INTEGRAL MANIFOLDS FOR NONLINEAR CIRCUITS

by

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Abstract

Although the theory of integral manifolds is well known among applied mathematicians as a powerful tool in nonlinear oscillations, it is relatively unknown, let alone applied, among circuit engineers. The purpose of this mostly tutorial paper is to illustrate the applications of integral manifolds to explain various nonlinear phenomena widely observed in nonlinear circuits. Numerous examples and graphical illustrations are included in order to present the theory with a minimum amount of mathematics.

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1. INTRODUCTION

Most of the qualitative methods of nonlinear analysis consist of reducing a given equation to a more simple form. One of these methods is a generalization of the slowly-varying amplitude and phase approach introduced by van der Pol [1]. This method has since been rigorously justified by Krylov, Bogoliubov, Mitropolski, and others [2-7]. Because this method appears to be rather mathematical and involved it has not been used by engineers until only recently [8-10]. Our purpose of this paper is to present a tutorial on this important method and to apply it to the analysis of some well-known electronic circuits. In particular, we shall show that many "heuristic" approximation techniques used by engineers in the analysis of nonlinear "circuits and systems involving some small parameters" (e.g. parasitic capacitance or inductance) can be given a rigorous foundation via the method of integral manifolds.

To illustrate the ideas behind the integral manifold approach, let us consider the well known van der Pol equation

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \varepsilon(1-x^2)y - \omega_0^2 x \end{aligned} \tag{1.1}$$

If $\varepsilon = 0$ equation (1.1) becomes linear and all solutions are of the form:

$$\begin{aligned} x(t) &= \rho \cos(\omega_0 t+\phi) \\ y(t) &= -\omega_0 \rho \sin(\omega_0 t+\phi) \end{aligned} \tag{1.2}$$

where ρ and ϕ are integration constants depending on initial conditions.

To analyze the case of $\varepsilon \neq 0$ (but small) assume that ρ and ϕ in (1.2) are time dependent. Rigorously speaking, we treat (1.2) as transformation of variables. Since ϕ is time dependent it is more natural to consider the phase: $\theta(t) \triangleq \omega_0 t + \phi(t)$ i.e.,

$$x = \rho \cos \theta, \quad y = -\omega_0 \rho \sin \theta \tag{1.3}$$

Applying transformation (1.3) to (1.1) we obtain

$$\begin{aligned} \dot{\theta} &= \omega + \frac{\varepsilon}{2} \{[1 - \frac{\rho^2}{2}]\sin 2\theta - \frac{\rho^2}{4} \sin 4\theta\} \\ \dot{\rho} &= \varepsilon \cdot \frac{\rho}{2} \{1 - \frac{\rho^2}{4} + \cos 2\theta + \frac{\rho^2}{4} \cos 4\theta\} \end{aligned} \tag{1.4}$$

Neglecting the terms whose average (in θ) is equal to zero one gets:

$$\dot{\theta} = \omega_0$$
$$\dot{\rho} = \varepsilon \frac{\rho}{2} \left(1 - \frac{\rho^2}{4}\right) \tag{1.5}$$

Since (1.5) is an approximation of (1.4), it is necessary to analyze how close are the solutions of (1.5) to the exact solutions. Let us first investigate the geometrical behavior of the solutions of (1.5). The two equations in (1.5) are independent. The solutions of the first are of the form $\theta(t) = \omega_0 t + \theta_0$, while the latter has for $\rho > 0$ the constant solution $\rho_0 = 2$. All other solutions (with positive initial conditions) tend to $\rho_0 = 2$. In terms of x and y we get

$$x(t) = \rho(t) \cos(\omega_0 t + \theta_0), \quad y(t) = -\omega_0 \rho(t) \sin(\omega_0 t + \theta_0)$$

where $\rho(t) \to 2$ as $t \to +\infty$.

The phase portrait in the (x,y)-plane is shown in Fig. 1.1(a). If we consider the trajectories of (1.5) in the (t,x,y)-space, we get the picture shown in Fig. 1.1(b). Note that the trajectories on the cylinder

$$c = \{(t,x,y) : x^2 + \frac{y^2}{\omega_0} = \rho_0^2\} \tag{1.6}$$

have the form

$$[x(t),y(t)] = [\rho_0 \cos(\omega_0 t + \theta_0), -\omega_0 \rho_0 \sin(\omega_0 t + \theta_0)].$$

Hence, for initial conditions on this cylinder c, equations (1.5) are equivalent to:

$$\dot{\theta}(t) = \omega_0$$
$$\rho(t) \equiv \rho_0 \tag{1.7}$$

We would like to know under what conditions will the behavior of solutions be similar to that of (1.5). Moreover, for the nonautonomous equation

$$\dot{x} = y$$
$$\dot{y} = \varepsilon(1-x^2)y - \omega_0^2 x + B \cos \omega t \tag{1.8}$$

we would also like to know under what conditions on B and ω can (1.7) be reduced to (1.5).

In general we cannot expect that solutions of (1.4) or (1.8) will be close to these of (1.5) over the infinite time interval even if they start from the same initial point. We shall show in Section 3 that for ε small (and if either B or $\frac{1}{\omega}$ is small in (1.8)) equation (1.1) (and also (1.8)) possesses in the neighborhood of the cylinder (1.6) a surface to which all other solutions must tend to, and the solutions on this surface are described by the equation:

$$\dot{\theta} = \omega_0 + \text{"small perturbation."}$$

2. DEFINITION AND EXAMPLES OF INTEGRAL MANIFOLDS

Consider the equation:

$$\dot{x} = X(x,t) \tag{2.1}$$

where x is an n-vector and $t \in (-\infty,+\infty)$. Throughout this paper we shall assume that the solutions of (2.1) are defined for $t \in (-\infty,+\infty)$ and that for any initial condition $x(t_0) = x_0$ (2.1) has a unique solution $x(t;t_0,x_0)$.

Definition 2.1. [4,5] A surface S in the (x,t)-space is called an integral manifold of (2.1) if any solution of (2.1) originating on S will remain on S for all t.

Let U_σ be a σ-neighborhood of S:

$$U_\sigma = \{(x,t) : \text{dist}[x,S] < \sigma\} \tag{2.2}$$

Definition 2.2. An integral manifold S is said to be isolated if there exists $\sigma > 0$ such that U_σ does not contain any other integral manifold except S.

Definition 2.3. An integral manifold S is said to be stable if for any $\sigma_0 > 0$ we can fix $\sigma_1 > 0$ such that any solution of (2.1) which originates in U_{σ_1} at time $t = t_0$ will remain in U_{σ_0} for $t > t_0$ and will tend to S as $t \to +\infty$.

Example 2.1. Consider the circuit shown in the Fig. 2.1. Kirchoff's laws yield:

$$\frac{di}{dt} = \frac{1}{L} v$$
$$\frac{dv}{dt} = \frac{1}{C} i + \frac{1}{C} i_R \tag{2.3}$$

Suppose that the nonlinear resistor v-i characteristic $i_R = g(v)$ has the following properties:

1. $g(v)$ is an odd and continuously differentiable function of v,
2. there exists v_0 such that $g(v) < 0$ for $0 < v < v_0$, and $g(v)$ is positive and monotone increasing for $v > v_0$.

It can be shown [5,11] that (2.3) has exactly one periodic orbit in the (v,i)-plane and that orbit is <u>asymptotically stable</u>. When the solutions of (2.3) are considered in the (v,i,t)-space then those which pass through the orbit γ (in the (v,i)-plane) form a cylinder S which appears to be a stable integral manifold of (2.3) as shown in Figs. 2.2(a) and (b).

<u>Example 2.2</u>. Consider the equation

$$\ddot{x} + \varepsilon f(x,\dot{x}) + \omega_0^2 x = 0 \tag{2.4}$$

This equation is a generalization of (1.1); namely, for $f(x,\dot{x}) = (1-x^2)\dot{x}$ we get the van der Pol equation. Also if the nonlinear function in example 2.1 is small, i.e., $g(v) \triangleq \varepsilon F(v)$, then (2.3) can be reduced to (2.4) with $x \triangleq v$, $\omega_0^2 \triangleq 1/LC$, and $\varepsilon f(x,\dot{x}) = \varepsilon f(v,\dot{v}) \triangleq \frac{1}{C}\frac{d}{dv}F(v)\cdot\dot{v}$.

The transformation:

$$\begin{aligned} x &= \rho\cos\theta \\ \dot{x} &= -\omega_0\rho\sin\theta \end{aligned} \tag{2.5}$$

when applied to (2.4) yields

$$\begin{aligned} \dot{\theta} &= \omega_0 + \frac{\varepsilon}{\omega_0}\cdot\frac{1}{\rho}\cdot f(\rho\cos\theta, -\omega_0\rho\sin\theta)\cos\theta \\ \dot{\rho} &= +\frac{\varepsilon}{\omega_0} f(\rho\cos\theta, -\omega_0\rho\sin\theta)\sin\theta \end{aligned} \tag{2.6}$$

Let us consider the following equation associated with (2.6):

$$\begin{aligned} \dot{\theta} &= \omega_0 \\ \dot{\rho} &= \frac{\varepsilon}{\omega_0} f_0(\rho) \end{aligned} \tag{2.7}$$

where $f_0(\rho) \triangleq \frac{1}{2\pi}\int_0^{2\pi} f[\rho\cos\theta, -\omega_0\rho\sin\theta]\sin\theta d\theta$

Equation (2.6) can be considered as a perturbation of (2.7). It will be shown in Section 3 that if there exists $\rho_0 > 0$ such that $f_0(\rho_0) = 0$ and $f'(\rho_0) < 0$ (i.e., ρ_0 is a constant and asymptotically-stable solution of the second equation in (2.7)), then for ε sufficiently small, equation (2.6) has a stable integral manifold in $(x,\dot{x},t)$-space. Moreover the manifold tends to the cylinder:

$$c = \{(x,\dot{x},t) : x^2 + \dot{x}^2 = \rho_0^2\}$$

as ε tends to zero.

Example 2.3. Consider equation (2.4) with $\varepsilon = 0$

$$\ddot{x} + \omega_0^2 x = 0 \tag{2.8}$$

This equation describes a lossless LC circuit shown in Fig. 2.3 with $x \triangleq v$, $\omega_0^2 \triangleq \frac{1}{LC}$. All solutions of (2.8) are of the form $x = A\cos(\omega_0 t+\phi)$. It is easy to see that (2.8) has a continuum of nonisolated cylindrical manifolds in the $(x,\dot{x},t)$ space as shown in Fig. 2.4.

Example 2.4. The nonlinear lossless LC circuit has a similar structure of integral manifolds as that shown in Fig. 2.4. Indeed, consider the circuit shown in Fig. 2.5, and let $i = g(\phi)$ be the characteristics of the nonlinear inductor, such that the "energy function":

$$G(\phi) \triangleq \int_0^\phi g(\phi)d\phi = \int_0^\phi i d\phi = \int_{-\infty}^t i\cdot v\cdot dt$$

increases monotonically to infinity when either $\phi \to +\infty$ or $\phi \to -\infty$.† Then all solutions of the equation:

$$\begin{aligned} \dot{\phi} &= v \\ \dot{v} &= -\frac{1}{C} g(\phi) \end{aligned} \tag{2.9}$$

must lie on nonisolated "cylindrical" manifolds of the form:

$$S = \{(\phi,v,t) : \frac{Cv^2}{2} + G(\phi) = \text{const},\ t \in R\}\ .$$

The manifolds are cylindrical because $G(\phi)$ increases monotonically to infinity with $|\phi|$, so that $\frac{Cv^2}{2} + G(\phi) = \text{const}$ does form a closed curve in (ϕ,v)-plane. Observe

†Physically this property means that the magnetic energy stored in the inductor increases with the absolute value of the magnetic flux.

that this curve needs no longer be an ellipse, as it is the case with the previous example.

Example 2.5. Josephson-junction

The point-contact or micro-bridge junctions can be modeled with the circuit shown in Fig. 2.6 [12,13]. In terms of dimensionless variables this circuit can be described by the equation

$$\begin{aligned} \dot{x} &= y \\ \beta\dot{y} &= \alpha - \sin x - y \end{aligned} \tag{2.10}$$

This equation was discussed in great detail by Andronov, Vitt, and Khaikin [14]. In particular, they have shown that for $\alpha > 1$ (or for $0 < \alpha < 1$ and β smaller than some critical value $\beta_0(\alpha)$), equation (2.10) has a stable and 2π-periodic (with respect to x) trajectory defined by $x(t)$ and $y(t) = \psi(x(t))$. Hence, in the (x,y,t)-space this equation has an integral manifold

$$S_0 = \{(x,y,t) : y = \psi(x),\ t \in R\} \tag{2.11}$$

as shown in Fig. 2.7. Moreover the motion on the manifold is described by the equation

$$\dot{x} = \psi(x) \tag{2.12}$$

For a more detailed discussion of this case see [14-16, 9].

When the parameter β (junction capacitance) is small, another approach for analyzing (2.10) is also possible. One does expect intuitively that the behavior of (2.10) should be similar to that of the equation:

$$\begin{aligned} \dot{x} &= y \\ 0 &= \alpha - \sin x - y \end{aligned} \tag{2.13}$$

i.e., one expects that the solutions of (2.9) should lie on the surface $\{(x,y,t) : y = \alpha - \sin x,\ t \in \mathbb{R}\}$ and their behavior is described by:

$$\dot{x} = \alpha - \sin x \tag{2.14}$$

This means that we can neglect the capacitance shown in Fig. 2.6. Assumptions of this kind were often made [16,17], their justification can be found in [19,4,9,10].

Example 2.6. Tunnel-junction

When Josephson effect is due to a semiconductor tunneling mechanism, we can no longer assume that the resistance in the circuit in Fig. 2.6 is linear. A more realistic model in this case is given by

$$\begin{aligned} \dot{x} &= y \\ \beta\dot{x} &= \alpha - \sin x - g(y) \end{aligned} \tag{2.15}$$

where the typical $g(y)$ characteristic is shown in Fig. 2.8.[†]

Since $g(y)$ is a one-to-one function we can expect as before that, for small β, the solutions of (2.15) will lie near to the surface

$$S = \{(x,y,t) : y = g^{-1}(\alpha - \sin x),\ x \in \mathbb{R},\ t \in \mathbb{R}\} \tag{2.16}$$

Motion on the surface S is described by

$$\dot{x} = g^{-1}(\alpha - \sin x) \tag{2.17}$$

3. TRANSFORMATION OF COORDINATES

Our next goal will be to present the conditions under which an integral manifold is preserved under small perturbations. In order to do this, we shall introduce a new coordinate system which is especially convenient for studying the behavior of trajectories in a neighborhood of the original cylinder.

Consider the autonomous system

$$\dot{x} = f_0(x) \tag{3.1}$$

together with the perturbed equation

$$\dot{x} = f_0(x) + \varepsilon f(t,x,\varepsilon) \tag{3.2}$$

where f_0 and f_1 are smooth[††] and bounded vector-valued functions, and ε is a small parameter.

Suppose that the autonomous system (3.1) possesses an asymptotically stable T-periodic solution $u(t)$. This solution gives rise to a closed orbit Γ (in the x-space) and an invariant cylinder S_0 (in the (t,x)-space) as shown in Fig. 3.1.

[†] We would like to thank Professor T. Van Duzer for informative discussions concerning this subject.

[††] A function is said to be <u>smooth</u> iff it is at least twice continuously differentiable

Since we are interested in the solutions close to Γ (respectively S_0), it will be convenient for us to introduce local coordinates in some neighborhood of Γ (respectively S_0). In terms of these new coordinates the perturbed equation (3.2) reduces to:

$$\dot{\theta} = \omega_0 + g_1(\theta,\rho) + \varepsilon g_2(t,\theta,\rho,\varepsilon) \tag{3.3}$$

$$\dot{\rho} = A(\theta)\rho + R_1(\theta,\rho) + \varepsilon R_2(t,\theta,\rho,\varepsilon) \tag{3.4}$$

Our approach consists of two steps:

1. We introduce a <u>moving</u> orthonormal system along the orbit Γ.
2. With this orthonormal system, we introduce new coordinates in which the perturbed equation will take the desired form.

3.1. <u>Moving orthonormal system</u> [5,20]

Let us parametrize Γ with $\theta \triangleq \omega t$, $\omega \triangleq \frac{2\pi}{T}$, i.e.,

$$\Gamma = \{x \in \mathbb{R}^n : x = u(\theta),\ \theta \in [0,2\pi)\}^{\dagger} \tag{3.5}$$

and let $v(\theta) \triangleq \left|\frac{du}{d\theta}\right|^{-1} \frac{du}{d\theta}$ denote the <u>unit vector</u> tangent to Γ.

In the 2-dimensional case the orthonormal system consists of two vectors

$$v(\theta) = [v_1(\theta),v_2(\theta)]^T \text{ and } \xi(\theta) = [-v_2(\theta),v_1(\theta)]^T$$

as shown in Fig. 3.2.

For the case $n \geq 2$ we can always find a unit vector $e_1 \in R^n$ such that, for any θ, $v(\theta)$ is never parallel to e_1 (i.e., $v(\theta) \neq \pm e_1$ for $\theta \in [0,2\pi)$). With a fixed e_1, let us choose $e_2,\ldots,e_n$ such that $e_1,\ldots,e_n$ form an orthonormal basis in $\mathbb{R}^n$ as shown in Fig. 3.3. Let us then transform, for any $\theta \in [0,2\pi)$, the whole system so that e_1 coincides with $v(\theta)$,†† and denote the transformed vectors $e_2,\ldots,e_n$ as $\xi_2(\theta),\ldots\ \xi_n(\theta)$. The vectors $\xi_2(\theta)\ldots\xi_n(\theta)$ span a space orthogonal to the orbit Γ, while $v(\theta)$, $\xi_2(\theta)\ldots\xi_n(\theta)$ constitute a <u>moving orthonormal system</u>.

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†More precisely $x = \hat{u}(\theta)$ where $\hat{u}(\theta)$ is defined as $\hat{u}(\theta) \triangleq u(\frac{\theta}{\omega}) = u(t)$.

††This can be done as follows: we fix an n-2 dimensional subspace S which is orthogonal to both e_1 and $v(\theta)$ and then rotate the system along S until e_1 coincides with $v(\theta)$ [5,20].

3.2. New coordinates

We are now in a position to introduce new coordinates θ, $\rho = [\rho_1,\ldots,\rho_{n-1}]$ via the formula

$$x = u(\theta) + P(\theta)\rho \tag{3.6}$$

where $P(\theta) \triangleq [\xi_2(\theta)\vdots\ldots\vdots\xi_n(\theta)]$

is an $n \times (n-1)$ matrix whose columns are the orthonormal vectors $\xi_2(\theta),\ldots,\xi_n(\theta)$.

It can be shown [5] that if $f_0(x)$ is smooth and if $\|\rho\|$ is small enough, then the transformation (3.6) is well defined. To obtain equations (3.3)-(3.4) we apply (3.6) to (3.2):

$$[u'(\theta) + P'(\theta)\rho]\cdot\dot{\theta} + P(\theta)\dot{\rho} = f_0[u(\theta) + P(\theta)\rho] + \varepsilon f_1[t,u(\theta) + P(\theta)\rho,\varepsilon] \tag{3.7}$$

where $\dot{} = \frac{d}{dt}$, $' = \frac{d}{d\theta}$.

For $\|\rho\|$ small enough the matrix

$$[u'(\theta) + P'(\theta)\rho\vdots P(\theta)] = [u' + P\rho\vdots\xi_2\vdots\ldots\vdots\xi_n]$$

is nonsingular so that $\dot{\rho}$ and $\dot{\theta}$ can be found from (3.7):

Solving (3.7) for $\dot{\theta}$ and $\dot{\rho}$ we obtain (3.3) and (3.4)

Examples of new coordinate systems:

Example 1(n=2). Consider the circle in R^2 as shown in Fig. 3.4.

$$u(\theta) = R_0\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$

then

$$v(\theta) = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}, \quad \xi(\theta) = -\begin{bmatrix}-\cos\theta\\ \sin\theta\end{bmatrix}$$

$$\Gamma = \{(x_1,x_2) \in R^2 : x_1 = R_0\cos\theta,\ x_2 = R_0\sin\theta,\ \theta \in [0,2\pi)\} \tag{3.8}$$

and formula (3.6) takes the form

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix} = R_0\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} - \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}\rho \tag{3.9}$$

As long as $\rho < R_0$, formula (3.9) gives a one-to-one correspondence between (x_1,x_2) and (θ,ρ). Moreover in this particular case, we can find $\theta = \theta(x_1,x_2)$, $\rho = \rho(x_1,x_2)$ explicitly; namely,

$$\rho = R_0 - \sqrt{x_1^2 + x_2^2}$$

$$\theta = \begin{cases} \frac{\pi}{2} & \text{for } x_1 = 0 \ , \ x_2 > 0 \\ \frac{3}{2}\pi & \text{for } x_1 = 0 \ , \ x_2 < 0 \\ \tan^{-1} \frac{x_2}{x_1} & \text{for } x_1 > 0 \\ \pi + \tan^{-1} \frac{x_2}{x_1} & \text{for } x_1 < 0 \end{cases}$$

Example 2.

Consider the same circle in R^3 as shown in Fig. 3.5(a). Choose $u(\theta) = R_0 \, [\cos\theta,\sin\theta,0]^T$ then $v(\theta) = [-\sin\theta,\cos\theta,0]^T$

The natural choice of e_1 in this case is $e_1 = [0, 0, 1]^T$. Then $e_2 = [1, 0, 0]^T$, $e_3 = [0, 1, 0]^T$ and

$$S(\theta) = \{(x_1,x_2,x_3) \in R^3 : [x_1,x_2,x_3]^T = [-\cos\theta,-\sin\theta,0]^T t, \ t \in R\}$$

Hence,† $\xi_2(\theta) = [\cos^2\theta, \sin\theta \cos\theta, \sin\theta]^T$

$\xi_3(\theta) = [\cos\theta \sin\theta, \sin^2\theta, -\cos\theta]^T$ as shown in Fig. 3.5(b).

†Explicit formulae for ξ_j are derived in [5,20]. In particular, we have: $\xi_j = e_j - (\lambda_j+\mu_j)e_1 + [\lambda_j+\mu_j(2\cos\gamma_1-1)]v$ where: $\lambda_j \triangleq \cos\gamma_1 \cos\gamma_j \sin^{-2}\gamma_1$, $\mu_j \triangleq \cos\gamma_j \sin^{-2}\gamma_1$, and γ_j is an angle between e_j and v, i.e., $\cos\gamma_j \triangleq e_j v$

Thus the transformation is of the form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = R_0 \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} + \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}$$

or equivalently,

$$\begin{aligned} x_1 &= (R_0 + \rho_1\cos\theta + \rho_2\sin\theta)\cos\theta \\ x_2 &= (R_0 + \rho_1\cos\theta + \rho_2\sin\theta)\sin\theta \\ x_3 &= \rho_1\sin\theta - \rho_2\cos\theta \end{aligned} \tag{3.10}$$

Let us come back to equations (3.3-3.4). If $\varepsilon = 0$ then $\rho = 0$ is clearly a solution of (3.4), and it corresponds to the integral manifold of the unperturbed system.

We shall ask now for conditions under which this manifold can be preserved for small ε. These conditions can be formulated [5,21] in terms of equations (3.3)-(3.4). The procedure however, is very long. In this paper, we choose a different approach based on the Floquet theory [4,5]. We shall reduce [Appendix A] equations (3.3)-(3.4) to the form:

$$\begin{aligned} \dot{\theta} &= \omega_0 + g_1(\theta,\rho) + \varepsilon g_2(t,\theta,\rho,\varepsilon) \\ \dot{\rho} &= A\rho + R_1(\theta,\rho) + \varepsilon R_2(t,\theta,\rho,\varepsilon) \end{aligned} \tag{3.11}$$

with A being a constant matrix.[†]

In the next chapter we shall present theorems on the existence of an integral manifold for equations in the form (3.11) as well as for some of their generalizations.

[†]Note that g_k and R_k, k = 1,2 and are different from those functions in (3.3)-(3.4). However, they play the same role and hence it is more logical to use the same notation.

4. EXISTENCE AND STABILITY OF INTEGRAL MANIFOLDS

4.1. Main theorem

Assume that[†]

$$\begin{aligned} \dot{\theta} &= \omega_0 + g_1(\theta,\rho) + \varepsilon g_2(t,\theta,\rho,\varepsilon) \\ \text{and} \quad \dot{\rho} &= A\rho + R_1(\theta,\rho) + \varepsilon R_2(t,\theta,\rho,\varepsilon) \end{aligned} \tag{4.1}$$

satisfy the following hypotheses:

H1. Functions g_1, g_2, R_1, R_2 are continuous and bounded for $t \in \mathbb{R}$, $\theta \in \mathbb{R}$ and $\|\rho\| < \delta_0$,[††] where δ_0 is an arbitrary constant.

H2. These functions are 2π-periodic in θ.

H3. $g_1(\theta,\rho) = O(\|\rho\|)$, $R_1(\theta,\rho) = O(\|\rho\|^2)$.[†††]

H4. g_1, g_2, R_1, R_2 are Lipschitzian in both θ and ρ with Lipschitz constants which tend to zero with $\varepsilon \to 0$.

H5. Eigenvalues of A have nonzero real parts.

Theorem 4.1. [22,4,5] If hypotheses H1-H5 are satisfied, then there exists a function $h(t,\theta,\varepsilon)$ continuous in all variables and 2π-periodic in θ, bounded by D_ε, and Lipschitzian in θ with a Lipschitz constant Δ_ε, where $D_\varepsilon \to 0$, $\Delta_\varepsilon \to 0$ with $\varepsilon \to 0$, such that the surface:

$$S_\varepsilon = \{(t,\theta,\rho) : \rho = h(t,\theta,\varepsilon),\ t \in \mathbb{R},\ \theta \in \mathbb{R}\} \tag{4.2}$$

is an integral manifold of (4.1)

Behavior of solutions on this manifold is obtained by solving the system:

$$\dot{\theta} = \omega_0 + g_1(\theta,h(t,\theta,\varepsilon)) + \varepsilon g_2(t,\theta,h(t,\theta,\varepsilon),\varepsilon) \tag{4.3}$$

with arbitrary $\theta(t_0) = \theta_0$.

Moreover:

(a) If g_2 and R_2 are periodic (respectively almost periodic) in t so is $h(t,\theta,\varepsilon)$.

(b) If g_1, g_2, R_1 and R_2 are smooth, so is $h(t,\theta,\varepsilon)$.

[†] Observe that (4.1) is the same as (3.11).

[††] For $\rho = [\rho_1 \dots \rho_n]$, $\|\rho\| \triangleq \left(\sum_{k=1}^{n} \rho_k^2\right)^{1/2}$.

[†††] We say that $f(x,\alpha) = O(\alpha^n)$, if $f(x,\alpha)/\alpha^n$ remains (uniformly in x) bounded as $\alpha \to 0$.

(c) If all the eigenvalues of A have negative real parts then the manifold S_ε is stable.

4.2. Generalizations and remarks

1. Theorem 4.1 remains valid also when $\theta(t)$ is a vector-valued function: $\theta = [\theta_1..\theta_k]^T \in \mathbb{R}^k$. In this case H2 requires that g_1, g_2, R_1, and R_2 are periodic functions with a vector-period $T_0 = [T_{01},\ldots,T_{0k}]^T$.† Obviously (4.3) is in this case a k-vector equation with $\omega_0 = [\omega_{01},\ldots\omega_{0k}]^T$. As an example, consider a system of weakly-coupled oscillators (example 5.2 in Section 5).

2. Theorem 4.1 remains valid if ω_0 (respectively a vector $[\omega_{01},\ldots,\omega_{0k}]^T$) continuously depend on ε.

3. Theorem 4.1 holds also for equations of the form:

$$\begin{aligned}\dot\theta &= \omega_0 + g_1(\theta,x,y) + \varepsilon g_2(t,\theta,x,y,\varepsilon)\\ \dot x &= Ax + X_1(\theta,x,y) + \varepsilon X_2(t,\theta,x,y,\varepsilon)\\ \dot y &= \varepsilon By + \varepsilon Y_1(\theta,x,y) + \varepsilon^2 Y_2(t,\theta,x,y,\varepsilon)\end{aligned} \tag{4.5}$$

where hypotheses H1-H4 hold for X_1, X_2, Y_1, and Y_2 (instead of R_1 and R_2), and H5 holds for A and B. In this case there exist $h(t,\theta,\varepsilon)$ and $f(t,\theta,\varepsilon)$ (with the properties stated in Theorem 4.1) which define an integral manifold:

$$S_\varepsilon = \{(t,\theta,x,y),\;\; x = h(t,\theta,\varepsilon),\;\; y = \varepsilon f(t,\theta,\varepsilon),\;\; t \in \mathbb{R},\ \theta \in \mathbb{R}\} \tag{4.6}$$

Integral manifolds are also preserved under perturbation which average to zero in the following sense: Consider

$$\dot\theta = \omega_0 + g_1(\theta,x,y) + \varepsilon g_2(t,\theta,x,y,\varepsilon) + g_3(t,\theta,x,y,\varepsilon) \tag{4.7a}$$

$$\dot x = Ax + X_1(\theta,x,y) + \varepsilon X_2(t,\theta,x,y,\varepsilon) \tag{4.7b}$$

$$\dot y = \varepsilon By + \varepsilon Y_1(\theta,x,y) + \varepsilon^2 Y_2(t,\theta,x,y,\varepsilon) + \varepsilon Y_3(t,\theta,x,y,\varepsilon) \tag{4.7c}$$

where

$$\lim \frac{1}{T}\int_0^T g_3(t+\tau,\theta+\omega_0\tau,x,y,\varepsilon)d\tau = 0,\ \lim_{T\to\infty} \frac{1}{T}\int_0^T Y_3(t+\tau,\theta+\omega_0\tau,x,y,\varepsilon)d\tau = 0 \tag{4.8}$$

† Without loss of generality one can assume that $T_{01} = T_{02} = \ldots = T_{0k} = 2\pi$.

and $Y_3(t,\theta,x,y,\varepsilon)$ and $g_3(t,\theta,x,y,\varepsilon)$ have continuous second partial derivatives with respect to θ and y. Other hypotheses are the same as before. In this case ω_0 (respectively: $[\omega_{01},\ldots,\omega_{0k}]^T$) may also depend on ε but there must exist a positive constant d (independent of ε) such that $\omega_{0\ell} \geq d$ for $\ell = 1,\ldots,k$. For more subtle results, see [22, p. 140].

4.3. Outline of the proof

The proof of theorem 4.1 consists of defining a family of candidates for integral manifold together with an appropriate transformation which maps this family into itself. An outline of this proof can be found in [22]. The detailed proof is given in [23,4] and, (for more general case) in [5]. For our purpose, it suffices to know that the transformation of the family of candidates depends on the matrix A, and that it is a contraction mapping. The integral manifold is obtained as a fixed point of this transformation via successive iterations.

The case when θ is a vector and when a vector y is present as in (4.5), is proved in exactly the same way. The proof of (4.7) is based on the so called Krylov-Bogoliubov transformation which replaces a function proportional to ε and with a zero average by a function proportional to ε^2. In example 5.1 below we shall illustrate another application of this trick.

Let us note that in general the matrix A of (4.1) which was obtained via Floquet's theorem is not known explicitly. Thus neither the function $h(t,\theta,\varepsilon)$ (and $f(t,\theta,\varepsilon)$ for (4.5) and (4.7) nor the right hand side of (4.3) are explicitly known. It is, however, possible to obtain important qualitative properties of the system without knowing the exact form of the right hand side of (4.3). (See examples in Section 5.)

Let us note also that in some important cases (examples 5.3-5.6 of Section 5) we can find explicitly the matrix A (and matrix B for (4.5) and (4.7)) and consequently also the manifold $h(t,\theta,\varepsilon)$ (approximately at least) and the equation (4.3).

5. ILLUSTRATIVE EXAMPLES

In this section we shall show how theorem 4.1 can be applied to various systems. In each example we shall prove that an integral manifold exists. Discussions on the behavior of the trajectories are postponed to Sections 6 and 7.

Examples 5.1 and 5.2 are of a general character. Second-order systems are discussed in Examples 5.3 a,b,c. Examples 5.4 and 5.5 show how the method works in higher-order systems. The circuit structure in the first example allows us to apply directly to theorem 4.1. (In the second example some additional transformations are required. Examples 5.6 and 5.7 are of a slightly different character. Here the integral manifold arises not from a closed orbit but from an unbounded trajectory which is "periodic on a cylinder."

5.1. Theorem on averaging

The method to be outlined below was invented by Krylov and Bogoliubov [2] to give a mathematically rigorous justification of van der Pol's intuitive approach [1]. Further development of their approach has given rise to the method of integral manifolds. Here we present their result as a special case of theorem 4.1.

Consider the equation:

$$\dot{x} = \varepsilon X(t,x) \tag{5.1}$$

Suppose that $X(t,x)$ is T-periodic in t and can be expanded in a Fourier series (uniformly in x)

$$X(t,x) = \sum_{n=-\infty}^{+\infty} X_n(x) e^{j \frac{2\pi}{T} t}$$

Assume that the equation $X_0(x) = 0$ has a solution x_0 such that all eigenvalues of $\left.\frac{\partial X_0}{\partial x}\right|_{x=x_0}$ have nonzero real parts.

Define now the function $W(x,t) \triangleq \sum_{n\neq 0} \frac{1}{jn\frac{2\pi}{T}} X_n(x) e^{jn \frac{2\pi}{T} t}$. Note that $W(x,t)$ has the property that $\frac{\partial W}{\partial t} = X(t,x) - X_0(x)$.

Applying the transformation

$$x = \xi + \varepsilon W(\xi,t) \tag{5.2}$$

to (5.1) we get

$$\dot{\xi} = \varepsilon X_0(\xi) + \text{"small" terms} \tag{5.3}$$

which is equivalent to (4.7) with θ and x absent, $y \triangleq \xi - x_0$ and $B \triangleq \left.\frac{\partial X_0}{\partial \xi}\right|_{\xi=x_0}$.†

Thus, the manifold S_ε reduces to a periodic trajectory which remains close to the constant solution $x(t) \equiv x_0$.

†The theorem on averaging and the transformation (5.2) were first given by Krylov and Bogoliubov [2] in 1934. The version presented here is due to Bogoliubov Mitropolski [3] it remains valid even if $X(t,x)$ is not periodic and $X_0(x)$ is defined as: $X_0(x) \triangleq \lim_{T\to\infty} \frac{1}{T}\int_0^T X(t,x)dt.$

Example 5.2. Weakly coupled oscillators

Consider the system

$$\ddot{x}_k + \omega_k^2 x_k = -\varepsilon f_k(t,x_1\ldots x_n,\dot{x}_1\ldots\dot{x}_n) \qquad k = 1,2,..n. \tag{5.4}$$

Applying the transformation

$$x_k := \rho_k \cos\theta_k, \quad \dot{x}_k = -\omega_k\rho_k \sin\theta_k \tag{5.5}$$

to (5.4) yields

$$\begin{aligned} \dot{\theta}_k &= \omega_k + \frac{1}{\rho_k}\frac{\varepsilon}{\omega_k} f_k(t,x_1\ldots x_n,\dot{x}_1\ldots\dot{x}_n)\cos\theta_k \\ \dot{\rho}_k &= \frac{\varepsilon}{\omega_k} f_k(t,x_1\ldots x_n,\dot{x}_1\ldots\dot{x}_n)\sin\theta_k. \end{aligned} \tag{5.6}$$

If the limit

$$\bar{f}_k(\rho_1..\rho_n) \triangleq \lim_{T\to\infty} \frac{1}{T}\int_0^T f_k[t\rho_1\cos(\omega_1 t+\theta_{01}),\ldots \rho_n\cos(\omega_n t+\theta_{0n}), -\omega_1\rho_1\sin(\omega_1 t+\theta_{01}),$$

$$\ldots, -\omega_n\rho_n\sin(\omega_n t+\theta_{0n})]\, e^{j(\omega_k t+\theta_{0k})} dt$$

exists and does not depend on $\theta_{01}\ldots\theta_{0n}$ for all $k = 1,\ldots,n$, then (5.6) can be represented as

$$\begin{aligned} \dot{\theta}_k &= \omega_k + \frac{1}{\rho_k}\frac{\varepsilon}{\Omega_k} \operatorname{Re} \bar{f}_k(\rho_1..\rho_n) + \varepsilon\hat{g}(t,\rho) \\ \dot{\rho}_k &= \frac{\varepsilon}{\omega_k} \operatorname{Im}\bar{f}_k(\rho_1..\rho_n) + \varepsilon\hat{f}(t,\rho) \end{aligned} \tag{5.7}$$

where $\hat{f}(t,\rho)$ and $\hat{g}(t,\rho)$ denote terms with zero time average. Hence, if there exist $\rho_0 = [\rho_{01},\ldots\rho_{0n}]$ such that $\operatorname{Im} f_k(\rho_{01},\ldots\rho_{0n}) = 0$ and $\rho_{0n} \neq 0$ for $k = 1,\ldots n$ and the Jacobian matrix $[\frac{\partial}{\partial\rho} \operatorname{Im} f_k(\rho_0)]$ has eigenvalues with nonzero real parts then (5.7) is a special case of (4.7) with $\theta \in R^n$, $\rho = y \in \mathbb{R}^n$, and with the vector x absent.

Let us consider now some important special cases of example 5.2:

Example 5.3(a). Simple nonlinear oscillator

The simplest circuit model of a nonlinear oscillator which is considered in Example 2.1 is shown in Fig. 2.1. This model is described by

$$\ddot{v} + \frac{1}{C} g'(v)\dot{v} + \frac{1}{LC} v = 0 \tag{5.8}$$

Suppose now that $\frac{1}{C}\, g'(v)$ can be represented as $\varepsilon f(v)$ where $f(v)$ is bounded and ε is a small parameter, while $\frac{1}{LC} \triangleq \omega_0^2$ does not change with ε,[†] thus (5.8) is special case of (5.4) with $k = 1$, $x \triangleq v$, $\omega_0^2 \triangleq \frac{1}{LC}$, $\varepsilon f(t,x,x) \triangleq \frac{1}{c}\, g'(v)\dot{v} = \varepsilon f(v)\dot{v}$ and it can be reduced to the form

$$\begin{aligned} \dot{\theta} &= \omega_0 + \varepsilon f(\rho \cos \theta) \sin \theta \cos \theta \\ \dot{\rho} &= \varepsilon f(\rho \cos \theta) \rho \sin^2 \theta \end{aligned} \tag{5.9}$$

where $v = \rho \cos \theta$.

If we take the time[††] average of (5.9) then it will reduce to:

$$\dot{\theta} = \omega_0 \;,\; \dot{\rho} = \varepsilon \bar{f}(\rho)$$

where

$$f(\rho) = \frac{1}{\pi} \int_0^{\pi} f(\rho \cos \theta) \rho \;\; \sin^2 \theta d\theta^{\dagger\dagger\dagger} \tag{5.10}$$

If the nonlinear resistor characteristic has an "N" shape[§], then the function $\bar{f}(\rho)$ can be easily found; namely, it is zero at $\rho = 0$, decreases for $\rho > 0$ until it reaches its minimum, and then increases to $+\infty$ as shown in Fig. 5.1(b). Thus there exists $\rho_0 \neq 0$ such that $\bar{f}(\rho_0) = 0$. The transformation $y \triangleq \rho - \rho_0$ reduces (5.9) to (4.7) with $\theta \in \mathbb{R}^1$, $y \in \mathbb{R}^1$, $B = \frac{d}{d\rho} \bar{f}(\rho_0)$, and with the vector x absent.

It follows from theorem 4.1 that (5.9) has an integral manifold:

$$\begin{aligned} S_\varepsilon = \{(t,v,\dot{v}):\;\; & v = [\rho_0 + h(\theta,\varepsilon)] \cos \theta, \\ & \dot{v} = -\omega_0 [\rho_0 + h(\theta,\varepsilon)] \sin \theta,\; \theta \in [0,2\pi],\; t \in \mathbb{R}\} \end{aligned} \tag{5.11}$$

Motion on this manifold is described by:

$$\dot{\theta} = \omega_0 + \varepsilon f\{[\rho_0 + h(\theta,\varepsilon)] \cos \theta\} \sin \theta \cos \theta \tag{5.12}$$

[†] We can either change both L and C to keep $\frac{1}{C}$ small and $\frac{1}{LC}$ constant, or change the slope of the nonlinear characteristic to obtain a small $g'(v)$.

[††] More exactly we put $\theta = \omega_0 \tau + \phi$ and average (5.9) with respect to τ.

[†††] Since $f(\theta \cos \theta) \sin \theta \cos \theta$ is an odd function, its integral over a 2π-interval is equal to zero.

[§] More exactly we require $g'(v)$ to decrease monotonically for $v < 0$, increase monotonically for $v > 0$ and to be negative at $v = 0$ (as shown in Fig. 5.1(a)). Note that the van der Pol equation with $g'(v) = -1+v^2$ satisfies these requirements.

Since (5.9) is autonomous the integral manifold does not depend on time. Moreover, since $h(\theta,\varepsilon)$ is small the manifold lies close to a cylinder of radius ρ_0 in the (t,v,v)-space (as shown in Figs. 5.2(a) and (b)). Observe that the van der Pol equation considered earlier in Section 1 is a special case of (5.9). Thus theorem 4.1 justifies using the approximate formula $v(t) = 2\cos(\omega_0 t+\phi_0)$, which can be obtained from (5.11) and (5.12) with $\varepsilon = 0$ and $g(v) = -v + \frac{v^3}{3}$.

b. Nonautonomous case

Consider the same circuit with a periodic current source (as shown in Fig. 5.3) described by

$$\ddot{v} + \frac{1}{C} g'(v)\dot{v} + \frac{1}{LC} v = \frac{1}{C}\frac{d}{dt} i_s(\omega t) \tag{5.13}$$

Suppose, as before, that $\frac{1}{C} g'(v)$ can be represented as $\varepsilon f(v)$, $\omega_0^2 \triangleq \frac{1}{LC}$, and moreover $\frac{1}{C}\frac{d}{dt} i_s(\omega t)$ can be represented as $\varepsilon\omega A p(\omega t)$. Introducing dimensionless time $\tau \triangleq \omega t$ we obtain:

$$v'' + \frac{\varepsilon}{\omega} f(v)v' + \left(\frac{\omega_0}{\omega}\right)^2 v = \varepsilon A p(\tau)$$

where $v' \triangleq \frac{d}{d\tau} v$.

Now, we can apply transformation (5.5) (with x, $\dot{x}$ replaced by v, v') to obtain:

$$\begin{aligned} \theta' &= \frac{\omega_0}{\omega} + \frac{\varepsilon}{\omega} f(\rho\cos\theta)\sin\theta\cos\theta - \frac{\varepsilon}{\rho} A p(\tau)\cos\theta \\ \rho' &= \frac{\varepsilon}{\omega} f(\rho\cos\theta)\rho\sin^2\theta - \varepsilon A p(\tau)\sin\theta \end{aligned} \tag{5.14}$$

If ε is small enough with respect to ω (and the nonlinear resistor characteristic has the same properties as those in example 5.3a) then theorem 4.1 holds and (5.13) has an integral manifold:

$$\begin{aligned} S_\varepsilon = \{(t,v,\dot{v}):\ & v = [\rho_0 + h(\omega t,\theta,\varepsilon,A)]\cos\theta, \\ & \dot{v} = -\omega_0[\rho_0 + h(\omega t,\theta,\varepsilon,A)]\sin\theta,\ \theta\in[0,2\pi],\ t\in\mathbb{R}\} \end{aligned} \tag{5.15}$$

Observe, that if ω is large then $\frac{\varepsilon}{\omega}$ is small and so is $\frac{\omega_0}{\omega}$. It follows from Remark 4 of Section 4 that theorem 4.1 may not hold in this case (unless $\varepsilon \ll \omega_0$).

A similar situtation also occurs in some autonomous system as shown in the following example.

c. A counterexample: Wien-bridge oscillator

A Wien-bridge oscillator and its' circuit model are shown in Figs. 5.4(a) and (b). The circuit can be described by:

$$\ddot{v} + \frac{1}{RC}(3-f'(v))\dot{v} + \frac{1}{(RC)^2} v = 0 \tag{5.16}$$

One is tempted to treat $\frac{1}{RC}$ both as the small parameter ε and the frequency ω_0 ($\omega_0 = \varepsilon = \frac{1}{RC}$) when applying theorem 4.1. Unfortunately ω_0 decreases to zero with ε and the theorem cannot be applied.†

d. A counterexample: Lossless LC oscillators

Observe that theorem 4.1 does not apply for circuits considered in Sections 2.3 and 2.4. Indeed these circuits possess families of nonisolated manifolds. The standard transformation (5.5) reduces (2.9) to

$$\dot{\theta} = \Phi(\rho,\theta)^{\dagger\dagger}$$

$$\dot{\rho} = 0$$

Hence, "the matrix A" is zero, hypothesis H5 is not satisfied and we cannot expect the manifolds to be preserved under small perturbation.

Example 5.4. Nonlinear oscillator weakly coupled with a linear dissipative circuit.

Consider the circuit shown in Fig. 5.5. This circuit is described by:

$$\begin{aligned} C\ddot{v} + g'(v)\dot{v} + \frac{L_1}{L_1L-M^2} v &= -B\omega \sin \omega t - \frac{M}{LL_1-M^2} Ri_1 \\ \frac{di_1}{dt} &= -\frac{M}{LL_1-M^2} v - \frac{LR}{LL_1-M^2} i_1 \end{aligned} \tag{5.17}$$

Assume that both the nonlinearity and the "perturbations" are small. More precisely, assume that we can introduce the following representation

$$\varepsilon f(v) \triangleq \frac{1}{C} g'(v), \quad \varepsilon a \triangleq \frac{B}{C}, \quad \varepsilon b \triangleq \frac{M}{LL_1-M^2} \approx \frac{M}{LL_1}$$

$$A \triangleq -\frac{LR}{L_1L-M^2} \approx \frac{R}{L_1}, \quad \omega_0^2 \triangleq \frac{L_1}{C(LL_1-M^2)} \approx \frac{1}{LC}$$

†Equation (5.16) possesses, however, an integral manifold, the existence of which can be proved via phase-plane methods [11,5].

††In the case of the linear equation (2.8) we have $g(\phi) = \phi/L$ and $\Phi(\rho,\theta) = \omega_0 \triangleq \frac{1}{\sqrt{LC}}$.

where ε can be made arbitrarily small while a, b, A and ω_0, even if dependent on ε, remain bounded (as functions of ε) both from below and from above by positive constants.

In terms of the new notation, equation (5.17) assumes the form:

$$
\begin{aligned}
&\ddot{v} + \varepsilon f(v)\dot{v} + \omega_0^2 v = -\varepsilon a\omega \sin \omega t - \varepsilon b \frac{R}{c} i_1 \\
&\frac{di_1}{dt} = Ai_1 - \varepsilon bv
\end{aligned}
\qquad (5.18)
$$

Let us denote $i_1 \triangleq x$ and introduce the standard transformation $v = \rho \cos \theta$, $\dot{v} = -\omega_0 \rho \sin \theta$ in (5.18) to obtain:

$$\dot{\theta} = \omega_0 + \varepsilon f(\rho \cos \theta)\sin \theta \cos \theta - \frac{\varepsilon}{\omega_0 \rho} (a\omega \sin \omega t + b \frac{R}{c} x)\cos \theta \qquad (5.19a)$$

$$\dot{\rho} = \varepsilon f(\rho \cos \theta)\rho \sin^2\theta - \frac{\varepsilon}{\omega_0} (a\omega \sin \omega t + b \frac{R}{c} x)\sin \theta \qquad (5.19b)$$

$$\dot{x} = Ax - \varepsilon b\rho \cos \theta \qquad (5.19c)$$

If $\omega \neq \omega_0$ then the time average of the r.h.s. of (5.19b) reduces to $\varepsilon\overline{f}(\rho) = \frac{\varepsilon}{\pi} \int_0^{\pi} f(\rho \cos \theta)\sin^2\theta d\theta$ and (5.19a,b,c) can be represented in the form (4.7) with $y := \rho-\rho_0$ (where $\overline{f}(\rho_0) = 0$, $\frac{d}{d\rho} \overline{f}(\rho_0) \neq 0$, $\rho_0 \neq 0$) $B \triangleq \frac{d}{d\rho} \overline{f}(\rho_0)$ where x and θ are scalars.

<u>Example 5.5</u>: <u>Colpitts-type oscillator</u>

This oscillator can be represented by the circuit shown in the Fig. 5.6. It is described by

$$
\begin{aligned}
v &= - \frac{1}{RC_2} v + \frac{1}{C_2} i \\
\dot{v}_1 &= - \frac{1}{C_1} g(v) - \frac{1}{C_1} i \\
\frac{d}{dt} i &= - \frac{1}{L} v + \frac{1}{L} v_1
\end{aligned}
\qquad (5.20)
$$

Suppose that $g(v) = \frac{C_1}{RC_2} v - LC_1 f(v)$ so that (5.20) assumes the equivalent form:

$$\frac{d}{dt} v = - \frac{1}{RC_2} v + \frac{1}{C_2} i \qquad (5.21a)$$

$$\frac{d}{dt} i = i' \qquad (5.21b)$$

$$\frac{d}{dt} i' = -\frac{C_1+C_2}{LC_1C_2} i + \varepsilon f(v) \tag{5.21c}$$

The form of (5.21) suggests that we are considering an oscillator in which the variable v has only a weak influence. However we cannot apply the theorem 4.1 directly because the "influence of i on v" is not "weak." In particular, (5.21a) is not in the form of (4.7b).

Thus before applying theorem 4.1 we must reduce (5.21) to the appropriate form:† Define first:

$$a \triangleq \frac{1}{RC_2}, \; b \triangleq \frac{1}{C_2}, \; \omega_0^2 \triangleq \frac{C_1+C_2}{LC_1C_2} \tag{5.22}$$

Note that for $\varepsilon = 0$ the right-hand side of (5.21) is linear and its eigenvalues are equal to a, $j\omega_0$, and $-j\omega_0$. Note also that the transformation

$$x_1 \triangleq v + \frac{ab}{a^2+\omega_0^2} i + \frac{b}{a^2+\omega_0^2} i', \; x_2 \triangleq \frac{1}{2} i + \frac{1}{2\omega_0} i', \; x_3 \triangleq \frac{1}{2} i - \frac{1}{2\omega_0} i' \tag{5.23}$$

reduces (5.21) to the form:

$$\begin{bmatrix} \dot{x}_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & -\omega_0 \\ 0 & \omega_0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + f(v) \begin{bmatrix} \frac{b}{a^2+\omega_0^2} \\ \frac{1}{2\omega_0} \\ -\frac{1}{2\omega_0} \end{bmatrix} \tag{5.24}$$

where $v = x_1 - \dfrac{b(a\omega_0+1)}{\omega_0(\omega_0^2+a^2)} x_2 - \dfrac{b(a\omega_0-1)}{\omega_0(\omega_0^2+a^2)} x_3$

The remaining procedure is standard; we introduce the "amplitude" ρ and the "phase" θ as follow: $x_2 \triangleq \rho\cos\theta$, $x_3 \triangleq \rho \sin\theta$. In terms of ρ and θ, (5.24) becomes

†Computations to this example were done by Mr. Mojaddad-Shahruz Shahram.

$$\dot{\theta} = \omega_0 - \frac{1}{\rho}\frac{\varepsilon}{2\omega_0} f(v)(\cos\theta + \sin\theta)$$

$$\dot{\rho} = \frac{\varepsilon}{2\omega_0} f(v)[\cos\theta - \sin\theta] \tag{5.25}$$

$$\dot{x}_1 = ax_1 + \frac{\varepsilon b}{a^2+\omega_0^2} f(v)$$

Assume that there exists $\rho_0 \neq 0$ such that $\overline{f}(\rho_0,0) = 0$ and $\frac{\partial}{\partial\rho}\overline{f}(\rho_0,0) < 0$, where

$$\overline{f}(\rho,x) \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} f[x_1 - \frac{b(a\omega_0+1)}{\omega_0(\omega_0^2+a^2)}\rho\cos\theta - \frac{b(a\omega_0-1)}{\omega_0(\omega_0^2+a^2)}\rho\sin\theta](\cos\theta - \sin\theta)d\theta \tag{5.26}$$

With $y := \rho-\rho_0$ and $B := \frac{d}{d\rho}\overline{f}(\rho_0,0)$, equation (5.25) is of the form (4.7) and theorem 4.1 holds. Hence, there exists an integral manifold

$$S_\varepsilon = \{(t,\theta,\rho,x):\ \rho = \rho_0 + \varepsilon h_1(\theta,\varepsilon),\ x = h_2(\theta,\varepsilon),\ t \in R,\ \theta \in [0,2\pi]\} \tag{5.27}$$

The solutions of (5.21) which lie on S_ε are of the form:

$$v(t) = h_2(\theta(t),\varepsilon) - \frac{b[\rho_0+\varepsilon h_1(\theta(t),\varepsilon)]}{\omega_0(\omega_0^2+a^2)}[(a\omega_0+1)\cos\theta(t) - (a\omega_0-1)\sin\theta(t)]$$

$$i(t) = [\rho_0 + \varepsilon h_1(\theta(t),\varepsilon)](\cos\theta(t) + \sin\theta(t)) \tag{5.28}$$

$$i'(t) = \frac{1}{\omega_0}[\rho_0 + \varepsilon h_1(\theta(t),\varepsilon)](\cos\theta(t) - \sin\theta(t))$$

In other words, the behavior of the solutions on the integral manifold is completely described by the phase $\theta(t)$ which can be determined from

$$\dot{\theta} = \omega_0 - \frac{1}{\rho_0+\varepsilon h_1(\theta,\varepsilon)}\frac{\varepsilon}{2\omega_0} f(v)(\cos\theta + \sin\theta) \tag{5.29}$$

where v is given by the first formula of (5.28).

Example 5.6. Josephson junction circuit [9]

The circuit shown in Fig. 2.6 is governed by the equation

$$C\frac{d^2\phi}{dt^2} + \frac{1}{R}\frac{d\phi}{dt} + I_c\sin(\frac{4\pi e}{h}\phi) = I_{dc} + I_{ac}\sin\nu t \tag{5.30}$$

This equation can be transformed into the dimensionless form:

$$\begin{aligned} \dot{x} &= y \\ \beta\dot{y} &= \alpha - \sin x - y + \varepsilon \sin \omega\tau \end{aligned} \tag{5.31}$$

It can be shown [14,15,9] that if $\varepsilon = 0^{\dagger}$ then for any $\beta > 0$ there exists a unique number $\alpha_0(\beta) \in (0,1]$ such that for $\alpha > \alpha_0(\beta)$ (and $\varepsilon = 0$) equation (5.31) has a stable invariant surface in the (t,x,y)-space (Fig. 2.7)

$$S_0 = \{(t,x,y): \quad y = \psi(x),\ x \in \mathbb{R},\ t \in \mathbb{R}\}.$$

Moreover, if $\varepsilon \neq 0$ is small, the surface persists and remains stable. To prove this we introduce new coordinates (see Fig. 5.7):

$$\begin{aligned} x &\triangleq \theta - \psi'(\theta)\rho \\ y &\triangleq \psi(\theta) + \rho \end{aligned} \tag{5.32}$$

In terms of these coordinates (5.31) assumes the form:

$$\begin{aligned} \dot{\theta} &= \psi(\theta) + G(\tau,\theta,\rho,\varepsilon) \\ \dot{\rho} &= A(\theta)\rho + F(\tau,\theta,\rho,\varepsilon) \end{aligned} \tag{5.33}$$

Note that (5.33) is not in the form of (4.7). However (5.33) can be simply reduced to this form by introducing a new phase variable $\phi(\theta) \triangleq \int_0^{\theta} \frac{1}{\psi(\tau)}\, d\tau$ such that

$$\dot{\phi} = \frac{d\phi}{d\theta}\,\dot{\theta} = \frac{1}{\psi(\theta)}\,[\psi(\theta) + G(\tau,\phi,\rho,\varepsilon)] = 1 + \frac{1}{\psi(\theta)}\,G(\tau,\theta(\phi),\rho,\varepsilon)$$

The transformation $\theta \to \phi$ is one-to-one due to the fact that $\psi(\theta)$ is a positive periodic function.

Now we have the equations

$$\begin{aligned} \dot{\phi} &= 1 + \overline{G}(\tau,\phi,\rho,\varepsilon) \\ \dot{\rho} &= \overline{A}(\phi)\rho + \overline{F}(\tau,\phi,\rho,\varepsilon) \end{aligned} \tag{5.34}$$

where $\overline{A}(\phi)$ is T_0-periodic with $T_0 = \int_0^{2\pi} \frac{ds}{\psi(s)}$.

$^{\dagger}\varepsilon$ is a dimensionless number equivalent for I_{ac}.

Hence, we can apply the Floquet transformation as described in Appendix A. Since in this particular case the "matrix" $\overline{A}(\phi)$ is actually one-dimensional, it is possible to find the transformation explicitly; namely, the fundamental solution of

$$\frac{d\rho}{d\phi} = \overline{A}(\phi)\rho$$

is

$$\phi(\phi,\theta) = e^{\int_0^{\phi} \overline{A}(s)ds} .$$

Hence,

$$B \triangleq \frac{1}{T_0}\int_0^{T_0} \overline{A}(s)ds$$

and

$$P(\phi) \triangleq e^{\int_0^{\phi}\overline{A}(s)ds - \frac{1}{T_0}\int_0^{T_0}\overline{A}(s)ds} \tag{5.35}$$

Defining the new amplitude

$$r = P^{-1}(\phi)\rho = e^{\frac{1}{T_0}\int_0^{T_0} \overline{A}(s)ds - \int_0^{\phi} \overline{A}(s)ds} \tag{5.36}$$

we obtain

$$\begin{aligned} \dot{\phi} &= 1 + \tilde{G}(\tau,\rho,r,\varepsilon) \\ \dot{r} &= Br + \tilde{F}(\tau,\phi,r,\varepsilon) \end{aligned} \tag{5.37}$$

where

$$\tilde{G}(\tau,\phi,r,\varepsilon) = \overline{G}(\tau,\phi,\rho,\varepsilon) .$$

and

$$\tilde{F}(\tau,\phi,r,\varepsilon) = \omega_0 P^{-1}(\phi)\ \overline{F}(\tau,\phi,\rho,\varepsilon) + G(\tau,\phi,\rho,\varepsilon)[\overline{A}(\phi) - \frac{1}{T_0}\int_0^{T_0} \overline{A}(s)ds]$$

are "small."

<u>Example 5.7</u>. <u>Josephson junction (general case)</u> [10]

Most Josephson weak-link junctions are described by the simplified equation (5.30). However, theoretical justification of this equation is far from complete. It is known for example that although the Josephson supercurrent is a periodic and odd function of the phase difference, it need <u>not</u> be sinusoidal [25,26]. Similarly, in metal junctions, we cannot assume the normal current to depend linearly on the voltage drop. Moreover, in general, it depends also on the phase difference.

Thus let us consider the more realistic equation:

$$\beta\ddot{x} + f(x,\dot{x}) = \alpha + \varepsilon p(\omega t) \tag{5.38}$$

where $f(x,\dot{x})^{\dagger}$ is smooth and 2π-periodic in x. The constant α and the periodic (almost periodic) function $p(\omega t)$ represents an external excitation.

In the important particular case of (5.38), we have $f(x,\dot{x}) = \dot{x} + s(x)$ i.e.,

$$\beta\ddot{x} + \dot{x} + s(x) = \alpha + \varepsilon p(\omega t) \tag{5.39}$$

where $s(x)$ is a smooth, odd, and 2π-periodic function of x.

Without loss of generality we can assume that $\alpha \geq 0$ and that the time average of $p(\omega t)$ is equal to zero.

We shall establish now conditions under which equation (5.38) and (5.39) possesses an integral manifold for "small" ε. Our approach will be similar to that of Example 5.6: first we find an invariant surface for an associated autonomous equation (i.e., for $\varepsilon = 0$). Next we apply theorem 4.1 to prove that this surface persists under small perturbations. The autonomous equations of this type were studied in [27,28]. Our approach will follow that of Barbashin and Tabueva [28].

Consider at first (5.39) with $\varepsilon = 0$.

Lemma 5.7.1 [28]

If $\varepsilon = 0$ and $s(x) < \alpha$ for all x, then for all $\beta > 0$ (5.39) has a stable invariant surface

$$S_0 = \{(t,x,y): \; y = \psi(x), \; x \in \mathbb{R}, \; t \in \mathbb{R}\} \tag{5.40}$$

where $\psi(x)$ is smooth, positive and 2π-periodic. Moreover this surface is unique. (See Fig. 2.7.)

Lemma 5.7.2. [28] If $\frac{1}{2\pi}\int_0^{2\pi} s(x)dx < \alpha$ and the equation $s(x) = \alpha$ has (in the interval $[0,2\pi)$) exactly two solutions x_1 and x_2 such that $s'(x_1) > 0$, $s'(x_2) < 0$, then there exists a critical value β_0 such that for $\beta > \beta_0$ equation (5.39) has a unique invariant surface described by (5.40).

$^{\dagger}f(x,\dot{x})$ includes the normal and quasiparticle currents, and the supercurrent. In the original Josephson paper it was calculated to be of the form: $f(x,y) = \sigma(y)\sin x + [\sigma_1(y) + \sigma_2(y)\cos x]y$. In the metal junction described in Section 2.6 $f(x,y) = \sin x + g(y)$.

In a similar way we shall discuss (5.38). Assume[†] that $f(x,y)$ is monotonically increasing in y and such that for any x we have (possibly equal to $\pm\infty$) $\lim_{y\to+\infty} f(x,y) > 0$ and $\lim_{y\to-\infty} f(x,y) < 0$. The following lemmas hold:

Lemma 5.7.3 [28] If $f(x,0) < \alpha$ for all x then equation (5.31) has, for $\varepsilon = 0$, a unique and globally stable invariant surface S_0 described by (5.40). □

Consider now the case when $f(x,0) = \alpha$ has n solutions in $[0,2\pi)$; namely, $0 \le x_1 < x_2 < \dots < x_{2n-1} < x_{2n} < 2\pi$.[††] Suppose that $\frac{\partial}{\partial x} f(x_i,0) \neq 0$ $i = 1\dots 2n$. To be specific let us assume that $\frac{\partial}{\partial x} f(x_i,0) > 0$ for $i = 2k-1$ and $\frac{\partial}{\partial x} f(x_i,0) < 0$ for $i = 2k$ $k = 1,\dots,n$. It can be easily shown [28] that the solution at points $(x_{2k},0)$ are saddles while $(x_{2k-1},0)$ are sinks. The neighborhood of each saddle is shown in Fig. 5.8, i.e., there are two separatrices converging toward it, one from below and one from above. Consider the separatrix from above (denoted S_{2k} in Fig. 5.11).

Lemma 5.7.4. [28] If none of the separatrices tends to $+\infty$ as $t \to -\infty$ (see Fig. 5.9(a)) and if there exists a trajectory which originates in the lower half plane ($y<0$) and does not converge to any of the equilibrium points (see Fig. 5.9(b)). Then (5.39) (with $\varepsilon=0$) possesses an invariant surface described with (5.40). Moreover all the solutions tend either to this manifold or to equilibrium points. □

Observe that all Lemmas (5.7.1)-(5.7.4) imply the existence of a surface S_0 given by formula (5.40). It should be noted, however, that the parametrizing function $y = \psi(x)$ (and so the surface) is different in each lemma. Now once the conditions for the existence of S_0 have been established, we can proceed exactly as in the previous example because the only properties of $\psi(x)$ that we need are smoothness, periodicity, and positiveness.

6. EQUATIONS ON THE INTEGRAL MANIFOLD

6.1. Introduction

One of the main advantages of the integral manifold theory is that the original equation can be reduced to the first-order scalar equation:

[†] These assumptions are natural generalization of the Josephson formula (see the previous footnote).

[††] Since $f(x,0)$ is smooth and periodic there must be an even number of these solutions.

$$\dot{\theta} = \omega_0 + g_1(\theta,h(t,\theta,\varepsilon)) + \varepsilon g_2(t,\theta,h(t,\theta,\varepsilon),\varepsilon) \tag{6.1}$$

In general all that we know about $h(t,\theta,\varepsilon)$ is that it exists, is periodic in θ, and is periodic (almost-periodic) in t. We shall see, however, that it is enough to obtain a lot of important qualitative information on (6.1) even without knowing its exact form.

The significance of (6.1) is that if the manifold is globally stable, then <u>any steady state trajectory</u> of the original system must lie on it, and consequently must be a solution of (6.1).

6.2. Applications of rotation number

We shall discuss now some qualitative properties of (6.1) which, we shall rewrite, for the sake of simplicity in the form

$$\dot{\theta} = \omega_0 + g(t,\theta,\varepsilon) \tag{6.2}$$

where $g \triangleq g_1+\varepsilon g_2$ is smooth[†] (in all the variables), T-periodic in t, 2π-periodic in θ, and $\sup_{t,\theta} |g(t,\theta,\varepsilon)| \to 0$ when $\varepsilon \to 0$[††]

<u>Case 1</u>.

Consider at first the case when $g(t,\theta,\varepsilon)$ in (6.2) does <u>not</u> depend on time[†††] and when $\theta(t)$ is scalar valued, i.e.,

$$\dot{\theta} = \omega_0 + g(\theta,\varepsilon) \tag{6.3}$$

(a) If there exists a θ_0 such that[§] $g(\theta_0,\varepsilon) = -\omega_0$ then θ_0 is a constant solution of (6.3). Its' stability properties can be found from the sign of $(\theta-\theta_0)\,[\omega_0 + g(\theta,\varepsilon)]$ for $\theta \neq \theta_0$ but close to it (see Fig. 6.1). Let us note that in terms of the original equations (3.2)[§§] the constant θ_0 corresponds to the constant solution of (3.2)

[†]A function is said to be smooth iff it is at least twice continuously differentiable.

[††]Since $g_1(\theta,\rho) = O(\rho)$, it follows that $g_1(\theta,h(t,\theta,\varepsilon))$ tends to zero whenever $h(t,\theta,\varepsilon)$ tends to zero.

[†††]Equations of autonomous sytems have such a form (see Examples 5.3a, 5.5).

[§]Recall that, ω_0 may depend (continuously) on ε. Theorem 4.1 also holds for (4.5) if ω_0 decreases to zero with ε. However, when (4.7) is considered, we must also assume ε to be much smaller than ω_0 (see Examples 5.3(b) and (c)).

[§§]In the case considered f_1 is independent of t, thus we have
$\dot{x} = f_0(x) + \varepsilon f_1(x,\varepsilon)$ and $\rho(t) = h(\theta(t),\varepsilon)$.

$$x_0 = u(\theta_0) + P(\theta_0)\, h(\theta_0,\varepsilon) \tag{6.4}$$

(b) If ω_0 remains "large" with respect to ε (which is usually the case) then for ε small enough the right-hand side of (6.3) is positive. It can be easily shown [Appendix C] that in this case the solutions of (6.3) assume the form:

$$\dot{\theta}(t) = \frac{2\pi}{T_\varepsilon}\, t + q(t) \tag{6.5}$$

where $q(t)$ is T_ε-periodic with $T_\varepsilon = \int_0^{2\pi} \frac{d\theta}{\omega_0 + g(\theta,\varepsilon)}$. In terms of the θ and ρ coordinates, this means that the "phase" θ is increasing, while the "amplitude" $\rho = h(\theta(t),\varepsilon)$ is T_ε-periodic in t.

Thus the original variable assumes the form

$$x(t) = u\left[\frac{2\pi}{T_\varepsilon}\, t + q(t)\right] + P\left[\frac{2\pi}{T_\varepsilon}\, t + q(t)\right] h\left[\frac{2\pi}{T_\varepsilon}\, t + q(t)\right] \tag{6.6}$$

and is also T_ε-periodic in t. Hence, in an autonomous system the solutions on the manifold exhibit two different behaviors:

Statement 6.1 (Fig. 6.2(a), (b))

a. If there exist at least one θ_0 such that $\omega_0 + g(\theta_0,\varepsilon) = 0$, then some solutions are constant and all the other solutions (on the manifold) tend to them either for $t \to +\infty$, or for $t \to -\infty$.

b. If $\omega_0 + g(\theta,\varepsilon) \neq 0$† for all ε, then all solutions on the manifold are T_ε-periodic, where $T_\varepsilon = \int_0^{2\pi} \frac{d\theta}{\omega_0 + g(\theta,\varepsilon)}$.

Case 2. Forced oscillations

Consider now an important case where θ is scalar valued and $g(t,\theta,\varepsilon)$ is T-periodic in t and 2π-periodic in θ. In this case (6.2) can be considered as an equation on a torus and the important concept of rotation number can be introduced [9,11,24].

Let us consider an arbitrary solution $\theta(t,\theta_0)$ of (6.2) such that $\theta_\varepsilon(0,\theta_0) = 0$ and define the limit

$$\mu \triangleq \frac{T}{2\pi} \lim_{t\to+\infty} \frac{\theta(t,\theta_0)}{t} \tag{6.7}$$

†The case $\omega_0 + g(\theta,\varepsilon) < 0$ can be discussed in the same way as $\omega_0 + g(\theta,\varepsilon) > 0$.

It can be shown [11,24] that this limit exists and does not depend on θ_0. Suppose now that the right hand side of (6.3) continuously depends on a parameter $\alpha^{\dagger}$. The rotation number is said to be <u>stable</u> [24] if it remains constant under small changes of α. It can be shown [24] that if μ is an irrational number then it can <u>never</u> be stable. (If it is rational, it can be stable provided that some additional conditions hold [24]). Thus a typical graph of μ as a function of α will be as shown in Fig. 6.3.† The above properties of rotation number can be used to explain the strange a.c.-characteristic observed in Josephson-junction circuits [9,10].

<u>Statement 6.2.</u> [11,24]

a. If the rotation number is <u>rational</u> (i.e., there exist integers M, N such that $\mu = \frac{M}{N}$), then (6.2) has at least one solution of the form

$$\theta(t) = \mu \frac{2\pi}{T} t + q(t) \tag{6.8}$$

where $q(\cdot)$ is Nt-periodic. ¤

Other solutions on the manifold are either of the form (6.8) (with q(t) possibly different but NT-periodic) or tend to some solution of the form (6.8) when $t \to +\infty$ (and also when $t \to -\infty$)

b. If μ is an <u>irrational</u> number then <u>any</u> solution on the manifold is of the form

$$\theta(t) = \mu \frac{2\pi}{T} t + \theta_0 + s(t, \mu \frac{2\pi}{T} t + \theta_0) \tag{6.9}$$

where $s(t,\theta)$ is T-periodic in t and 2π-periodic in θ. Thus, for irrational μ, $s(t, \mu \frac{2\pi}{T} t + \theta_0)$ is an almost-periodic (but not periodic) function of t.

Note that if we put $\mu = \frac{M}{N}$ in (6.9) then $s(t, \frac{M}{N} \frac{2\pi}{T} t + \theta_0)$ is NT-periodic in t. Hence (6.8) can be considered as a special case of (6.9)

Recall that the original variables are of the form

$$x(t) = u(\theta(t)) + P(\theta(t))\, h(t, \theta(t), \varepsilon)$$

It follows that

(a) if μ is rational, then x(t) is an NT-periodic function of t;

(b) if μ is irrational, then x(t) is almost-periodic with basic frequencies $\frac{2\pi}{T}$ and $\mu \frac{2\pi}{T}$.

$^{\dagger}\mu$ is known to be a continuous but <u>not</u> necessarily Lipschitzian function of α [24].

If case (a) holds we say that the solution is <u>synchronized</u> with a period of oscillations equal to some multiple of the period of the external forcing frequency.

The above properties of rotation number (see Fig. 6.3) explain what happens when some parameter is slowly varied. Namely there appear synchronization zones (constant steps in the Fig. 6.3) of varying lengths sandwiched between zones where the oscillation frequency (or basic frequencies) changes with the parameter.

<u>Remark</u>: The problem of crucial importance is to determine <u>whether a given rational rotation number is stable</u>. This problem can be answered only partially: there are criteria of stability of μ which can be expressed in terms of the so called "Poincaré map" [24], however it is impossible in general to express this criterion in terms of the r.h.s. of the differential equation. The only case where this can be done is when some small-parameter assumptions are satisfied. We shall discuss this case in <u>Section 6.3</u>.

6.3. <u>Synchronization via the theorem on averaging</u>

Throughout this section we assume that ε is small with respect to ω_0 and that equations (6.2) can be rewritten in the form:†

$$\dot{\theta} = \omega_0 + \varepsilon g(t,\theta,\varepsilon) \tag{6.10}$$

Since $g(t,\theta,\varepsilon)$ is T-periodic in t and 2π-periodic in θ, it can be expanded into a Fourier series

$$g(t,\theta,\varepsilon) = \sum_{m,n} g_{m,n} e^{j(m\omega t+n\theta)} \tag{6.11}$$

where $\omega \triangleq \frac{2\pi}{T}$.

Without loss of generality we can assume $g_{00} = 0$. If this is not the case we can include εg_{00} into ω_0 so that $\omega_\varepsilon \triangleq \omega_0 + \varepsilon g_{00}$ and discuss (6.10) with ω_ε.

Take now integers M and N†† such that $\frac{M}{N}\omega-\omega_0$ is of the order ε, let us write it as

$$\omega_0 = \frac{M}{N}\omega + \varepsilon\Delta \tag{6.12}$$

†Since $g(t,\theta,\varepsilon)$ is smooth in ε and $g(t,\theta,0) = 0$ we can represent it as $g(t,\theta,\varepsilon) = \varepsilon g(t,\theta,\eta_\varepsilon)$ where $\eta_\varepsilon \in (0,\varepsilon)$, i.e., we can represent $g(t,\theta,\varepsilon)$ in the form (6.10).

††For discussion of their existence see the Remarks in this section.

Introduce the new variable

$$\phi \triangleq \theta - \frac{M}{N}\omega t \tag{6.13}$$

which satisfies the equation:

$$\dot{\phi} = \varepsilon[\Delta + g(t, \frac{M}{N}\omega t + \phi, \varepsilon)] \tag{6.14}$$

Taking the time average of (6.14) we obtain

$$\dot{\phi} = \varepsilon[\Delta + \bar{g}(\phi)] \tag{6.15}$$

with $\bar{g}(\phi) = \sum_L g_{LM_1-LN} e^{-jLN\phi}$.

Suppose that there exists ϕ_0 such that

$$\Delta + g(\phi_0) = 0 \quad \text{and} \quad \bar{g}'(\phi_0) < 0^{\dagger} \tag{6.16}$$

Then it follows from the theorem on averaging (Section 5.1) that (6.14) has a stable NT-periodic solution $\phi(t)$ which remains close to ϕ_0 for $t \in (-\infty,+\infty)$ and is unique in some neighborhood of ϕ_0. Thus $\theta(t) = \frac{M}{N}\omega t + \phi(t)$ and the original solution

$$x(t) = u[\frac{M}{N}\omega t + \phi(t)] + P[\frac{M}{N}\omega t + \phi(t)]\, h(t, \frac{M}{N}\omega t + \phi(t), \varepsilon) \tag{6.17}$$

is NT-periodic. Hence, our system has synchronized steady-state oscillations. Now if the paramerer $\varepsilon^{\dagger\dagger}$ is slowly varying, then the solution $\phi(t)$ may also change slowly but it remains NT-periodic as long as the conditions (6.16) hold. In other words, the system remains synchronized as long as (6.16) are satisfied. This kind of behavior was already discussed in the previous section and it is illustrated by constant steps in Fig. 6.3. The last result says that we shall remain on a constant step as long as (6.16) is satisfied.

6.4. Trajectories outside of the manifold

In the previous sections we have discussed trajectories on the manifold. Let us now consider (4.1). It can be shown [4,5] that if the hypotheses H1-H5 hold and if the eigenvalues of A have negative real parts, then all trajectories

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†If $\bar{g}'(\phi) \neq 0$ for some $\phi = \phi_{01}$ such that $\Delta + \bar{g}(\phi_{01}) = 0$ then it follows from the continuity and periodicity of $\bar{g}(\phi)$ that there exists (at least one) $\phi_{02} \neq \phi_{01}$ such that $\Delta + \bar{g}(\phi_{02}) = 0$.

††It applies also to any parameter on which the right hand side of (6.14) depend continuously.

close to the manifold tend to it as $t \to \infty$. Moreover for each such trajectory Γ, there exists a trajectory Γ_1 on the manifold such that $\Gamma \to \Gamma_1$ as $t \to \infty$.

The typical situations for the autonomous and the forced (not necessarily synchronized) case are shown in Fig. 6.4.

7. EXAMPLES OF EQUATIONS ON MANIFOLDS

7.1. Simple oscillator

Let us return to the circuit of Example 5.3 which is described by:

$$\ddot{v} + \varepsilon f(v)\dot{v} + \omega_0^2 v = \varepsilon A\omega p(\omega t) \tag{7.1}$$

Under the conditions stated in Section 5.3, equation (7.1) has an invariant surface.

a) Autonomous case $A = 0$

If the forcing term is absent then the manifold is a "cylinder" (Fig. 5.3(b)) and the motion on it is described by (5.11) i.e., by an equation of the form:

$$\dot{\theta} = \omega_0 + \varepsilon g(\theta,\varepsilon) \tag{7.2}$$

Since ε is "small" with respect to ω_0 the theory presented in Section 6.2 (case 1(b)) can be applied. Thus the phase θ is of the form

$$\theta(t) = \omega_\varepsilon t + q(t) \tag{7.3}$$

where $q(t)$ is $\frac{2\pi}{\omega_\varepsilon}$-periodic and $\omega_\varepsilon \to \omega_0$ as $\varepsilon \to 0$ (compare (6.5)). Thus for initial conditions picked on the manifold the voltage is a $\frac{2\pi}{\omega_\varepsilon}$-periodic function of the form

$$v(t) = [\rho_0 + h(\omega_\varepsilon t+q(t),\varepsilon)]\cos[\omega_\varepsilon t + q_\varepsilon(t)] \tag{7.4}$$

If initial conditions are chosen outside the manifold then the corresponding solution tends (as $t \to \infty$) toward a function of the form (7.4)

b) Weakly-forced oscillation $A \neq 0$, εA "small"

If ε is small, then (7.1) has an invariant manifold (see (5.15)) the motion on which is described by:

$$\theta' = \frac{\omega_0}{\omega} + \varepsilon g(\tau,\theta,\varepsilon) \tag{7.4}$$

where $\tau \triangleq \omega t$ and $\theta \triangleq \frac{d\theta}{d\tau}$,

$$g(\tau,\theta,\varepsilon) = \frac{1}{\omega} f\{[\rho_0 + h(\tau,\theta,\varepsilon,\varepsilon A)] \cos\theta\}\sin\theta\cos\theta + \frac{A}{\rho_0+h(\tau,\theta,\varepsilon,A)} p(\tau)\cos\theta$$

and $h(\tau,\theta,\varepsilon,A)$ is given by (5.15).

It follows from Section 6.2 that the solution of (7.4) is of the form (6.9) i.e., it is either periodic with a frequency $\frac{M}{N}\omega$ (commensurate with the forcing frequency), or almost periodic with basic frequencies ω and $\mu\omega$ where μ is an irrational rotation number. We shall now apply the results of Section 6.3 to check which solutions are stable.

As in the previous section we introduce

$$\varepsilon\Delta \triangleq \frac{\omega_0}{\omega} - \frac{M}{N} \quad \text{and} \quad \phi(\tau) \triangleq \theta(\tau) - \frac{M}{N}\tau .$$

In the new notation (7.4) takes the form:

$$\phi' = \varepsilon\Delta - \varepsilon g(\tau, \frac{M}{N}\tau + \phi,\varepsilon) \tag{7.6}$$

Consider now the solutions of the averaged equation (7.6). Note first that

$$g(\tau,\theta,0) = \frac{1}{\omega} f[\rho_0\cos\theta]\sin\theta\cos\theta - \frac{A}{\rho_0} p(\tau)\cos\theta$$

and that the average of the first term is zero, while the average of the second term is nonzero only if $N = 1$ (in both cases $\theta = \frac{M}{N}\tau+\phi$ and the averaging is with respect to time τ).

Thus the constant solutions of the averaged equation (7.6) can be found from

$$\Delta = \frac{A}{\rho_0} P\cos(\phi-\alpha) \tag{7.7}$$

where P and α denotes amplitude and phase of the M-th harmonic of $p(\tau)$, i.e., $p(\tau) = \sum_m p_m e^{jm\tau}$ and $p_M = P\,e^{j\alpha}$. Thus as long as A is "large enough" $(AP>\Delta\rho_0)$ equation (7.7) has in the $[0,2\pi)$ interval a pair of solutions, one of which approximates $2\pi M$-periodic and stable solution of (7.6).[†] The second solution of (7.7) approximates the unstable solution of (7.6). Thus if $N = 1$, $p_M = Pe^{j\alpha} \neq 0$ and $AP > \rho_0\Delta$ then the equation on the manifold (7.4) has a stable solution of the form:

$$\theta(\tau) = M\tau + \phi(\tau) \tag{7.8}$$

[†]Existence of which is granted by the theorem on averaging (Section 5.1).

where $\phi(\tau)$ is $M2\pi$-periodic.

Thus under the above assumptions, there exists a pair of $M\frac{2\pi}{\omega}$ -periodic (in real time t) solutions of (7.1) one of which is stable and the other is unstable. Both solutions are of the form

$$v(t) = \rho_0 + h[t, \theta(\omega t), \varepsilon, A]\cos\ \theta(\omega t) \tag{7.9}$$

where $\theta(\omega t)$ is given by (7.8).

All other solutions originating either on the manifold or outside of it tend toward (7.9) (the stable one) as shown in Fig. 6.4(b).

Note that the inequality $AP > \rho_0|\Delta|$ has a simple physical interpretation; namely εAP is the amplitude of the M-th harmonic of the forcing term, where $\varepsilon\Delta = \frac{\omega_0 - M}{\omega}$ provides a measure of the amount of detuning between the M-th harmonic and the self-frequency. The inequality $AP > \rho_0|\Delta|$ says that, as long as theory from the previous section is applicable, the amount of detuning must be "small" and that it can increase with the amplitude of the forcing term. The dashed zones in Fig. 7.1 shows zones in the (ω,A)-parameter plane where the theorem on averaging is valid and where synchronization holds.

Let us summarize our results:

a) If the forcing term is absent (A=0) then the integral manifold is a cylinder and all trajectories on it are of the same form (Fig. 6.2(b)).

b) If the system is forced then the integral manifold is periodic in time. There may appear stable synchronized oscillations on it. In particular, it is the case when $\omega_0 - M\omega$ is "small" and $p_M \neq 0$ (Fig. 6.4(b)).

Example 7.2. Oscillator weakly coupled to the dissipative circuit

Consider the circuit in Fig. 5.5, discussed earlier in Section 5.4. Equation (5.17) has an integral manifold. Moreover, the equation on the manifold (5.19(a)) with x and ρ replaced by $x = g(t,\theta,\varepsilon)$ and $\rho = \rho_0 + h(t,\theta,\varepsilon))$ is similar in form to (7.4), and the same results, as in the previous example, hold: if the forcing term is absent then the integral manifold is cylindrical and all solutions on it differ only by a phase-shift (Fig. 6.2(b)).

If the forcing term is present (and it has only one harmonic in the case considered) then the integral manifold is periodic in time. Moreover, if $\omega_0 - \omega$ is "small," then tere exists a (unique) synchronized solution (Fig. 6.4(b)).

Example 7.3. Colpitts-type oscillator

The autonomous system in Fig. 5.6 has a cylindrical manifold. The dynamics on the manifold is described by an equation of the form (7.2) and is shown in Figs. 6.2(a) and (b). All solutions which originate outside of the manifold S tend to S as shown in Fig. 6.4(a).

Example 7.4. Josephson-junction circuits

Consider (5.38):

$$\beta\ddot{x} + \dot{x} + s(x) = \alpha - \varepsilon p(\omega t) \tag{7.10}$$

Equation (5.30) which describes the circuit shown in Fig. 5.8 is a particular case of (5.38).

As before we can consider the autonomous (ε=0) and forced ($\varepsilon \neq 0$) circuits and the theory presented in Sections 6.1 and 6.2 is immediately applicable.

If $\varepsilon = 0$, then the equation on the manifold is of the form

$$\dot{\theta} = \psi(\theta) \tag{7.11}$$

It can be shown [14] that $\psi(\theta)$ is positive thus $\theta(t) = \mu t + q(t)$, moreover μ can be shown [9] to be a strictly increasing function of α.as shown in Fig. 7.2(a).

If $\varepsilon \neq 0$, $\theta(t)$ is described by (6.2) and $\theta(t) = \mu\omega t + s(t,\mu\omega t+\theta_0) + \theta_0$. Here, μ also increases with α but now (at some rational value) it can remain constant as a function of α as shown in Fig. 7.2(b).

The relationship between μ and α predicted above was first observed experimentally.† The above analysis provides a rigorous explanation of this exotic phenomenon.

ACKNOWLEDGEMENT

One of the authors (M.O) gratefully acknowledges his debt to Professor J. Kudrewicz who introduced him into the theory of integral manifolds.

†The rotation number is proportional to the average voltage across the junction, and this voltage was experimentally measured.

APPENDIX

A. Floquet transformation

Consider

$$\dot{\theta} = \omega_0 + g(t,\theta,\rho,\varepsilon)$$
$$\dot{\rho} = A(\theta)\rho + f(t,\theta,\rho,\varepsilon) \tag{A1}$$

where $A(\theta+T_0) = A(\theta)$.

Let the matrix $\phi(\theta,\theta_0)$ be the fundamental solution of

$$\frac{dx}{d\theta} = A(\theta)x \tag{A2}$$

Since $A(\theta)$ is periodic and $\phi(\theta+T_0,\theta_0)$ is a matrix solution of (A2), there exists a nonsingular constant matrix C such that

$$\phi(\theta+T_0,\theta_0) = C\phi(\theta,\theta_0)$$

On the other hand, $C = \phi(\theta+T_0,\theta) = \phi(T_0,0)$. Since C is nonsingular, there exists a matrix B such that $e^{BT_0} = \phi(T_0,0)$. Let us define

$$P(\theta) \triangleq \phi(\theta,0)e^{-B\theta} \tag{A3}$$

Note that $P(\theta+T_0) = P(\theta)$ and that the transformation $x \triangleq P(\theta)z$ when applied to (A2) yields

$$\frac{dz}{d\theta} = Bz \ .$$

We shall apply now a similar transformation to equations (A1):

$$\rho \triangleq \frac{1}{\omega_0} P(\theta)r \tag{A4}$$

Thus

$$\dot{\rho} = \frac{1}{\omega_0}\frac{d}{d\theta}P(\theta)\cdot\dot{\theta}\cdot r + \frac{1}{\Omega}P(\theta)\dot{r}$$

$$= \frac{1}{\omega_0}\left[\frac{d\phi}{d\theta}e^{-B\theta} - \phi\cdot Be^{-B\theta}\right][\omega_0 + g(t,\theta,\rho,\varepsilon)]\, r + \frac{1}{\omega_0}P(\theta)\dot{r}$$

and from (A1)

$$\frac{1}{\omega_0} P(\theta)\dot{r} = A(\theta)\cdot P(\theta)\cdot r + f(t,\theta,\rho,\varepsilon)$$

$$- \frac{1}{\omega_0} [A(\theta)\phi e^{-B\theta} - \phi B e^{-B\theta}] [\omega_0 + g(t,\theta,\rho,\varepsilon)]\cdot r$$

It follows that

$$\dot{r} = Br + F(t,\theta,r,\varepsilon) \tag{A5}$$

where

$$F(t,\theta,r,\varepsilon) \triangleq \omega_0 P^{-1}(\theta)\ f(t,\theta,\ \frac{1}{\omega_0} P(\theta)r,\varepsilon)$$

$$- P^{-1}(\theta)[A(\theta)\phi e^{-B\theta} - \phi B e^{-B\theta}]\ g(t,\theta,\ \frac{1}{\omega_0} P(\theta)r,\varepsilon)\cdot r$$

$$= \omega_0 P^{-1}(\theta)\ f(t,\theta,\ \frac{1}{\omega_0} P(\theta)r,\varepsilon)$$

$$- [P^{-1}(\theta)A(\theta)P(\theta) - B]\ g(T,\theta,\ \frac{1}{\omega_0} P(\theta)r,\varepsilon)r\ ,$$

thus (A1) can be reduced to:

$$\dot{\theta} = \omega_0 + g(t,\theta,r,\varepsilon)$$

$$\dot{r} = Br + F(t,\theta,r,\varepsilon)$$

with the constant matrix B.

B. Reduction of (3.2) to (3.3) and (3.4)

Consider:

$$[u^1(\theta) + P^1(\theta)\rho]\dot{\theta} + P(\theta)\dot{\rho} = f_0[u(\theta) + P(\theta)\rho] + \varepsilon f_1(t,u(\theta)+P(\theta)\rho,\varepsilon) \tag{B1}$$

To obtain an equation on $\dot{\theta}$ let us project both sides of (B1) onto $v(\theta)$:

$$[v^T(\theta)\ u^1(\theta) + v^T(\theta)P^1(\theta)\rho]\dot{\theta} + v^T(\theta)P(\theta)\dot{\rho} = v^T(\theta)f_0[u(\theta) + P(\theta)\rho]$$

$$+ \varepsilon v^T(\theta)f_1(t,u(\theta) + P(\theta)\rho,\varepsilon).$$

Note that by definition $v(\theta) = u^1(\theta)/|u^1(\theta)|$, $P(\theta) = [\xi_2 \vdots \cdots \vdots \xi_n]$

$$v^T(\theta)u^1(\theta) = |u^1(\theta)|$$

and

$$v^T(\theta)\ P(\theta) = 0.$$

Define $a(\theta,\rho) \triangleq [|u'(\theta)| + v^T(\theta)P'(\theta)\rho]$

so

$$\dot{\theta} = \frac{v^T(\theta)}{a(\theta,\rho)} f_0[u(\theta) + P(\theta)\rho] + \varepsilon \frac{v^T(\theta)}{a(\theta,\rho)} f_1(t,u+P\rho,\varepsilon) \tag{B2}$$

Since

$$u'(\theta) = \frac{du}{d\theta} = \frac{du}{dt}\;\frac{dt}{d\theta} = \frac{1}{\omega_0}\,u = \frac{1}{\omega_0} f_0(u)$$

and

$$v(\theta) = u'(\theta)/|u(\theta)|$$

we get

$$v^T(\theta)\; u'(\theta) = |u(\theta)|$$

and

$$|u(\theta)| = \frac{1}{\omega_0} v^T(\theta)\; f_0(u(\theta))\;.$$

Thus

$$a(\theta,0) = |u'(\theta)| = v^T(\theta)u'(\theta) = \frac{v^T(\theta)}{\omega_0} f_0(u(\theta))$$

and

$$\frac{v^T(\theta)}{a(\theta,\varepsilon)} f_0(u(\theta)+P(\theta)\rho) = \omega_0 + v^T(\theta)\left[\frac{f_0(u(\theta)+P(\theta)\rho)}{a(\theta,\rho)} - \frac{f_0(u(\theta))}{a(\theta,0)}\right]$$

Hence, equation (B2) assumes the form:

$$\dot{\theta} = \omega_0 + g_1(\theta,\rho) + \varepsilon g_2(t,\theta,\rho,\varepsilon) \tag{B3}$$

with

$$g_1(\theta,\rho) \triangleq v^T(\theta)\left[\frac{f_0(u(\theta)+P(\theta)\rho)}{a(\theta,\rho)} - \frac{f_0(u(\theta))}{a(\theta,0)}\right] = O(\|\rho\|)$$

and

$$g_2(t,\theta,\rho,\varepsilon) \qquad \frac{v^T(\theta)}{a(\theta,\rho)} f_1(t,u(\theta)+P(\theta)\rho,\varepsilon)$$

In a similar way we can obtain an equation on ρ: we project both sides of (B1) onto $P(\theta)$:

$$P^T(\theta)\; u'(\theta)\dot{\theta} + P^T(\theta)\; P'(\theta)\rho\dot{\theta} + P^T(\theta)\; P(\theta)\dot{\rho} = P^T(\theta)\; f_0(u(\theta)+P(\theta)\rho) + \varepsilon P^T(\theta)\; f_1(t,u(\theta)+P(\theta)\rho,\varepsilon) \tag{B4}$$

Since $\xi_2(\theta), \ldots \xi_n(\theta)$ are mutually orthonormal and orthogonal to $u'(\theta) = \frac{1}{\omega_0} f_0(u(\theta))$, we get $P^T(\theta)\; P(\theta) = I$, $P^T(\theta)u'(\theta) = 0$, $P^T(\theta)\; f_0(u(\theta)) = 0$. Hence, equation (B4) reduces to

$$\dot{\rho} = -P^T(\theta)\, P'(\theta)\rho\dot{\theta} + P^T(\theta)[f_0(u(\theta)+P(\theta)\rho) - f_0(u(\theta))] + \varepsilon P^T(\theta)\, f_1(t,u(\theta)+P(\theta)\rho,\varepsilon)$$

Replacing $\dot{\theta}$ by the r.h.s. of equation (B3), we get

$$\dot{\rho} = A(\theta)\rho + R_1(\theta,\rho) + \varepsilon R_2(t,\theta,\varepsilon) \tag{B5}$$

where

$$A(\theta) \triangleq P^T(\theta)[\tfrac{d}{dx} f_0(u(\theta))\, P(\theta) - \omega_0 P'(\theta)]$$

$$R_1(\theta,\rho) \triangleq P^T(\theta)[f_0(u(\theta)+P(\theta)\rho) - f_0(u(\theta)) - \tfrac{d}{dx} f_0(u(\theta))\, P(\theta)$$

$$- P'(\theta)\cdot\rho\cdot g_1(\theta,\rho)] = O(\|\rho\|^2)^{\dagger}$$

$$R_2(t,\theta,\varepsilon) \triangleq P^T(\theta)[f_1(t,u(\theta)+P(\theta)\rho,\varepsilon) + P'(\theta)\cdot\rho\cdot g_2(t,\theta,\rho,\varepsilon)]$$

$$= P^T(\theta)\left[I - P'(\theta)\rho\,\frac{v^T(\theta)}{a(\theta,\rho)}\right] f_1(t,u(\theta)+P(\theta)\rho,\varepsilon)$$

C. <u>"Running Periodic" Solutions</u>

Consider

$$\dot{\theta} = f(\theta) \tag{C1}$$

where $f(\theta)$ is a positive, continuous and 2π-periodic function. Let $\theta(t;\theta_0)$ be a solution of (C1) satisfying the initial condition $\theta(t_0;\theta_0) = \theta_0$.

Observe that $\theta(t;\theta_0)$ is strictly increasing (since $\dot{\theta} \geq \inf_{\theta} f(\theta)>0$) and there exists a finite time T such that $\theta(t_0+T;\theta_0) = \theta_0 + 2\pi$. Moreover T does not depend on θ_0 or t_0. Indeed $T = \int_{t_0}^{T+t_0} dt = \int_{t_0}^{T+t_0} \frac{\dot{\theta}\,dt}{f(\theta)} = \int_{\theta_0}^{2\pi+\theta_0} \frac{d\theta}{f(\theta)} = \int_0^{2\pi} \frac{d\theta}{f(\theta)}$.

Define now $q(t) \triangleq \theta(t,\theta_0) - \frac{2\pi}{T} t$. Clearly $q(t)$ is T periodic and $\theta(t) = \frac{2\pi}{T} t + q(t)$.

---

† $g_1(\theta,\rho) \triangleq v^T(\theta)\left[\frac{f_0(u(\theta)+P(\theta)\rho)}{a(\theta,\rho)} - \frac{f_0(u(\theta))}{a(\theta,0)}\right]$ is the same as in equation(B3).

REFERENCES

[1] B. van der Pol, "Forced oscillations in a circuit with nonlinear resistance," Phil. Mag. 3 (1927) pp. 65-80, Proc. I.R.E. 22 (1934) pp. 1051-1086.

[2] N. Krylov and N. N. Bogoliubov, "Introduction to nonlinear mechanics," Am. Math. Studies, No. 11, P.U.P., Princeton 1947.

[3] N. N. Bogoliubov and Y. A. Mitropolskii, Asymptotic Methods in the Theory of Nonlinear Oscillations, Gordon and Breach, New York, 1961.

[4] Y. A. Mitropolskii and O. B. Lykova, Integral Manifolds in Nonlinear Mechanics, Nauka, Moscow, 1973 (in Russian).

[5] J. K. Hale, Ordinary Differential Equations, R. Krieger, New York, 1980.

[6] S. P. Diliberto, "Perturbation theorems for periodic surfaces I, II. Rend. Circ. Mat. Palermo 9, (1960) pp. 265-299.

[7] R. J. Sacker, "A new approach to the perturbation theory of invariant surfaces" Comm. Pure Math. 18 (1965) pp. 717-732.

[8] J. Kudrewicz and M. Odyniec, "Slowly Modulated Oscillations in a Nonlinear Multiresonance System," Proc. of EECTD'80 European Conference on Circuit Theory and Design, Warsaw, Poland, Sept. 2-5, 1980.

[9] M. Odyniec and L. O. Chua, "Josephson junction circuit analysis via integral manifolds," IEEE Trans. on Circuits and Systems, CAS-30, May 1983.

[10] M. Odyniec and L. O. Chua, "Josephson junction circuit analysis via integral manifolds Part II" Memorandum No. UCB/ERL M82/46.

[11] E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.

[12] T. Van Duzer and C. W. Turner, Principles of Semiconductive Devices and Circuits, Elsevier, New York 1981.

[13] A. Barone and G. Paterno, Physics and Applications of the Josephson Effect J. Wiley & Sons 1982.

[14] A. A. Andronov, S. E. Khaikin, and A. A. Vitt, Theory of Oscillators, Pergamon Press, New York, 1966.

[15] M. Levi, F. C. Hoppensteadt, and W. L. Miranker, "Dynamics of the Josephson junction," Quarterly of Appl. Math., pp. 167-198, July 1978.

[16] A. A. Abidi and L. O. Chua, "On the dynamics of Josephson-junction circuits," Electronic Circuits and Systems, vol. 3, pp. 186-200, July 1979.

[17] K. K. Likharev and B. T. Ulrich, "Josephson Junction Circuits and Applications" Moscow 1978 (in Russian).

[18] J. R. Waldram, A. B. Pippard, and J. Clarke, "Theory of the Current-Voltage Characteristics of SNS Junctions and Other Superconducting Weak Links," Phil. Trans. Roy. Soc. London, vol. 268A (1970) pp. 265-287.

[19] Ya. S. Baris, V. I. Fodchuk, "Bounded solutions in nonlinear singularly perturbed systems via integral manifolds," Ukr. Math. Journal, Vol. 22, No. 1, pp. 3-11, 1970 (in Russian).

[20] M. Urabe, Nonlinear Autonomous Oscillations, Academic Press, New York, 1967.

[21] B. Montandon, "Almost periodic solutions and integral manifolds for weakly nonlinear nonconservative systems," J. Differential Equations, Vol. 12, 417-425, 1972.

[22] J. K. Hale, Oscillations in Nonlinear Systems, McGraw-Hill, New York 1963.

[23] J. K. Hale, "Integral Manifolds of Perturbed Differential Systems," Ann. Math. 73 (1961) pp. 496-531.

[24] V. A. Pliss, Nonlocal Problems of the Theory of Oscillations, Academic Press, New York, 1966.

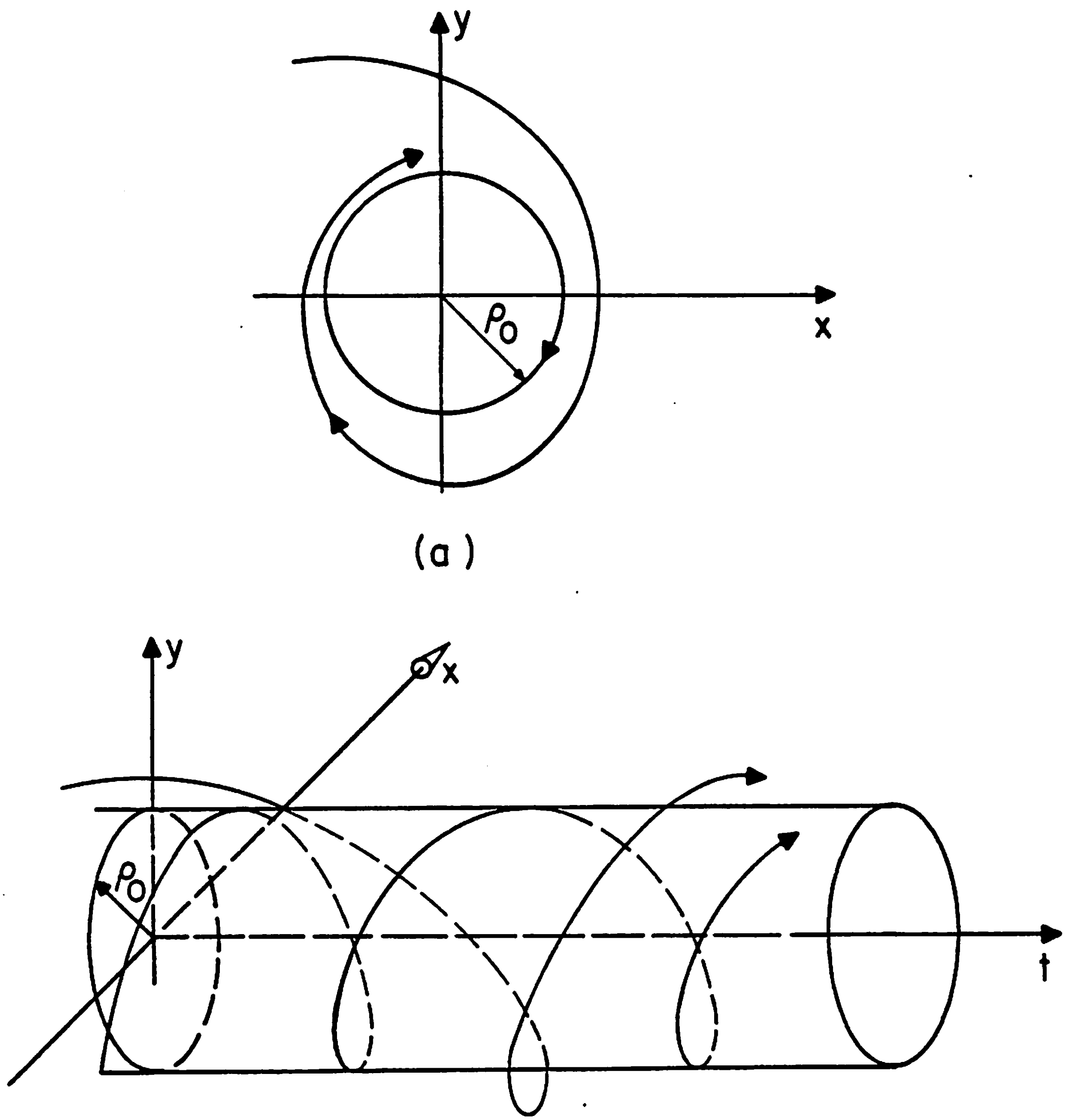
[25] B. D. Josephson, "Supercurrents through barriers," Advan. Phys. vol. 14 (1965) pp. 419-451.

[26] H. Lubbig, "Progress in physics of Josephson junctions," SQUID'80, Berlin, New York 1980.

[27] G. Sansone and R. Conti, Nonlinear Differential Equations, Pergamon Press, New York, 1964.

[28] E. A. Barbashin and V. A. Tabueva, Dynamical Systems with a Cylindrical Phase Space, Moscow 1969 (in Russian).

FIGURE CAPTIONS



(b)

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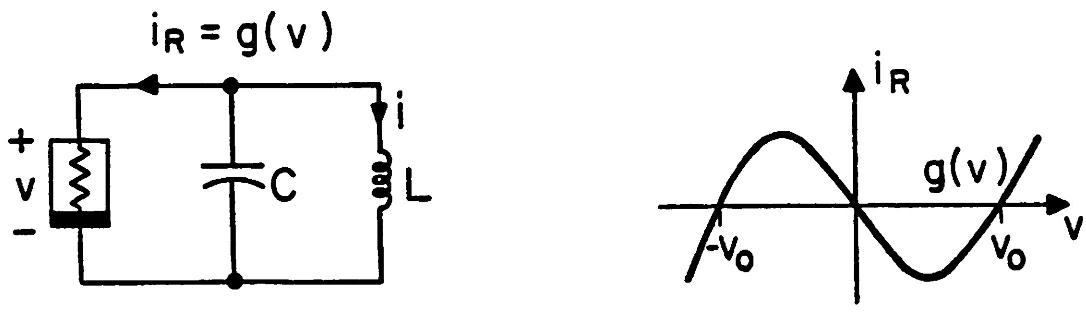

(a) (b)

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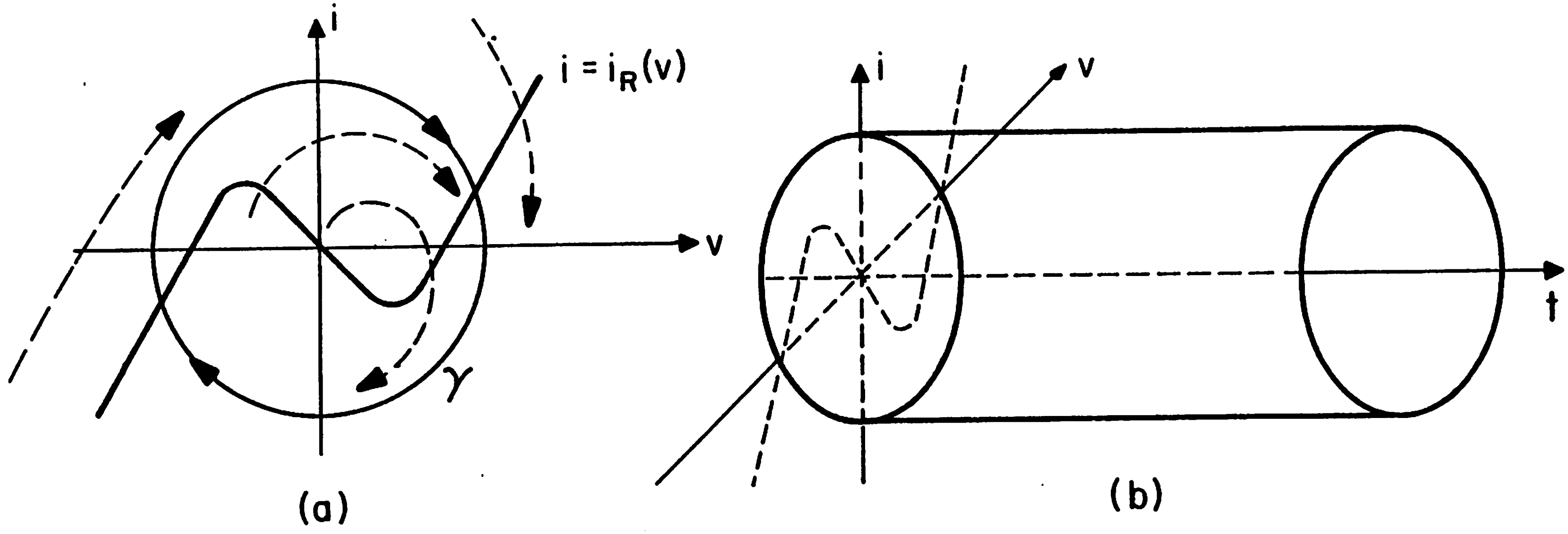

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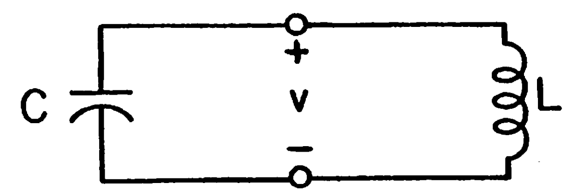

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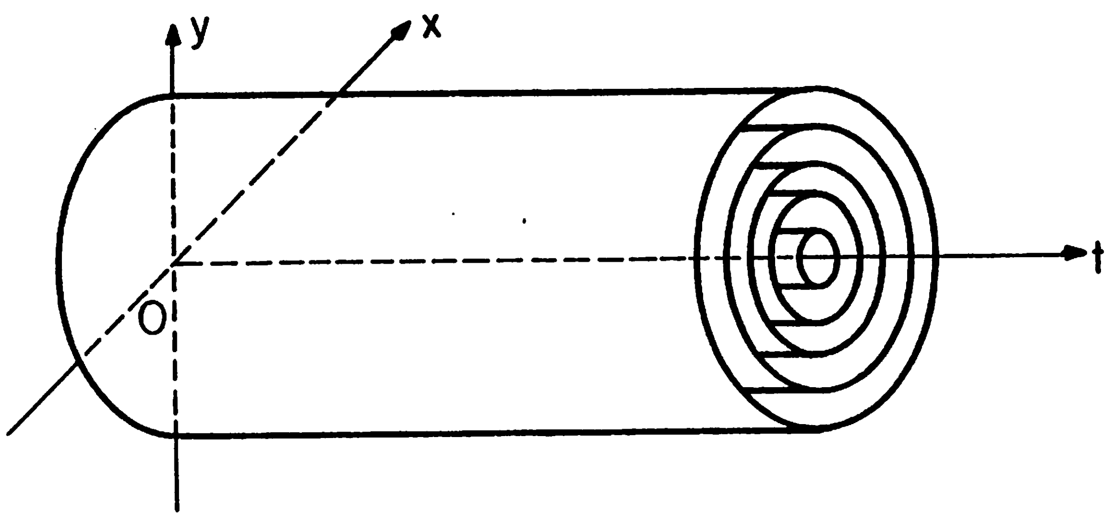

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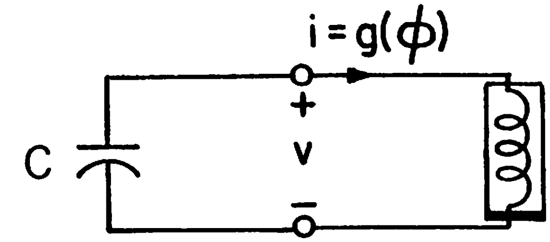

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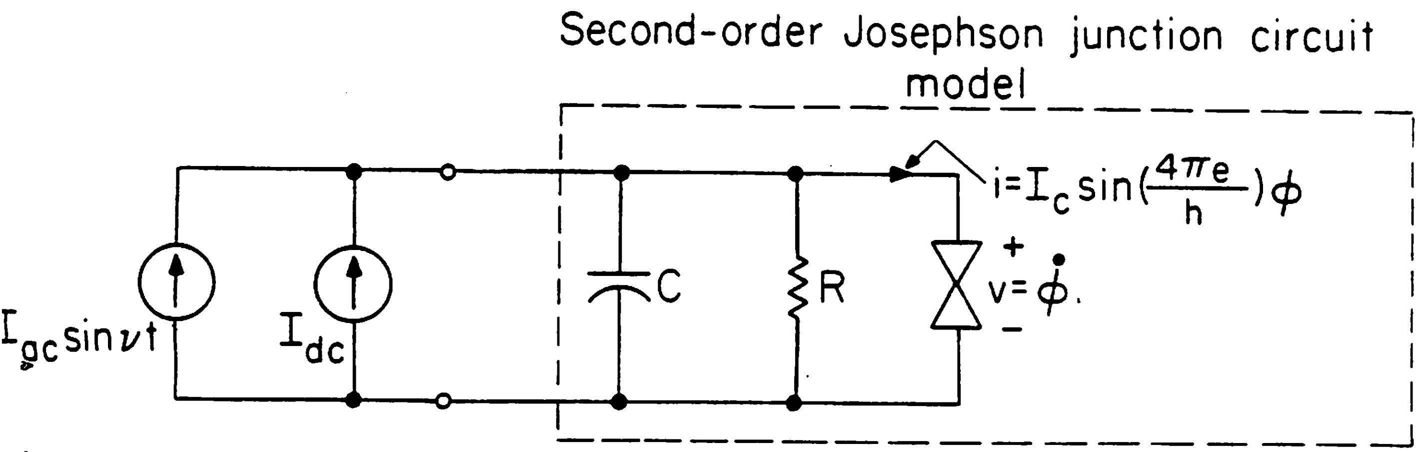

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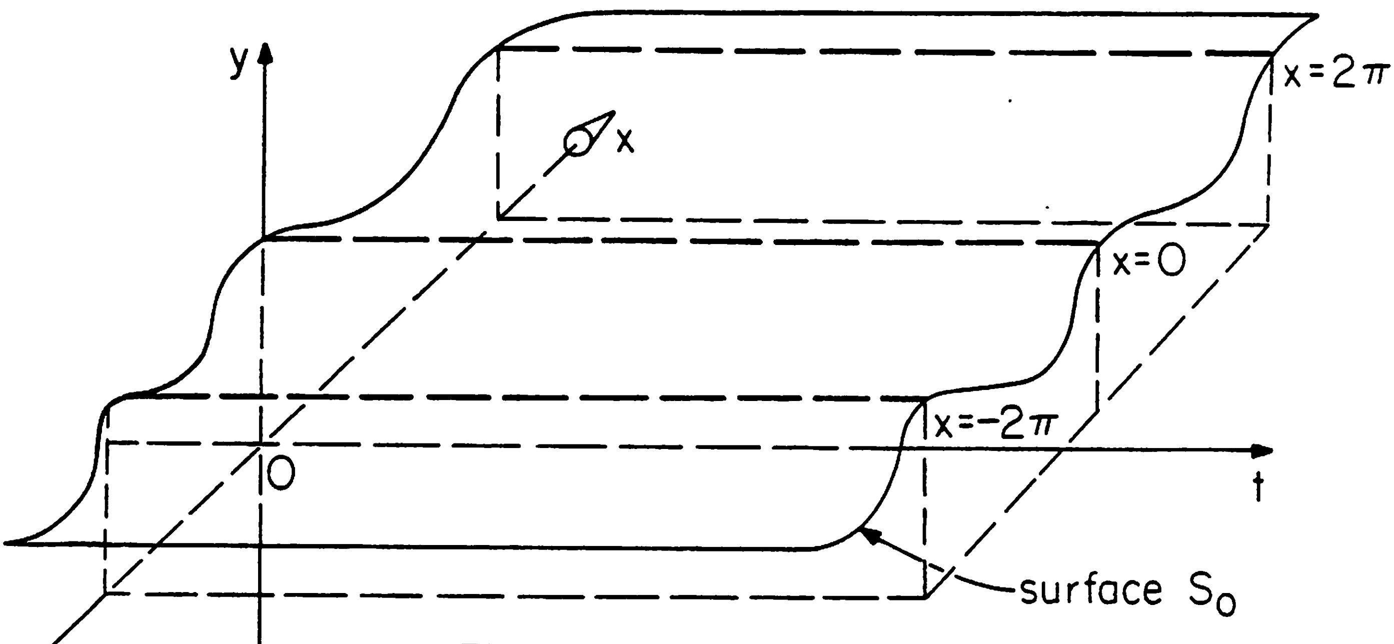

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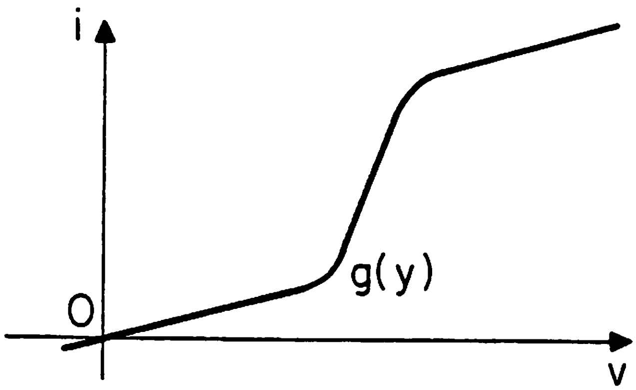

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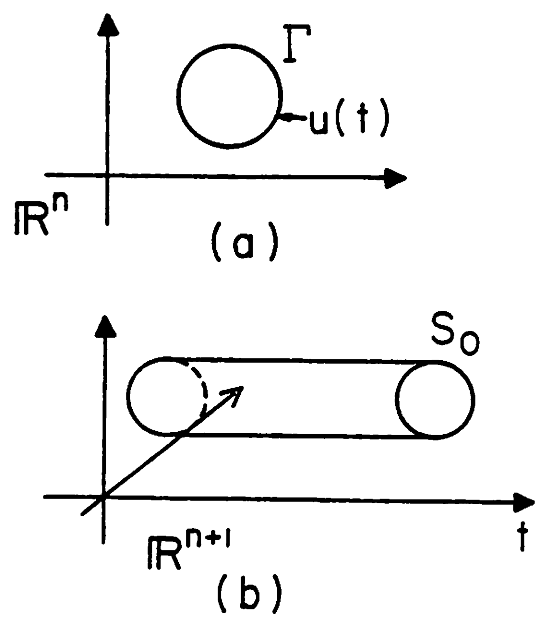
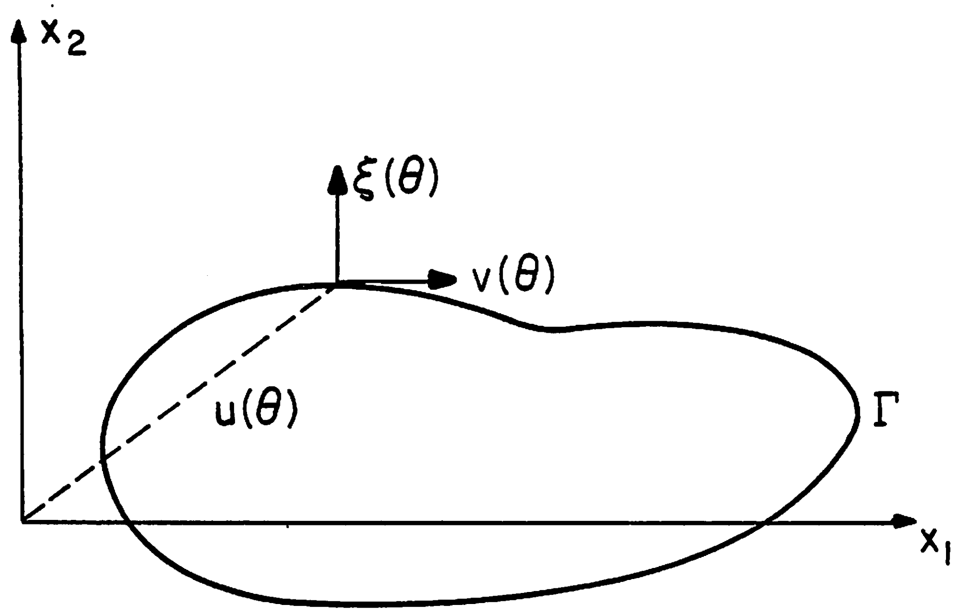
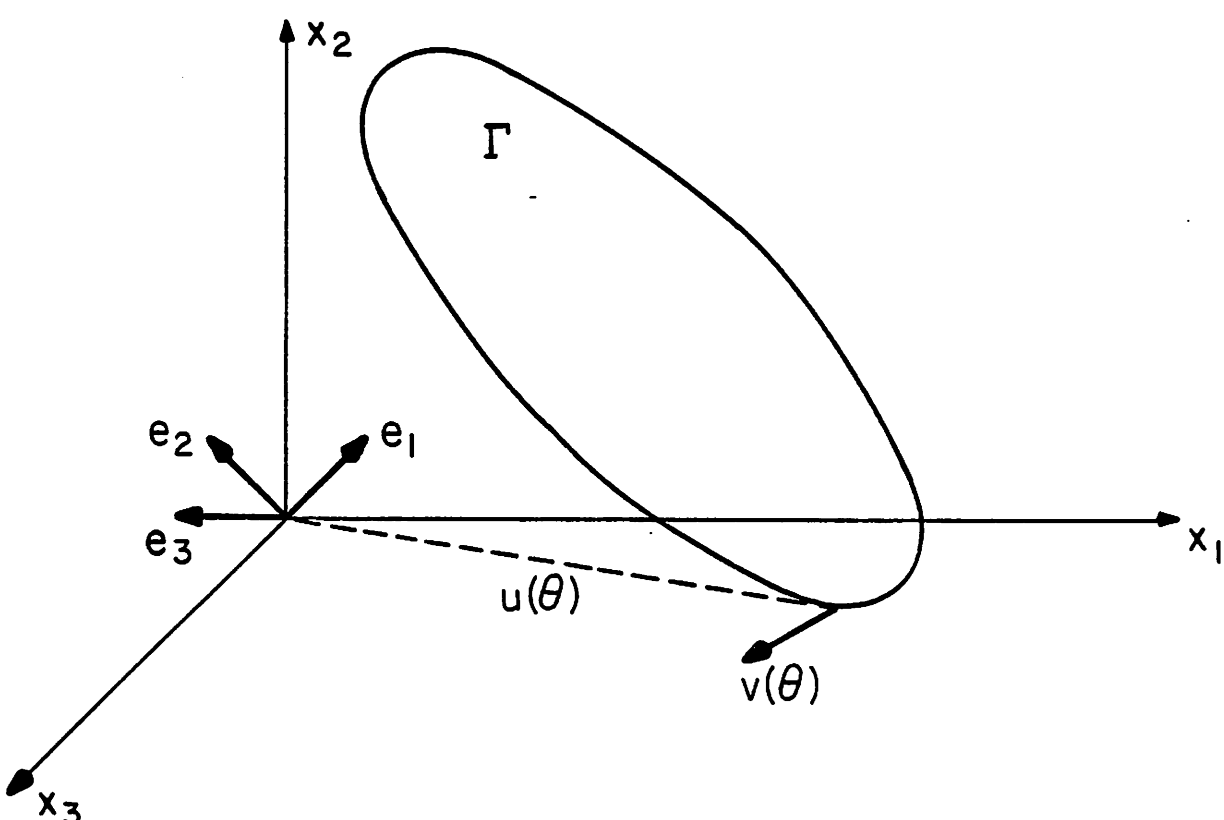


Fig. 3.1



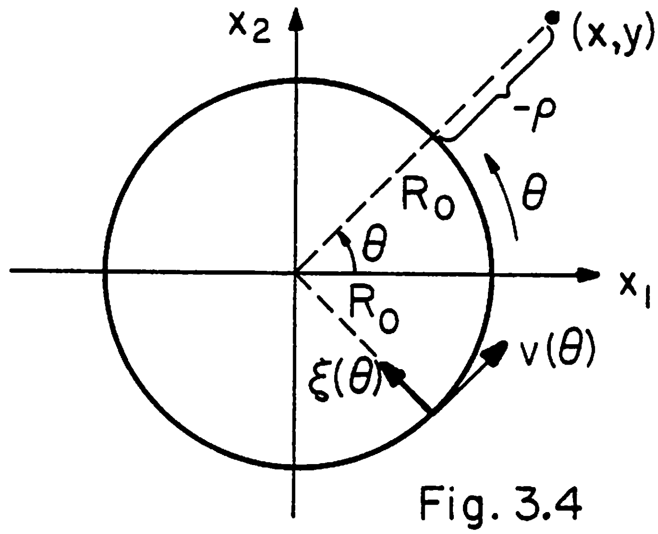
x_2
$\xi(\theta)$
$v(\theta)$
$u(\theta)$
Γ
x_1

Fig. 3.2



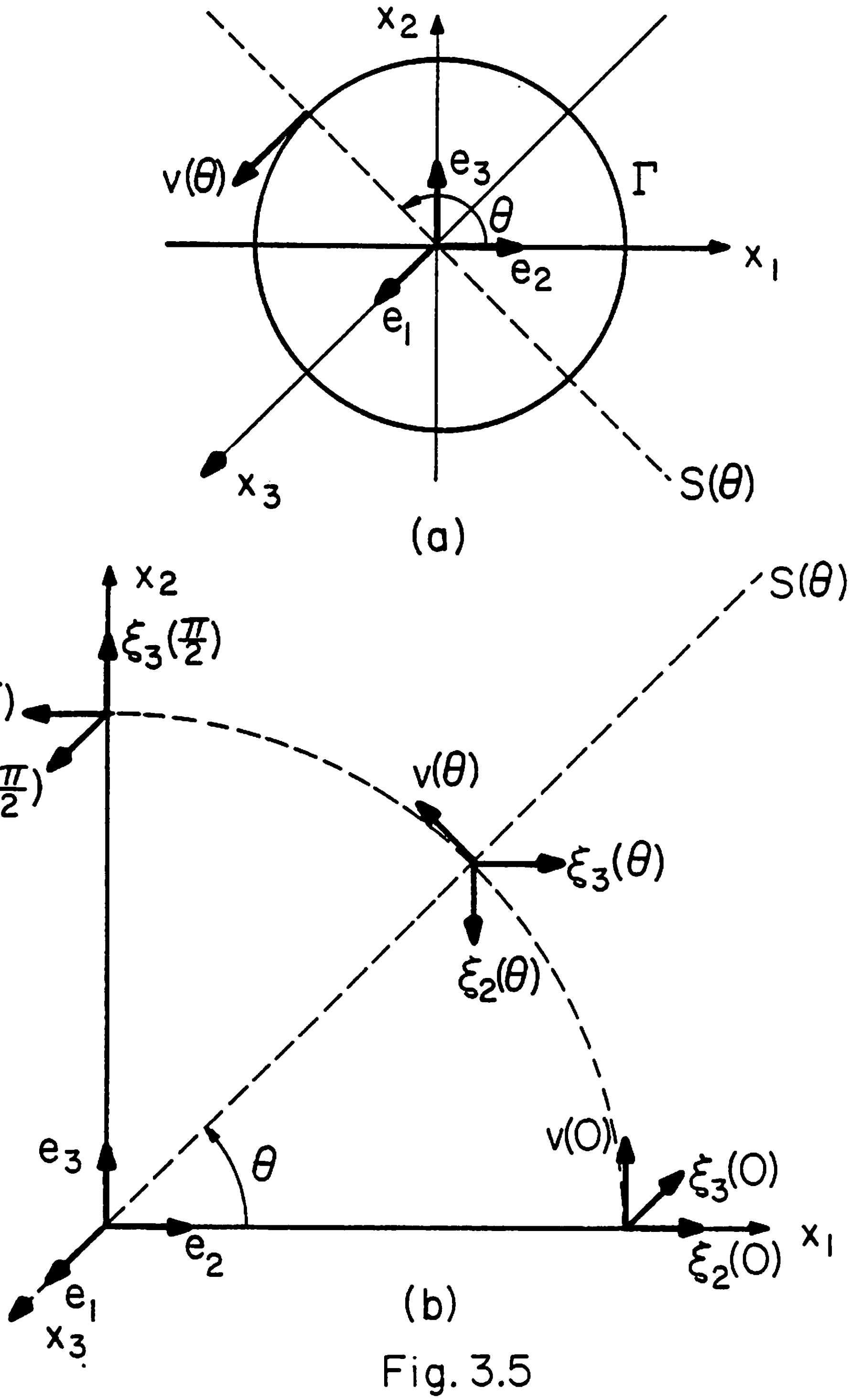
x_2
Γ
e_2
e_1
e_3
x_1
$u(\theta)$
$v(\theta)$
x_3

Fig. 3.3



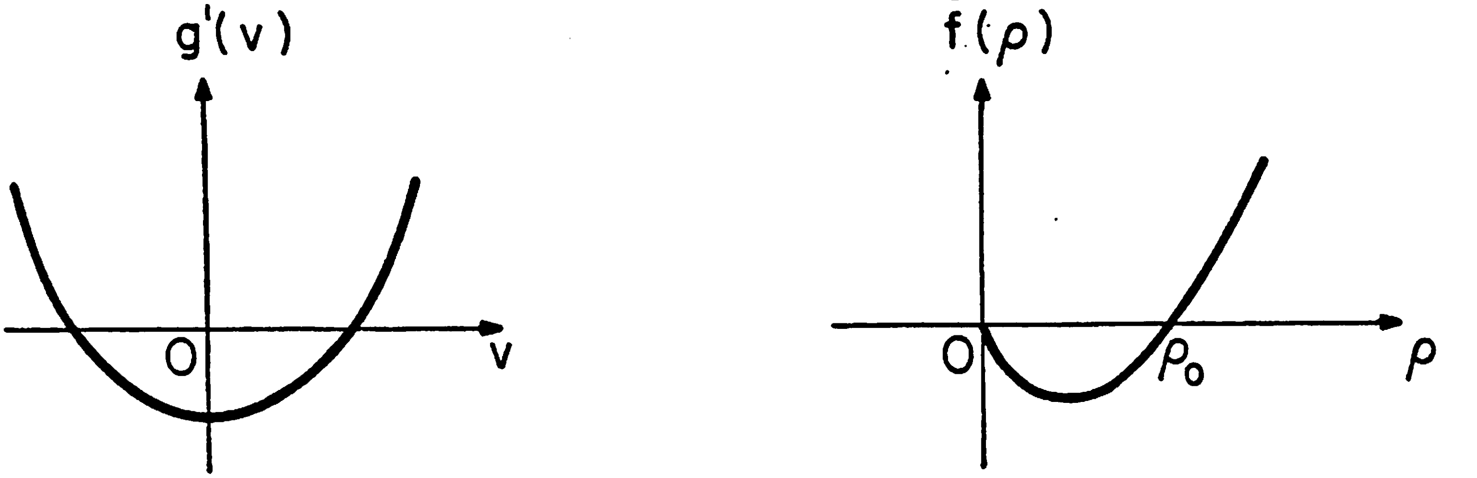
x_2
(x,y)
$-\rho$
R_0
θ
θ
x_1
R_0
$v(\theta)$
$\xi(\theta)$

Fig. 3.4



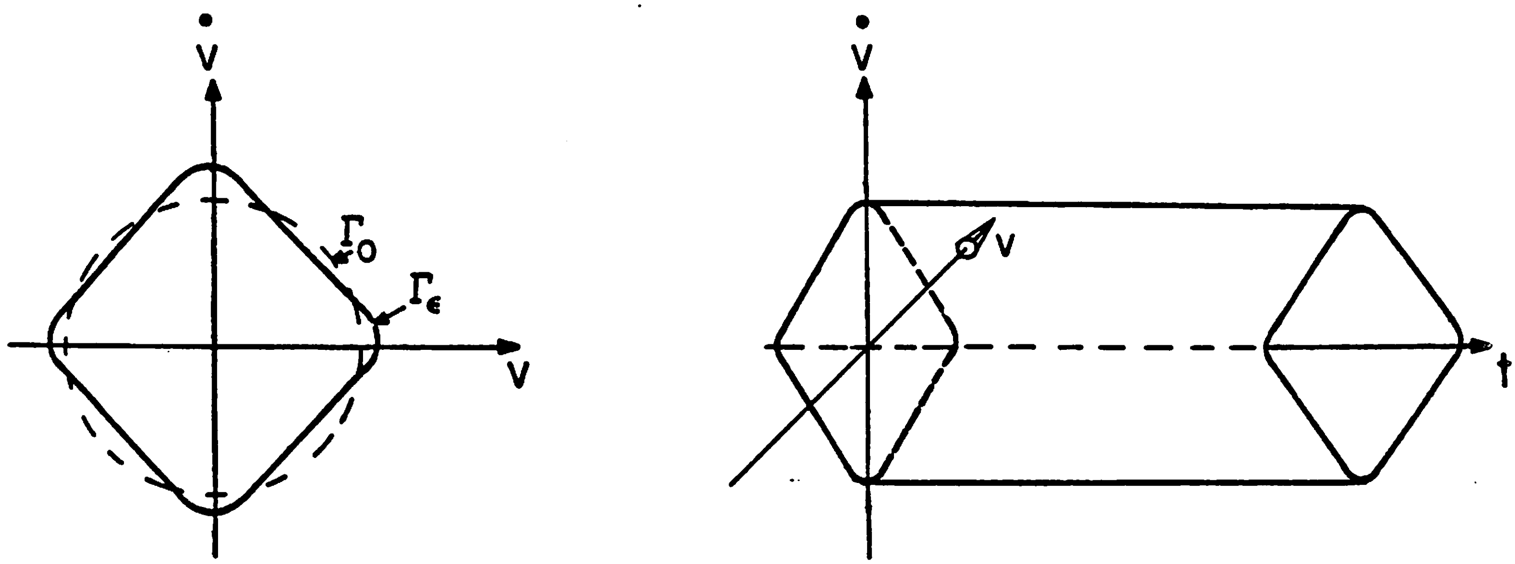
x_2
$v(\theta)$
e_3
Γ
θ
x_1
e_2
e_1
x_3
$S(\theta)$
(a)
x_2
$S(\theta)$
$\xi_3(\frac{\pi}{2})$
$v(\frac{\pi}{2})$
$\xi_2(\frac{\pi}{2})$
$v(\theta)$
$\xi_3(\theta)$
$\xi_2(\theta)$
e_3
θ
$v(0)$
$\xi_3(0)$
x_1
e_2
$\xi_2(0)$
e_1
x_3
(b)

Fig. 3.5



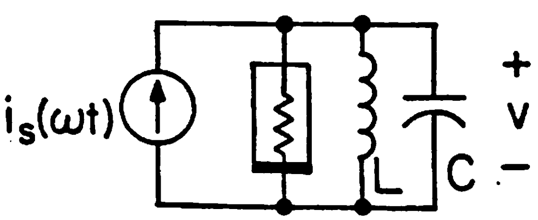
(a) (b)

Fig. 5.1



(a) (b)

Fig. 5.2



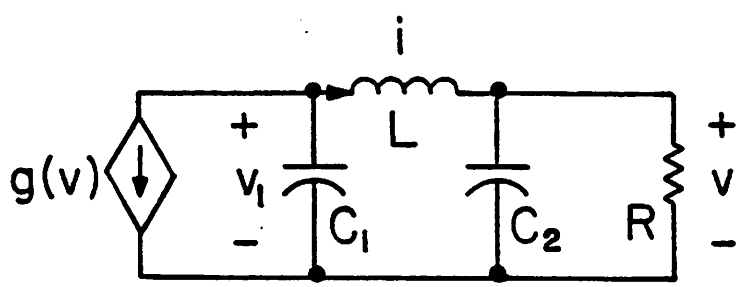
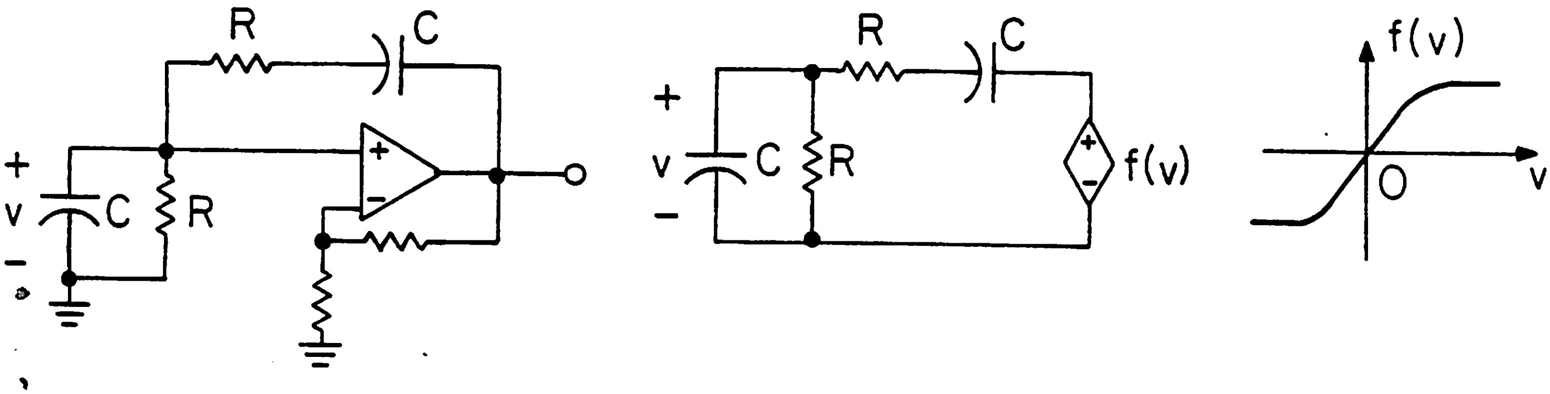


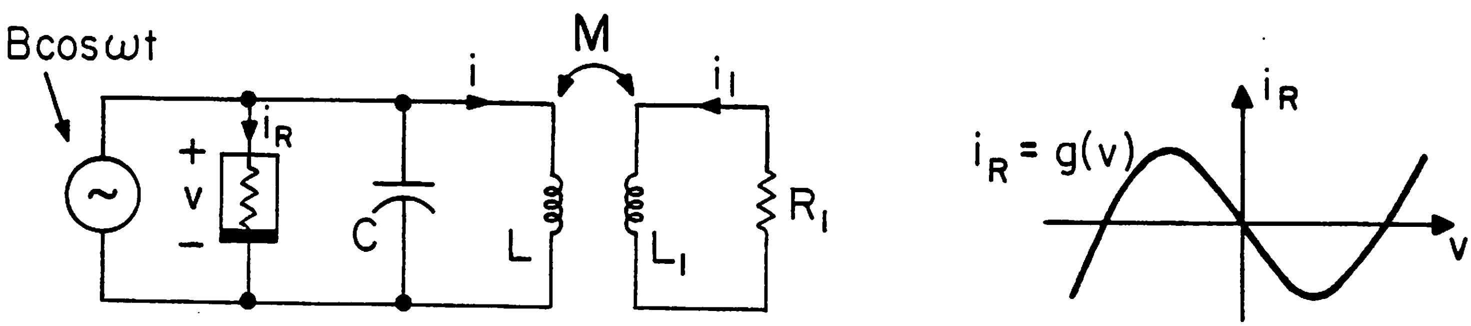
Fig. 5.3

Fig. 5.6



(a) (b)

Fig. 5.4



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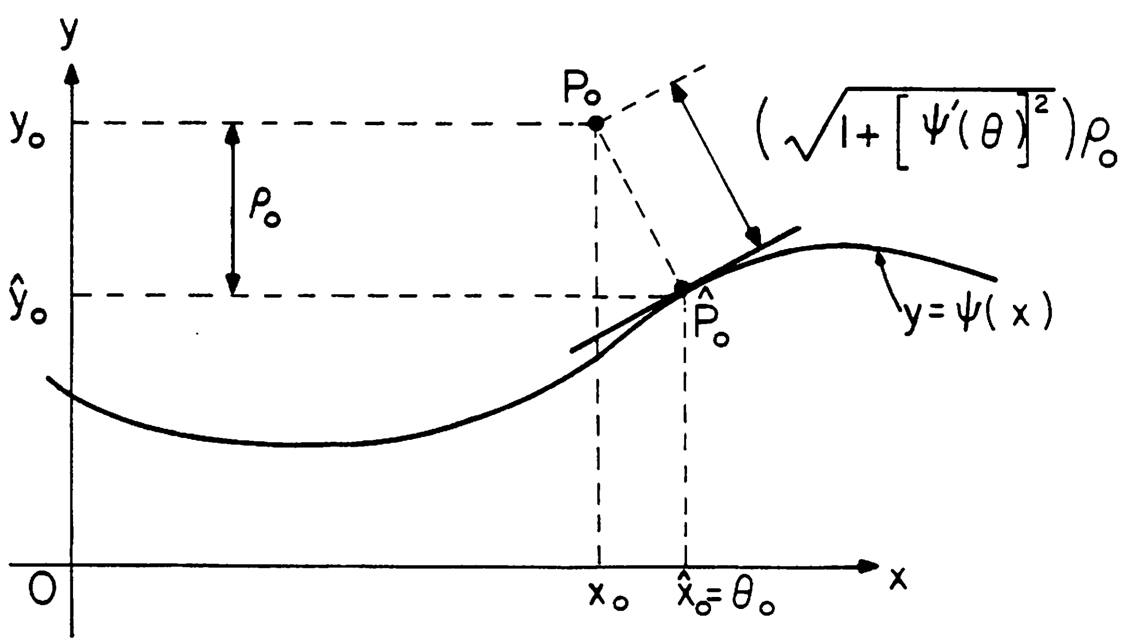

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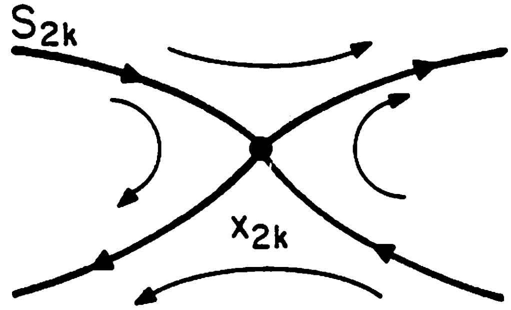

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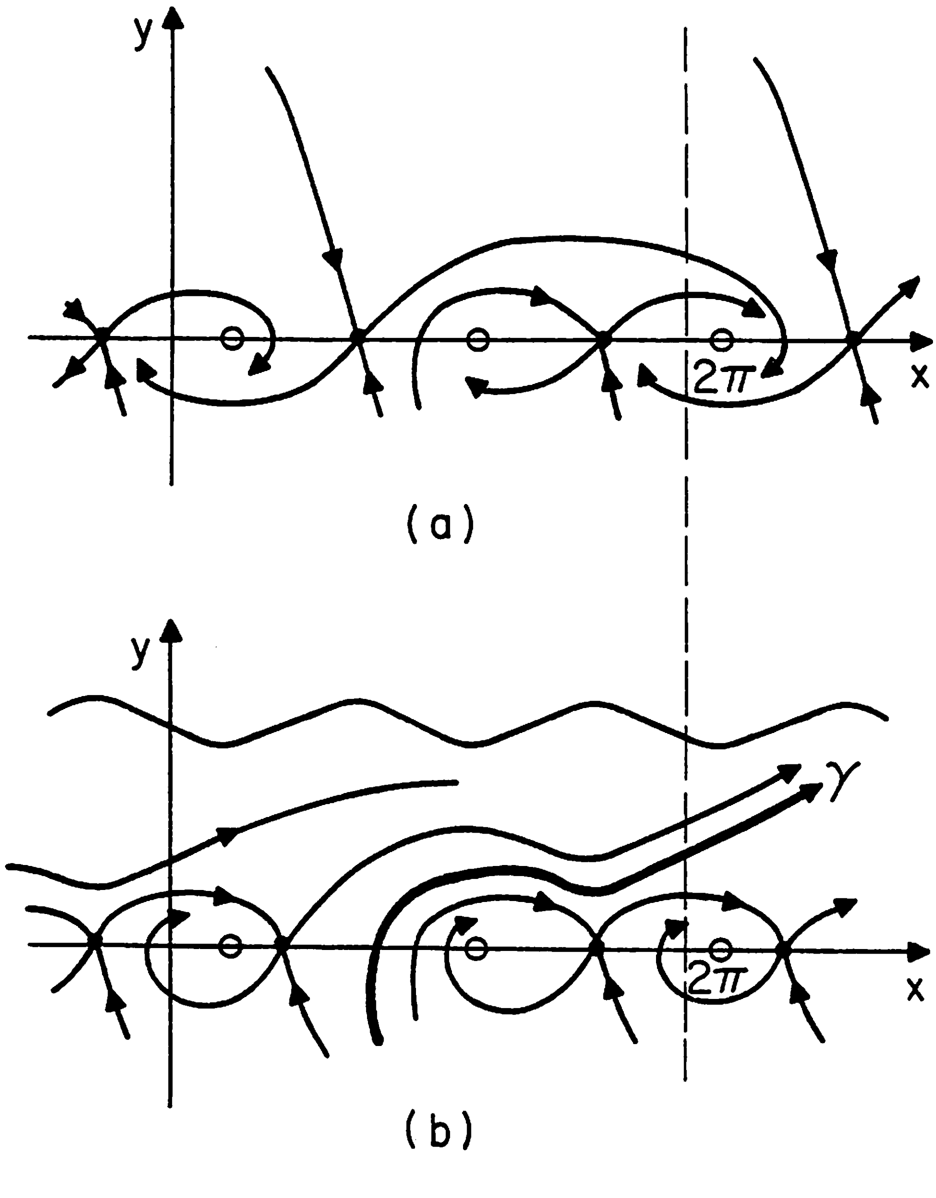

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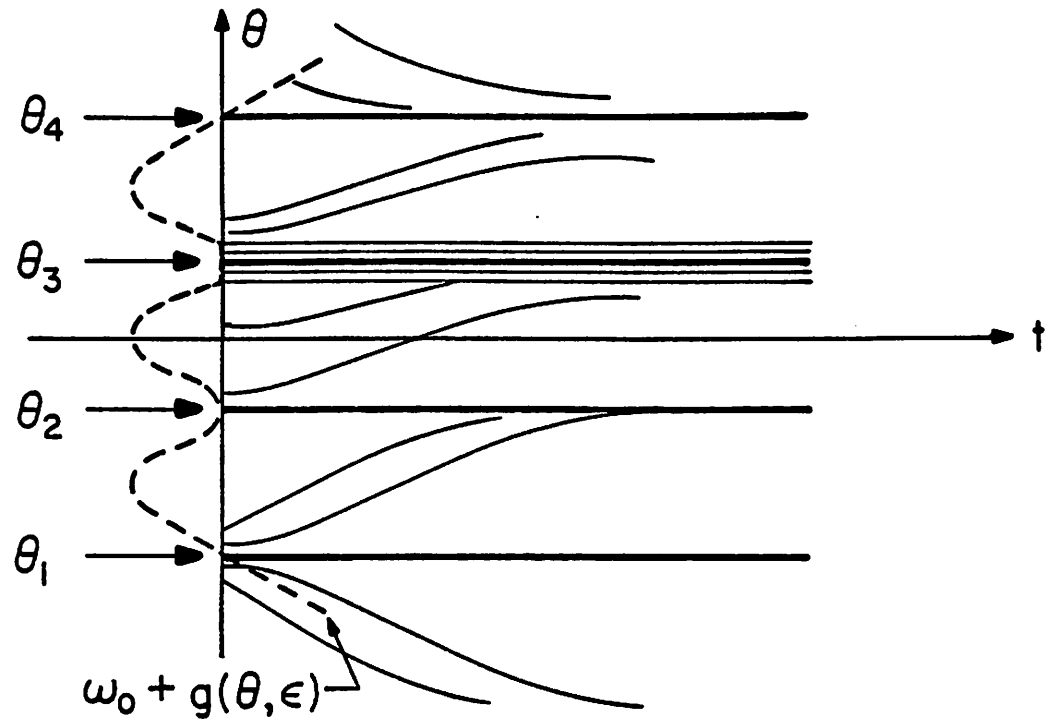
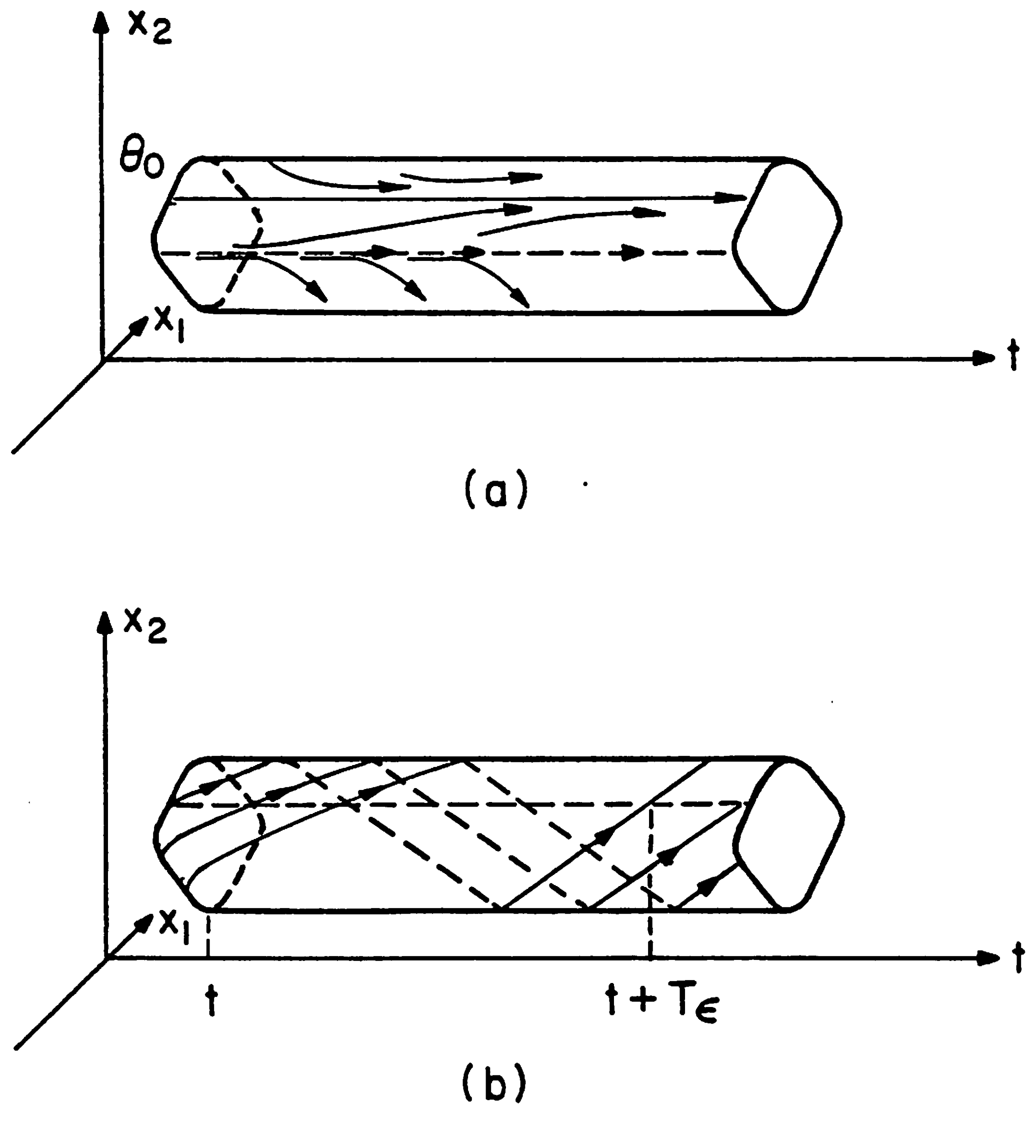


θ
θ_4
θ_3
θ_2
θ_1
t
$\omega_0 + g(\theta,\epsilon)$

Fig. 6.1



x_2
θ_0
x_1
t
(a)
x_2
x_1
t
t
$t+T_\epsilon$
(b)

Fig. 6.2

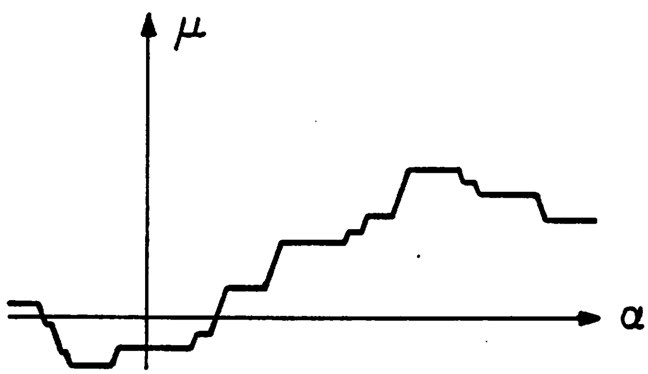
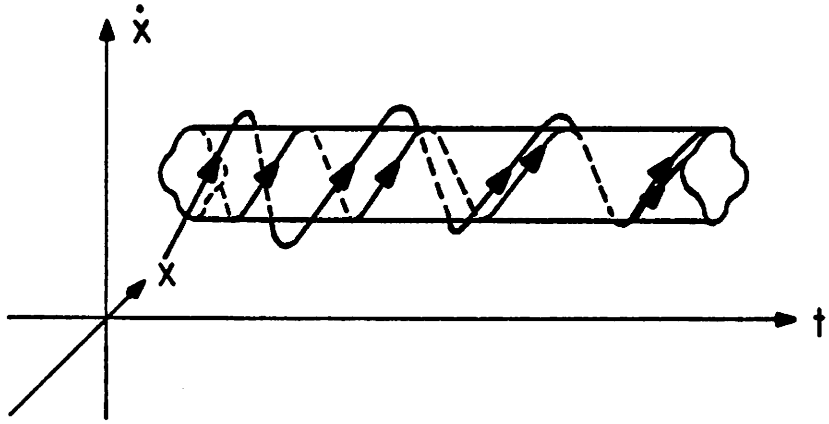
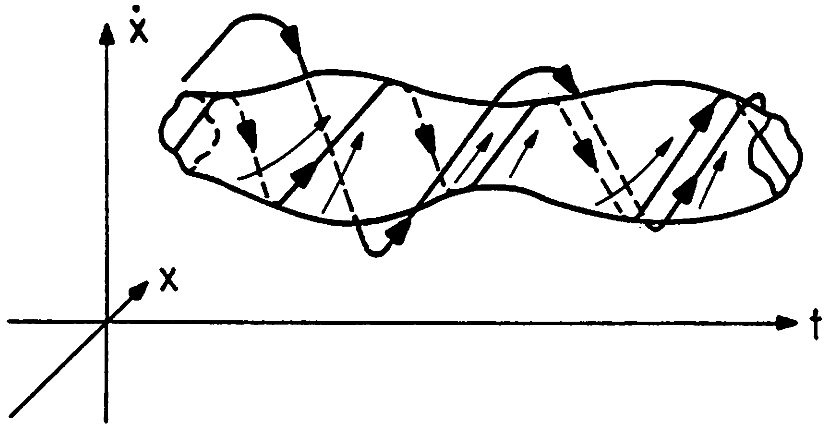


Fig. 6.3



(a)



(b)

Fig. 6.4

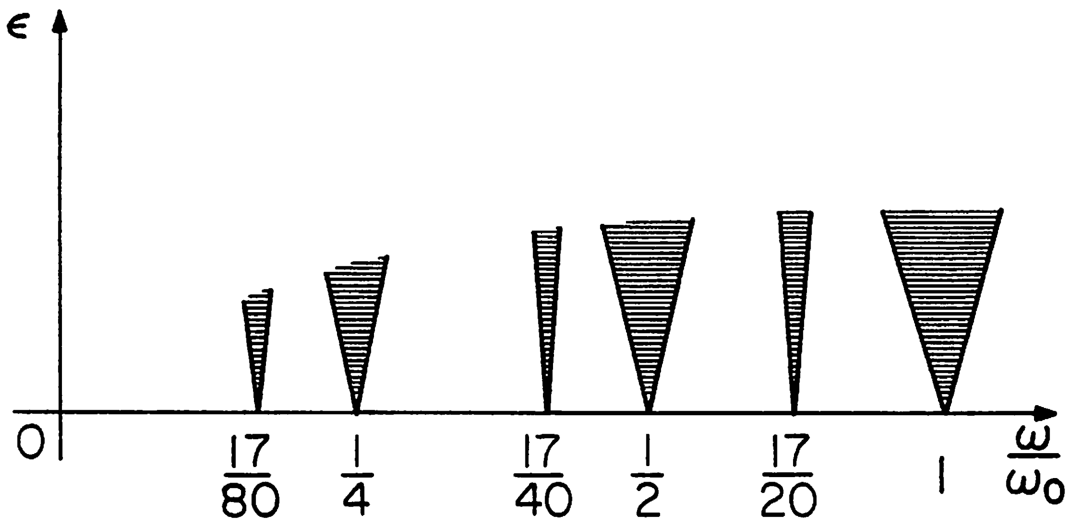
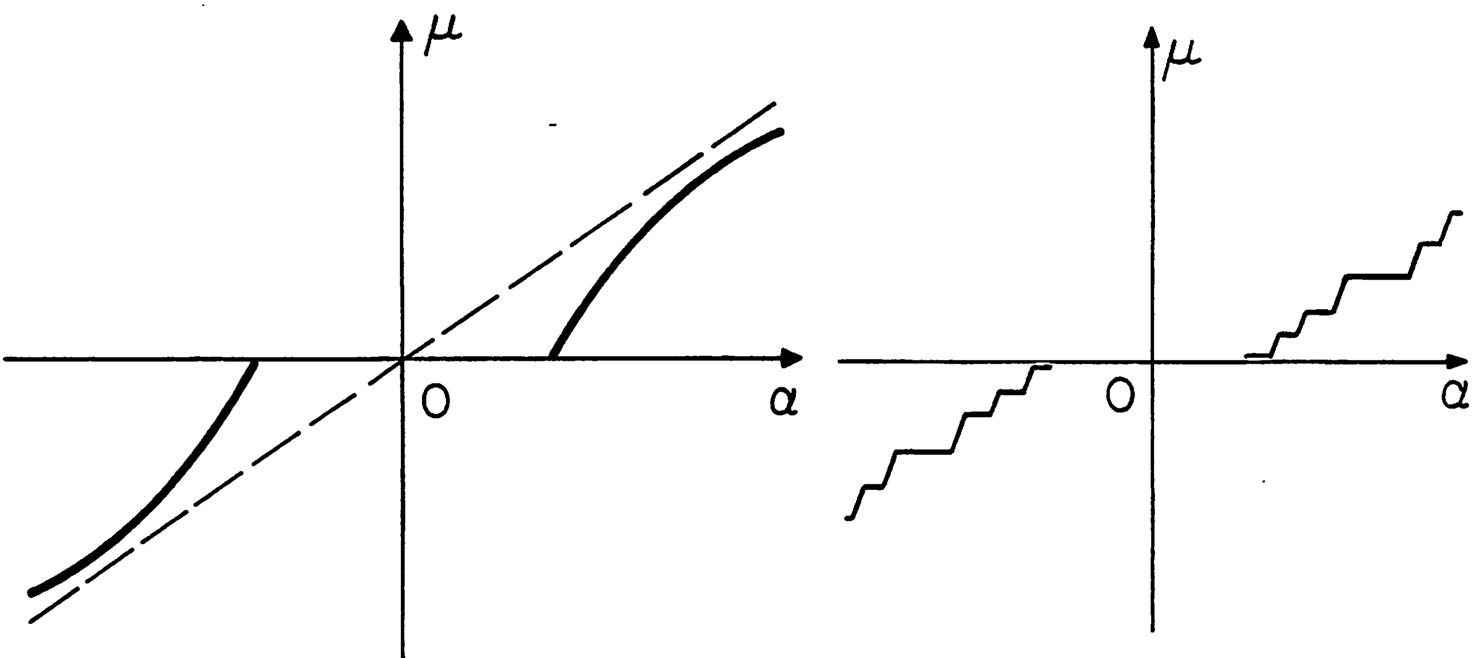


Fig. 7.1



(a) $\epsilon=0$ (b) $\epsilon \neq 0$

Fig. 7.2