour and make it look as if something much simpler is happening.

In the rest of this paper, we are going to analyse a mathematical model for threshold synchronization and describe some experiments with real electronic circuits that use it. To make the paper accessible to readers who are not circuit theorists, we will begin by describing how threshold synchronization works. It should then be possible to skip the detailed circuit description (section 2) without too much loss of continuity.

Suppose we have a system that produces a steadily rising output $x(t)$ until some upper threshold b is reached, then produces a steadily falling output until some lower threshold a is reached, and then restarts with a rising output as in Figure 1.1a (drawn with $a = 0$). To simplify matters, we will work with linearly rising and falling outputs throughout the paper, but we will show that this restriction can be dispensed with. Thus

$$\dot{x} = \begin{cases} I_0 & \text{if} \quad \dot{x}>0 \ , \quad x\le b \\ -\dfrac{I_0}{\alpha} & \text{if} \quad \dot{x}<0 \ , \quad x\ge a \end{cases}$$

where $a < b$. Let us add a narrow periodic pulse $d(t)$ to the output, with period slightly shorter than the natural period. If the system causes $\dot{x}$ to change from positive to negative whenever the total signal $x(t) + d(t)$ reaches the upper threshold, we can have the situation in Figure 1.1b where the combined signal $x + d$ has the same period as the driving signal d. (This case only arises if the driving pulses are large enough and the periods are close enough, of course.) The equation is now

$$\dot{x}(t^+) = \begin{cases} I_0 & \text{if } \dot{x}(t) > 0 \text{ and } x(t) + d(t) < b \, , \\ & \text{or if } x(t) + d(t) \le a \\ -\dfrac{I_0}{\alpha} & \text{if } \dot{x}(t) < 0 \text{ and } x(t) + d(t) > a \\ & \text{or if } x(t) + d(t) \ge b \, . \end{cases} \tag{1.1}$$

where t^+ denotes $\lim\limits_{\varepsilon \downarrow 0}(t+\varepsilon)$.

This elementary mathematical model is an extremely accurate description of the behaviour of a host of circuits: to take only one example, the emitter - coupled astable multivibrator to be discussed in section 2. It will turn out that if the return from the upper threshold is too slow ($\alpha>1$, in fact), synchronization is lost and the output becomes truly non-periodic. The system turns out then to have an interesting relationship with pseudo-random number generators.

§2. Circuit model

An astable multivibrator usually consists of a capacitor connected across a resistive (memoryless) nonlinear circuit. This circuit, usually made of resistors, transistors and a dc power supply, charges and discharges the capacitor periodically to produce the desired oscillation. Figure 2.1 shows one of these circuits. In this circuit, transistors Q_5 and Q_6 act approximately as two current sources which always draw constant currents I_1 and I_2, respectively. D_{z_1} and D_{z_2} are two Zener diodes with Zener voltage V_z. When Q_1 is on and Q_2 is off, V_2 is equal to V_{cc} which makes V_x equal to $V_{cc}-V_z-2V_{BE(on)}$, where $V_{BE(on)} \approx 0.6V$ is the base-to-emitter voltage drop for the transistor in the active region. Current I_1+I_2 flows through R_{c_1} so V_1 is equal to $V_{cc}-(I_1+I_2)R_{c_1}$ (R_{c_1} is chosen to be small enough so that Q_1 is not saturated.) Assume V_y is greater than $V_1-V_z-2V_{BE(on)}$ then Q_2 remains off so current I_1 flows through C_0. This causes V_y to decrease and will eventually turn on Q_2. As soon as Q_2 is on, V_2 will drop sharply so Q_1 will be cut off. V_y will jump to $V_{cc}-V_z-2V_{BE(on)}$ but V_{xy} cannot change because of the capacitor C_0. Now V_x is greater than $V_2-V_z-2V_{BE(on)}$ so Q_2 remains off and the process is reversed. This mechanism results in a nearly triangular waveform across C_0 and nearly square waveforms of V_1 and V_2.

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† All voltages are measured with respect to ground.

Instead of the above analysis, this oscillation can also be studied in a more general manner by modelling the transistor circuit (excluding C_0) as a nonlinear resistor. Within the range when all transistors work as they are designed to, the current-voltage relationship looking from terminals x-y (with the capacitor C_0 removed) can be obtained and is shown in Figure 2.2. The branch AB corresponds to Q_1 off, Q_2 on and CD corresponds to Q_1 on Q_2 off. The dashed line BC corresponds to the case when both Q_1 and Q_2 are on. Since the current flowing into x will go through Q_1, R_{c_1}, R_{c_2}, Q_2 , and comes out from y , V_1-V_2 is proportional to the current by $R_{c_1}+R_{c_2}$. Because $V_x-V_1 \approx V_y-V_2 \approx 2V_{BE(on)}+V_z$, the characteristic between BC is approximately a straight line with slope $-1/(R_{c_1}+R_{c_2})$. Notice that the previous analysis shows that corners B and C occur at $V_{xy}=-(I_1+I_2)R_{c_1}$ and $V_{xy}=(I_1+I_2)R_{c_1}$ respectively (when Q_1 and Q_2 change states) and the straight line joining them does have the slope $-1/(R_{c_1}+R_{c_2})$. Though we did not carefully analyse the exact shapes of the corners, it is clear that they have little effect on the oscillation so we simply join B and C with a straight line. The non-zero slopes of AB and CD result from the imperfect performance of the transistor current sources and are usually negligible.

This circuit and the resistor-capacitor model motivated our study but from now on we shall analyze the further idealized and simplified model shown in Figure 2.3a and b.

Here the nonlinear resistor charges capacitor C_0 with current I_0 until it reaches point C ($V_R=b$). An instantaneous transition from C to A is assumed to occur at this point and thereafter resistor R begins to discharge C_0 with a current equal to $-I_0/\alpha$. When it reaches B ($V_R=0$), another instantaneous transition from B to D is assumed to occur and R starts charging C_0 again. This idealized model of an "astable" oscillator enables us to study the effect when it is triggered by external narrow pulses as shown in Figure 2.4. The behavior of this circuit model is characterized by equation (1.1) where x corresponds to the capacitor voltage.

§3. Analysis of the model

We will now look in detail at the behavior of the threshold-triggering model. We shall assume the lower threshold a is zero and both the signal $x(t)$ and the triggering pulse $d(t)$ are non-negative, so only the upper threshold matters. We assume the pulse is so narrow (compared to its period) that we can write it as

$$d(t) = \begin{cases} c & t = np \\ 0 & \text{otherwise} \end{cases}$$

where n is any integer and $p > 0$ is the period.

When $x(t) + d(t)$ exceeds b, x starts to decrease (because the capacitor in the circuit is switched from charging to discharging). A possible waveform is shown in Figure 3.1; notice that the ratio of the falling and rising times is still always α, as in the non-triggered case, merely because the slopes remain the same and the lower limit is always 0. However, the times themselves change, allowing the time for a cycle to range from a possible maximum of $q = b(1+\alpha)/I_0$ down to a minimum of $q' = (b-c)(1+\alpha)/I_0$ (see Figure 3.1; the existence of q' stems from the fact that triggering is impossible if $x(t)+d(t) < b-c$.)

Let us assume the driving signal's period p is slightly longer than the free running period q. (If p is slightly shorter than q, we always get synchronization eventually.) Suppose also that $p-q$ is less than c/I_0, the time for x to rise from its lowest triggerable value to its maximum.

With these assumptions, a possible waveform of $x+d$ is plotted in Figure 3.2. If $p-q$ is small, the circuit will not usually be triggered again on the cycle immediately after a successful triggering: instead, it will free-run with the triggering pulse shifting a distance $p-q$ on each cycle. From Figure 3.2, it is easy to see that the next successful triggering occurs when

$$(p-q)k \geq \alpha t_n + (b-c)/I_0 \qquad (3.1)$$

where k is the smallest integer that satisfies (3.1). The assumption that $p-q < c/I_0$ ensures a pulse <u>will</u> occur in the first region where it could cause switching; if $p-q > c/I_0$ the region could be skipped, which would complicate the analysis but could not give rise to any more complicated behavior.

It follows at once that

$$t_{n+1} = (p-q)k-\alpha t_n \qquad (3.2)$$

and this recurrence relation describes the system fully. However, k depends on t_n and it is more convenient to remove it from the discussion; it turns out to be easy to relate the conclusions derived without the presense of k to the qualitative behavior when k is present. To remove k we notice that

$$\alpha t_n + (b-c)/I_0 = (k-1)(p-q) + (\alpha t_n+(b-c)/I_0)\bmod(p-q) \; . \qquad (3.3)$$

This can be obtained either from the picture or by considering (3.1) together with

$$(p-q)(k-1) < \alpha t_n + (b-c)/I_0 .$$

Now subtract (3.3) from (3.2) to give

$$t_{n+1} - (b-c)/I_0 = (p-q)-(\alpha t_n + (b-c)/I_0)\bmod(p-q) \qquad (3.4)$$

or setting, $\tau_n = t_n-(b-c)/I_0$,

$$\tau_{n+1} = (p-q) - (\alpha\ \tau_n + \beta)\bmod(p-q) \qquad (3.5)$$

where β is a constant which we may take to lie in $[0,p-q)$ without loss of generality because of the modulo $(p-q)$ operation. There is also no loss of generality in choosing units in which $p-q=1$, so we can write $\tau_{n+1} = f(\tau_n)$ where

$$f(\tau) = 1-(\alpha\ \tau + \beta)\bmod 1 . \qquad (3.6)$$

Equation (3.6) is our new system equation. Before analyzing it in detail, we get a hint of its nature by noticing its similarity to the popular linear congruence method for generating pseudo-random number sequences [3]. Even though the details are different (we assume τ_n , α and β are real rather than rational) we expect something of the same degree of unpredictability in the motion of the system.

Note that if the rising and falling parts of the waveform are not straight lines, equation (3.6) still holds except that the fall time $\alpha\tau$ is replaced by a function $\alpha(\tau)$, and the constant in (3.1) is no longer $(b-c)/I_0$. The important thing is that the fall time is still a function of the rise time. We will return to this later.

Figure 3.2a shows the effect of the mapping $\tau_{n+1} = f(\tau_n)$ for a typical f with $\alpha>1$. The auxiliary line $f(\tau) = \tau$ serves the usual purpose of allowing us to follow iterates of the mapping, as explained in the figure caption. The two fixed points are unstable and the typical trajectory shown is pushed outwards in the domain of one point until it crosses the discontinuity into the domain of the other. Then it is pushed outwards once more until it crosses the discontinuity again, and so on.

It is obvious that the motion is complicated, but it is not clear at once why it could not be periodic with a long period, or perhaps be almost-periodic. We will soon show that neither of these is possible if $\alpha>1$, but first let us dispose of the cases $\alpha<1$ and $\alpha=1$.

3.1 The case $\alpha \leq 1$

When $\alpha < 1$, the map (3.6) is shown in Figure 3.3; it can be seen that the equation $\tau = f(\tau)$ has at least one fixed point, and that because $\alpha < 1$, f is a contraction on each of its straight-line segments. Consequently, the fixed points are stable attractors of the discrete dynamical system $\tau_{n+1} = f(\tau_n)$, and every initial state is attracted to one of them.

It is then trivial to see that these fixed points correspond to periodic solutions of the original system, the value of k being fixed for any one fixed point.

3.2 The case $\alpha = 1$

When $\alpha = 1$, we get

$$\begin{aligned}
\tau_{n+2} &= 1 - (\tau_{n+1} + \beta)\mathrm{mod}\ 1 \\
&= 1 - (1-(\tau_n + \beta) + \beta)\mathrm{mod}\ 1 \\
&= -(-\tau_n)\mathrm{mod}\ 1 \\
&= \tau_n
\end{aligned}$$

because $0 \leq \tau_n < 1$. This means the circuit will be synchronized to an even multiple of p for all initial conditions except those for which $\tau_n = \tau_{n+1}$, in which case synchronization to odd multiples is possible.

3.3 The case $\alpha > 1$

The interesting case is when $\alpha > 1$. We can see at once that all the fixed points of f are now unstable;

moreover, iterates f^m of f consist of straight line segments with slope $(-\alpha)^m$ and so fixed points of all periods are unstable. So there can be no stable periodic solutions in the original system.

What is more, the mapping is always locally expansive: nearby points get pulled further and further apart, until eventually they find themselves on opposite sides of a discontinuity of f, corresponding to being in different switching cycles of the original system. (We will go into more detail on this shortly.)

Figure 3.5 shows a possible map; because $\alpha > 1$, there is at least one discontinuity. We are going to show that this implies every almost-periodic solution must be periodic. (Since we know there can be no stable periodic solutions, this shows there can certainly be no stable almost-periodic ones.)

<u>Theorem</u> Every almost-periodic orbit of $\tau_{n+1} = f(\tau_n)$, where f is defined by (3.6), is periodic.

<u>Proof</u> Let $\{\tau_n\}$ be an almost periodic orbit. Then for every $\varepsilon > 0$ there is an $N > 0$ such that $|\tau_{j+N} - \tau_j| < \varepsilon$ for all j. Define $\varepsilon_0 = |\tau_N - \tau_0|$. We show that this must be zero, proving that $\{\tau_n\}$ is actually periodic.

Choose $\varepsilon > 0$ so small that $\alpha\varepsilon << 1$. Consider the sequence $\varepsilon_j = |\tau_{N+j} - \tau_j|$. Suppose there is a point of discontinuity of f between τ_{N+j} and τ_j for some j; then since $\alpha\varepsilon_j << 1$, ε_{j+1} is nearly 1, which violates $\varepsilon_{j+1} < \varepsilon$. So there can be no point of discontinuity between

τ_{N+j} and τ_n for any j, which means we must have $\varepsilon_{j+1} = \alpha^j \varepsilon_0$. But since $\alpha > 1$, $\varepsilon_{j+1} < \varepsilon$ for all j implies $\varepsilon_0 = 0$; that is, the orbit is periodic.

Remark If $\alpha\tau$ is replaced by a function $\alpha(\tau)$ as described earlier, the theorem still holds if the function is non-negative and Lipschitz with minimum Lipschitz constant strictly greater than 1. The same applies to the separation proposition which follows.

Returning now to Figure 3.2a, we see that the situation must be rather complicated. If we let $\{r_n, s_n\}$ be the sequence of numbers of iterates spent in the domains of repulsion of the fixed points, so a point starting in the left side of the picture takes r_1 time intervals to enter the right side, spends s_1 intervals there, re-enters the left side for r_2 intervals, and so on. A slight variation on the proof of the above theorem shows us that two points starting arbitrarily close together must have different sequences: we will call them kneading sequences though the stretching and folding of our interval is not the usual dough-like operation.

Separation proposition

Any two distinct points $\tau, \tau' \in [0,1)$ have different kneading sequences.

Proof Consider the sequence $\varepsilon_j = |\tau_j - \tau'_j|$ with $\tau_0 = \tau$ and $\tau'_0 = \tau'$. Suppose that there are no discontinuous points of f between τ_j and τ'_j for all j from 0 to k (i.e. the kneading sequences match up to the k^{th} iteration), so we must have $\varepsilon_{k+1} = \alpha^k \varepsilon_0$. Since $\alpha > 1$, ε_{k+1} eventually becomes so large that τ and τ' lie on opposite sides of a discontinuity, so the kneading sequence then differ.

This proposition shows that the trajectories of two distinct points τ,τ' must split apart onto different sides of a discontinuous point of f for at least once. We shall call this "separation by order 1".† The points may eventually come close together again but the proposition shows that unless they coincide, they must split again by order 1. Thus any two trajectories must either coincide after a finite time or split with order 1 infinitely often. They may not approach each other asymptotically.

In the case when α is an integer, we can in principle give a complete picture of the history of any point using a counting argument [4,5]. The trick is to make a base-α expansion of the unit interval, and to show that the expansion of any point contains its complete history (kneading sequence) in full view. When α is not an integer, the problem is that while every point has some expansion in base $[\alpha]+1$ that contains its kneading sequence, not every expansion to this base corresponds to a point on the interval. This makes it difficult to attach the expansion to the point. Here is how things work for integral values of α. Note, incidentally, that integral α has nothing to do with our choice $p-q=1$, and to make this clear we go back to (3.2) which was

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† The reason we talk about "order 1" is that if two points are on opposite sides of a discontinuity, either they are already far apart or they will be so on the next iteration. That is, given any $\delta > 0$ sufficiently small and τ,τ' on opposite sides of a discontinuity, either $|\tau-\tau'| > \delta$ or $|f(\tau)-f(\tau')| = O(1-\delta)$.

$$t_{n+1} = (p-q)k-\alpha t_n$$

where k is an integer.

Define $x_n = [t_n/(p-q)]\bmod 1$ and suppose x_0 has a base α expansion: that is,

$$x_0 = \sum_{j=1}^{\infty} x_{0j}\ \alpha^{-j} \qquad \text{where} \qquad 0 \le x_{0j} < \alpha .$$

It follows from (3.2) and the fact that α is an integer, that

$$x_{n+1} = (-\alpha\ x_n)\bmod 1$$

which implies

$$x_1 = 1- \sum_{k=1}^{\infty} x_{0,(k+1)}\alpha^{-k} ,$$

$$x_2 = \sum_{k=1}^{\infty} x_{0,k+2}\alpha^{-k} ,$$

$$x_{2n} = \sum_{k=1}^{\infty} x_{0,k+2n}\alpha^{-k}$$

$$x_{2n+1} = 1 - \sum_{k=1}^{\infty} x_{0,k+2n+1}\alpha^{-k} .$$

Thus there is a one-to-one correspondence between the α-nary expansion of x_0 and the sequence $\{x_n\}$. The above equations show that at each iterate, the expansion loses its first digit and is complemented modulo α . Thus in the case $\alpha = 2$,

$$\begin{aligned} .0101\ldots. &\to .1010\ldots.. \text{ (drop first digit)} \\ &\to .0101\ldots.. \text{ (subtract from 1)} . \end{aligned}$$

We have found a fixed point. (It is unstable, of course). In fact, since rational numbers exist we can find fixed points of any order m , i.e. points for which $x_{n+m} = x_n$ for all n , but $x_{n+\ell} \neq x_n$ for $0 < \ell < m$. Moreover, irrational number exist, so there are non-periodic points which are certainly not asymptotically periodic.

To examine the history of any point, write down its α-nary expansion and complement the odd numbered digits modulo α . The digits then give, in order, the kneading intervals visited. Since any two distinct points must differ in their base α expansions by at least one digit, we see another interpretation of the separation proposition.

4. Experiment

To illustrate and confirm the chaotic behavior predicted in section 3, a simple experimental circuit which simulates, almost exactly, the idealized I_R-V_R characteristic in Figure 3.2b, is built and studied. The circuit is shown in Figure 4.1. A standard integrated circuit module NE555 performs the switching between charging and discharging. The output of the NE555 jumps from 15V down to 0 whenever the threshold input reaches 10V and jumps from 0 to 15V whenever the trigger input falls below 5V. Two transistors act as current sources. When the output of NE555 is high, Q_1 overcomes Q_2 and charges C_0. Otherwise Q_1 is off and C_0 is discharged by Q_2. The other NE555 generates the triggering signal which is added to the oscillator via an operational amplifier.

The experiment was performed with $\alpha \approx 1.5$, $q \approx 1\text{ms}$, $p \approx 1.1\text{ms}$, $b \approx 12\text{V}$, $a \approx 5\text{V}$, and $c \approx 2\text{V}$. These values are not critical. The real technical problem is to obtain meaningful displays of v_c using an oscilloscope.

Oscilloscopes can be triggered by the input. That is, the display will always start at the time when the input rises up to (or, at the user's option, falls down to) a certain adjustable level, namely, the trigger level. By triggering, oscilloscopes are able to display steady pictures of periodic waveforms. When requested by the user, a scope makes its attempts to be triggered, but it often fails when the input is chaotic. Even if the scope is triggered, the picture may still not be steady for non-periodic inputs.

Thus a chaotic input usually results in a messy display which is not meaningful to anyone. This may be one reason why chaotic behaviors of electronic circuits are seldom observed, let alone reported.

For this circuit, we are able to obtain meaningful pictures by properly selecting the trigger level. Since the waveform consists of triangles resting on one level (5V), we can adjust the trigger level so that the display always starts from a certain voltage of the rising portion of triangles (i.e. slightly larger than 5V, positive slope), and so obtain a picture consisting of overlapping triangles. Moreover, if we choose the sweep speed of the scope so that no more than one triangle is displayed every time it is triggered, we shall obtain a picture as plotted in Figure 4.2. Here A is the trigger level of the scope, the band BC consists of triangles being triggered at different levels by the pulses, and D is the free running triangle. Since the time range that a triangle can be triggered is p-q, the width of BC is approximately p-q. Notice that in order to see the entire band BC, the trigger level of the scope must be less than the smallest triangle. In fact, if the level is set between C and D, we shall see only the free-running triangle which may mislead us to conclude that the waveform is periodic. Several real pictures taken from the scope are shown in Figure 4.3. Figure 4.4 shows the broad spectrum of the resulting triangular waveform.

5. Conclusions

We have discussed a simple mathematical model and an experimental circuit that is well described by the model. The real circuit is described by a differential equation rather than a difference equation, with the equation taking the form

$$\dot{x} = g(x+d,y) \ ,$$
$$\mu\dot{y} = h(x+d,y)$$

for $x \in \mathbb{R}^1$, $y \in \mathbb{R}^n$, and small $\mu \in \mathbb{R}$. The good agreement between theory and experiment indicates that the output of the differential system is chaotic. Certainly, it has a broad spectrum and is sensitive to initial conditions. We believe that just as the qualitative properties of the real system are close to those of the model, so other features of the model (such as straight line segments in the waveform) are more important for ease of analysis than they are for determining the general features of the behavior. (Recall that we showed in section 3 that $\alpha\tau$ could be replaced by a function $\alpha(\tau)$ without affecting our conclusions.)

If we accept that the mathematical model is realistic, or if we regard it as worth studying in its own right, then we discover interesting features of our system. Different initial conditions give rise to quite different futures, giving the sensitivity and broad spectrum that we require. Specifically, we have proved our model has the following features when $\alpha > 1$.

(a) There are no stable periodic solutions of any order.

(b) There is no almost-periodic solution that is not periodic.

(c) There are unstable periodic solutions. If α is an integer, these exist for all possible periods.

(d) Any two trajectories either coincide exactly after a finite time, or are separated by order 1 infinitely often.

These features are akin to but are not quite the same as the features found in the systems of first order difference equations which are usually discussed (see, for example, Li and York [6]). For example, the discontinuities in our system remove <u>all</u> stable periodic solutions, whereas the usual one-hump functions tend to have stable periodic solutions even in their chaotic regions [7].

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Figure Captions

Fig. 1.1a Output $x(t)$ increases until $x(t) = b$, then decreases until $x(t) = 0$, and repeats. Ratio of slopes is α. (Here $a = 0$).

Fig. 1.1b Slope changes when $x + d$ reaches b from below or $a(=0)$ from above. Period of $x + d$ is locked on to the period of d.

Fig. 2.1 An emitter-coupled astable multivibrator.

Fig. 2.2 The current-voltage characteristic of the transistor circuit measured across terminals $x - y$.

Fig. 2.3a The circuit model of the astable oscillator.

Fig. 2.3b The idealized I_R-V_R characteristic of the nonlinear resistor.

Fig. 2.4a The circuit model of the triggered astable oscillator.

Fig. 2.4b Narrow periodic trigger pulses.

Fig. 3.1 A possible triggered waveform. Pulses are $d(t)$; when $x(t) + d(t) > c$, $\dot{x}$ changes sign. It is still true that $\dot{x} = I_0$ or $-I_0/\alpha$ always.

Fig. 3.2 A possible triggered waveform using assumptions mentioned in the text. The n^{th} successful triggering occurred at time t_n measured relative to the preceding time when $x(t) = 0$.

Fig. 3.2a The effect of $f(\tau)$. τ_0 is mapped to $f(\tau_0)$. The line $f(\tau) = \tau$ reflects $f(\tau_0)$ into τ-axis. This reflected point becomes τ_1 and the process continues. The mapping and reflecting proceed along the arrows.

Fig. 3.3 A possible map with $\alpha < 1$.

Fig. 3.4 A possible $f^2(\tau)$.

Fig. 3.4 A possible map with $\alpha > 1$.

Fig. 4.1 The experimental circuit.

Fig. 4.2 A plot of overlapping triangles observed on the scope.

Fig. 4.3a A picture of overlapping triangles.

Fig. 4.3b,c Amplified BC portion of the overlapping triangles. Pictures are obtained with slightly different p . They indicate a great change in the distribution of triangles with a very small change of p .

Fig. 4.4 The spectrum of a typical waveform of v_c .

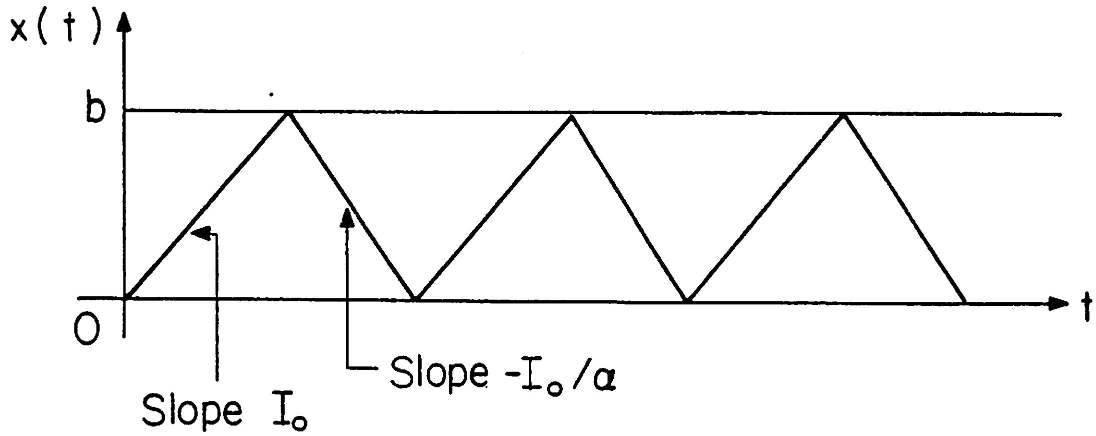


Fig. 1.1a

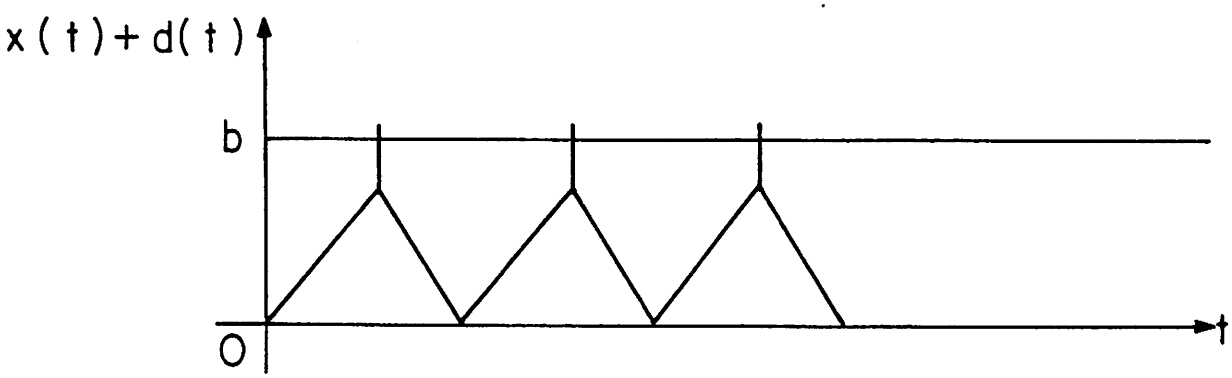
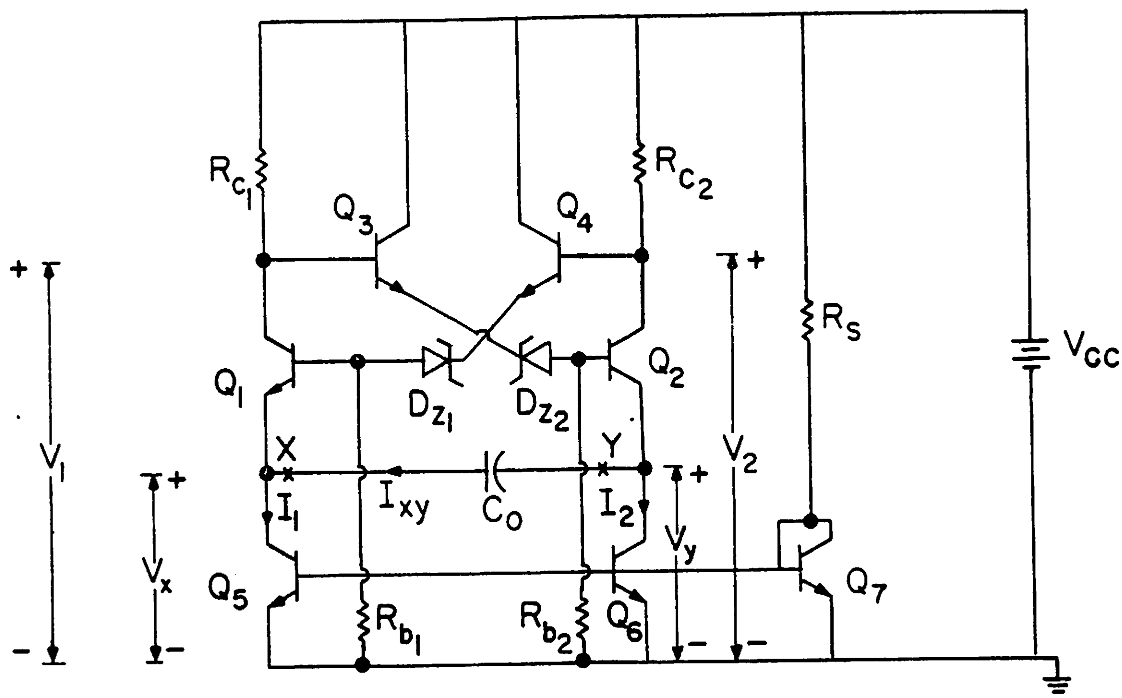
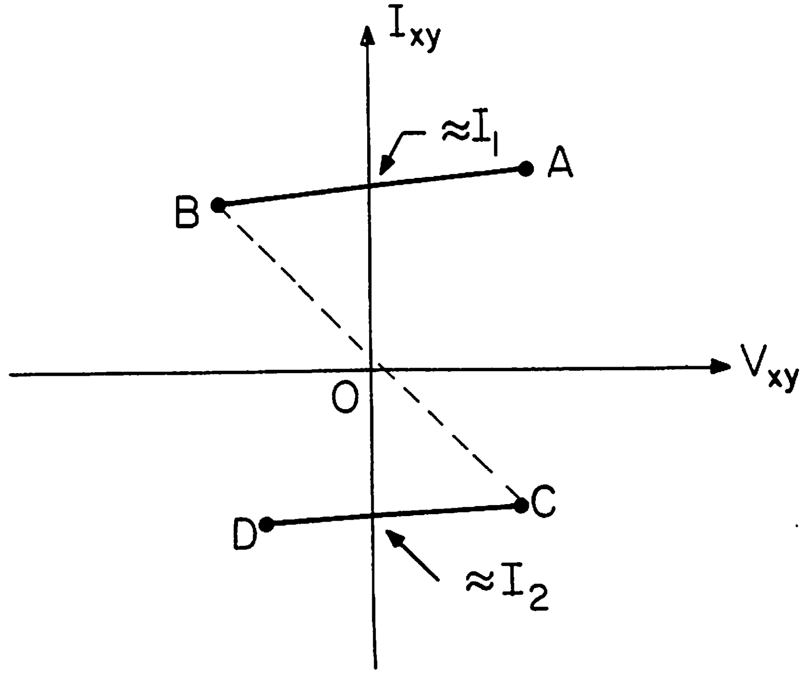


Fig. 1.1b



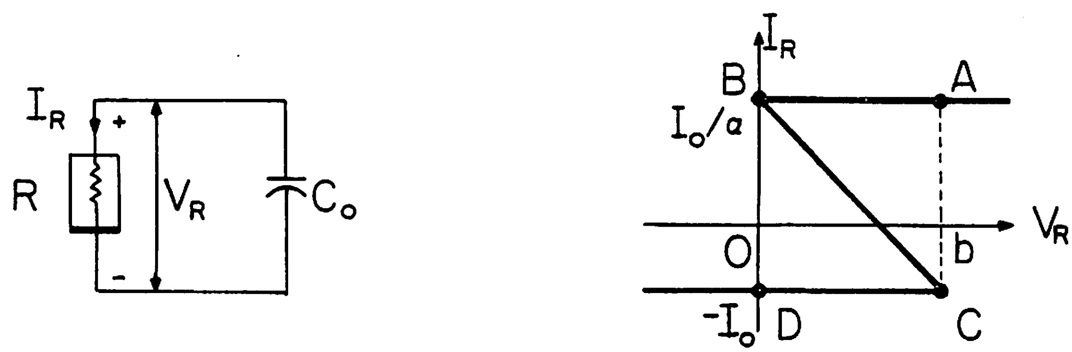
R_{c_1}
R_{c_2}
Q_3
Q_4
+
+
R_S
V_{CC}
Q_1
D_{Z_1}
D_{Z_2}
Q_2
V_1
X
Y
V_2
+
I_{xy}
C_0
+
I_1
I_2
V_y
V_x
Q_5
Q_7
R_{b_1}
R_{b_2}
Q_6
-
-
-
-

Fig. 2.1



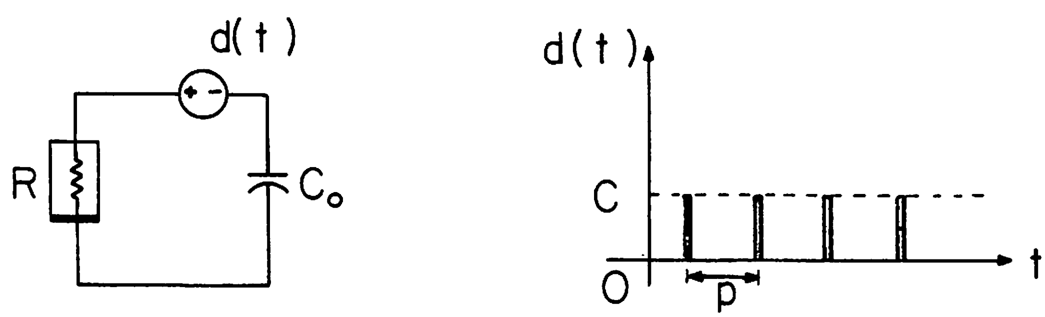
I_{xy}
$\approx I_1$
A
B
V_{xy}
O
C
D
$\approx I_2$

Fig. 2.2



(a) (b)

Fig. 2.3



(a) (b)

Fig. 2.4

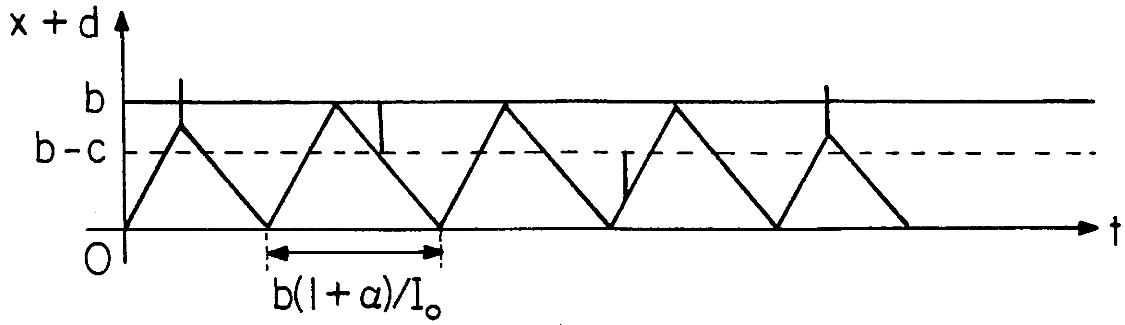


Fig. 3.1

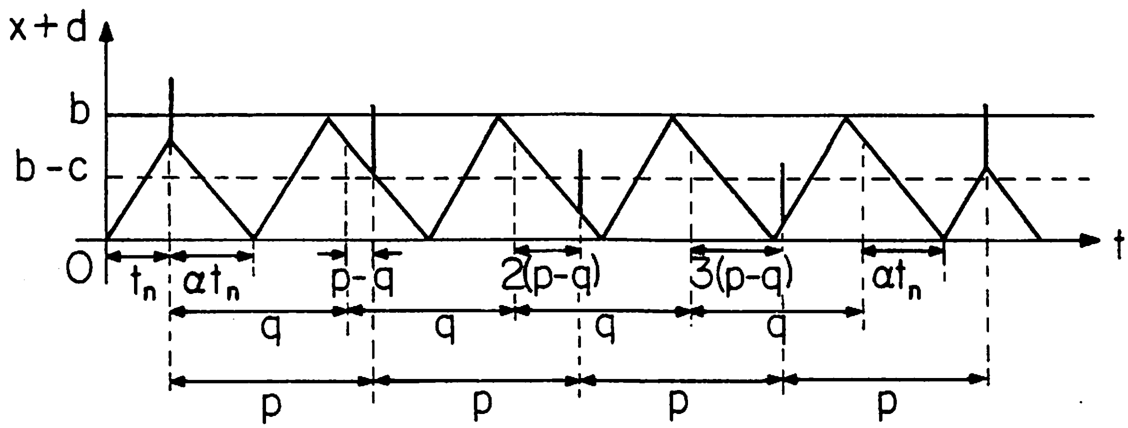


Fig. 3.2a

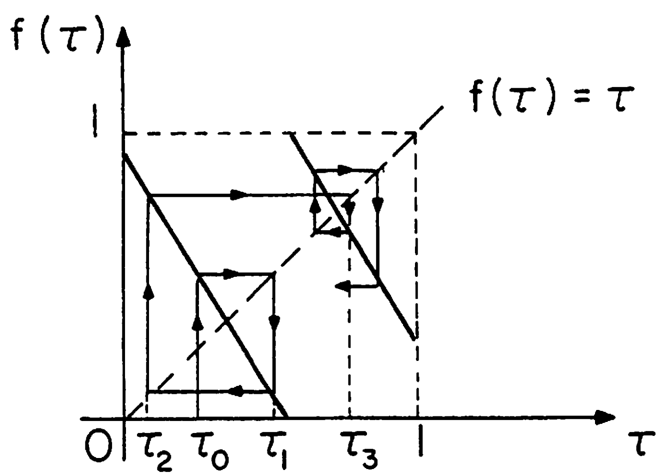


Fig. 3.2b

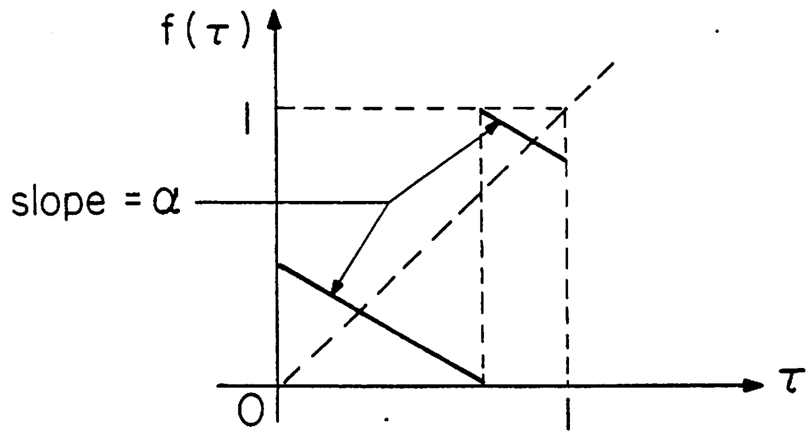


Fig. 3.3

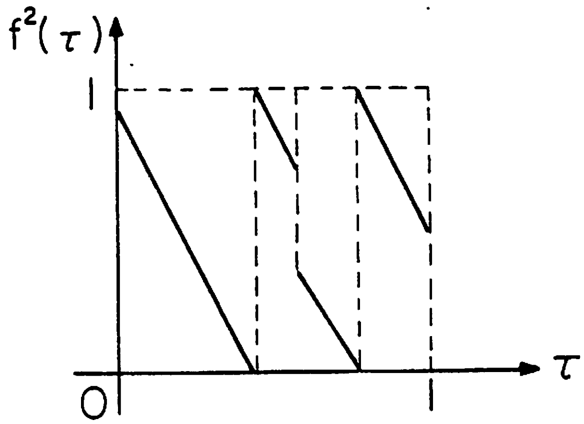


Fig. 3.4

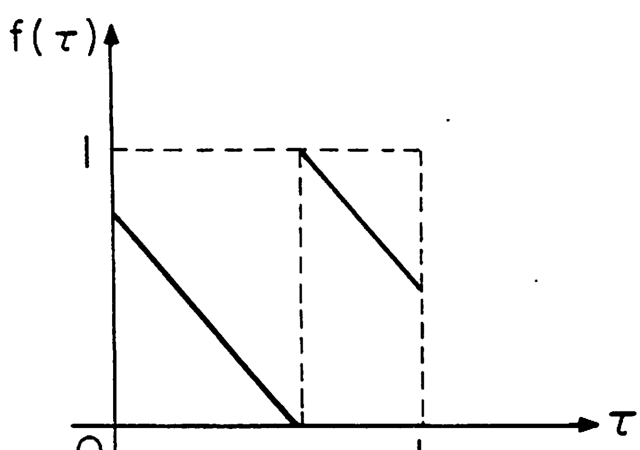
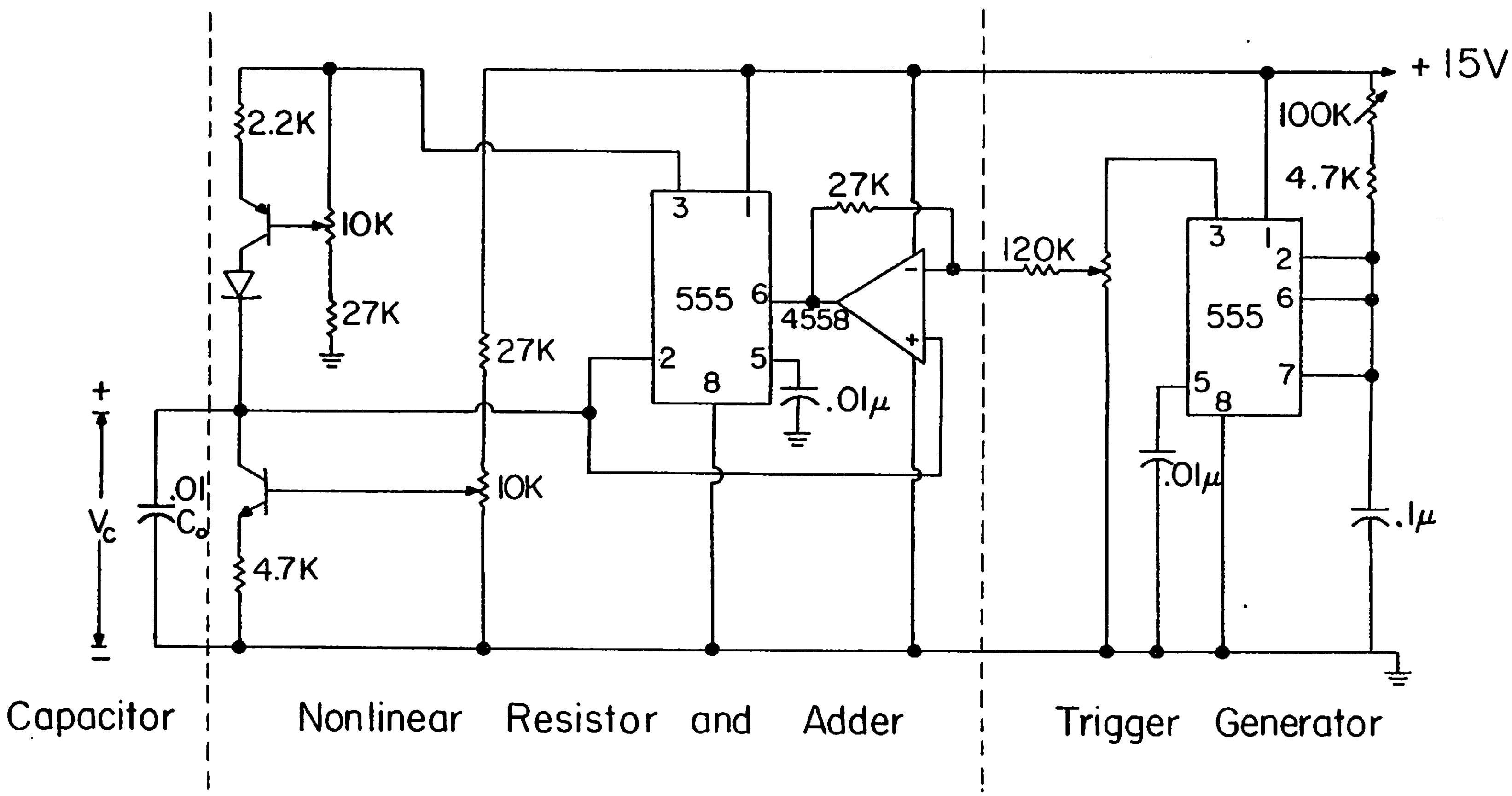


Fig. 3.5



+ 15V
2.2K
10K
27K
27K
10K
4.7K
.01
C_o
+
V_c
−
3
1
555
6
2
5
8
4558
27K
.01μ
120K
100K
4.7K
3
1
2
6
555
5
8
7
.01μ
.1μ
Capacitor
Nonlinear Resistor and Adder
Trigger Generator

Fig. 4.1

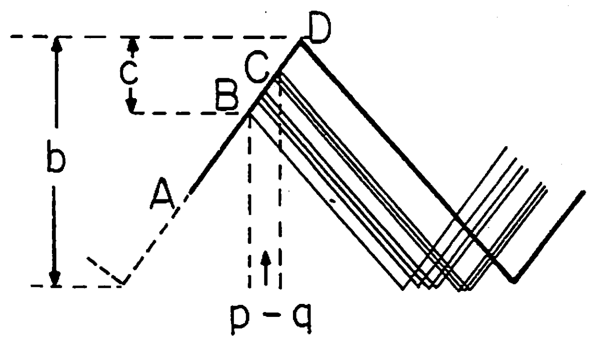
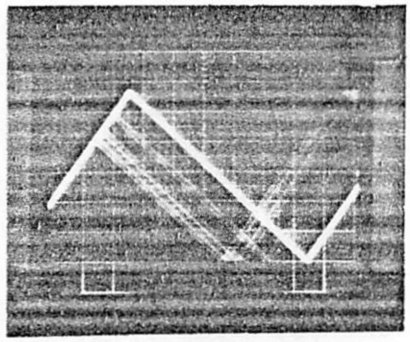
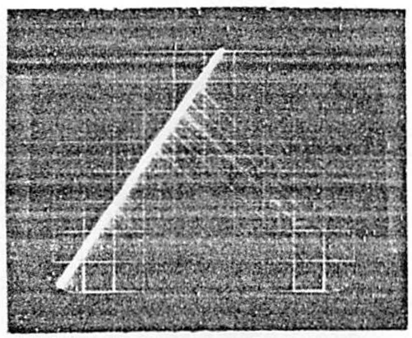


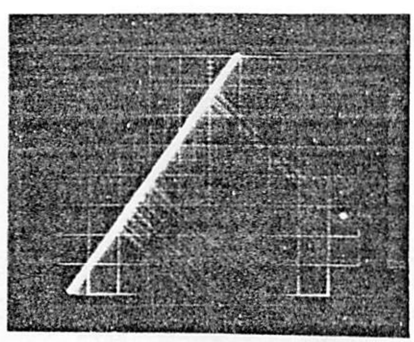
Fig. 4.2



(a)



(b)



(c)

Fig. 4.3

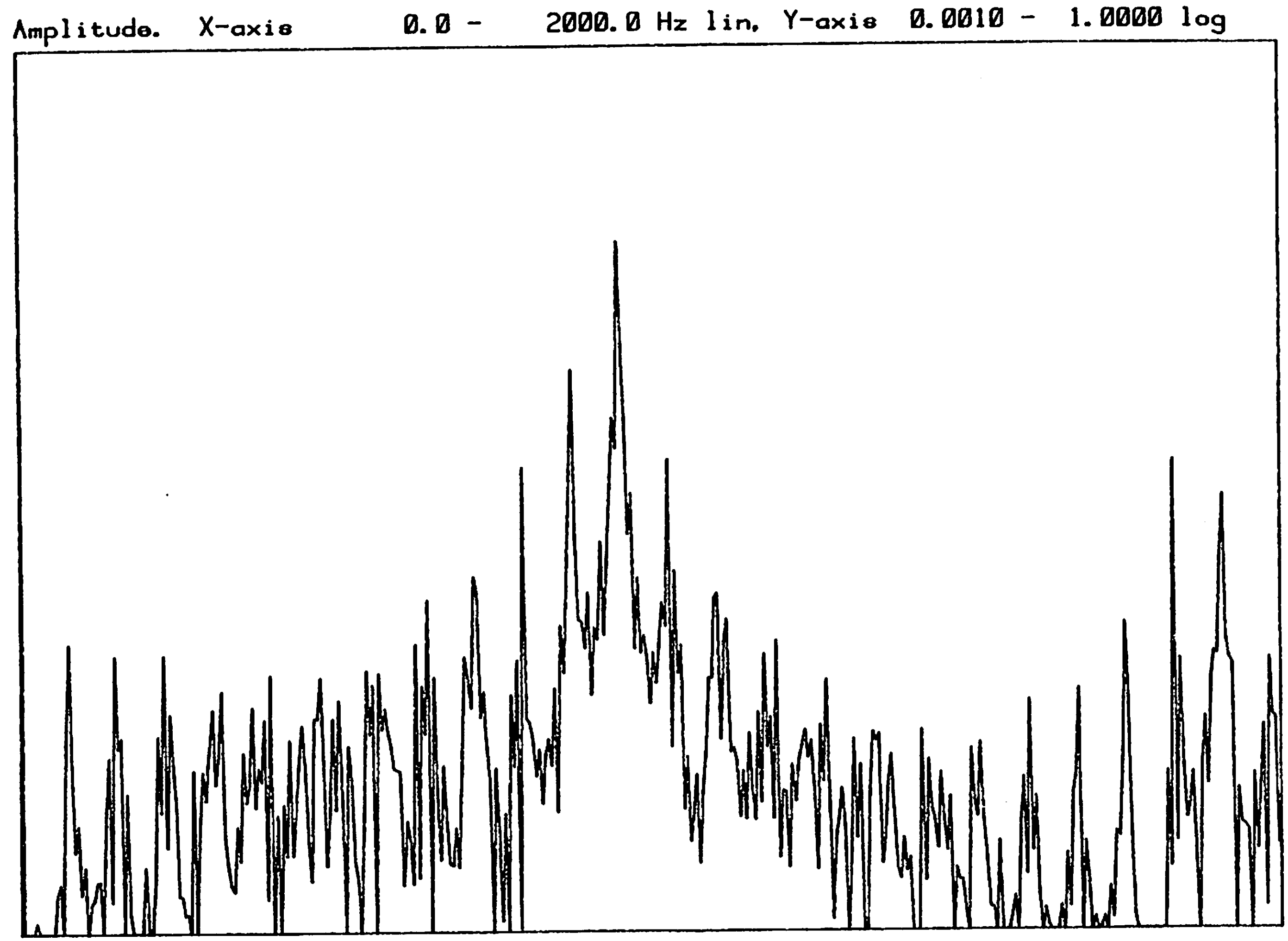


Fig. 4.4