ON PARAMETER CONVERGENCE IN ADAPTIVE CONTROL

by

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Abstract

It is well known that the parameter error as well as the model-plant mismatch error in a model reference adaptive scheme tends exponentially to zero iff a certain sufficient richness condition holds for signals inside the time-varying plant control loop. In this paper we give conditions on the reference signal (the exogenous input to the adaptive loop) - namely, that it have as many spectral lines as there are unknown parameters, in order to guarantee parameter convergence.

Key words: Model Reference Adaptive Systems, Parameter Convergence, Sufficient Richness, Persistent Excitation

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Section 1. Problem Statement

In recent work [1,2,8] on continuous time model reference adaptive systems, it has been shown that under a suitable choice of adaptive control law the output of the controlled plant y_p asymptotically tracks the output y_M of a stable reference model, despite the fact that the parameter error vector may not converge to zero (indeed, it may not converge at all). Consider, for example, the case when r(t) is a step. In this case it may be shown that the parameter error vector converges, not necessarily to zero but to a value such that the (asymptotic) closed loop plant transfer function matches the model transfer function at D.C. (0 rad/sec). This observation suggests the following intuitive argument: assuming that the parameter vector does converge, the plant loop is "asymptotically time invariant." If the input r has spectral lines at frequencies $\nu_1, \ldots, \nu_N$, we expect y_p will also; since $y_p \to y_M$, we "conclude" that the asymptotic closed loop plant transfer function matches the model transfer function at $s = j\nu_1, \ldots, j\nu_N$. If N is large enough, this implies that the asymptotic closed loop transfer function is precisely the model transfer function so that the parameter error converges to zero. It is the purpose of this paper to make this intuitive argument formal.

Results that have appeared in the literature on parameter error convergence (notably [3,4,5,13]) have established the uniform asymptotic and (equivalently) the exponential stability of the adaptive schemes under a certain sufficient richness condition. As is widely recognized, the principal drawback to this condition is that it applies to a certain vector of signals w(t) appearing inside the time varying feedback loop around the unknown plant. As a result, it is presently impossible to determine a priori whether a given reference input will result in a sufficiently rich w(t) and subsequent parameter error convergence to zero. In this paper, we remedy this deficiency. Specifically, we show that when the reference input (which is the exogenous input to the adaptive system) has as many spectral lines as there are unknown parameters, then the output error $y_p - y_M$ and parameter error converge to zero exponentially. We also sketch how prior parameter and plant-model state error bounds can be used along with the methods of [4] to give an estimate of the rate of exponential convergence.

We agree with the authors of [12] that the issue of parameter convergence is important, not just for its own sake, but as a first step in tackling important questions like robustness to unmodelled dynamics, slowly-time varying plants, etc. that have recently been raised (e.g. [9,10]).

The organization of the paper is as follows: Section 2 briefly describes the model reference adaptive system; in Section 3, we state and prove our main result for the relative degree 1 case, in Section 4, we discuss the extension to the higher relative degree cases. Section 5 contains concluding remarks.

Section 2. The Model Reference Adaptive System

To fix notation, we briefly review the model reference adaptive system of Narendra, Valavani, et al. [1,2]. The single-input single-output plant is assumed to be represented by a transfer function

$$\hat{W}_p(s) = k_p \frac{\hat{n}_p(s)}{\hat{d}_p(s)} \tag{2.1}$$

where $\hat{n}_p(s)$, $\hat{d}_p(s)$ are relatively prime monic polynomials of degree m, n respectively and k_p is a scalar. The following are assumed known about the plant transfer function:

(A1) The degree of the polynomial $\hat{d}_p$, i.e. n is known.

(A2) The relative degree of $\hat{W}_p$, i.e. (n-m) is known.

(A3) The sign of k_p is known (say, + without loss of generality).

(A4) The transfer function $\hat{W}_p$ is assumed to be minimum phase, i.e., $\hat{n}_p$ is Hurwitz.

Remark: (A1) may be replaced by the weaker assumption that an upper bound on the degree of $\hat{d}_p$ is known. We use (A1) here for simplicity.

The objective of adaptive control is to build a dynamic compensator so that the plant output asymptotically matches that of a stable reference model $\hat{W}_M(s)$ with input r(t), output $y_M(t)$ and transfer function

$$\hat{W}_M(s) = k_M \frac{\hat{n}_M(s)}{\hat{d}_M(s)} \tag{2.2}$$

where $\hat{n}_M$, $\hat{d}_M$ are monic polynomials of degree m*, n* respectively $k_M > 0$. Since our interest in this paper is in parameter convergence we will assume n* = n, m* = m. We do not, however, need $\hat{n}_M$ and $\hat{d}_M$ to be relatively prime. If we denote the input and output of the plant u(t) and $y_p(t)$ respectively, the objective may be stated as: choose u(t) such that as $t \to \infty$ $y_p(t) - y_M(t) \to 0$.

2.1. Relative Degree 1 Case

By suitable prefiltering, if necessary, we may assume that the model $\hat{W}_M(s)$ is strictly positive real. The adaptive scheme in this case is as shown in Figure 1.

The dynamic compensation blocks F_1, F_2 are identical one input, (n-1) output systems, each with transfer function

$$(sI-\Lambda)^{-1}b; \qquad \Lambda \in \mathbb{R}^{(n-1)x(n-1)}, \qquad b \in \mathbb{R}^{(n-1)}$$

where Λ is chosen so that the eigenvalues of Λ are the zeros of $\hat{n}_M$. We assume that the pair (Λ,b) is in controllable canonical form so that

$$(sI-\Lambda)^{-1}b = \frac{1}{\hat{n}_M(s)}\begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n-2} \end{bmatrix} \tag{2.3}$$

The adaptive gains $c \in \mathbb{R}^{n-1}$ are in the pre-compensator block for the purpose of cancelling the plant zeros and replacing them by the model zeros, $d \in \mathbb{R}^{n-1}$, $d_0 \in \mathbb{R}$ in the feedback compensator for the purpose of assigning the plant poles. The adaptive gain c_0 adjusts the overall plant gain. Thus, the vector of 2n adjustable parameters denoted θ is

$$\theta^T = [c_0, c^T, d_0, d^T].$$

If the signal vector $w \in \mathbb{R}^{2n}$ is defined by

$$w^T = [r, v^{(1)^T}, y_p, v^{(2)^T}] \tag{2.4}$$

we see that the input to the plant u is given by

$$u = \theta^T w. \tag{2.5}$$

It may be verified that there exists a unique constant $\theta^* \in \mathbb{R}^{2n}$ such that when $\theta = \theta^*$, the transfer function of the plant plus controller $= \hat{W}_M(s)$. Further, it has been shown that under the update law

$$\dot{\theta} = - e_1 w \tag{2.6}$$

then $\lim_{t\to\infty} e_1(t) = 0$ provided $r(t)$ is bounded. Further, all signals in the loop, viz. $u(t)$, $v^{(1)}(t)$, $v^{(2)}(t)$, $y_p(t)$, $y_M(t)$ are bounded. Define the parameter error $\phi = \theta - \theta^*$. Then we have from [1] that

$$\phi \in L^2 \cap L^\infty,\ \dot{\phi} \in L^\infty \text{ and } \dot{\phi} \to 0 \text{ as } t \to \infty$$

However, we cannot say anything as yet about the convergence of $\phi(t)$ and hence of $\theta(t)$.

2.2. Relative Degree 2 Case

In this case $\hat{W}_M$ cannot be chosen positive real; however, we may assume (using suitable prefiltering, if necessary) that $\exists\ L(s) = (s+\delta)$, with $\delta > 0$ such that $\hat{W}_M\hat{L}$ is positive real. The scheme of Figure 1 is modified (see [1])* by replacing each of the gains θ_i, viz. c_0, d_0, c, d by the gains $\hat{L}\theta_i\hat{L}^{-1}$ which in turn is given by

$$\hat{L}\theta_i\hat{L}^{-1} = \theta_i + \dot{\theta}_i\hat{L}^{-i} \qquad i = 1, \ldots, 2n.$$

We now define the signal vector

$$\zeta^T(t) = [\hat{L}^{-1}r, \hat{L}^{-1}v^{(1)}, \hat{L}^{-1}y_p, \hat{L}^{-1}v^{(2)}] \tag{2.6}$$

*Λ is now chosen to be an exponentially stable, with the zeros of $\hat{n}_M$ a subset of the eigenvalues of Λ.

$$\dot{\theta} = - e_1\zeta \qquad (2.7)$$

yields that $e_1(t) \to 0$ as $t \to \infty$ provided $r(t)$ is bounded.

2.3. The Case of Relative Degree ≥ 3

As in Section (2.2), pick a stable Hurwitz polynomial $\hat{L}$ so that $\hat{L}\hat{W}_M$ is positive real. The trick used in Section 2.2, namely, to replace each θ_i by $\hat{L}\theta_i\hat{L}^{-1}$ is no longer possible since $\hat{L}\theta_i\hat{L}^{-1}$ depends on second and (possibly higher) derivatives of θ_i. To obtain a positive real error equation we retain the original configuration of Figure 1, and augment the model output by

$$\hat{W}_M\hat{L}[\theta^T\hat{L}^{-1}-\hat{L}^{-1}\theta^T]w$$

as shown in Figure 2. In addition to obtain $\dot{\phi} \in L^2$ and thereby prove stability of the adaptive scheme, we add an additional quadratic term to y_a to get the total augmented model output y_a

$$y_a = \hat{W}_M\hat{L}\{[\theta^T\hat{L}^{-1}-\hat{L}^{-1}\theta^T]w-\alpha\zeta^T\zeta\} \qquad (2.8)$$

where $\alpha > 0$ and ζ is defined in (2.6). The update law

$$\dot{\theta} = - e_1\zeta \qquad (2.7)$$

yields that as $t \to \infty$, $e_1(t) \to 0$, $y_a(t) \to 0$ so that $y_M(t) \to y_p(t)$. As before, the parameter error ϕ satisfies

$$\phi \in L^2 \cap L^\infty,\ \dot{\phi} \in L^\infty \text{ and } \dot{\phi} \to 0 \text{ as } t \to \infty$$

Again, nothing can be said about the convergence of $\phi(t)$.

Section 3. Spectral Lines and Sufficient Richness in the Relative Degree 1 Case

Consider the adaptive system of Section 2.1 for the case of relative degree 1. We noted that the control law of (2.5) with the adaptive law of (2.6) yield that

$$\lim_{t\to\infty} e_1(t) = 0$$

provided r(t) is bounded. Without additional conditions, however, we cannot guarantee

$$\lim_{t\to\infty} \theta(t) = \theta^*$$

(or in fact that θ converges at all). It has been shown by Morgan and Narendra [3], Anderson [4] Kreisselmeier [5] that $e_1(t) \to 0$, $\theta(t) \to \theta^*$ exponentially iff the signal vector w(t) is sufficiently rich, in the following sense: $\exists\ \delta > 0,\ \alpha > 0$ such that $\forall\ s \in \mathbb{R}_+$

$$\int_s^{s+\delta} w(t)w^T(t)dt \geq \alpha I. \tag{3.1}$$

Recall from the definition of w(t) in (2.5) that it contains signals $v^{(1)}(t)$, $v^{(2)}(t)$, $y_p(t)$ generated inside the time varying feedback loop around the unknown plant. Conditions on the reference input r(t) required for (3.1) to hold are to our knowledge so far, unknown. In the remainder of this section we will show that if r(t) has 2n spectral lines (in the sense that will be made precise), then we have exponential convergence of $e_1(t)$ to 0 and $\theta(t)$ to θ^*. The proof is in two steps.

Step 1 consists of transcribing the condition (3.1) into an analogous condition for the model, which is a linear time-invariant system.

Step 2 consists of showing that the condition analogous to (3.1) for the model is obtained when the reference signal r(t) has 2n spectral lines. We now discuss these steps in detail:

For Step 1, redraw Fig. 1 as shown in Fig. 3 with the model represented (in non-minimal form) as the plant with dynamic compensator and $\theta = \theta^*$. The signal vector $w_M \in \mathbb{R}^{2n}$ in the model-loop is given by

$$w_M^T = [r, v_M^{(1)}, y_M, v_M^{(2)}]$$

We have that as $t \to \infty$, $w_M \to w$. Hence, it seems reasonable to expect that if w is sufficiently rich then so is w_M. The foregoing is indeed true if $\dot{w}$ and $\dot{w}_M$ are bounded. However, we will use no supplementary assumptions on w, w_M but rather the conclusion from Narendra-Valavani [1] that $w(\cdot) - w_M(\cdot) \in L^2$. Further, it follows from their proof (specifically, Equations 16, 17, 18 of [1]) that

$$\|w(\cdot)-w_M(\cdot)\|_2 \leq K_0(\|\theta(0)-\theta^*\|+\|x_M(0)-x_p(0)\|+\|v^{(1)}(0)-v_M^{(1)}(0)\| + \|v^{(2)}(0)-v_M^{(2)}(0)\|) \qquad (3.2)$$

where x_M, x_p are the state variables in minimal representations for the plant in the model loop, plant loop respectively. Hence, from prior bounds on the parameter error, and initial state errors a bound on the L_2 norm of $w(\cdot) - w_M(\cdot)$ is obtained. Further, from [1], it follows that $\exists K_2$ such that

$$\|w(t)\|, \|w_M(t)\| \leq K_2 \qquad \forall t. \qquad (3.3)$$

The bound K_2 depends as before on $\|\theta(0)-\theta^*\|$, $\|x_m(0)-x_p(0)\|$ $\|v^{(1)}(0)-v_M^{(1)}(0)\|$, $\|v^{(2)}(0)-v_M^{(2)}(0)\|$. We now have

Theorem 3.1

Suppose $\|w(t)\|$, $\|w_M(t)\| \leq K_2$ and $\|w(\cdot)-w_M(\cdot)\|_2 = K_1 < \infty$. Then, $w(t)$ is sufficiently rich $\Leftrightarrow$ $w_M(t)$ is sufficiently rich.

Proof: The argument is symmetric between w and w_M. Hence, we only show $\Rightarrow$. w sufficiently rich implies that $\exists\ \alpha, \delta > 0$ such that $\forall\ s \in \mathbb{R}_+, z \in \mathbb{R}^{2n}$

$$z^T[\int_s^{s+\delta} ww^T dt]z \geq \alpha z^T z \tag{3.4}$$

Iterating on (3.4) p times we get that $\forall\ p \in \mathbb{Z}_+$

$$z^T[\int_s^{s+p\delta} ww^T dt]z = \int_s^{s+p\delta} (z^T w)^2 dt \geq \alpha p z^T z \tag{3.5}$$

Now, note that

$$(z^T w)^2 - (z^T w_M)^2 = z^T(w-w_M)\cdot z^T(w+w_M) \leq z^T z\cdot 2K_2\|w-w_M\|$$

Hence

$$\int_s^{s+p\delta} (z^T w)^2 - (z^T w_M)^2 dt \leq z^T z\ 2K_2 \int_s^{s+p\delta} \|w-w_M\| dt \tag{3.6}$$

But, by Cauchy-Schwarz

$$\int_s^{s+p\delta} \|w-w_M\| dt \leq (p\delta)^{1/2} \int_s^{s+p\delta} \|w-w_M\|^2 dt \leq K_1 (p\delta)^{1/2} \tag{3.7}$$

Using (3.7) in (3.6), and (3.4), we have that $\forall\ p \in \mathbb{Z}_+$,

$$z^T[\int_s^{s+p\delta} w_M w_M^T dt]z \geq z^T z(\alpha p - 2K_2K_1(p\delta)^{1/2}).$$

Choose p_0 sufficiently large so that

$$\bar{\alpha} := \alpha p_0 - 2K_2K_1(p_0 s)^{1/2} > 0$$

and define $\bar{\delta} = p_0\delta$. Then we have that $\forall\ s \in \mathbb{R}_+$,

$$[\int_s^{s+\bar{\delta}} w_M w_M^T dt] \geq \bar{\alpha} I \tag{3.8}$$

Thus w_M is sufficiently rich. □

<u>Remark</u>: We have shown that we have exponential convergence of parameter error and $e_1(t)$ provided that w_M is sufficiently rich (i.e., (3.8) holds). This completes Step 1.

<u>Step 2</u>. We now give conditions on $r(t)$ so that $w_M(t)$ is sufficiently rich, using the classical concept of a spectral line (see Wiener [6]).

<u>Definition 3.2</u>. A function $u(t)$: $\mathbb{R}_+ \to \mathbb{R}^n$ is said to have a <u>spectral line at frequency ν of amplitude $\hat{u}(\nu)$</u> $\in \mathbb{C}^n$ iff

$$\frac{1}{T}\int_s^{s+T} u(t)e^{-j\nu t}dt \tag{3.9}$$

converges to $\hat{u}(\nu)$ as $T \to \infty$, uniformly in s. When $\hat{u}(\nu) \neq 0$ we will say <u>u has a spectral line at ν</u>.

Remark: u does <u>not</u> have to be <u>almost periodic</u> to have a spectral line at frequency ν_0; for example (3.9) need not converge for $\nu \neq \nu_0$.

The following lemma is immediate:

Lemma 3.3. Let $u(t)$, $y(t)$ be the input and output, respectively, of a stable linear time-invariant system with transfer function $\hat{L}(s)$ (and arbitrary initial condition). If u has a spectral line at frequency ν then so does y, with amplitude

$$\hat{y}(\nu) = \hat{L}(j\nu)\hat{u}(\nu) \tag{3.10}$$

Remark: Since the initial condition contributes a decaying exponential to $y(t)$ it does not appear in (3.10). $\hat{y}(\nu)$ in (3.10) may be zero if $\hat{L}(s)$ has a zero on the imaginary axis.

The second lemma is key to our main result:

Lemma 3.4. Let $x(t) \in \mathbb{R}^N$ have spectral lines at frequencies $\nu_1, \nu_2, \ldots, \nu_N$. Further, let $\{\hat{x}(\nu_1), \hat{x}(\nu_2), \ldots, \hat{x}(\nu_N)\}$ be linearly independent in $\mathbb{C}^N$. Then, $x(t)$ is sufficiently rich, i.e., $\exists\ \alpha, \delta > 0$ such that $\forall\ s \in \mathbb{R}_+$

$$\int_s^{s+\delta} xx^T dt \geq \alpha I. \tag{3.11}$$

Proof: Define the NxN matrix $X(s,\delta)$ by

$$X(s,\delta) := \frac{1}{\delta}\int_s^{s+\delta} \begin{bmatrix} e^{-j\nu_1 t} \\ \vdots \\ e^{-j\nu_N t} \end{bmatrix} x^T(t)dt.$$

and the NxN matrix X_0 which is the (uniform in s) limit of $X(s,\delta)$ as $\delta \to \infty$

$$X_0 := \begin{bmatrix} \hat{x}^T(\nu_1) \\ \vdots \\ \hat{x}^T(\nu_N) \end{bmatrix}$$

By hypothesis X_0 is non-singular. Hence for δ sufficiently large $X(s,\delta)$ is invertible and $\|X(s,\delta)^{-1}\| \le 2\|X_0^{-1}\|$ for $\delta \ge \delta^*$ and all s. Now for $z \in \mathbb{R}^N$ with $\|z\| = 1$, and any $\nu \in \mathbb{R}$ we have

$$\frac{1}{\delta}\int_s^{s+\delta} (x^T z)^2 dt = \frac{1}{\delta}\int_s^{s+\delta} |x^T z e^{-j\nu t}|^2 dt$$

$$\ge |\frac{1}{\delta}\int_s^{s+\delta} x^T z e^{-j\nu t} dt|^2 \text{ (by Jensen's inequality)} \tag{3.12}$$

Using (3.12) for $\nu = \nu_1, \nu_2, \ldots, \nu_N$ we have

$$\frac{1}{\delta}\int_s^{s+\delta} (x^T z)^2 dt \ge \frac{1}{N}\sum_{k=1}^{N} |\frac{1}{\delta}\int_s^{s+\delta} x^T z e^{-j\nu_k t} dt|^2$$

$$= \frac{1}{N}\|X(s,\delta)z\|^2$$

$$\ge \frac{1}{N}\|X(s,\delta)^{-1}\|^{-2} \qquad \text{for } \delta \ge \delta_*$$

$$\ge \frac{1}{4N}\|X_0^{-1}\|^{-2}$$

Equation (3.11) now holds with $\delta = \delta_*$ and $\alpha = \frac{1}{4N}\|X_0^{-1}\|^{-2} > 0$. □

We now apply Lemmas (3.3), (3.4) to prove the main result of this section.

Theorem 3.5

Suppose $r(t)$ has spectral lines at $\nu_1, \nu_2, \ldots, \nu_N$. Then $w_M(t)$ is sufficiently rich.

Remark: Once we have shown $w_M(t)$ sufficiently rich, Theorem (3.2) guarantees that $w(t)$ is also sufficiently rich which in turn guarantees exponential convergence of $e_1(t)$ to 0 and $\theta(t)$ to θ^*.

Proof: Recall that $w_M^T(t)$ is $[r, v_M^{(1)T}, y_M, v_M^{(2)T}]$. We derive the transfer function from $r(t)$ to $w_M^T(t)$; using (2.3)

$$\hat{Q}^T(s) = [1, \frac{\hat{W}_M}{\hat{W}_p}\cdot\frac{1}{\hat{n}_M}, \frac{\hat{W}_M}{\hat{W}_p}\cdot\frac{s}{\hat{n}_M}, \ldots, \frac{\hat{W}_M}{\hat{W}_p}\cdot\frac{s^{n-2}}{\hat{n}_M}, \hat{W}_M, \frac{\hat{W}_M s}{\hat{n}_M}, \ldots, \frac{\hat{W}_M s^{n-2}}{\hat{n}_M}]$$

$$= \frac{k_M}{k_p\hat{n}_p\hat{d}_M}[\frac{k_p\hat{n}_p\hat{d}_M}{k_M}, \hat{d}_p, \ldots, \hat{d}_p s^{n-2}, k_p\hat{n}_p\hat{n}_M, k_p\hat{n}_p, \ldots, k_p\hat{n}_p s^{n-2}] \tag{3.13}$$

Since the plant is minimum phase and the model is stable the transfer function $\hat{Q}(s)$ in (3.31) is stable. Neglecting the initial conditions (which do not, anyhow, contribute to the spectral lines of $w_M(t)$) we have

$$w_M^T = \hat{Q}^T r(t).$$

Now, the (n+1)th entry of $\hat{Q}$ has numerator polynomial $\hat{n}_p\hat{n}_M$ with $\hat{n}_M$ of degree (n-1). Further the first entry of $\hat{Q}$ has numerator polynomial $\hat{n}_p\hat{d}_M$ with $\hat{d}_M$ of degree n. Compare these terms with the last (n-1) entries of $\hat{Q}$, viz., $\hat{n}_p, \ldots, \hat{n}_p s^{n-1}$. Using constant row operations then we can write

$$w_M = T\bar{w} = T \cdot \frac{1}{\hat{n}_p \hat{d}_M} \begin{bmatrix} \hat{d}_p \\ \vdots \\ \hat{d}_p s^{n-2} \\ \hat{n}_p \\ \vdots \\ \hat{n}_p s^{n-2} \\ \hat{n}_p s^{n-1} \\ \hat{n}_p s^n \end{bmatrix} r(t) \tag{3.14}$$

for some $T \in \mathbb{R}^{2n\times 2n}$, a non-singular matrix. It follows that w_M is sufficiently rich iff $\bar{w}$ is sufficiently rich. Now by Lemma 3.3 $\bar{w}$ has spectral lines at $\nu_1, \ldots, \nu_{2n}$ of amplitude

$$\frac{1}{\hat{n}_p(j\nu_i)\hat{d}_M(j\nu_i)} \begin{bmatrix} \hat{d}_p(j\nu_i) \\ \vdots \\ \hat{d}_p(j\nu_i)(j\nu_i)^{n-2} \\ \hat{n}_p(j\nu_i) \\ \vdots \\ \hat{n}_p(j\nu_i)(j\nu_i)^n \end{bmatrix} \qquad i = 1, \ldots, 2n$$

By Lemma (3.4) we need only show that these vectors are linearly independent. If not, $\exists$ a row vector $[\beta \vdots \gamma]$ with $\beta^T \in \mathbb{R}^{n-1}$, $\gamma^T \in \mathbb{R}^{n+1}$ such that

$$[\beta \vdots \gamma] \begin{bmatrix} \hat{d}_p(j\nu_1) & \cdots & \hat{d}_p(j\nu_{2m}) \\ \vdots & & \\ \hat{d}_p(j\nu_1)(j\nu_1)^{n-2} & \cdots & \hat{d}_p(j\nu_{2n})(j\nu_{2n})^{n-2} \\ \hat{n}_p(j\nu_1) & \cdots & \hat{n}_p(j\nu_{2n}) \\ \vdots & & \vdots \\ \hat{n}_p(j\nu_1)(j\nu_1)^n & \cdots & \hat{n}_p(j\nu_{2n})(j\nu_{2n})^n \end{bmatrix} = 0 \tag{3.15}$$

Defining $\hat{\beta}(s) = \beta_1 + \beta_2 s + \ldots + \beta_{n-1}s^{n-1}$ and $\hat{\gamma}(s) = \gamma_1 + \gamma_2 s + \ldots + \gamma_{n+1}s^n$, we may write (3.15) as

$$\hat{\beta}(s)\hat{d}_p(s) + \hat{\gamma}(s)\hat{n}_p(s) = 0 \text{ at } s = j\nu_1, \ldots, j\nu_{2n} \tag{3.16}$$

The polynomial in (3.16) has degree (2n-1) so we conclude that it is identically 0 and

$$\hat{\beta}\hat{d}_p \equiv - \hat{\gamma}\hat{n}_p.$$

But, since $\hat{n}_p$ and $\hat{d}_p$ are coprime (by assumption) the zeros of $\hat{\beta}$ must include those of $\hat{n}_p$. But this is impossible since $\hat{\beta}$ has degree n-2 and $\hat{n}_p$ has degree (n-1). This establishes the contradiction. Thus $\bar{w}$ and hence w_M are sufficiently rich. □

Comments: (1) We say that r(t) is persistently exciting at frequencies $\nu_1, \ldots \nu_{2n}$ if it has spectral lines at these frequencies. We have shown that when the reference input is persistently exciting at as many frequencies as there are unknown parameters, then w(t) is sufficiently rich resulting in exponential parameter and error convergence.

(2) $r(t)$ does not have to be almost periodic [7] to satisfy the conditions of Theorem 3.5. It need only have spectral lines at 2n frequencies. Further the strength of the spectral lines figures only in an estimate of the rate of exponential convergence (which may be derived using the techniques of [4]). In particular a low intensity persistently exciting signal (i.e., having 2n spectral lines) may be added to the $r(t)$ that needs to be tracked in the model to guarantee parameter convergence - see also remark 6 below.

(3) It is not widely appreciated in the literature that parameter convergence may not occur (even to an incorrect value), unless the signal $w(t)$ is sufficiently rich. If it were known that $\lim_{t\to\infty} \theta(t)$ exists, a more elementary proof can be given - though the convergence proven need not be either exponential or uniform.

(4) The hypothesis of the theorem can be weakened. For instance, we do not need $r(t)$ to have spectral lines at $\nu_1, \ldots, \nu_{2n}$; it is adequate that

$$\lim\sup \left| \frac{1}{T} \int_s^{s+T} r(t) e^{-j\nu_k t} dt \right| > 0 \text{ uniformly in } s$$

for $k = 1, \ldots, 2n$.

(5) Most periodic functions (specifically, those having at least 2n non-zero Fourier coefficients) for $r(t)$ yield exponential parameter convergence.

(6) Our estimate for the rate of convergence of the parameter error given the magnitude of the spectral line would (in principle) proceed as follows: use the estimates of Lemma (3.4) to obtain the α, δ in the

definition of sufficient richness for w_M. Then, use the prior bounds on parameter and initial error to bound the L^2 difference between w and w_M, and obtain using Theorem (3.1) the α, δ in the definition of sufficient richness for w. From here, the techniques of [4] may be used to obtain a (conservative!) rate of convergence estimate.

Section 4. Parameter Convergence when the Relative Degree ≥ 2

Consider first the relative degree 2 Case of Section 2.2. In this case, the sufficient richness condition for exponential parameter and error convergence is on the signal vector $\zeta(t)$ of (2.6), i.e. $\exists\alpha$, $\delta > 0$, $\forall\, s \in \mathbb{R}_+$

$$\int_s^{s+\delta} \zeta\zeta^T dt \geqq \alpha I. \tag{4.1}$$

Even though the adaptive scheme has changed, redraw the model exactly as in Figure 3. Now define from the w_M of the model the signal vector

$$\zeta_M^T = [\hat{L}^{-1}r, \hat{L}^{-1}v_M^{(1)^T}, \hat{L}^{-1}y_M, \hat{L}_M^{-1}v_M^{(2)^T}] \tag{4.2}$$

i.e. ζ_M is obtained by filtering each component of w_M through the stable system with transfer function $\hat{L}^{-1}$. Now, if r(t) has 2n spectral lines we have by Theorem 3.5 that $\hat{w}_M(\nu_1)$, $\hat{w}_M(\nu_2)$, ..., $\hat{w}_M(\nu_{2n})$ are linearly independent. From the definition of ζ_M in (4.2) and the fact that $\hat{L}^{-1}(s)$ is stable, it follows that

$$\hat{\zeta}_M(\nu_i) = \frac{1}{\hat{L}(j\nu_i)} \hat{w}_M(\nu_i) \quad i = 1, \ldots, 2n$$

are linearly independent. Hence ζ_M is sufficiently rich.

Further, the stability proof [1] yields that $\zeta(\cdot) - \zeta_M(\cdot) \in L^2$, so that ζ is sufficiently rich thereby guaranteeing exponential parameter convergence.

Now consider the scheme of Figure 2 for the relative degree ≥ 3 case. Redraw the model as in Figure 3 and define ζ_M as in (4.2) above. The same argument, as above, yields that when r has 2n spectral lines then ζ_M is sufficiently rich. Further since (see [2] for the proof)

$$w(\cdot) - w_M(\cdot) \in L^2$$

and $\hat{L}(s)$ is stable, it follows that

$$\zeta(\cdot) - \zeta_M(\cdot) \in L^2$$

so that ζ is sufficiently rich as well. This guarantees parameter error convergence.

Thus, we see that for each of the model Reference Adaptive Schemes of [1,2] it follows that <u>r(t) has 2n spectral lines</u> $\Rightarrow$ <u>exponential parameter convergence</u>. Further, given prior bounds on the parameters and plant states, an estimate of the rate of convergence can be given.

5. Concluding Remarks

We have shown that continuous time MRAS systems exhibit parameter convergence when the reference input r(t) has 2n spectral lines. The same result also holds for the discrete time algorithm of Narendra-Lin [11] as well, with the obvious modification in the definition of spectral lines for discrete-time signals.

Further, we feel that the machinery of generalized harmonic analysis will be useful in other problems in adaptive control as well, indeed it is well suited to the analysis of asymptotically linear line invariant systems. We conclude by proving the following interesting proposition:

Proposition 5.1

Let $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ be a controllable pair and let the input u to the system

$$\dot{x} = Ax + bu$$

have n spectral lines. Then, if A is exponentially stable, x is sufficiently rich.

Proof: By suitable change of coordinates we may assume that (A,b) are in controllable canonical form so that the transfer function from u to x is

$$\frac{1}{\hat{p}(s)} \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n-1} \end{bmatrix} \quad \text{with } \hat{p}(s) = \det(sI-A)$$

Since A is exp. stable, so is this transfer function. If u has spectral lines at $\nu_1, \ldots, \nu_n$ then so does x. The spectral amplitudes are

$$\hat{x}(\nu_i) = \frac{\hat{u}(\nu_i)}{\hat{p}(j\nu_i)} \begin{bmatrix} 1 \\ \vdots \\ (j\nu_i)^{n-1} \end{bmatrix} \qquad i = 1, \ldots, n.$$

But the $\hat{x}(\nu_i)$ are linearly independent since

$$\det \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \\ (j\nu_1)^{n-1} & \cdots & (j\nu_n)^{n-1} \end{bmatrix} = \pm \prod_{i<j} (j\nu_i - j\nu_j) \neq 0$$

By Lemma 3.4, then, x is sufficiently rich. □

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References

1. Narendra, K. S., Valavani, L. S., "Stable Adaptive Controller Design - Direct Control," IEEE Trans. on Auto. Control, Vol. AC-23 (1978) pp. 570-583.
2. Narendra, K. S., Lin, Y-M., Valavani, L. S., "Stable Adaptive Controller Design, Part II: Proof of Stability ," IEEE Trans. on Auto Control, Vol. AC-25 (1980) pp. 440-448.
3. Morgan, A. P., Narendra, K. S., "On the Uniform Asymptotic Stability of Certain Linear Non-Autonomous Differential Equations," SIAM J. Control and Opt., Vol. 15 (1977) pp. 5-24.
4. Anderson, B. D. O., "Exponential Stability of Linear Equations Arising in Adaptive Identification," IEEE Trans. on Auto Control, Vol. AC-22 (1977), pp. 83-88.
5. Kreisselmeier, G., "Adaptive Observers with Exponential Rate of Convergence," IEEE Trans. on Auto. Control, Vol. AC-22 (1977) pp. 2-9.
6. Wiener, N., "Generalized Harmonic Analysis," Acta Mathematica Vol. 55 (1930), pp. 117-258.
7. Corduneanu, C., Almost Periodic Functions, Interscience Publishers, John Wiley, 1968.
8. Morse, A. S., "Global Stability of Parameter-Adaptive Control Systems," IEEE Trans. on Auto Control, Vol. AC-25 (1980), pp. 433-440.
9. Rohrs, C. E., Adaptive Control in the Presence of Unmodelled Dynamics, Ph.D. Thesis, MIT, Nov. 1982, also L.I.D.S. report no. TH-1254.

10. Ioannou, P. A., Robustness of Model Reference Adaptive Schemes with Respect to Modelling Errors, Ph.D. Thesis, Univ. of Illinois, Urbana-Champaign, August 1982, also D. C. Report no. 53.

11. Narendra, K. S. and Lin Y-H., "Stable Discrete Adaptive Control," IEEE Trans. Auto. Control, Vol. AC-25 (1980) pp. 456-461.

12. Anderson, B. D. O. and Johnstone, R. M., "Adaptive Systems and Time-Varying Plants," Int. J. of Control, Vol. 37 (1983) pp. 367-377.

13. Yuan, J. S-C. and Wonham, W. M., "Probing Signals for Model Reference Identification," IEEE Trans. Auto. Control, Vol. AC-22 (1977) pp. 530-538.

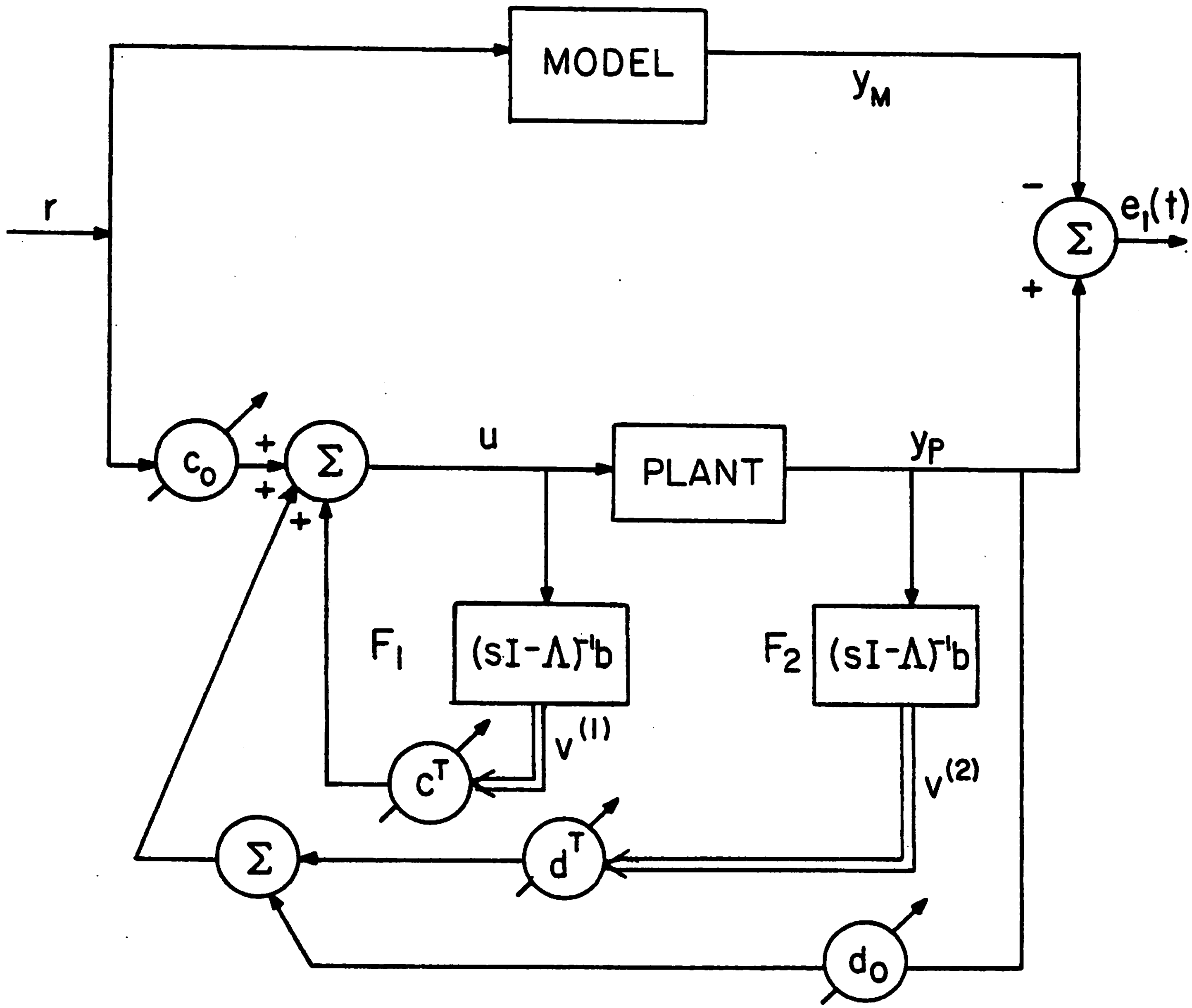

Fig. 1. The adaptive system for the relative degree 1 case.

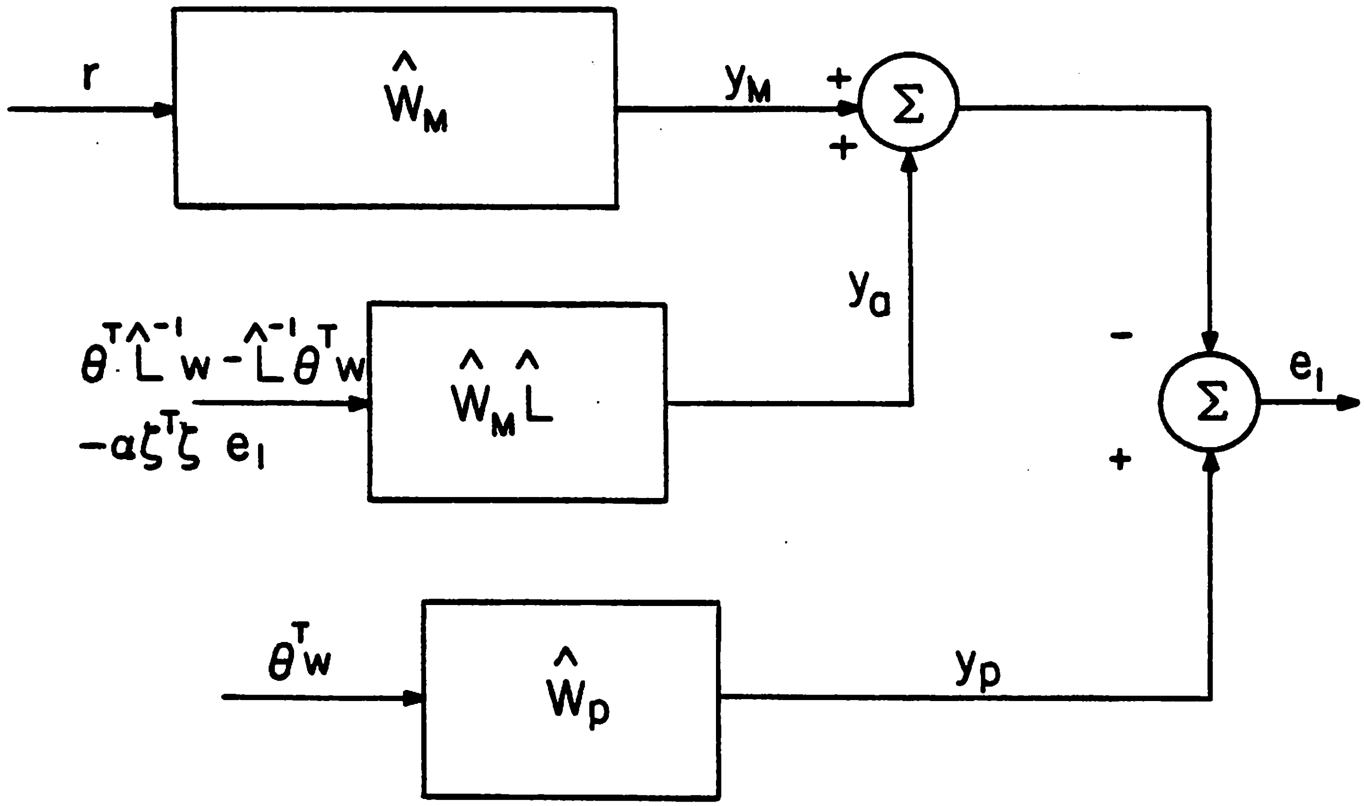

Fig. 2. Schematic of the adaptive system when relative degree $\geq$ 3.

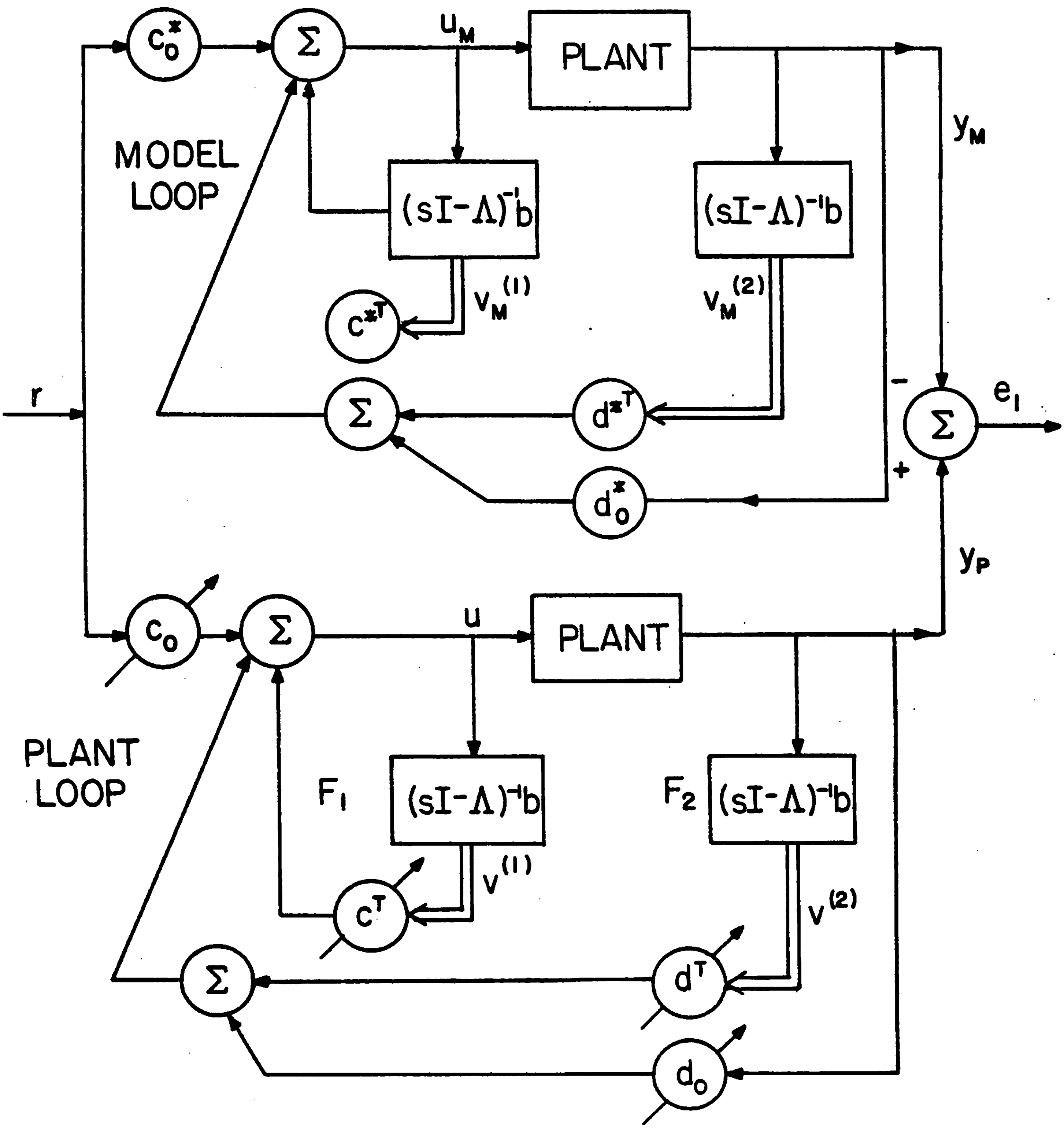

Fig. 3. The adaptive system of Figure 1 with a new representation for the model.