UNIQUENESS OF CIRCUITS AND SYSTEMS CONTAINING ONE NONLINEARITY

by

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Uniqueness of Circuits and Systems Containing One Nonlinearity *

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ABSTRACT

We study systems containing *one* memoryless nonlinearity. We show that two such systems have the same I/O operator only when they are related by simple scaling, delay, and loop transformations. The theory is applied to one-port networks containing one nonlinear element.

1. Introduction

In [1] the authors considered systems consisting of a memoryless nonlinearity sandwiched between two linear time-invariant (LTI) operators. We showed that if two such systems have the same I/O operator then one can be got from the other by scaling the LTI operators and memoryless nonlinearity, and possibly redistributing some delay between the LTI operators. Thus such systems are *essentially unique*, in the sense that the I/O operator determines the nonlinearity and the pre- and post- LTI filters up to scaling and delays.

In this paper we continue our study of systems which are interconnections of LTI and memoryless operators. We consider systems containing *one* nonlinearity, possibly in a feedback loop, and show that these systems too are essentially unique, in this case modulo *scaling, delays, and loop tranformations* (theorem 3). Using this fact we show that the I/O maps realizable with some common structures for nonlinear systems (we have called these the cascade, Lur'e, and complementary Lur'e structures) are completely disjoint. This raises the possibility of *determining internal structure from I/O measurements*.

In section 7 we apply the theory to one-port networks containing one nonlinear element and show that two such networks are equivalent, that is, look the same from the external port, only if they are related in a simple way (theorem 4).

2. Notation and Foundations

In order to easily accomodate *memoryless nonlinearities* we extend the usual Volterra series formalism slightly to allow *measures* as kernels. This will allow memoryless operators as well as operators like

$$Au(t) = \int u(t-\tau)^2 h(\tau) d\tau$$

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which are called 'Volterra-like' by Sandberg [6,7], and arise in interconnections of memoryless and LTI operators. In fact the operators we allow are included in even more general formalisms, e. g. that of DeFigueiredo and Dwyer [8,9].

Let $<a>$ be a sequence where the nth term a_n is a *symmetric bounded measure supported on* $(R^+)^n$. Define

$$\mathbf{Rad}<a> = [\limsup_{n\to\infty} \|a_n\|^{1/n}]^{-1}$$

where $\|a_n\| = |a_n|(R^{+n})$. Then if $\mathbf{Rad}<a> = \rho > 0$, $<a>$ defines an operator A on B_ρ, the open ball of radius ρ in L^∞, into L^∞ given by*

$$Au(t) = \sum_{n=1} \int \cdots \int u(t-\tau_1)\ldots u(t-\tau_n)da_n(\tau_1,\ldots,\tau_n) \tag{2.1}$$

Unless otherwise stated, we will only consider operators of the form (2.1). We call a_n the nth *time domain kernel* of A; we'll use more often its Laplace transform

$$A_n(s_1,\ldots,s_n) \triangleq \int \cdots \int \exp-(s_1\tau_1+\ldots+s_n\tau_n)da_n(\tau_1,\ldots,\tau_n)$$

which is analytic and bounded in $(C^+)^n \triangleq \{ s | \mathrm{Re}s_k > 0,\ 1 \le k \le n \}$. A_n will be called the nth *kernel* of A, and we will use the notational convention that whenever, say B is an operator of the form (2.1), $B_n(s_1,\ldots,s_n)$ will denote its nth kernel.

A is LTI if $A_n = 0$, $n > 1$ and in this case we write its only nonzero kernel $A_1(s)$ as $A(s)$. For example e^{-sT} will denote both an *analytic function* and the $T-$ second delay *operator*. Conversely if $A_1 = 0$ then we say A is *strictly nonlinear*.

The part of (2.1) due to the masses or 'delta functions' at the origin in the a_n will be called the *memoryless part* of A; formally $\mathbf{MP}A$ is the operator defined by

$$(\mathbf{MP}A)_n \triangleq a_n(\{0\})$$

We develop some of the properties of **MP** in the appendix. If $\mathbf{MP}A = A$ then we say A is *memoryless*, and then we'll also use $A(\cdot)$ to denote the associated function $:R \to R$ given by $A(x) \triangleq \sum A_n x^n$ (the A_n are constants here).

I is as usual the identity operator with kernels

$$I_n = \begin{cases} 1 & n=1 \\ 0 & n>1 \end{cases}$$

If A is memoryless *and* LTI, it has the form αI for some real constant α; we will simply write it as α. For example $\alpha B \beta$ is the operator defined by

$$(\alpha B \beta)u = \alpha B(\beta u)$$

*Just as convolution with a bounded measure is a bounded map from L^p into L^p or $C^{(k)}$ into $C^{(k)}$, A also maps B_ρ in $C^{(k)}$ into $C^{(k)}$, if you prefer these signal spaces.

where α and β are just real numbers on the right hand side.

In the sequel H will always denote a LTI operator, F a memoryless operator, and N a memoryless strictly nonlinear operator.

Finally, we list a few facts we'll use in the paper. If A and B are operators, then:

Fact 1: $A=B$ if and only if $A_n=B_n$ for all n. Note that $A=B$ asserts equality of operators, whereas $A_n=B_n$ asserts equality of functions analytic in $(C^+)^n$. This is sometimes called the *uniqueness theorem.*

Fact 2: $A+B$ and AB (composition of A and B) are operators with kernels $(A+B)_n=A_n+B_n$ and

$$(AB)_n = \mathbf{SYM} \sum_{m=1}^{n} \left[\sum_{\substack{i_1,\ldots,i_m \geq 1 \\ i_1+\ldots+i_m=n}} \right] A_m(s_1+\ldots+s_{i_1}, \ldots, s_{n+1-i_m}+\ldots+s_n) \cdot$$

$$\cdot B_{i_1}(s_1,\ldots,s_{i_1}) \cdots B_{i_m}(s_{n+1-i_m},\ldots,s_n)$$

where **SYM** symmetrizes a function on $(C^+)^n$:

$$\mathbf{SYM} f \overset{\Delta}{=} (n!)^{-1} \sum_{\sigma \in S_n} f(s_{\sigma 1},\ldots,s_{\sigma n})$$

The n in **SYM** can be determined by context; it is the order of the kernel on the left hand side of the equation. When one of the operators is LTI the composition formula simplifies to:

$$(AH)_n(s_1,\ldots,s_n) = A(s_1,\ldots,s_n)H(s_1) \cdots H(s_n)$$

$$(HA)_n(s_1,\ldots,s_n) = H(s_1+\ldots+s_n)A(s_1,\ldots,s_n)$$

Fact 3: If A is strictly nonlinear, $I+A$ has an inverse (near 0) which is an operator in our sense. In particular, $\mathbf{Rad}[(I+A)^{-1}]>0$.

3. Problem Set-up

We will be concerned with systems which are *stable* interconnections of various LTI operators $H_k(s)$ and *one* memoryless nonlinear operator $F(\cdot)$ (see figure 1). Specifically, we assume that the *linearized* system ($F(\cdot)$ replaced by F_1) is *internally stable.** Under this assumption we may extract N, the *strictly nonlinear* part of F, collect the rest of the system into a 2-input 2-output LTI operator H, and redraw figure 1 as figure 2. Here

$$H = \begin{bmatrix} H_{yu} & H_{yd} \\ H_{xu} & H_{xd} \end{bmatrix}$$

and the overall I/O operator S is therefore

$$S = H_{yu} + H_{yd}N(I-H_{xd}N)^{-1}H_{xu} \tag{3.1}$$

*By internally stable we mean that if we inject a signal u into a summing node placed anywhere in the system, and pick off an output y from anywhere in the system, the resulting map $\Phi: u \to y$ is LTI in our sense (in particular $\Phi(s) \neq (1-s)^{-1}$, s, etc.).

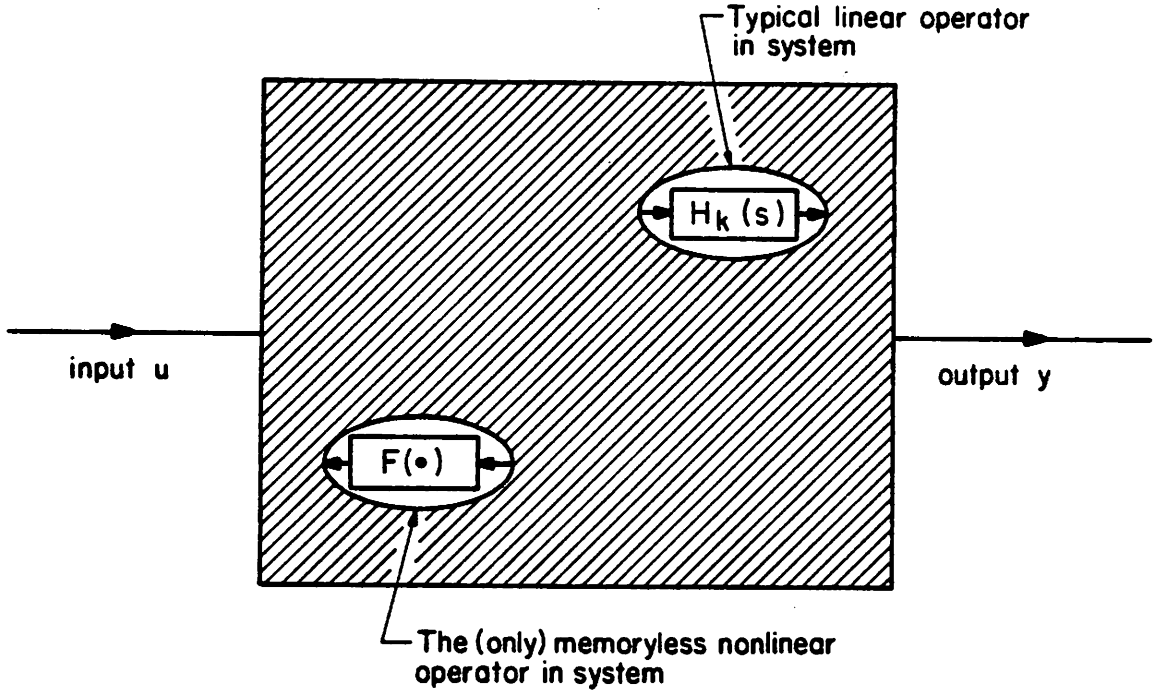


fig 1: System which is interconnection of various LTI operators $H_k(s)$ and *one* memoryless nonlinear operator $F(\cdot)$.

Incidently this form is a special case of the class of systems Sandberg considers in [6,7]**. We now ask the question, under what conditions could two systems of this form have the *same* I/O operator?

4. System Transformations

We first describe three system transformations which leave the I/O operator S unchanged, *scaling*, *delay*, and *loop* transformations.

Scaling transformations: Let α and β be nonzero real constants. Consider the system shown in figure 3. It clearly has I/O operator S independent of α and β. That is, if

$$\tilde{H}_{yu} = H_{yu} \qquad \tilde{H}_{yd} = \beta H_{yd}$$

$$\tilde{H}_{zu} = \alpha H_{zu} \qquad \tilde{H}_{zd} = \alpha\beta H_{zd}$$

$$\tilde{N} = \beta^{-1} N \alpha^{-1}$$

then $\tilde{S} = S$.

Proof: Obvious from figure 3, or more formally:

$$\tilde{S} = H_{yu} + \beta H_{yd}\beta^{-1}N\alpha^{-1}(I-\alpha\beta H_{zd}\beta^{-1}N\alpha^{-1})^{-1}\alpha H_{zu}$$

**This form occurs whenever a system is decomposed into two subsystems, one of which is linear. In the notation of [6,7], we consider the special case where all the operators are SISO, N is memoryless strictly nonlinear, and A, B, C, and D are given by convolution with bounded measures. Not all LTI bounded causal operators: $L^\infty \to L^\infty$ are given by convolution with bounded measures, though all the ones of engineering interest are.

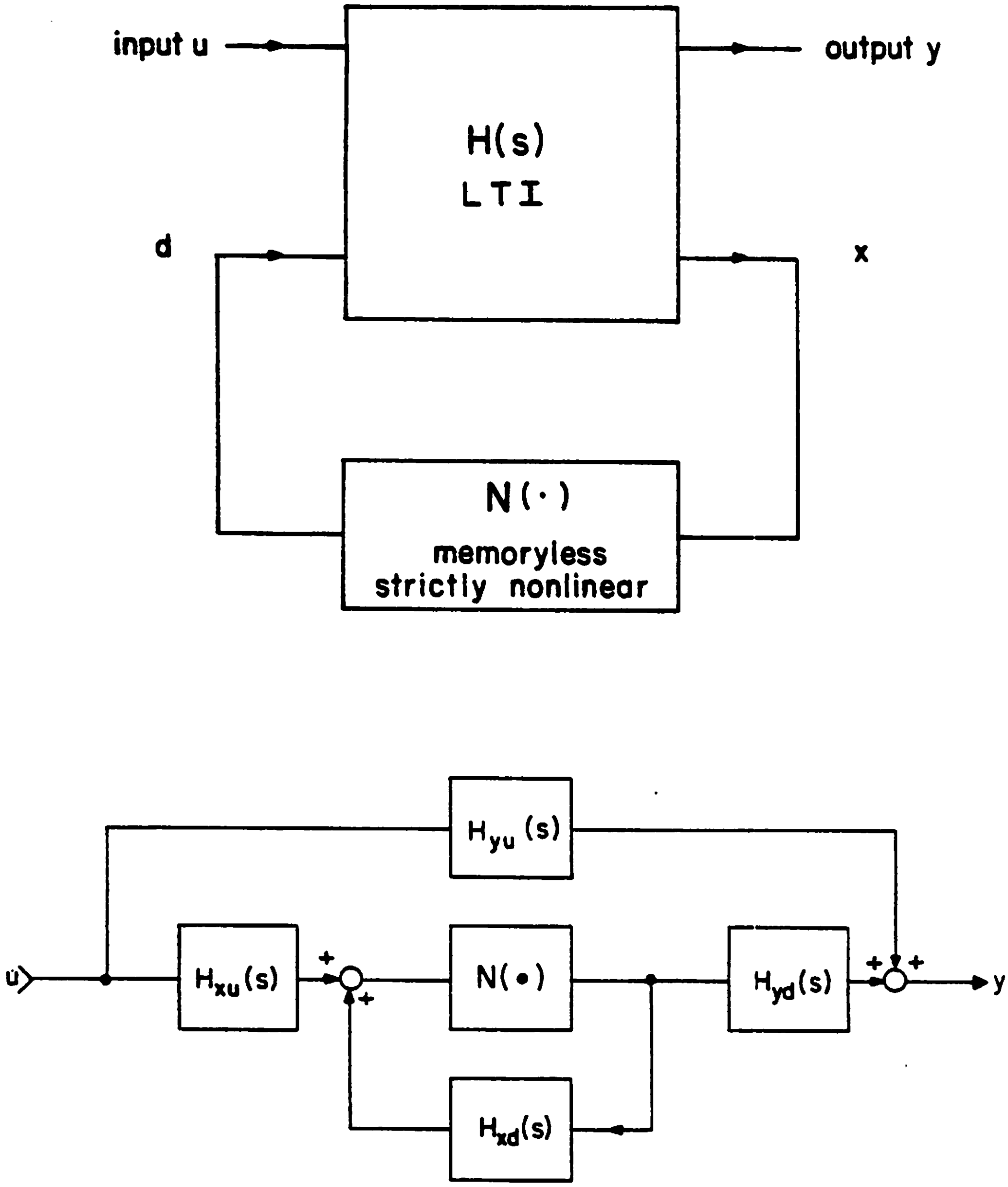


fig.2: (a) System redrawn as 2-input 2-output LTI operator $H(s)$ and ***strictly nonlinear*** memoryless operator $N(\cdot)$. (b) Block diagram.

$$= H_{yu} + H_{yd}N((I - \alpha H_{xd}N\alpha^{-1})\alpha)^{-1}\alpha H_{xu}$$

since β commutes with H_{yd} and H_{xd} and $B^{-1}A^{-1} = (AB)^{-1}$ generally. Carefully distributing the α we get

$$= H_{yu} + H_{yd}N(\alpha - \alpha H_{xd}N)^{-1}\alpha H_{xu}$$

$$= H_{yu} + H_{yd}N(I - H_{xd}N)^{-1}H_{xu} = S$$

after extracting the α on the left and using $(AB)^{-1} = B^{-1}A^{-1}$ again.

Delay transformations: When T is such that

$$\tilde{H}_{xu} = e^{-sT}H_{xu} \qquad \tilde{H}_{yd} = e^{sT}H_{yd}$$

Fig. 3 (a)

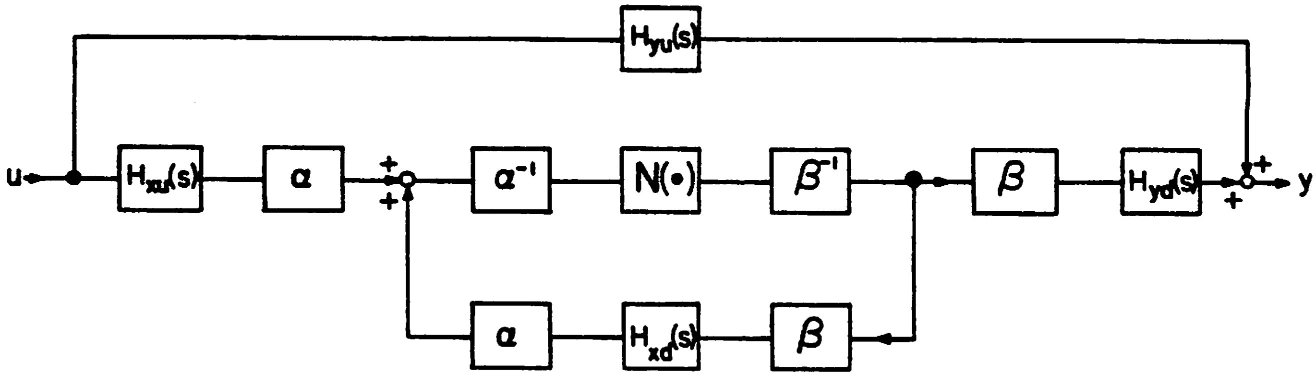


Fig. 3 (b)

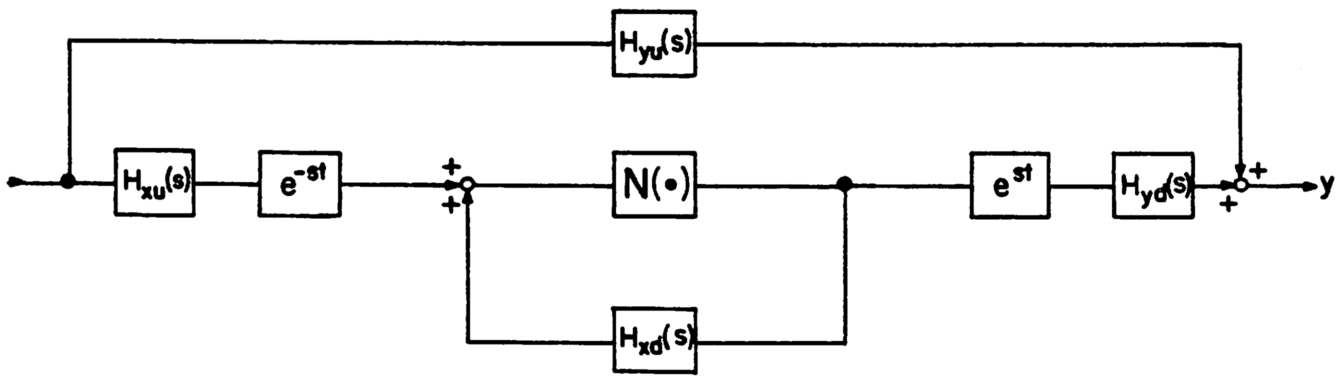


fig.3: (a) Scaling transformation of system in figure 2. The I/O operator is independent of α and β. (b) Delay transformation: any time delay in H_{xu} and H_{yd} can be distributed arbitrarily between them.

are operators of the form we consider (i.e. still causal), then

$$\tilde{H}_{yd} N(I-H_{xd}N)^{-1}\tilde{H}_{xu} = H_{yd}N(I-H_{xd}N)^{-1}H_{xu}$$

(See figure 3b.) This follows from the time invariance of $N(I-H_{xd}N)^{-1}$ and is trivial to verify.

Loop transformations: Let k be any real constant and consider the feedback subsystem shown in figure 4. The I/O operator of the subsystem shown in figure 4b is independent of k, that is, if

$$\tilde{H}_{xd} = H_{xd} + k \qquad \tilde{N} = N(I+kN)^{-1}$$

then

$$\tilde{N}(I-\tilde{H}_{xd}\tilde{N})^{-1} = N(I-H_{xd}N)^{-1}$$

and thus $\tilde{S} = S$ if $\tilde{H}_{yu} = H_{yu}$, $\tilde{H}_{xu} = H_{xu}$, and $\tilde{H}_{yd} = H_{yd}$. Note that the transformed subsystem has the same structure: a strictly nonlinear memoryless operator with LTI feedback around it. By facts 2 and 3 of section 2, $\tilde{N}$ has a positive radius of convergence. We leave to the reader the proof that $\tilde{N}$ is strictly nonlinear and that the transformed subsystem has the same I/O operator.

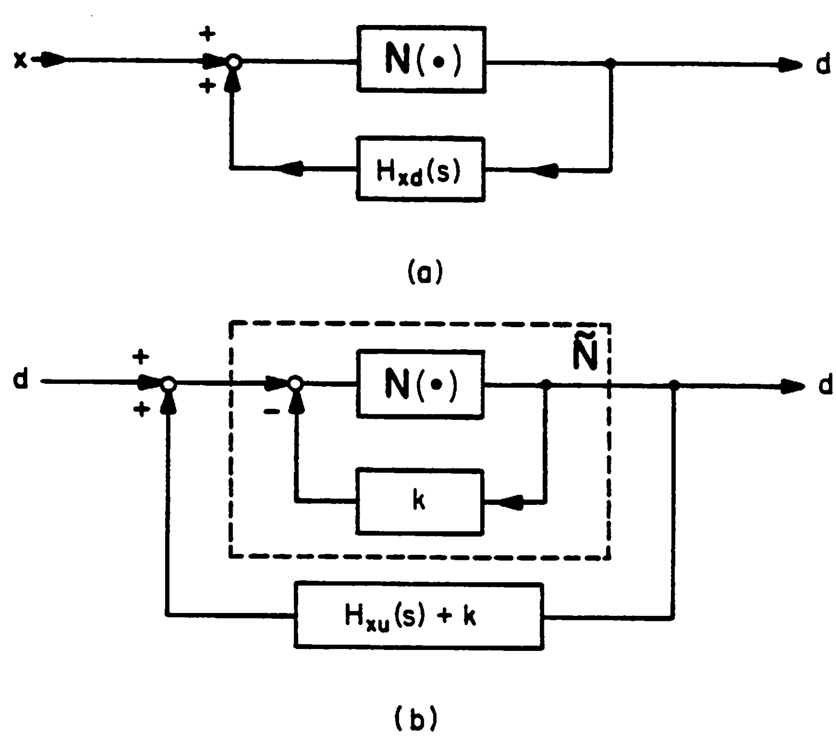


fig.4: Loop transformation of the feedback subsystem. The transformed subsystem in (b) has the same form as the original subsystem: a strictly nonlinear memoryless operator with LTI feedback around it.

It will be convenient to say that the subsystem in figure 4a is *normalized* if $\mathbf{MP}H_{xd} = 0$. Since $\mathbf{MP}\tilde{H}_{xd} = \mathbf{MP}H_{xd} + k$, any subsystem of the form in figure 4a can be brought to an equivalent *normalized* subsystem by a loop transformation with $k = -\mathbf{MP}H_{xd}$. This normalization has an intuitive interpretation: a normalized H_{xd} has some sort of response 'delay' or 'smoothness': its step response is continuous at $t = 0$.

5. Statement and Proof of Main Theorems

In this section we will show that if two systems as in figure 2 have the same I/O operator, then the systems are related by a scaling, delay, and loop transformation. Thus the transformations described in the last section are the *only* transformations which preserve the I/O operator. We first develop some results concerning the feedback subsystem shown in figure 4a.

Lemma 1: Let $G = N(I-HN)^{-1}$, where H is LTI, $\mathbf{MP}H = 0$, and N is memoryless strictly nonlinear. Then $\mathbf{MP}G = N$.

Intuitively, there is some 'delay' in the feedback loop (the subsystem is normalized), so that only the feedforward path N contributes to the memoryless part of the closed loop operator G.

Proof: Deferred to appendix.

We will need to explicitly compute a few kernels of the subsystem:

Lemma 2: Let $G = N(I-HN)^{-1}$, where H is LTI and N is memoryless with first nonvanishing term N_k, that is, $N_i = 0, \ 1 \le i < k, \ N_k \ne 0$. Then:

$$G_1 = \cdots = G_{k-1} = 0$$

$$G_k = N_k, \cdots G_{2k-2} = N_{2k-2}$$

$$G_{2k-1} = N_{2k-1} + kN_k^2 \mathbf{SYM}\, H(s_1+...+s_k)$$

Thus the first $2k-2$ terms of the closed loop operator G are simply those of N, as if the feedback were not present. We have to look at the kernel of order $2k-1$ to even detect the presence of the feedback H.

Proof: Deferred to appendix.

We are now ready to state and prove

Theorem 1: Suppose two *normalized* systems of the form (3.1) have the same I/O operator. Formally, suppose

$$\tilde{H}_{yu} + \tilde{H}_{yd}\tilde{N}(I-\tilde{H}_{xd}\tilde{N})^{-1}\tilde{H}_{xu} = H_{yu} + H_{yd}N(I-H_{xd}N)^{-1}H_{xu} \tag{5.1}$$

where the H's are LTI, the N's are memoryless strictly nonlinear, $\mathbf{MP}\tilde{H}_{xd} = \mathbf{MP}H_{xd} = 0$, and S is not linear.

Then there are real constants T and nonzero α and β such that

$$\tilde{H}_{yu} = H_{yu} \qquad \tilde{H}_{yd} = \beta e^{sT} H_{yd}$$

$$\tilde{H}_{xu} = \alpha e^{-sT} H_{xu} \qquad \tilde{H}_{xd} = \alpha\beta H_{xd}$$

$$\tilde{N} = \beta^{-1} N \alpha^{-1}$$

Proof of theorem 1: From lemma 2 $\tilde{H}_{yd}\tilde{N}(I-\tilde{H}_{xd}\tilde{N})^{-1}\tilde{H}_{xu}$ and $H_{yd}N(I-H_{xd}N)^{-1}H_{xu}$ are strictly nonlinear so the first kernel of (5.1) is:

$$\tilde{H}_{yu} = H_{yu}$$

Subtracting this term from (5.1) yields

$$\tilde{H}_{yd}\tilde{N}(I-\tilde{H}_{xd}\tilde{N})^{-1}\tilde{H}_{xu} = H_{yd}N(I-H_{xd}N)^{-1}H_{xu} \tag{5.2}$$

N is not zero, for then S would be linear, so suppose N_k is the first nonzero term in N. Then by lemma 2 the first nonzero kernel in (5.2) is:

$$\tilde{H}_{yd}(s_1+...+s_k)\tilde{N}_k\tilde{H}_{xu}(s_1)...\tilde{H}_{xu}(s_k) = H_{yd}(s_1+...+s_k)N_k H_{xu}(s_1)...H_{xu}(s_k) \tag{5.3}$$

In particular, $\tilde{N}$ also starts at the kth term. Since S is not linear (5.3) is not identically zero. We claim there are real T and nonzero β, α with

$$\tilde{H}_{yd} = \beta e^{sT} H_{yd} \qquad \tilde{H}_{xu} = \alpha e^{-sT} H_{xu} \tag{5.4}$$

This is proved in Boyd and Chua [1], so we will give an abbreviated argument here. Find an open ball D in $(C^+)^n$ in which (5.3) does not vanish. In D define

$$Q(s_1,\ldots,s_n) \overset{\Delta}{=} \ln\left[\frac{\tilde{H}_{yd}}{H_{yd}}(s_1+\ldots+s_n)\right] \tag{5.5}$$

$$= \ln\left[\frac{H_{zu}}{\tilde{H}_{zu}}(s_1)\cdots\frac{H_{zu}}{\tilde{H}_{zu}}(s_n)\frac{N_k}{\tilde{N}_k}\right] \tag{5.6}$$

From (5.5) and (5.6) we have

$$\frac{\partial^2 Q}{\partial s_1 \partial s_2} = \left[\ln\frac{\tilde{H}_{yd}}{H_{yd}}\right]''(s_1+\ldots+s_n) = 0$$

Thus in D and therefore in all of $(C^+)^n$

$$\left[\ln\frac{\tilde{H}_{yd}}{H_{yd}}\right](s_1+\ldots+s_n) = \gamma(s_1+\ldots+s_n) + T$$

for some constants γ and T. Hence

$$\tilde{H}_{yd}(s) = \beta e^{sT} H_{yd}(s)$$

where $\beta = \exp\gamma$. Substituting this back into (5.3) yields the other half of (5.4).

We now claim that (5.2) and (5.4) imply

$$\tilde{N}(I-\tilde{H}_{zd}\tilde{N})^{-1} = \beta^{-1}N(I-H_{zd}N)^{-1}\alpha^{-1} \tag{5.7}$$

which is what we would conclude if we pre- and post- operated on (5.2) with $\tilde{H}_{yd}^{-1}$ and $\tilde{H}_{zu}^{-1}$, respectively. To see that (5.7) is true even when $\tilde{H}_{yd}$ and $\tilde{H}_{zu}$ are not invertible, consider the nth kernel of (5.2). Find an open ball in $(C^+)^n$ where $H_{yd}(s_1+\ldots+s_n)$ and $H_{zu}(s_1)\ldots H_{zu}(s_n)$ do not vanish. Then *in that ball* we have, using (5.4):

$$\left\{\tilde{N}(I-\tilde{H}_{zd}\tilde{N})^{-1}\right\}_n(s_1,\ldots,s_n) = \beta^{-1}\alpha^{-n}\left\{N(I-H_{zd}N)^{-1}\right\}_n(s_1,\ldots,s_n) \tag{5.8}$$

Consequently (5.8) holds in all of $(C^+)^n$ and the nth kernels of (5.7) agree. This is true for all n, so (5.7) follows.

Now we look at the memoryless part of (5.7); by lemma 1

$$\mathbf{MP}[\tilde{N}(I-\tilde{H}_{zd}\tilde{N})^{-1}] = \tilde{N} = \mathbf{MP}[\beta^{-1}N(I-H_{zd}N)^{-1}\alpha^{-1}] = \beta^{-1}N\alpha^{-1}$$

By the last part of lemma 2 and (5.7)

$$\tilde{N}_{2k-1} + k\tilde{N}_k^2\mathbf{SYM}\tilde{H}_{zd}(s_1+\ldots+s_k) = \beta^{-1}\alpha^{1-2k}[N_{2k-1} + kN_k^2\mathbf{SYM}H_{zd}(s_1+\ldots+s_k)]$$

Cancelling $\tilde{N}_{1-2k} = \beta^{-1}\alpha^{2k-1}\tilde{N}_{2k-1}$ and dividing by $k\tilde{N}_k^2$ yields

$$\mathbf{SYM}\tilde{H}_{zd}(s_1+\ldots+s_k) = \frac{N_k^2}{\beta\alpha^{2k-1}\tilde{N}_k^2}\mathbf{SYM}H_{zd}(s_1+\ldots+s_k) = \alpha\beta\mathbf{SYM}H_{zd}(s_1+\ldots+s_k)$$

For $s \in C^+$, we evaluate this last equation at $s_1=\ldots=s_k=s/k$ to get

$$H_{xd}(s) = \alpha\beta H_{xd}(s)$$

which completes the proof of theorem 1.

In the next section we'll need

Remark: Under the hypotheses of theorem 1, $\tilde{H}_{xd} = \alpha\beta H_{xd}$ and $\det\tilde{H} = \alpha\beta\det H$.

Theorem 2: Suppose two systems of the form in figure 2 have the same I/O operator. Then there are real constants α, β, T, and γ such that (using previous notation)

$$\tilde{H}_{yu} = H_{yu} \qquad \tilde{H}_{yd} = \beta e^{sT} H_{yd}$$

$$\tilde{H}_{xu} = \alpha e^{-sT} H_{xu} \qquad \tilde{H}_{xd} = \alpha\beta H_{xd} + \gamma$$

$$\tilde{N} = \beta^{-1} N \alpha^{-1} (I + \gamma\beta^{-1} N \alpha^{-1})^{-1}$$

That is, the two systems are related by a scaling, loop, and delay transformation.

Proof: We first normalize the systems by loop transformations. Let $k = -\mathbf{MP}H_{xd}$ and $\tilde{k} = -\mathbf{MP}\tilde{H}_{xd}$. Then theorem 1 applies with H_{xd} replaced by $H_{xd} + k$, N replaced by $N(I+kN)^{-1}$, and similarly for the tilde'd expressions. Three of the conclusions above pop out immediately from theorem 1; we also conclude

$$\tilde{H}_{xd} + \tilde{k} = \alpha\beta(H_{xd} + k) \tag{5.9}$$

$$\tilde{N}(I+\tilde{k}\tilde{N})^{-1} = \beta^{-1} N (I+kN)^{-1} \alpha^{-1} \tag{5.10}$$

Letting $\gamma = \alpha\beta k - \tilde{k}$ in (5.9) yields the fourth conclusion of theorem 2. To get the last conclusion requires some work. In general if $B = A(I+A)^{-1}$ then $A = B(I-B)^{-1}$, so from (5.10) we have

$$\tilde{k}\tilde{N} = \tilde{k}\beta^{-1} N (I+kN)^{-1} \alpha^{-1} [I - \tilde{k}\beta^{-1} N (I+kN)^{-1} \alpha^{-1}]^{-1}$$

Dividing by $\tilde{k}$ and carefully moving the $(I+kN)^{-1}\alpha^{-1}$ into the bracketed expression we get

$$\begin{aligned} \tilde{N} &= \beta^{-1} N [\alpha + \alpha k N - \tilde{k}\beta^{-1} N]^{-1} \\ &= \beta^{-1} N \alpha^{-1} [I + \alpha k N \alpha^{-1} - \tilde{k}\beta^{-1} N \alpha^{-1}]^{-1} \\ &= \beta^{-1} N \alpha^{-1} [I + \gamma\beta^{-1} N \alpha^{-1}]^{-1} \end{aligned}$$

which is the last conclusion of theorem 2.

6. Structural Uniqueness

Theorems 1 and 2 allow us to determine under what conditions two systems (or one-port networks) containing one nonlinearity have the same I/O operator (port (v,i) pairs). These systems are often described, perhaps after simplification such as lumping together cascaded LTI operators, by a simple structure like those in figure 5. Of course these systems can be put in the general form considered in the last section, but a structure like those in figure 5 is usually a more natural description. Indeed the individual boxes often correspond to parts of the actual

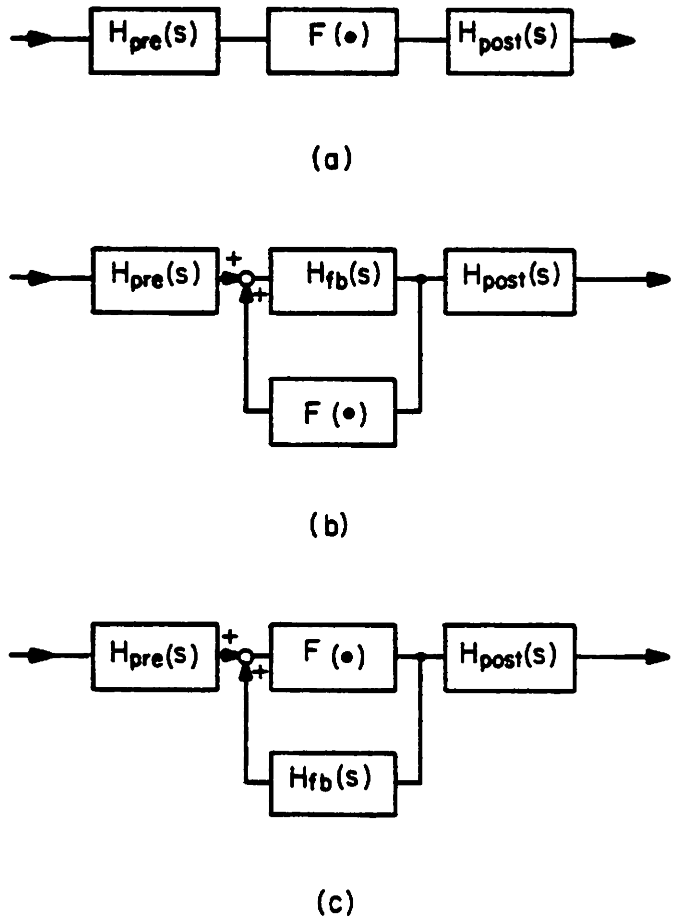


fig.5: Three structures for systems with one nonlinearity. (a) Cascade structure, (b) Lur'e structure, and (c) Complementary Lur'e structure. Except for trivial cases, the I/O operators of these structures are *completely disjoint.* From I/O measurements we could determine *which* structure such a system has.

physical system being modelled. So we now rephrase our original question in terms of these structures: when can two systems as in figure 5 have the same I/O operator? We'll now show that except for the trivial case when the system is linear, the realizable I/O operators for these different structures are completely disjoint, that is, no system with one structure can have the same I/O operator as a system with a different structure.

In fact we could expand the list of structures in figure 5, for example, by taking the output (via H_{post}) from the output of $F(\cdot)$ in (5b) or the input of $F(\cdot)$ in (5c): we only intend the next theorem to *illustrate* what we call *structural uniqueness.*

Theorem 3: Consider the three structures shown in figure 5, where F is memoryless and the H's are as usual LTI. Suppose F and H_{fb} are not constant, H_{pre} and H_{post} are not identically zero, and H_{fb} is strictly proper, that is $H_{fb}(\infty) \overset{\Delta}{=} \lim_{s \to \infty} H_{fb}(s) = 0$.

Then two such systems each with structure (a), (b), or (c) have the same I/O operator *if and only if*

(I) they have the *same structure;* and furthermore

(II) the corresponding operators are related by scaling, and possibly shuttling some delay between H_{pre} and H_{post}.

Proof: We transform the systems into the form considered in the previous section and apply theorem 1. Let $N = F - F_1$, the strictly nonlinear part of F, and let $K(s) = (1 - F_1 H_{fb})^{-1}$. Then in the notation of section 3 the systems of figure 5 have nonlinearity N and H-matrices

$$H_{(a)} = \begin{bmatrix} H_{pre} F_1 H_{post} & H_{post} \\ H_{pre} & 0 \end{bmatrix}$$

$$H_{(b)} = \begin{bmatrix} H_{pre} H_{fb} K H_{post} & K H_{fb} H_{post} \\ H_{pre} K H_{post} & K H_{fb} \end{bmatrix}$$

$$H_{(c)} = \begin{bmatrix} H_{pre} F_1 K H_{post} & K H_{post} \\ H_{pre} K & K H_{fb} \end{bmatrix}$$

Note that the strict properness of the H_{fb} guarantees that these systems are normalized, so by the remark after theorem 1 we have:

[A] any system with the same I/O operator as (a) has $H_{xd} = 0$, and

[B] any system with the same I/O operator as (b) has $\det H = 0$.

Thus a system with structure (b) or (c) could have the same I/O operator as (a) only if H_{pre} or H_{post} were zero, a contradiction. If a system with structure (c) has the same I/O operator as (b), then by [B] $\det H_{(c)} = H_{pre} H_{post} K = 0$, again a contradiction. This establishes conclusion [I].

Conclusion (II) for the stucture (a) is the main theorem of Boyd and Chua [1] and follows immediately from theorem 1 applied to $H_{(a)}$, so we omit the proof. The proofs for the other two structures are similar, so we'll just give the proof of (II) for (c). Assume two systems with structure (c) have the same I/O operator. Then from theorem 1 there are α, β, and T such that:

$$\begin{bmatrix} \tilde{H}_{post} \tilde{F}_1 \tilde{K} \tilde{H}_{pre} & \tilde{H}_{post} \tilde{K} \\ \tilde{H}_{pre} \tilde{K} & \tilde{H}_{fb} \tilde{K} \end{bmatrix} = \begin{bmatrix} H_{post} F_1 K H_{pre} & \beta e^{sT} H_{post} K \\ \alpha e^{-sT} H_{pre} K & \alpha\beta H_{fb} K \end{bmatrix} \tag{6.1}$$

$$\tilde{N} = \beta^{-1} N \alpha^{-1}$$

Thus $H_{yu} H_{xd} (H_{yd} H_{xu})^{-1}$ is:

$$\tilde{H}_{fb} \tilde{F}_1 = H_{fb} F_1$$

so $K = \tilde{K}$. Cancelling K from (6.1) yields

$$\begin{bmatrix} \tilde{H}_{post} \tilde{F}_1 \tilde{H}_{pre} & \tilde{H}_{post} \\ \tilde{H}_{pre} & \tilde{H}_{fb} \end{bmatrix} = \begin{bmatrix} H_{post} F_1 H_{pre} & \beta e^{sT} H_{post} \\ \alpha e^{-sT} H_{pre} & \alpha\beta H_{fb} \end{bmatrix}$$

So $\tilde{F}_1 = \alpha^{-1}\beta^{-1} F_1$. Coupled with $\tilde{N} = \beta^{-1} N \alpha^{-1}$ this implies

$$\tilde{F} = \beta^{-1} F \alpha^{-1}$$

and we've shown the systems differ only by scaling and shuttling delay between H_{pre} and H_{post}.

Theorem 3 has implications for black box modelling of systems having a structure like those in figure 5. It implies that *from I/O measurements alone* it is possible, in principle, to determine *which internal structure* such a system has. Furthermore we can determine the internal blocks H_{pre}, $N(\cdot)$, etc. up to scaling and possibly delay factors. From lemma 2 and the proof of theorem 3 we could construct explicit probing signals which distinguish the structures.

Of course, the differences in the I/O maps of the different structures may be subtle, or in some cases unmeasurable. For example if a system is very nearly second order, that is, its third and higher order kernels are very small, then *it may as well be modelled by the cascade structure of figure 5a*, since we need to measure the kernel of order three to observe the effects of the feedback (lemma 2). A similar statement holds for odd systems with unmeasurable fifth and higher order kernels.

7. Application to Circuit Theory

Suppose we have a one-port network **N** which contains *one* nonlinear element, say a voltage controlled nonlinear resistor **R** with characteristic $i=\hat{\imath}(v_R)$, as in figure 6a. We extract the incremental conductance g at 0 of **R** and partition **N** into a *linear two-port* $\mathbf{N}_{lin}$ and a *strictly nonlinear resistor* $\mathbf{R}_{snl}$, as in figure 6b. The network equations are then:

$$v_1 = Z_{11}i_1 + Z_{12}i_2$$

$$v_2 = Z_{21}i_1 + Z_{22}i_2$$

$$i_2 = -G(v_2)$$

where $[Z_{ij}]$ is the impedance matrix of $\mathbf{N}_{lin}$ and $i = G(v) = \hat{\imath}_R(v) - gv$ is the constitutive relation of $\mathbf{R}_{snl}$. These equations have the same form as those describing the system we have already studied: the I/O operator S corresponds to the (nonlinear) *impedance operator* Φ of our network **N**, and the matrix H corresponds to the impedance matrix of the linear two-port $\mathbf{N}_{lin}$.

If Z is an operator in our sense, theorem 2 applies and we have:

Theorem 4: Suppose two one-ports **N** and $\tilde{\mathbf{N}}$ as in figure 6 have the same (v,i) pairs, and are not linear. Then there are α, β, T, and r such that

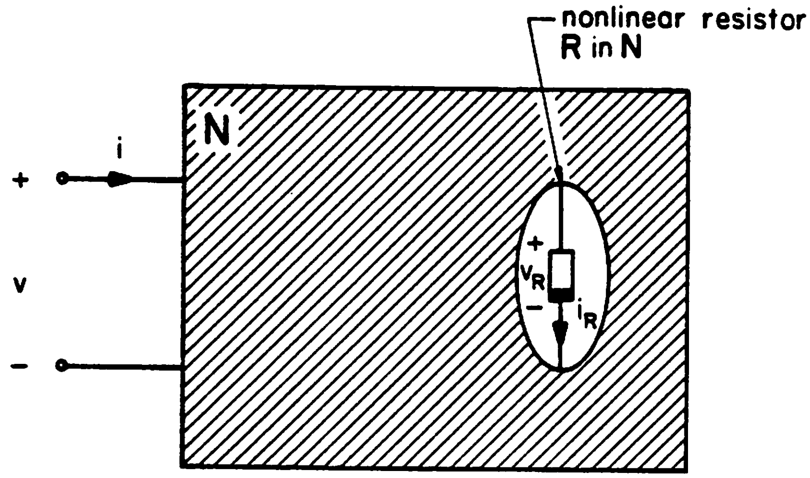
$$\tilde{Z} = \begin{bmatrix} 1 & 0 \\ 0 & \alpha e^{-sT} \end{bmatrix} Z \begin{bmatrix} 1 & 0 \\ 0 & \beta e^{sT} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -r \end{bmatrix} \tag{7.1}$$

and the strictly nonlinear resistors are related by

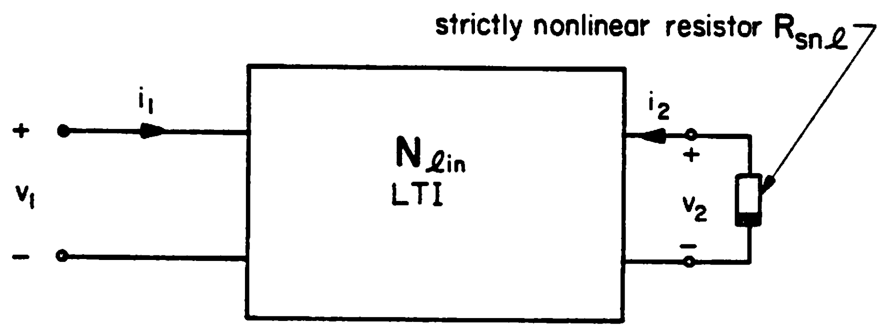
$$\tilde{G} = \beta^{-1} G \alpha^{-1} (I + r\beta^{-1} G \alpha^{-1})^{-1} \tag{7.2}$$

For the case T=0 this has the interpretation shown in figure 7.

If in addition $\mathbf{N}_{lin}$ and $\tilde{\mathbf{N}}_{lin}$ are reciprocal (for example, if they contain only two terminal elements and transformers) then T=0 and α=β in (7.1). In figure 7 the scalors are then



(a)



(b)

fig.6: (a) One-port network **N** containing *one* nonlinear element, a resistor **R** in this case. (b) **N** partitioned into a LTI 2-port $\mathbf{N}_{lin}$ and a strictly nonlinear resistor $\mathbf{R}_{snl}$.

transformers and the networks are related as in figure 8. **Proof:** If **N** and $\tilde{\mathbf{N}}$ have the same (v,i) pairs, they have the same impedance operator: (7.1) and (7.2) are the conclusions of theorem 2. Suppose the two-ports are reciprocal. Then (7.1), $Z=Z^T$, and $\tilde{Z}=\tilde{Z}^T$ imply

$$\alpha e^{-sT} Z_{12}(s) = \beta e^{sT} Z_{12}(s)$$

Since Z_{12} is not identically zero, $\alpha\beta^{-1}=\exp(2sT)$, hence $T=0$ and $\alpha=\beta$.

Of course by using another representation (say, admittance) for $\mathbf{N}_{lin}$ we can handle current controlled resistors. Similarly if the original resistor **R** had been a *flux-controlled inductor* with $i=\hat{\imath}_L(\varphi)$ we could rewrite the network equations as*

$$v_1 = Z_{11}i_1 + Z_{12}i_2$$

*$s^{-1}Z_{21}$ is sometimes not an operator in our sense, and in fact the same can be said for Z_{21} itself. But the previous theorems still hold with relaxed assumptions on H_{yu}, H_{xu}, and H_{yd}; they can be e.g. s or s^{-1}.

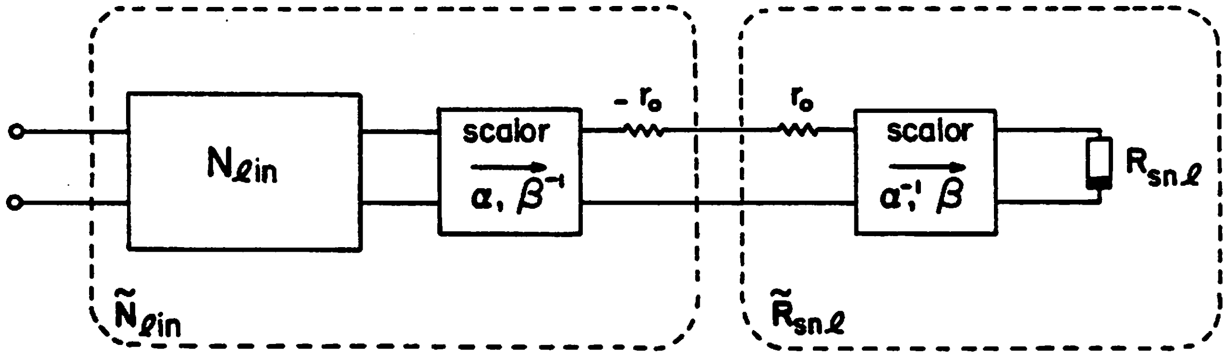


fig. 7: Relation between one-ports as in figure 6 which are port-equivalent. A (γ,δ) scalor is defined by $v_{out} = \gamma v_{in}$ and $i_{out} = -\delta i_{in}$ (see [12]).

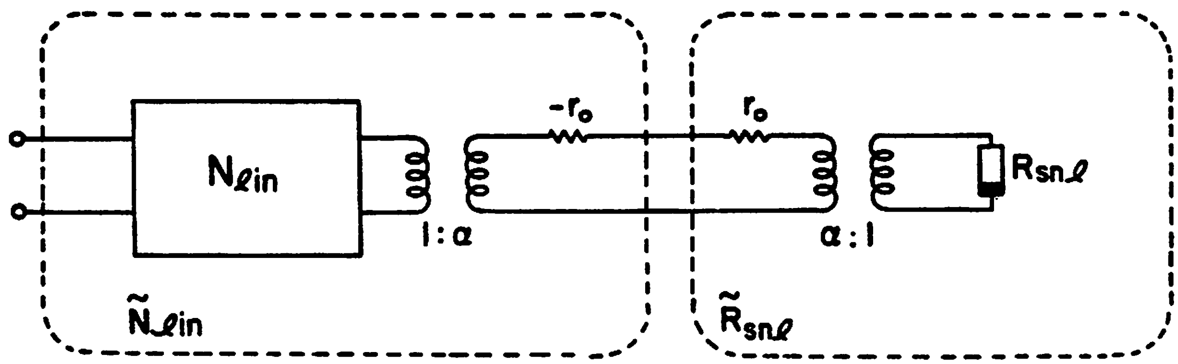


fig. 8: If $\mathbf{N}_{lin}$ and $\tilde{\mathbf{N}}_{lin}$ are reciprocal, the relation of figure 7 simplifies to that shown here.

$$\varphi_2 = s^{-1} Z_{21} i_1 + s^{-1} Z_{22} i_2$$

$$i_2 = -S(\varphi_2)$$

where $S(\cdot)$ is the strictly nonlinear part of $\hat{i}_L$. The conclusions of Theorem 4 then hold with G and $\tilde{G}$ replaced by S and $\tilde{S}$.

We will continue our study of uniqueness in nonlinear circuits in a future paper.

8. Acknowledgement

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9. References

[1] S. Boyd and L. O. Chua, "Uniqueness of a Basic Nonlinear Structure", IEEE Trans. Circuits Syst., vol. CAS-30 #10, Oct 1983.

[2] W. W. Smith and W. J. Rugh, "On the Structure of a Class of Nonlinear Systems", IEEE Trans. Autom. Contr., vol. AC-19, p701-706, Dec 1974.

[3] S. L. Baumgartner and W. J. Rugh, "Complete Identification of a Class of Nonlinear Systems

from Steady State Frequency Response", IEEE Trans. Circuits Syst., vol. CAS-22 #9, p753-759, Sept 1975.

[4] E. M. Wysocki and W. J. Rugh, "Further Results on the Identification Problem for the Class of Nonlinear Systems S_M ", IEEE Trans. Circuits Syst., vol. CAS-23 #11, p664-670, Nov 1976.

[5] T. R. Harper and W. J. Rugh, "Structural Features of Factorable Volterra Systems", IEEE Trans. Aut. Control, vol AC-21 #6, p822-832, Dec 1976.

[6] I. W. Sandberg, "Expansions for Nonlinear Systems", Bell System Technical Journal, vol 61, p159-200, Feb 1982.

[7] I. W. Sandberg, "Volterra-like Expansions for Solutions of Nonlinear Integral Equations and Nonlinear Differential Equations", IEEE Trans. Circuits Syst., vol CAS-30 #2, p68-77, Feb 83.

[8] R. DeFigueiredo and T. A. Dwyer, "A Best Approximation Framework and Implementation for Simulation of Large Scale Nonlinear Systems", IEEE Trans. Circuits Syst., vol CAS-27 #11, p1005-1014, Nov 1980.

[9] R. DeFigueiredo, "A Generalized Fock Space Framework for Nonlinear System and Signal Analysis", IEEE Trans. Circuits Syst., vol CAS-30 #10, Oct 1983.

[10] L. O. Chua and C. Y. Ng, "Frequency Domain Analysis of Nonlinear Systems: General Theory", IEE Journal of Electr. Circuits and Systems vol 3 #4, p165-185, July 1979.

[11] L. O. Chua and C. Y. Ng, "Frequency Domain Analysis of Nonlinear Systems: Formulation of Transfer Functions", IEE Journal of Electr. Circuits and Systems vol 3 #6, p257-267, Nov 1979.

[12] L. O. Chua, "The Linear Transformation Converter and its Applications to the Synthesis of Networks", IEEE Trans. Circuit Theory, vol CT-17, p584-594, Nov 1970.

Appendices

A1. The Memoryless Part of an Operator

The main purpose of this section is to prove lemma 1. While a direct proof is possible we think the approach here is more interesting. We start with a theorem which gives an intuitive interpretation to **MP**A.

Theorem A1: Suppose $u(t)=0,\ t<0,\ \lim_{t\to 0^+} u(t)$ exists (we'll call this limit $u(0^+)$), and $\|u\|<\mathbf{Rad}A$. Then $(Au)(0^+)$ exists and $(Au)(0^+)=(\mathbf{MP}A)(u(0^+))$. Thus **MP**$A$ is the part of A which 'reacts instantaneously'.

Proof: Let $y(t)=(Au)(t)$. Then $y(t)=\sum_{n=1} y_n(t)$ where

$$
\begin{aligned}
y_n(t) &= \int\cdots\int u(t-\tau_1)\ldots u(t-\tau_n)\,da_n(\tau_1,\ldots\tau_n)\\
&= a_n(\{0\})u(t)^n + \int_{R^{+n}-\{0\}}\cdots\int u(t-\tau_1)\ldots u(t-\tau_n)\,da_n(\tau_1,\ldots\tau_n)
\end{aligned}
$$

Hence

$$
y(t) = \mathbf{MP}A(u(t)) + \sum_{n=1}\int_{R^{+n}-\{0\}}\cdots\int u(t-\tau_1)\cdots u(t-\tau_n)\,da_n(\tau_1,\ldots,\tau_n) \tag{A1.1}
$$

Now the sum in (A1.1) is bounded by

$$
\sum_n \|u\|^n |a_n|\big((0,t]^n\big) \tag{A1.2}
$$

Since the the summand in (A1.2) is summable and *decreases* as $t\to 0^+$, monotone convergence tells us that (A1.2) tends to zero as $t\to 0^+$ and hence the sum in (A1.1) also converges to zero as $t\to 0^+$. Since **MP**A is analytic near 0,

$$
\lim_{t\to 0^+} y(t) = \mathbf{MP}A(u(0^+))
$$

Theorem A2: $\mathbf{MP}(A+B)=\mathbf{MP}A+\mathbf{MP}B$ and $\mathbf{MP}(AB)=\mathbf{MP}A\ \mathbf{MP}B$.

Thus **MP** maps dynamic operators into memoryless ones, *preserving addition and composition.* This generalizes the fact that $\mu\to\mu(\{0\})$ is an algebra homomorphism of the bounded measures on R^+ with convolution into R. We should mention that *causality* is crucial here, and also that the analogous theorem for discrete time operators is obvious.

Proof: For $|\alpha|$ small ($<\min(\mathbf{Rad}A,\mathbf{Rad}B)$) let $u(t)=\alpha 1(t)$, a step of height α. Then from $((A+B)u)(0^+) = Au(0^+)+Bu(0^+)$ and theorem A1

$$
\mathbf{MP}(A+B)(\alpha) = \mathbf{MP}A(\alpha)+\mathbf{MP}B(\alpha)
$$

which proves the first assertion; similarly $Bu(0^+)=\mathbf{MP}B(\alpha)$ so $ABu(0^+)=\mathbf{MP}A(\mathbf{MP}B(\alpha))$. By theorem A1 $ABu(0^+)=\mathbf{MP}(AB)(\alpha)$, hence

$$\mathbf{MP}(AB)(\alpha) = \mathbf{MP}A\,\mathbf{MP}B(\alpha)$$

establishing theorem A2.

Theorem A3: If A is invertible, then $\mathbf{MP}(A^{-1}) = (\mathbf{MP}A)^{-1}$.

Proof: $I = \mathbf{MP}I = \mathbf{MP}(AA^{-1}) = (\mathbf{MP}A)(\mathbf{MP}(A^{-1}))$, hence $\mathbf{MP}(A^{-1}) = (\mathbf{MP}A)^{-1}$.

Now we can give

Proof of lemma 1: In lemma 1 we have $G = N(I-HN)^{-1}$, where H is LTI, $\mathbf{MP}H=0$, and N is memoryless. By theorems A1 and A2 $\mathbf{MP}(I-HN)=I$; now using theorems A3 and A2 we have $\mathbf{MP}[N(I-HN)^{-1}] = \mathbf{MP}N = N$.

A2. Proof of lemma 2

Recall that $G=N(I-HN)^{-1}$, where H is LTI and N is memoryless strictly nonlinear with first nonvanishing kernel N_k. We first derive a recursive expression for G_n. Since HN is strictly nonlinear, $I-HN$ is invertible ($\mathbf{Rad}[(I-HN)^{-1}] > 0$), hence so is $G = N(I-HN)^{-1}$. Taking the nth kernel of $G(I-HN) = N$ yields

$$[G(I-HN)]_n = N_n$$

Expanding the left expression using the composition formula:

$$N_n = \mathbf{SYM} \sum_{m=1}^{n} \left[\sum_{\substack{i_1,\dots,i_m \geq 1 \\ i_1+\dots+i_m=n}} \right] G_m(s_1+\dots+s_{i_1}, \ \dots \ , s_{n+1-i_m}+\dots+s_n) \cdot$$

$$\cdot (I-HN)_{i_1}(s_1,\dots,s_{i_1}) \cdots (I-HN)_{i_m}(s_{n+1-i_m},\dots,s_n)$$

For $n=1$ this gives $G_1=0$, hence the $m=1$ term doesn't contribute. The $m=n$ term is simply $G_n(s_1,\dots,s_n)$; rearranging the equation above we get a recursive formula for G_n given by

$$G_n(s_1,\dots,s_n) = N_n - \mathbf{SYM} \sum_{m=2}^{n-1} \left[\sum_{\substack{i_1,\dots,i_m \geq 1 \\ i_1+\dots+i_m=n}} \right] G_m(s_1+\dots+s_{i_1}, \ \dots \ , s_{n+1-i_m}+\dots+s_n) \cdot$$

$$\cdot (I-HN)_{i_1}(s_1,\dots,s_{i_1}) \cdots (I-HN)_{i_m}(s_{n+1-i_m},\dots,s_n)$$

We can now prove lemma 2.

Proof of lemma 2: From the recursive formula for G_n we see that if $G_i=0$, $i<n$, and $N_n=0$, then $G_n=0$. Thus $G_n=0$, $n=1\dots k-1$. The outer sum can therefore start at $m=k$. Now we claim that the smallest n for which sum doesn't vanish is $n=2k-1$. By hypothesis,

$$(I-HN)_i = \begin{cases} 1 & i=k \\ 0 & 1<i<k \end{cases}$$

The product $(I-HN)_{i_1} \cdots (I-HN)_{i_m}$ will vanish unless each i_j is one or $\geq k$. Since at least one

$i_j > 1$, the smallest $n = \sum_{j=1}^{m} i_j$ for which the sum can contribute occurs when $m=k$, one i_j is k, and the others are 1. Thus $n=m-1+k=2k-1$. The sum then contains only the k derangements of $(k,1,\ldots 1)$, so $G_i = N_i$, $i<2-1$ and

$$G_{2k-1} = N_{2k-1} + kN_k^2 \mathrm{SYM} H(s_1+\ldots+s_k)$$

using $G_k = N_k$ and $(I-HN)_k = -H(s_1+\ldots+s_k)N_k$. So lemma 2 is proved.