INPUT-OUTPUT DESCRIPTION OF LINEAR SYSTEMS WITH MULTIPLE TIME SCALES

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Abstract

We study the multiple time scales structure of linear systems of the form

$$\dot{x} = A_0(\varepsilon)x + B_0(\varepsilon)u$$

$$y = C_0(\varepsilon)x$$

with a view to obtaining "approximate" lower order transfer functions valid at different time scales. Our development includes the classical two time scales case as well. We use our results to study the positive realness of linear systems with multiple time scales in terms of the positive realness of the reduced order transfer functions.

Keywords: multiple time scales, singular perturbation, positive real.

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Section I. Introduction

It has been widely recognized that weak couplings are responsible for the multiple time scales evolution of linear systems. The study of systems evolving at multiple time scales is simplified by studying reduced order models of these systems valid at specific time scales. Such reduced order models are obtained by assuming that the dynamics at faster time scales has reached its equilibrium and that at slower time scales has yet to evolve. In Coderch et al. [1], the multiple time scales structure of the autonomous linear system

$$\dot{x} = A_0(\varepsilon)x$$

is studied. Motivated by their results we study linear, multiple time scale input-output systems of the form

$$\begin{aligned} \dot{x} &= A_0(\varepsilon)x + B_0(\varepsilon)u \\ y &= C_0(\varepsilon)x \end{aligned} \tag{1.1}$$

with a view to obtaining approximate, lower order transfer functions valid at different time scales. We relate the time scale behavior at these separate time scales to the overall input-output behavior of the system (1.1).

There is an extensive literature on two-time scale singularly perturbed systems (see Kokotovic [3] for a review) and an input-output time scales decomposition has been suggested in the work of Porter and Shenton [4], and Luse and Khalil [5]. Our own work has been motivated by the paper of Saksena and Kokotovic [6], who show (in the two-time scale case) that the system (1.1) has a strictly positive real transfer function if the reduced order slow transfer function and the fast transfer function are strictly positive real. Our contribution in this paper is as follows: We utilize the methods of Coderch, et al. [1], which we review in Section 2 to obtain the input-output description of linear systems with two time scales in a form resembling that of Luse and Khalil [5], it however extends naturally to the multiple time scale case (Section 3). We use this description in Section 4 to generalize the results of Kokotovic and Saksena [6] on the positive realness of two-time scale transfer functions. Our method of proof is, however, quite different. Finally, in Section 5, we define and discuss the input-output behavior at different time scales of the system (1.1) and positive realness of multiple time scale linear systems.

The reason for the interest in the positive realness of multiple time scale systems is its relevance to the study of the robustness of adaptive control schemes to unmodelled dynamics. Roughly speaking we have shown here that positive realness of a reduced order transfer function at one time scale is preserved only if all the faster time scale dynamics (unmodelled dynamics) are also positive real! This implies non-robustness of the positive realness of a transfer function to unmodelled dynamics.

Section 2. Mathematical Preliminaries and Review

In this section we review briefly (a) some facts from the perturbation theory of linear operators (from Kato [2]) and (b) some of the results on the multiple time scales decomposition of autonomous linear differential equations (from Coderch et al. [1]) of the form

$$\dot{x} = A(\varepsilon)x$$

with $A(\varepsilon)$, an analytic function of ε.

2.1. Perturbation Theory for Linear Operators

Consider a linear map; $T : \mathbb{C}^n \to \mathbb{C}^n$. $\sigma(T)$, the set of all eigenvalues of T is called the spectrum of T. The function $R(\xi,T) : \mathbb{C} - \sigma(T) \to \mathbb{C}^{n\times n}$ defined by

$$R(\xi,T) = (T-\xi I)^{-1}$$

is called the <u>resolvent</u> of T. The resolvent of T is an analytic function with singularities at $\lambda_k \in \sigma(T)$, $k = 1, \ldots, s$. The Laurent series of $R(\xi,T)$ at λ_k has the form

$$R(\xi,T) = -(\xi-\lambda_k)^{-1}P_k - \sum_{i=1}^{m_k-1} (\xi-\lambda_k)^{-i-1}D_k^i + \sum_{i=0}^{\infty} (\xi-\lambda_k)^i S_k^{i+1} \tag{2.1}$$

where

$$P_k := \frac{-1}{2\pi i}\int_{\Gamma_k} R(\xi,T)d\xi \in C^{n\times n} \tag{2.2}$$

(with Γ_k a positively oriented contour enclosing λ_k but no other eigenvalue of T) is a projection (i.e. $P_k^2 = P_k$) called the <u>eigenprojection</u> of λ_k;

$$m_k := \dim R(P_k)$$

is the <u>algebraic multiplicity</u> of λ_k.

$$D_k := \frac{-1}{2\pi i}\int_{\Gamma_k} (\xi-\lambda_k)R(\xi,T)d\xi. \tag{2.3}$$

is the <u>eigennilpotent</u> (i.e., $D_k^{m_k} = 0$) for λ_k, and

$$S_k = \frac{1}{2\pi i}\int_{\Gamma_k} (\xi-\lambda_k)^{-1}R(\xi,T)d\xi. \tag{2.4}$$

It is known that

$$P_k P_\ell = \delta_{k\ell} P_k \tag{2.5}$$

$$C^n = R(P_1) \oplus \ldots \oplus R(P_s) \tag{2.6}$$

and that $R(P_k)$ is the <u>generalized eigenspace</u> for the eigenvalue λ_k. Further, the <u>spectral representation</u> of T is

$$T = \sum_{k=1}^{s} (\lambda_k P_k + D_k). \tag{2.7}$$

An eigenvalue λ_k is said to be <u>semisimple</u> if its associated <u>eigennilpotent</u> D_k is zero.

We now discuss the perturbation of a linear operator $T(\varepsilon)$ of the form

$$T(\varepsilon) = T + \sum_{n=1}^{\infty} \varepsilon^n \, T^{(n)} \quad \varepsilon \in [0,\varepsilon_0] \tag{2.8}$$

Here (2.8) is assumed to be an absolutely convergent power series expansion. The eigenvalues of $T(\varepsilon)$ satisfy

$$\det(T(\varepsilon)-\xi I) = 0 \tag{2.9}$$

This is an algebraic equation in ξ whose coefficients are ε-analytic. From analytic function theory it follows that the roots of (2.9) are branches of analytic functions of ε with only algebraic singularities. Hence the number of (distinct) eigenvalues of $T(\varepsilon)$ is a constant s, independent of ε, except at some isolated values of ε. Without loss of generality let $\varepsilon = 0$ be such an exceptional point and further let it be the only such point in $[0,\varepsilon_0]$. In the neighborhood of the exceptional point, the eigenvalues of $T(\varepsilon)$ can be expressed by s distinct, analytic functions: $\lambda_1(\varepsilon), \ldots, \lambda_s(\varepsilon)$. These may be grouped as

$$\{\lambda_1(\varepsilon), \ldots, \lambda_p(\varepsilon)\}, \{\lambda_{p+1}(\varepsilon), \ldots, \lambda_{p+\ell}(\varepsilon) , \ldots$$

so that each group has a Puiseux series of the form (written below for the first group)

$$\lambda_h(\varepsilon) = \lambda + \alpha_1\omega^h\varepsilon^{1/p} + \alpha_2\omega^{2h}\varepsilon^{2/p} + \ldots \qquad h = 1, \ldots, p$$

where λ is an eigenvalue of the unperturbed operator T and $\omega = \exp\{i2\pi/p\}$. Each group is called a cycle, λ is called the center of the cycle and the group of eigenvalues having λ as center is called the λ-group splitting at $\varepsilon = 0$. The perturbations of the resolvent and eigenprojection are discussed next:

Proposition 2.1 [1], [2]

If $\xi \notin \sigma(T)$, then $\xi \notin \sigma(T(\varepsilon))$ for $\varepsilon \in [0,\varepsilon_0]$ and

$$R(\xi,T(\varepsilon)) = R(\xi,T) + \sum_{n=1}^{\infty} \varepsilon^n R^{(n)}(\xi) \tag{2.10}$$

where

$$R^{(n)}(\xi) = \sum_{\substack{\nu_1+\ldots+\nu_p=n \\ \nu_i\geq 1}} (-1)^p R(\xi,T)T^{(\nu_1)}R(\xi,T)T^{(\nu_2)} \ldots T^{(\nu_p)}R(\xi,T) \tag{2.11}$$

the sum being taken over all integers p and $\nu_1, \ldots, \nu_p \geq 1$ satisfying $\nu_1 + \ldots + \nu_p = n$. The series (2.10) is uniformly convergent on compact subsets of $\mathbb{C} - \sigma(T)$.

Let λ be an eigenvalue of $T = T(0)$ with (algebraic) multiplicity m. Let Γ be a closed contour in $\mathbb{C} - \sigma(T)$ enclosing λ but no other eigenvalues of T. From the proposition 2.1 above, it follows that for ε small enough

$R(\xi, T(\varepsilon))$ is well defined for $\xi \in \Gamma$ (i.e., there are no eigenvalues of $T(\varepsilon)$ on Γ). Further, the matrix

$$P(\varepsilon) = \frac{-1}{2\pi i} \int_{\Gamma} R(\xi, T(\varepsilon)) d\xi \tag{2.12}$$

is a projection which is equal to the sum of the eigenprojections for all the eigenvalues of $T(\varepsilon)$ lying inside Γ. Using (2.10) and integrating term by term (recall uniform convergence from Proposition 2.1), we have

$$P(\varepsilon) = P + \sum_{n=1}^{\infty} \varepsilon^n P^{(n)} \text{ for } \varepsilon \in [0, \varepsilon_0] \tag{2.13}$$

where

$$P = \frac{-1}{2\pi i} \int_{\Gamma} R(\xi, T) d\xi \tag{2.14}$$

and

$$P^{(n)} = \frac{-1}{2\pi i} \int_{\Gamma} R^{(n)}(\xi) d\xi. \tag{2.15}$$

$P(\varepsilon)$ is called the <u>total projection</u> and $R(P(\varepsilon))$ the <u>total eigenspace</u> for the λ-group of eigenvalues of $T(\varepsilon)$.

The following proposition is useful in the results of Section 2.2:

<u>Proposition 2.2</u>

Let λ be an eigenvalue of $T = T(0)$ of algebraic multiplicity m and $P(\varepsilon)$ be the total projection for the λ-group of T. Then

$$\frac{(T(\varepsilon) - \lambda I)P(\varepsilon)}{\varepsilon} = \frac{-1}{2\pi i \varepsilon} \int_{\Gamma} (\xi - \lambda) R(\xi, T(\varepsilon)) d\xi = \frac{D}{\varepsilon} + \sum_{n=0}^{\infty} \varepsilon^n \tilde{T}^{(n)} \tag{2.18}$$

where D is the eigennilpotent for Γ and $\tilde{T}^{(n)}$ is given by

$$\tilde{T}^{(n)} = -\sum_{p=1}^{n+1}(-1)^p \sum_{\substack{\nu_1+\ldots+\nu_p=n+1 \\ k_1+\ldots+k_{p+1}=p-1 \\ \nu_i\geq 1, k_j\geq -m+1}} S^{(k_1)}T^{(\nu_1)}S^{(k_2)}\ldots S^{(k_p)}T^{(\nu_p)}S^{(k_{p+1})} \qquad (2.19)$$

with $S^{(0)} = -P$, $S^{(-k)} = -D^k$ for $k > 0$ and

$$S^{(k)} = [\frac{1}{2\pi i}\int_{\Gamma}(\xi-\lambda)^{-1}R(\xi,T)d\xi]^k \text{ for } k > 0$$

Corollary 2.3

If $\lambda = 0$ is a semi-simple eigenvalue of $T = T(0)$ in Proposition (2.2) above we have

$$\frac{T(\varepsilon)P(\varepsilon)}{\varepsilon} = \sum_{n=0}^{\infty} \varepsilon^n \tilde{T}^{(n)}$$

with the $\tilde{T}^{(n)}$ defined as above, with $\lambda = 0$.

2.2. Multiple Time Scales Structure of Autonomous Linear Systems

We state here the results of Coderch et al. [1] for the multiple time scales structure of the autonomous linear system

$$\dot{x} = A_0(\varepsilon)x \qquad x \in \mathbb{R}^n \qquad (2.20)$$

with

$$A_0(\varepsilon) = \sum_{p=0}^{\infty} \varepsilon^p A_{0p} \qquad \varepsilon \in [0,\varepsilon_0] \qquad (2.21)$$

The matrix $A_0(\varepsilon)$ is assumed to be semistable for each $\varepsilon \in [0,\varepsilon_0]$ (i.e., all eigenvalues in $\mathring{\mathbb{C}}_-$ except for perhaps a semi-simple eigenvalue at 0), with constant rank n for $\varepsilon \in]0,\varepsilon_0]$. As is shown in [1] the system (2.20) exhibits multiple time scales iff

$$\text{rank } A_0(0) < n$$

Further, heuristically there is a connection between the time scales evolution of (2.20) and the eigenvalues of $A(\varepsilon)$. In particular, eigenvalues of order ε^k are symptomatic of system behavior at time scale t/ε^k. We discuss how to make this connection precise. Since $A_0(0)$ has some zero eigenvalues, we can define $P_0(\varepsilon)$ to be the total projection matrix for the zero group of eigenvalues of $A_0(\varepsilon)$. Define

$$A_1(\varepsilon) := \frac{A_0(\varepsilon)P_0(\varepsilon)}{\varepsilon}\,. \tag{2.22}$$

Then if $A_0(0) = A_{00}$ has semi-simple null structure - SSNS (i.e., 0 is a semi-simple eigenvalue of A_{00}) the matrix $A_1(\varepsilon)$ has series expansion of the form

$$A_1(\varepsilon) = \sum_{p=0}^{\infty} \varepsilon^p A_{1p} \tag{2.23}$$

by Corollary (2.3). Intuitively, $A_1(\varepsilon)$ represents the part of $A_0(\varepsilon)$ having eigenvalues which <u>are at most $O(\varepsilon)$</u> - i.e., the zero group of eigenvalues of $A_0(\varepsilon)$ - corresponding to slower dynamics of (2.20). If the first term of the series (2.23), namely A_{10}, has semi-simple null structure it follows that

$$A_2(\varepsilon) := \frac{P_1(\varepsilon)A_1(\varepsilon)}{\varepsilon} = \frac{P_1(\varepsilon)P_0(\varepsilon)A_0(\varepsilon)}{\varepsilon^2}$$

where $P_1(\varepsilon)$ is the total projection for the zero group of eigenvalues of $A_1(\varepsilon)$, has series expansion

$$A_2(\varepsilon) = \sum_{p=0}^{\infty} \varepsilon^p A_{2p}$$

Proceeding recursively in this fashion and assuming successively, that A_{20}, A_{30}, ..., i.e., the leading terms of $A_2(\varepsilon)$, $A_3(\varepsilon)$, ... have SSNS, we define $A_k(\varepsilon)$, $P_k(\varepsilon)$ for $k = 0, \ldots, m$. The recursion ends when

$$A_{m+1}(\varepsilon) = 0 \text{ or equivalently } \sum_{k=0}^{m} \operatorname{rank} A_{k0} = n. \tag{2.16}$$

The assumption that $A_{00}, A_{10}, \ldots, A_{m0}$ have SSNS is referred to as the <u>multiple semi-simple null structure</u> (MSSNS) assumption. Under this assumption, $A_0(\varepsilon)$ represents the fast (time scale t) dynamics, $A_1(\varepsilon)$ the next slower (time scale t/ε) dynamics, $A_2(\varepsilon)$ the following slower (time scale t/ε^2) dynamics, ... Further, the total number of time scales is the stopping point of the above recursion, i.e., m. The following proposition establishes some important properties of the $A_k(\varepsilon)$, $P_k(\varepsilon)$ and the complementary projection to $P_k(\varepsilon)$, i.e., $Q_k(\varepsilon) = I - P_k(\varepsilon)$.

<u>Proposition 2.4</u>.

For ε small enough, including zero and $k = 1, \ldots, m$

(i) $P_i(\varepsilon)P_j(\varepsilon) = P_j(\varepsilon)P_i(\varepsilon) \quad i,j = 0, \ldots, m$

(ii) $Q_i(\varepsilon)Q_j(\varepsilon) = 0 \quad i \neq j \quad i,j = 0, \ldots, m$

(iii) $$\mathbb{C}^n = \mathcal{R}(Q_0(\varepsilon)) \oplus \ldots \oplus \mathcal{R}(Q_k(\varepsilon)) \oplus \mathcal{R}(P_0(\varepsilon)\ldots P_k(\varepsilon)) \tag{2.25}$$

(iv) $$\operatorname{rank} Q_k(\varepsilon) = \operatorname{rank} A_{k0} \tag{2.26}$$

and for ε small enough but not zero

(v) $$Q_k(\varepsilon)A_0(\varepsilon) = \varepsilon^k Q_k(\varepsilon)A_k(\varepsilon) = \varepsilon^k A_k(\varepsilon)Q_k(\varepsilon) = A_0(\varepsilon)Q_k(\varepsilon). \tag{2.27}$$

<u>Remarks</u>: By (2.27) $\mathcal{R}(Q_0(\varepsilon))$ is the generalized eigenspace of order 1 eigenvalues of $A_0(\varepsilon)$, $\mathcal{R}(Q_1(\varepsilon))$ is the generalized eigenspace of the order ε eigenvalues of $A_0(\varepsilon)$ (i.e., the generalized eigenspace of the

order 1 eigenvalues of $A_1(\varepsilon)$), ... (2.25) then is a decomposition of $\mathbb{C}^n$ into eigenspaces corresponding to different orders of eigenvalues of (2.25).

Next, we study the evolution of (2.20) at different time scales:

<u>Definition 2.5</u> (Time Scale Behavior of (2.20))

Consider (2.20) and let $\alpha(\varepsilon)$ be an order function (i.e., $\alpha: [0,\varepsilon_0] \to \mathbb{R}_+$, continuous with $\alpha(0) = 0$ and monotone increasing). $x(t)$ in (2.20) is said to have <u>well defined behavior at time scale $t/\alpha(\varepsilon)$</u> if there exists a bounded, continuous matrix function $Y(t)$ such that, for any $\delta > 0$ and $T < \infty$,

$$\lim_{\varepsilon \downarrow 0} \sup_{t \in [\delta, T]} \|\exp\{A_0(\varepsilon)t/\alpha(\varepsilon)\} - Y(t)\| = 0 \tag{2.28}$$

It is said to have <u>trivial behavior at time scale $t/\alpha(\varepsilon)$</u> if $Y(t)$ can be chosen to be zero in (2.28). □

$A_0(\varepsilon)$ is said to satisfy the multiple semi-stability (MSST) condition, if (i) $A_0(\varepsilon)$ satisfies the MSSNS condition and (ii) the matrices A_{k0}, $k = 0, \ldots, m$ are semi-stable (i.e., all eigenvalues in $\mathring{\mathbb{C}}_-$ except for perhaps a semi-simple eigenvalue at 0). The time scale behavior of (2.20) when $A_0(\varepsilon)$ satisfies the MSST condition is given by.

<u>Theorem 2.6</u> [1]

Let $A_0(\varepsilon)$ satisfy the MSST condition. Then,

(i) $\lim_{\varepsilon \downarrow 0} \sup_{\delta \le t \le T} \|\exp\{A_0(\varepsilon)t/\varepsilon^k\} - \Phi_k(t)\| = 0$

$$\forall\ \delta > 0,\ T < \infty,\ k = 0, 1, \ldots, m-1.$$

(ii) $\lim_{\varepsilon \downarrow 0} \sup_{\delta \le t < \infty} \|\exp\{A_0(\varepsilon)t/\varepsilon^m\} - \Phi_m(t)\| = 0$

where $\Phi_k(t)$ is given by:

$$\Phi_k(t) = Q_k \exp\{A_{k0}t\} + P_0 \cdots P_k$$

$$= P_0 \cdots P_{k-1} \exp A_{k0}t \qquad k = 0, \ldots, m. \qquad \square$$

Section 3. Input-Output Description of Two-Time Scale Linear Systems

In this section we study transfer functions of two-time scale linear, time-invariant systems of the form

$$\left.\begin{aligned} \dot{x} &= A_{11}x + A_{12}z + B_1u & x \in \mathbb{R}^n,\ u \in \mathbb{R}^{n_i} \\ \varepsilon\dot{z} &= A_{21}x + A_{22}z + B_2u & z \in \mathbb{R}^m \\ y &= C_1x + C_2z + Du & y \in \mathbb{R}^{n_0} \end{aligned}\right\} \tag{3.1}$$

We first describe the classical [3] analysis of (3.1) and then use the machinery of Section 2 to derive the desired results. The singularly perturbed approximation to (3.1) obtained by setting $\varepsilon = 0$ yields (with A_{22} <u>assumed non-singular</u>):

$$\left.\begin{aligned} \dot{x} &= A_rx + B_ru \\ y &= C_rx + D_ru \end{aligned}\right\} \tag{3.2}$$

with $A_r := A_{11} - A_{12}A_{22}^{-1}A_{21}$, $B_r := B_1 - A_{12}A_{22}^{-1}B_2$, $C_r := C_1 - C_2A_{22}^{-1}A_{21}$ and $D_r := D - C_2A_{22}^{-1}B_2$. The system (3.2) is of order n and along with the algebraic relation

$$z = -A_{22}^{-1}A_{21}x$$

provides an approximation to the trajectories of the full order (m+n) system (3.1) on compact intervals of time bounded away from the origin provided that $\sigma(A_{22}) \subset \mathring{\mathbb{C}}_-$, i.e., A_{22} is <u>exponentially stable</u>. We also associate with (3.1) the fast system (time scale εt) by rescaling time $\tau = t/\varepsilon$ and then formally taking the limit $\varepsilon \downarrow 0$, namely

$$\left.\begin{aligned} \frac{dz}{d\tau} &= A_{22}z + B_2u \\ y &= C_2z + Du \end{aligned}\right\} \tag{3.3}$$

we denote the transfer function of the reduced system (3.2), $\hat{H}_r(s)$,

$$\hat{H}_r(s) := C_r(sI-A_r)^{-1}B_r + D_r \tag{3.4}$$

and the transfer function of the fast system (3.3), $\hat{H}_f(s)$

$$\hat{H}_f(s) := C_2(sI-A_{22})^{-1}B_2 + D. \tag{3.5}$$

We study the relation between $\hat{H}_r(s)$, $\hat{H}_f(s)$ and the transfer function of the full system $\hat{H}_\varepsilon(s)$. First note that from the definition of D_r we have

$$\lim_{s\to\infty} \hat{H}_r(s) = D_r = \lim_{s\to 0} \hat{H}_f(s) \tag{3.6}$$

A laborious calculation yields that

$$\hat{H}_\varepsilon(s) = \hat{H}_r(s) + \hat{H}_f(\varepsilon s) - D_r + O(\varepsilon) \tag{3.7}$$

The $O(\varepsilon)$ is a matrix whose elements are uniformly of order ε in the common domain of definition of $\hat{H}_r(s)$, $\hat{H}_\varepsilon(s)$, $\hat{H}_f(\varepsilon s)$. The exact form of the $O(\varepsilon)$ term is unintuitive and is omitted. Expressions (3.6) and (3.7) taken together are suggestive - the overall transfer function $\hat{H}_\varepsilon$ is (up to an error of order ε), the sum of the reduced order transfer function $\hat{H}_r(s)$, the frequency scaled fast transfer function $H_f(\varepsilon s)$, minus the d.c. value of $\hat{H}_f(s)$ (or equivalently the high frequency asymptote of $\hat{H}_r(s)$). The form (3.7) has also been derived by Luse and Khalil [5].

From our standpoint the $O(\varepsilon)$ term is unwieldy and the preceding technique does not generalize to the multiple time scale case. Hence, we use the techniques of Section 2 to derive an alternate expression for $\hat{H}_\varepsilon(s)$.

To convert the system (3.1), which is singular at $\varepsilon=0$ to the form studied in Section 2.2, consider the undriven form of (3.1) in the $\tau = t/\varepsilon$ time scale

$$\begin{pmatrix} x' \\ z' \end{pmatrix} = \begin{pmatrix} \varepsilon A_{11} & \varepsilon A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} =: A_0(\varepsilon) \begin{pmatrix} x \\ z \end{pmatrix} \tag{3.8}$$

with x', z' representing $\frac{dx}{d\tau}$, $\frac{dz}{d\tau}$ respectively. Then we have

$$A_0(\varepsilon) = \begin{pmatrix} 0 & 0 \\ A_{21} & A_{22} \end{pmatrix} + \varepsilon \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix} \tag{3.9}$$

Under the assumption that A_{22} is nonsingular we see that A_{00} the first term of (3.9) has SSNS with n eigenvalues at 0. Let $P_0(\varepsilon)$ be the projection onto the zero group of eigenvalue of $A_0(\varepsilon)$. Then, using formulae (2.13)-(2.15) of Section 2 for $P_0(\varepsilon)$ we see that

$$P_0(\varepsilon) = P_0(0) + \sum_{n=1}^{\infty} \varepsilon^n P_0^{(n)} \tag{3.10}$$

with

$$P_0(0) = \frac{-1}{2\pi i} \int_\Gamma \begin{bmatrix} -\xi I & 0 \\ A_{21} & A_{22}-\xi I \end{bmatrix}^{-1} d\xi.$$

$$= \frac{-1}{2\pi i} \int_\Gamma \begin{bmatrix} -\xi^{-1} I & 0 \\ (A_{22}-\xi I)^{-1} A_{21} \xi^{-1} & (A_{22}-\xi I)^{-1} \end{bmatrix} d\xi$$

$$= \begin{bmatrix} I & 0 \\ -A_{22}^{-1} A_{21} & 0 \end{bmatrix} \tag{3.11}$$

A somewhat longer calculation using (2.10), (2.11) yields

$$P_0^{(1)} = -\frac{1}{2\pi i} \int_\Gamma R^{(1)}(\xi) d\xi$$

$$= \begin{bmatrix} -A_{12}A_{22}^{-2}A_{21} & -A_{12}A_{22}^{-1} \\ \begin{array}{l} -A_{22}^{-2}A_{21}A_{11} \\ +A_{22}^{-2}A_{21}A_{12}A_{22}^{-1}A_{21} \\ +A_{22}^{-1}A_{21}A_{12}A_{22}^{-2}A_{21} \end{array} & A_{22}^{-1}A_{21}A_{12}A_{22}^{-1} \end{bmatrix} \tag{3.12}$$

(3.10), (3.11), (3.12) yield the form of $P_0(\varepsilon)$ up to terms of order ε^2.

We block diagonalize the system (3.8) using the change of basis $T : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ defined by

$$T(\varepsilon) := \left[P(\varepsilon) \begin{bmatrix} I_{nxn} \\ \hline 0_{mxn} \end{bmatrix} \;\vdots\; (I-P(\varepsilon)) \begin{bmatrix} 0_{nxm} \\ \hline I_{mxm} \end{bmatrix} \right]^{-1} \tag{3.13}$$

Using (3.11) we see that at $\varepsilon = 0$

$$T(0) = \begin{bmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & I \end{bmatrix}^{-1} \tag{3.14}$$

which is well defined. By continuity then, $T(\varepsilon)$ is well defined in some interval $[0,\varepsilon_0]$. By construction the first n columns of $T(\varepsilon)^{-1}$

span the generalized eigenspace of the 0-group of eigenvalues. The next m columns span the generalized eigenspace of the order 1 eigenvalues.

Thus

$$T(\varepsilon)A(\varepsilon)T(\varepsilon)^{-1} = \left[\begin{array}{c|c} \varepsilon\bar{A}_r(\varepsilon) & 0 \\ \hline 0 & \bar{A}_{22}(\varepsilon) \end{array}\right] \tag{3.15}$$

From (3.14) it is immediate that $\bar{A}_{22}(0) = A_{22}$. However, in order to obtain $\bar{A}_r(0)$ we must use the form of $T(\varepsilon)$ accurate up to terms of order ε^2, i.e., use equations (3.11) and (3.12) to get that

$$T^{-1}(\varepsilon) := \begin{bmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} -A_{12}A_{22}^{-2}A_{21} & A_{12}A_{22}^{-1} \\ -A_{22}^{-2}A_{21}A_{11} + A_{22}^{-2}A_{21}A_{12}A_{22}^{-1}A_{21} + A_{22}^{-1}A_{21}A_{12}A_{22}^{-2}A_{21} & -A_{22}^{-1}A_{21}A_{12}A_{22}^{-1} \end{bmatrix} + 0(\varepsilon^2) \tag{3.16}$$

and

$$T(\varepsilon) = \begin{bmatrix} I & 0 \\ A_{22}^{-1}A_{21} & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} A_{12}A_{22}^{-2}A_{21} & A_{12}A_{22}^{-1} \\ A_{22}^{-2}A_{21}A_{11} - A_{22}^{-2}A_{21}A_{12}A_{22}^{-1}A_{21} & 0 \end{bmatrix} + 0(\varepsilon^2) \tag{3.17}$$

Using (3.16) and (3.17) in (3.15) we find that

$$\bar{A}_r(0) = A_{11} - A_{12}A_{22}^{-1}A_{21} \tag{3.18}$$

Now the system (3.1) in the τ time scale is

$$\begin{bmatrix} x' \\ z' \end{bmatrix} = A_0(\varepsilon) \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} \varepsilon B_1 \\ B_2 \end{bmatrix} u$$

$$y = [C_1 \quad C_2] \begin{bmatrix} x \\ z \end{bmatrix} + Du$$

Using the transformation matrix $T(\varepsilon)$ of (3.17) and defining $\begin{bmatrix} \tilde{x} \\ \tilde{z} \end{bmatrix} = T(\varepsilon) \begin{bmatrix} x \\ z \end{bmatrix}$, and converting back to the t-time scale we see that the diagonalized system satisfies

$$\left.\begin{aligned} \begin{bmatrix} \dot{\tilde{x}} \\ \varepsilon\dot{\tilde{z}} \end{bmatrix} &= \begin{bmatrix} \bar{A}_r(\varepsilon) & 0 \\ 0 & \bar{A}_{22}(\varepsilon) \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{z} \end{bmatrix} + \begin{bmatrix} \bar{B}_r(\varepsilon) \\ \bar{B}_2(\varepsilon) \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{z} \end{bmatrix} \\ y &= [\bar{C}_r(\varepsilon) \quad \bar{C}_2(\varepsilon)] \begin{bmatrix} \tilde{x} \\ \tilde{z} \end{bmatrix} + Du \end{aligned}\right\} \tag{3.19}$$

with $\bar{B}_r(0) = B_1 - A_{12}A_{22}^{-1}B_2$, $\bar{B}_2(0) = B_2$, $\bar{C}_r(0) = C_1 - C_2A_{22}^{-1}A_{21}$, $\bar{C}_2(0) = C_2$. Now, by inspection of (3.19) we have

$$\hat{H}_\varepsilon(s) = \bar{C}_r(\varepsilon)(sI-\bar{A}_r(\varepsilon))^{-1}\bar{B}_r(\varepsilon) + \bar{C}_2(\varepsilon)(\varepsilon sI-\bar{A}_{22}(\varepsilon))^{-1}\bar{B}_2(\varepsilon) + D \tag{3.20}$$

The form (3.20) is very similar to (3.7). In particular, the last two terms constitute (up to an error of order ε) $\hat{H}_f(\varepsilon s)$. At $\varepsilon = 0$ they also yield $D - C_2A_{22}^{-1}B_2 = D_r$. The form (3.20) is more convenient for analytic purposes since all the matrices in (3.20) are continuous functions of ε (cf. the results of Section 4). Further, this block diagonalization technique is useful for systems with multiple time scales (cf. Section 5).

Section 4. Positive Realness of Two Time Scale Systems

Recently, in problems of adaptive control, the robustness of the strict positive realness condition has been a topic of discussion. In this section we discuss the positive realness of a square ($n_i=n_0$) two time scale transfer function $\hat{H}_\varepsilon(s)$ in terms of the positive realness of $\hat{H}_r(s)$ and $\hat{H}_f(s)$. Our results are a generalization of related results derived by other methods in Saksena and Kokotovic [6] in the special case that $\hat{H}_f(s)$ is a positive constant. Our techniques enable the discussion in this section to be generalized to the multiple time scale case.

Definition 4.1 [7]

A matrix $\hat{Z}(s) \in \mathbb{R}(s)^{n_i \times n_i}$ is said to be positive real (PR) if $\hat{Z}(s)$ is analytic in $\mathring{\mathbb{C}}_+$, any purely imaginary pole $j\omega_0$ of $\hat{Z}(s)$ is simple, with associated residue matrix positive semi-definite Hermitian and for all other $j\omega$ $(\omega \in \mathbb{R})$, $\hat{Z}^*(j\omega) + \hat{Z}(j\omega)$ is positive semi-definite Hermitian.

Definition 4.2

A matrix $\hat{Z}(s) \in \mathbb{R}(s)^{n_i \times n_i}$ is said to be strictly positive real (SPR) if $\hat{Z}(s)$ is analytic in $\mathbb{C}_+$, the closed right half plane, and for all $\omega \in \mathbb{R}$, $\hat{Z}^*(j\omega) + \hat{Z}(j\omega)$ is positive definite Hermitian.

The key to our results is the following theorem.

Theorem 4.1

Let $A(\varepsilon) \in \mathbb{R}^{n \times n}$, $B(\varepsilon) \in \mathbb{R}^{n \times n_i}$, $C(\varepsilon) \in \mathbb{R}^{n_i \times n}$ be continuous functions of ε and $D \in \mathbb{R}^{n_i \times n_i}$. Then the transfer function

$$\hat{H}_\varepsilon(s) = C(\varepsilon)(sI - A(\varepsilon))^{-1}B(\varepsilon) + D \tag{4.1}$$

is SPR for $\varepsilon \in [0,\varepsilon^*]$ if $\hat{H}_0(s)$ is SPR.

Proof: It is easy to see that $\hat{H}_0(s)$ analytic in the right half plane implies that $\hat{H}_\varepsilon(s)$ is analytic in $\mathbb{C}_+$ for ε small enough. Thus, we only need show

$$\hat{H}_\varepsilon^*(j\omega) + \hat{H}_\varepsilon(j\omega) > 0 \qquad \forall\ \omega \in \mathbb{R}.$$

First we claim that given $\omega_0 > 0$ $\exists$ $\gamma > 0$ such that

$$\hat{H}_0^*(j\omega) + \hat{H}_0(j\omega) > \frac{\gamma I}{\omega^2} \text{ for } |\omega| > \omega_0. \tag{4.2}$$

(4.2) is true when $n_i = 1$ (i.e., the scalar case) since in that case the SPR condition implies that the proper rational function $\hat{H}_0$ can be of relative degree 0 or 1. For the general case consider $x^*\hat{H}_0(j\omega)x$ for $x \in \mathbb{C}^n$. If (4.2) is false, then for fixed ω_0 there exist sequences $\{x_i, x_i \in \mathbb{C}^{n_i}, \|x_i\|=1\}$ and $\{\omega_i : |\omega_i| \geq \omega_0\}$ such that

$$\omega_i^2 x_i^*[\hat{H}_0(j\omega_i)+\hat{H}_0^*(j\omega_i)]x_i \rightarrow 0 \text{ as } i \rightarrow \infty \tag{4.3}$$

Compactness of the unit ball in $\mathbb{C}^{n_i}$ implies that we may assume $\{x_i\}$ is a convergent sequence with limit $\bar{x} \in \mathbb{C}^{n_i}$. Therefore from (4.3);

$$\omega_i^2 \bar{x}^*[\hat{H}_0(j\omega_i)+\hat{H}_0^*(j\omega_i)]\bar{x} \rightarrow 0 \text{ as } i \rightarrow \infty \tag{4.4}$$

But this is a contradiction since $\bar{x}^*\hat{H}_0(j\omega)\bar{x}$ is a scalar SPR transfer function. (Either $\{\omega_i\}$ has a bounded convergent subsequence so that there exist finite $\bar{\omega} \in \mathbb{R}$ with

$$\bar{x}^*[\hat{H}_0(j\bar{\omega})+\hat{H}_0^*(j\bar{\omega})]\bar{x} = 0$$

or $\{\omega_i\}$ is unbounded and (4.4) violates condition (4.2) for scalar SPR transfer functions).

Now using the Laurent series for $(j\omega I-A(0))^{-1}$ in (4.2) we have for $|\omega| \geq \bar{\omega}_0$, large enough

$$D + D^T - \frac{1}{\omega^2}\{C(0)A(0)B(0)+(C(0)A(0)B(0))^T\} \geq \frac{\gamma I}{\omega^2} \tag{4.5}$$

neglecting higher order terms on the left hand side. Continuity in ε and the fact that D is <u>constant</u> implies that

$$D+D^T-\frac{1}{\omega^2}\{C(\varepsilon)A(\varepsilon)B(\varepsilon)+(C(\varepsilon)A(\varepsilon)B(\varepsilon))^T\}\geq\frac{\gamma I}{2\omega^2} \tag{4.6}$$

for $|\omega|\geq\bar{\omega}_0$ and $\varepsilon\in[0,\varepsilon_0]$. Now, we can add on the extra terms inthe Laurent series for $(j\omega I-A(\varepsilon))^{-1}$ to (4.6) to obtain that

$$\hat{H}_\varepsilon(j\omega)+\hat{H}_\varepsilon^*(j\omega)>\frac{\gamma I}{3\omega^2} \tag{4.7}$$

for $|\omega|>\omega_0$ and $\varepsilon\in[0,\varepsilon_0]$. In the compact region $|\omega|\leq\omega_0$, positive definiteness of $\hat{H}_0(j\omega)+\hat{H}_0^*(j\omega)$ implies that $\exists\bar{\gamma}$ such that

$$\hat{H}_0(j\omega)+\hat{H}_0^*(j\omega)>\bar{\gamma}I \text{ for } |\omega|<\omega_0$$

and continuity in ε yields,

$$\hat{H}_\varepsilon(j\omega)+\hat{H}_\varepsilon^*(j\omega)>\frac{\bar{\gamma}I}{2} \text{ for } |\omega|<\omega_0. \tag{4.8}$$

Combining (4.7), (4.8) yields the desired conclusion. □

<u>Corollary 4.2</u>. Let the assumptions of Theorem (4.1) be in effect except that D is a function of ε with $D(0)+D(0)^*>0$. Then the conclusion of Theorem (4.1) still holds.

<u>Theorem 4.3</u>

Consider the two time scale system of (3.1) with $n_i=n_0$. Further, let the reduced order system $\hat{H}_r(s)$ of (3.4) be PR, analytic in $\mathbb{C}_+$, except perhaps for a simple pole at $s=0$ and the fast transfer function $\hat{H}_f(s)$ be SPR. Then, the transfer function $\hat{H}_\varepsilon(s)$ of (3.1) is PR.

<u>Proof:</u> From the setup of (3.1) we see that $\hat{H}_\varepsilon(0)=\hat{H}_r(0)$ and $\lim_{s\to 0}s\hat{H}_\varepsilon(s)=\lim_{s\to 0}s\hat{H}_r(s)$. Thus, in the case that $\hat{H}_r$ has a simple pole at $s=0$, $\hat{H}_\varepsilon$ also has only a simple pole at $s=0$ with the same residue as

$\hat{H}_r$. Hence, $\hat{H}_\varepsilon$ is analytic in $\mathring{\mathbb{C}}_+$ and the only pole on the $j\omega$-axis is a simple pole at $s = 0$ with positive semi-definite residue matrix (same as $\hat{H}_r$). Hence to prove that $\hat{H}_\varepsilon$ is PR for ε small enough, we only need show that

$$\hat{H}_\varepsilon^*(j\omega) + \hat{H}_\varepsilon(j\omega) \geq 0$$

for ε small enough.

Consider the expression (3.20) for $\hat{H}_\varepsilon(j\omega)$. For $\omega \in$ any compact interval $[-\omega_1, \omega_1]$, $\hat{H}_\varepsilon(j\omega)$ is close to

$$\bar{C}_r(\varepsilon)(sI-\bar{A}_r(\varepsilon))^{-1}\bar{B}_r(\varepsilon) + \bar{C}_2(\varepsilon)(-\bar{A}_{22}(\varepsilon))^{-1}\bar{B}_2(\varepsilon) + D \tag{4.9}$$

which in turn is a perturbation of

$$C_r(sI-A_r)^{-1}B_r + D_r = \hat{H}_r(s) \tag{4.10}$$

i.e., $\bar{C}_r(0) = C_r$, $\bar{A}_r(0) = A_r$, $\bar{B}_r(0) = B_r$ and $D - \bar{C}_2(0)\bar{A}_{22}(0)\bar{B}_2(0) = D_r$. Further, since $\hat{H}_f(s)$ is SPR and $\hat{H}_f(0) = D_r$ (equation (3.6)) we have that $D_r + D_r^T > 0$. Since (4.10) is SPR, (4.9) satisfies the conditions of Corollary 4.2 and we have that (4.9) is SPR. Consequently, for any $\bar{\omega}_1$ $\exists\ \bar{\varepsilon}_1$ small enough such that

$$\hat{H}_\varepsilon(j\omega)^* + \hat{H}_\varepsilon(j\omega) > 0 \qquad \forall\ \omega \in [-\bar{\omega}_1, \bar{\omega}_1], \quad \forall\ \varepsilon \in [0, \bar{\varepsilon}_1] \tag{4.11}$$

Also $\hat{H}_f(s)$ SPR implies from Theorem 4.1 that the transfer function

$$\bar{C}_2(\varepsilon)\ (\varepsilon sI-\bar{A}_{22}(\varepsilon))^{-1}\bar{B}_2(\varepsilon) + D =: \hat{\bar{H}}_f(\varepsilon, s)$$

is SPR. Further from the estimates of Theorem 4.1,

$$\hat{\bar{H}}_f(\varepsilon, j\omega) + \hat{\bar{H}}_f^*(\varepsilon, j\omega) > \frac{\nu I}{1+\varepsilon^2\omega^2}\ ,\ \omega \in \mathbb{R} \tag{4.12}$$

for some $\nu, \varepsilon \in [0, \varepsilon_0]$. Also, in (4.9) we see that $C_r(\varepsilon)\ (sI-A_r(\varepsilon))^{-1}B_r(\varepsilon)$

is strictly proper so that $\exists\ \nu_1, \omega_1, \varepsilon_1, > 0$ such that $\forall\ |\omega| \geq \omega_1,\ \varepsilon \in [0,\varepsilon_1]$

$$C_r(\varepsilon)(j\omega I-A_r(\varepsilon))^{-1}B_r(\varepsilon) + \{C_r(\varepsilon)(j\omega I-A_r(\varepsilon))^{-1}B_r(\varepsilon)\}^* < \frac{\nu_1 I}{\omega^2} \tag{4.13}$$

Given ν and ν_1 the values of ε_1 and ω_1 can be revalued (i.e., ε_1 smaller and ω_1 larger) so that

$$\frac{\nu_1}{\omega^2} < \frac{1}{2}\frac{\nu}{1+\varepsilon^2\omega^2} \qquad \text{for } |\omega| > \omega_1,\ \varepsilon \in [0,\varepsilon_1].$$

Seeing that $\hat{H}_\varepsilon(j\omega)$ is the sum of the left hand side of (4.12), (4.13) we get

$$\hat{H}_\varepsilon(j\omega) + \hat{H}_\varepsilon^*(j\omega) > 0 \tag{4.14}$$

for $|\omega| > \omega_1$, and $\varepsilon < \min(\varepsilon_0,\varepsilon_1)$. Combining (4.11) and (4.14) yields that $\hat{H}_\varepsilon$ is PR for ε small enough. □

Corollary 4.4

If both $\hat{H}_f(s)$ and $\hat{H}_r(s)$ are SPR, the full order system $\hat{H}_\varepsilon(s)$ of (3.1) is SPR for $\varepsilon \in [0,\varepsilon_0]$. □

The following examples illustrate the necessity of the conditions of Theorem (4.3).

Example 4.1 (Fast System Zero)

$$\begin{pmatrix} \dot{x} \\ \varepsilon\dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = (1 \quad 0)\begin{pmatrix} x \\ z \end{pmatrix}$$

$\hat{H}_r(s) = \frac{1}{s+1}$ is SPR, but $\hat{H}_f(s) \equiv 0$ (not SPR). However, $\hat{H}_\varepsilon(s) = \frac{1}{s(\varepsilon s+1)+1}$

is not SPR since its relative degree is 2.

Example 4.2 (Fast System PR but not SPR)

$$\begin{pmatrix} \dot{x} \\ \varepsilon\dot{z} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} u$$

$\hat{H}_r(s) = \frac{3}{s+1}$ which is SPR, but $\hat{H}_f(s) = \frac{s}{s+1}$ which is PR (but not SPR).
$\hat{H}_\varepsilon(s) = \frac{3+\varepsilon s^2}{\varepsilon s^2+s+1}$ so that

$$\text{Re } \hat{H}_\varepsilon(j\omega) = \frac{(3-\varepsilon\omega^2)(1-\varepsilon\omega^2)}{(-\varepsilon\omega^2+1)^2+\omega^2} .$$

This is negative for ω satisfying $3-\varepsilon\omega^2 > 0 > 1-\varepsilon\omega^2$.

Example 4.3 (Reduced System PR, but not SPR)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \varepsilon\dot{z} \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ z \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} u$$

$$y = (0 \quad -1 \quad 1) \begin{pmatrix} x_1 \\ x_2 \\ z \end{pmatrix}$$

$\hat{H}_f(s) = \frac{1}{s+1}$ is SPR and $\hat{H}_0(s) = \frac{(s+1)^2}{s^2+1}$ is PR but not SPR. However the poles of the full system satisfy $\lambda^3 + \frac{\lambda^2}{\varepsilon} - (1+\frac{1}{\varepsilon})\lambda + \frac{1}{\varepsilon} = 0$, showing that the full system is unstable for small ε.

Section 5. Multiple-Time Scale Linear Systems

We consider in this section input-output descriptions of linear systems of the form

$$\dot{x} = A_0(\varepsilon)x + B_0(\varepsilon)u$$
$$y = C_0(\varepsilon)x \tag{5.1}$$

with $A_0(\varepsilon) \in \mathbb{R}^{nxn}$, $B_0(\varepsilon) \in \mathbb{R}^{nxn_i}$ and $C_0(\varepsilon) \in \mathbb{R}^{n_0xn}$. Under the MSST assumption of Section 2 we generalize definition 2.5:

<u>Definition 5.1</u> (Time Scale Behavior of (5.1)

The system (5.1) is said to have <u>well defined behavior at time scale $t/\alpha(\varepsilon)$</u> (with $\alpha(\varepsilon)$ an order function) if there exists a bounded continuous matrix function $Y(t)$ such that for any $\delta > 0$, $T > \infty$,

$$\lim_{\varepsilon \downarrow 0} \sup_{t \in [\delta, T]} \left\| \frac{1}{\alpha(\varepsilon)} C_0(\varepsilon) \exp\{A_0(\varepsilon)t/\alpha(\varepsilon)\} B_0(\varepsilon) - Y(t) \right\| = 0 \tag{5.2}$$

It is said to have <u>trivial behavior at time scale $t/\alpha(\varepsilon)$</u> if $Y(t)$ can be chosen to be zero in (5.2). □

By the MSST condition, $A_0(\varepsilon)$ also satisfies MSSNS so that by Proposition (2.4):

$$A_0(\varepsilon) = \sum_{i=0}^{m} \varepsilon^i Q_i(\varepsilon) A_i(\varepsilon) \tag{5.3}$$

Using (5.3) and the fact that

$$\mathbb{R}^n = R(Q_0(\varepsilon)) \oplus \ldots \oplus R(Q_m(\varepsilon)) \tag{2.25}$$

and $Q_i(\varepsilon)Q_j(\varepsilon) = 0$, we obtain

$$\left.\begin{aligned} Q_i(\varepsilon)\dot{x} &= \varepsilon^i A_i(\varepsilon) Q_i(\varepsilon)x + Q_i(\varepsilon)B_0(\varepsilon)u \\ y_i &= C_0(\varepsilon)Q_i(\varepsilon)x \\ y &= \sum_{i=0}^{m} y_i \end{aligned}\right\} \tag{5.4}$$

equivalent to (5.1). To find conditions under which we obtain (m+1) subsystems with time scale behavior at time scales of order t, t/ε, t/ε^2, ..., t/ε^m respectively, consider a change of basis $T(\varepsilon) \in \mathbb{R}^{n\times n}$ constructed as follows:

First, from Proposition (2.4), we have that rank $Q_k(\varepsilon)$ = rank $A_{k0} =: \rho_k$ for ε small enough. Now choose matrices M_k, $k = 0, 1, \ldots, m$ of dimensions $n \times \rho_k$ such that $Q_k(0)M_k$ is full rank (ρ_k), and define

$$T(\varepsilon) := [Q_0(\varepsilon)M_0 \vdots \cdots \vdots Q_{m-1}(\varepsilon)M_{m-1} \vdots Q_m(\varepsilon)M_m]^{-1} \tag{5.5}$$

$T(\varepsilon)$ in (5.5) is well defined for ε small enough, since it is well defined at $\varepsilon = 0$. By the definition of the projection operators $Q_i(\varepsilon)$, the ρ_i column vectors of $Q_i(\varepsilon)M_i$, $i = 0, \ldots, m$ are an independent set of vectors spanning the 'non-zero group' eigenspace of $A_i(\varepsilon)$, $i = 0, \ldots, m$. Thus, $T(\varepsilon)A_0(\varepsilon)T(\varepsilon)^{-1}$ is of the form

$$\left[\begin{array}{c|c|c|c} \bar{A}_0(\varepsilon) & 0 & 0 & 0 \\ \hline 0 & \varepsilon\bar{A}_1(\varepsilon) & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 \\ \vdots & \vdots & & \vdots \\ \hline 0 & 0 & 0 & \varepsilon^m\bar{A}_m(\varepsilon) \end{array}\right] \tag{5.6}$$

where $\bar{A}_k(\varepsilon) \in \mathbb{R}^{\rho_k \times \rho_k}$ are stable, for ε small enough by the MSST assumption. Define matrices $\bar{B}_k(\varepsilon) \in \mathbb{R}^{\rho_k \times n_i}$ and $\bar{C}_k(\varepsilon) \in \mathbb{R}^{n_0 \times \rho_k}$, by

$$T(\varepsilon)B_0(\varepsilon) = \begin{bmatrix} \bar{B}_0(\varepsilon) \\ \vdots \\ \bar{B}_m(\varepsilon) \end{bmatrix} \tag{5.7}$$

and

$$C_0(\varepsilon)T^{-1}(\varepsilon) = [\bar{C}_0(\varepsilon) \vdots \cdots \vdots \bar{C}_m(\varepsilon)] \tag{5.8}$$

Note that $\bar{B}_k(\varepsilon)$, $\bar{C}_k(\varepsilon)$, $\bar{A}_k(\varepsilon)$ are analytic in ε. Thus, (5.1) may be decoupled into the (m+1) subsystems

$$\left.\begin{aligned} \dot{x}_i &= \varepsilon^i \bar{A}_i(\varepsilon)x_i + \bar{B}_i(\varepsilon)u, \; x_i \in \mathbb{R}^{\rho_i} \\ y_i &= \bar{C}_i(\varepsilon)x_i \\ y &= \sum_{i=0}^{m} y_i \end{aligned}\right\} \tag{5.9}$$

We now give conditions under which they have well defined behavior at time scale t/ε^i. Consider the impulse response of the ith subsystem of (5.9) at time scale t/ε^i (i.e. Definition (5.1)) given by

$$\frac{1}{\varepsilon^i} \bar{C}_i(\varepsilon)\exp\{\bar{A}_i(\varepsilon)t\}\bar{B}_i(\varepsilon) \tag{5.10}$$

A sufficient condition for (5.10) to have a limiting value as $\varepsilon \downarrow 0$ uniformly for $t \in [\delta,T]$ is that

$$\bar{C}_i(\varepsilon)\bar{A}_i^j(\varepsilon)\bar{B}_i(\varepsilon) \sim O(\varepsilon^i) \qquad j = 0,\ldots,\rho_i-1 \tag{5.11}$$

At <u>faster time scales</u> i.e., t/ε^k for $k < i$, the impulse response of the ith subsystem is

$$\frac{1}{\varepsilon^k} \bar{C}_i(\varepsilon)\exp\{\bar{A}_i(\varepsilon)\varepsilon^{i-k}t\}\bar{B}_i(\varepsilon) \tag{5.12}$$

(5.12) has a uniform limit as $\varepsilon \downarrow 0$ for $t \in [\delta,T]$ if

$$\bar{C}_i(\varepsilon)\bar{B}_i(\varepsilon) \sim O(\varepsilon^k) \tag{5.13}$$

(5.13) in turn is implied by (5.11). In fact if (5.11) holds the uniform limit of (5.12) is 0. At <u>slower time</u> scales i.e., t/ε^k for $k > i$, (5.12) has the uniform limit 0 as $\varepsilon \downarrow 0$ for $t \in [\delta,T]$ if $\bar{A}_i(\varepsilon)$ has its eigenvalues in $\mathring{\mathbb{C}}_-$. This in turn is implied by MSST assumption.

Thus, under the MSST assumption and condition (5.11) for $i = 0, \ldots, m$, the subsystems of (5.9) have well defined time scale behavior at all time scales t/ε^k. However, the ith subsystem has <u>non-trivial time-scale behavior only at the time scale t/ε^i</u>. We summarize these results in a proposition:

<u>Proposition 5.2</u>

Consider the decomposition of (5.1) into the form (5.9). Further, assume that $A_0(\varepsilon)$ satisfies the MSST condition and that condition (5.11) holds for $i = 0, \ldots, m$. Then, we have

$$\text{(i)} \quad \lim_{\varepsilon \downarrow 0} \sup_{\delta \le t \le T} \left\| \frac{1}{\varepsilon^k} C_0(\varepsilon) \exp\{A_0(\varepsilon) t/\varepsilon^k\} B_0(\varepsilon) - \phi_k(t) \right\| = 0 \tag{5.14}$$

$\forall\ \delta > 0,\ \forall T < \infty,\ k = 0, \ldots, m-1$

$$\text{(ii)} \quad \lim_{\varepsilon \downarrow 0} \sup_{\delta \le t < \infty} \left\| \frac{1}{\varepsilon^m} C_0(\varepsilon) \exp\{A_0(\varepsilon) t/\varepsilon^m\} B_0(\varepsilon) - \phi_m(t) \right\| = 0 \tag{5.15}$$

$\forall\ \delta > 0$. In (5.14), (5.15) $\phi_k(t)$ is given by the pointwise limit

$$\phi_k(t) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^k} \bar{C}_k(\varepsilon) \exp\{\bar{A}_k(\varepsilon) t\} \bar{B}_k(\varepsilon)$$

for $k = 0, \ldots, m$.

$$\text{(iii)} \quad \left\| C_0(\varepsilon) \exp\{A_0(\varepsilon) t\} B_0(\varepsilon) - \sum_{k=0}^{m} \varepsilon^k \phi_k(\varepsilon^k t) \right\| = o(1) \tag{5.16}$$

uniformly in t.

<u>Remarks</u>: (i) The conditions (5.11), namely

$$\bar{C}_i(\varepsilon) \bar{A}_i^j(\varepsilon) \bar{B}_i(\varepsilon) \sim O(\varepsilon^i) \qquad j = 0, \ldots, \rho_i - 1 \tag{5.11}$$

for $i=0,\ldots,m$ are only sufficient conditions for Proposition 5.2 to hold. A simpler set of conditions that implies (5.11) is

$$\left.\begin{array}{ll} \bar{B}_i(\varepsilon) \sim O(\varepsilon^i) & i = 0, \ldots, m \\ \text{or} & \\ \bar{C}_i(\varepsilon) \sim O(\varepsilon^i) & i = 0, \ldots, m \end{array}\right\} \tag{5.17}$$

(ii) The conditions (5.11), (5.17) are on the matrices $\bar{A}_i(\varepsilon)$, $\bar{B}_i(\varepsilon)$, $\bar{C}_i(\varepsilon)$. In order to obtain conditions on the original system matrices we need to consider the decomposition (5.4) and the impulse response

$$\frac{1}{\varepsilon^i} C_0(\varepsilon)Q_i(\varepsilon)\exp\{\varepsilon^i A_i(\varepsilon)t\}Q_i(\varepsilon)B_0(\varepsilon) \tag{5.18}$$

By the construction of $T(\varepsilon)$ (equation 5.5) it follows that

$$T(\varepsilon)A_i(\varepsilon)T(\varepsilon)^{-1} = \begin{bmatrix} 0 & & & & & & \\ & \ddots & & & & \bigcirc & \\ & & 0 & & & & \\ & & & \bar{A}_i(\varepsilon) & & & \\ & & & & \varepsilon\bar{A}_{i+1}(\varepsilon) & & \\ & \bigcirc & & & & \ddots & \\ & & & & & & \varepsilon^{m-i}\bar{A}_m(\varepsilon) \end{bmatrix}, \tag{5.19}$$

$$C_0(\varepsilon)Q_i(\varepsilon)T(\varepsilon)^{-1} = [0,\ldots,0,\ \bar{C}_i(\varepsilon),\ldots,0], \tag{5.20}$$

and

$$T(\varepsilon)Q_i(\varepsilon)B(\varepsilon) = \begin{bmatrix} 0 \\ \vdots \\ \bar{B}_i(\varepsilon) \\ \vdots \\ 0 \end{bmatrix} \tag{5.21}$$

Using (5.19), (5.20), (5.21) we obtain

$$\frac{1}{\varepsilon^i} C_0(\varepsilon)Q_i(\varepsilon)\exp\{\varepsilon^i A_i(\varepsilon)t\}Q_i(\varepsilon)B_0(\varepsilon) = \frac{1}{\varepsilon^i} C_i(\varepsilon)\exp\{\bar{A}_i(\varepsilon)\varepsilon^i t\}\bar{B}_i(\varepsilon) \tag{5.22}$$

From (5.22) it follows that,

$$C_0(\varepsilon)Q_i(\varepsilon)A_i^j(\varepsilon)Q_i(\varepsilon)B_0(\varepsilon) = \bar{C}_i(\varepsilon)\bar{A}_i^j(\varepsilon)\bar{B}_i(\varepsilon) \qquad j = 0, 1, \ldots$$

Consequently the condition equivalent to equation (5.11) to ensure the existence of a uniform limit as $\varepsilon \downarrow 0$ of (5.22) is

$$C_0(\varepsilon)Q_i(\varepsilon)A_i^j(\varepsilon)Q_i(\varepsilon)B_0(\varepsilon) \sim 0(\varepsilon^i) \tag{5.23}$$

$$i = 0, \ldots, m \qquad j = 0, \ldots, \rho_i - 1.$$

<u>Example 5.3</u> (Three time scale system)

$$\dot{x} = \left[\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}\right] x + \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \varepsilon\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right] u \tag{5.24}$$

$$y = x.$$

The system (5.24) can be shown* to satisfy the MSST condition and diagonalized to yield

$$\dot{x} = \begin{bmatrix} -1+0(\varepsilon^3) & 0 & 0 \\ 0 & -\varepsilon+0(\varepsilon^3) & 0 \\ 0 & 0 & -\varepsilon^2+0(\varepsilon^3) \end{bmatrix} x + \begin{bmatrix} 1+0(\varepsilon^3) \\ \varepsilon+0(\varepsilon^3) \\ -\varepsilon^2+0(\varepsilon^3) \end{bmatrix} u$$

i.e., $\bar{B}_1(\varepsilon) = \varepsilon + 0(\varepsilon^3), \bar{B}_2(\varepsilon) = -\varepsilon^2+0(\varepsilon^3)$. Thus the system satisfies the conditions (5.17) which imply (5.11) for well defined time scale behavior at time scales t, t/ε, t/ε^2.

Conditions (5.11) allow the internal time-scale structure of $A_0(\varepsilon)$ (obtained under the MSST condition) to be reflected in the input-output

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*We discuss computational issues in Section 6.

behavior of (5.1). In general, there will be circumstances under which there are fewer time scales in the input-output description than in the internal dynamics of the system. Consider the following examples 5.4 and 5.5:

<u>Example 5.4</u> (Two time scale system)

$$\dot{x} = \left[\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}\right] x + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} u \tag{5.25}$$

$$y = [(1 \quad 0 \quad 0) + \varepsilon(0 \quad 1 \quad 0)]x. \tag{5.26}$$

Note that the system (5.25) has the same $A_0(\varepsilon)$ as (5.24) so that the internal dynamics contain three time scales t, t/ε, t/ε^2. On diagonalizing the system we obtain

$$\dot{x} = \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & -\varepsilon & 0 \\ 0 & 0 & -\varepsilon^2 \end{bmatrix} + 0(\varepsilon^3)\right) x + \left(\begin{bmatrix} 1-\varepsilon \\ 1 \\ 1-\varepsilon-\varepsilon^2 \end{bmatrix} + 0(\varepsilon^3)\right) u \tag{5.27}$$

$$y = ([1+\varepsilon^2,\varepsilon,\varepsilon]+0(\varepsilon^3))x$$

Thus $\bar{B}_0(\varepsilon) = 1-\varepsilon+0(\varepsilon^3)$, $\bar{C}_0(\varepsilon) = 1 + \varepsilon^2+0(\varepsilon^3)$ implying well defined time scale behavior at time scale t. Further, $\bar{B}_1(\varepsilon) = 1 + 0(\varepsilon^3)$, $\bar{C}_1(\varepsilon) = \varepsilon \neq 0(\varepsilon^3)$ implying well defined non-trivial time scale behavior at time scale t/ε. However $\bar{B}_2(\varepsilon) = 1-\varepsilon-\varepsilon^2+0(\varepsilon^3)$, $\bar{C}_2(\varepsilon) = \varepsilon$ implying that the system does not satisfy (5.11) for well defined time scale behavior at t/ε^2. However

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \bar{C}_2(\varepsilon)\exp \bar{A}_2(\varepsilon)\varepsilon^2 t/\varepsilon\}\bar{B}_2(\varepsilon)$$

exists uniformly for $t \in [\delta, T]$ and is equal to 1. Thus from Definition (5.1), we can conclude that the system (5.25) has well defined time scale behavior at only two time scales t and t/ε. □

Example 5.4 shows that when the condition (5.11) is not satisfied the input-output description of the system can have fewer time-scales than the internal dynamics of the system.

Example 5.5 (One time-scale system)

$$\dot{x} = \left[\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}\right] x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u \tag{5.28}$$

$$y = (1 \quad 1 \quad 1)x. \tag{5.29}$$

The $A_0(\varepsilon)$ is the same as that of examples 5.3, 5.4 and so has three time scales t, t/ε, t/ε^2 in its internal dynamics. On diagonalizing the system we get (5.27) and

$$y = ([1 \quad 1 \quad 1] + O(\varepsilon))x. \tag{5.30}$$

Note that the conditions (5.11) are met by the system (5.27), (5.30) only at the time scale t. Further

$$\lim_{\varepsilon \downarrow 0} \bar{C}_1(\varepsilon)\exp\{\bar{A}_1(\varepsilon)\varepsilon t\}B_1(\varepsilon)$$

and

$$\lim_{\varepsilon \downarrow 0} \bar{C}_2(\varepsilon)\exp\{\bar{A}_2(\varepsilon)\varepsilon^2 t\}\bar{B}_2(\varepsilon)$$

exist uniformly on $[\delta, 1]$ and are both 1. Thus by definition (5.1), the system (5.28), (5.29) has well defined time scale behavior only at time scale t. □

Condition (5.11) is a sufficient condition for internal time-scales to be reflected into external time scales. In examples 5.4, 5.5 we have shown that when (5.11) is not satisfied, the input-output system can have fewer than the internal time scales. The following example shows that if the conditions (5.11) are not met at a specific time scale, say t/ε^k, then behavior at faster time scales is t/ε^j for $j < k$ may also not exist:

<u>Example 5.6</u>

Consider a diagonalized system with three time scales

$$\dot{x} = \begin{bmatrix} \bar{A}_0(\varepsilon) & 0 & 0 \\ 0 & \varepsilon\bar{A}_1(\varepsilon) & 0 \\ 0 & 0 & \varepsilon^2\bar{A}_2(\varepsilon) \end{bmatrix} x + \begin{bmatrix} \bar{B}_0(\varepsilon) \\ \varepsilon\bar{B}_1(\varepsilon) \\ B_2(\varepsilon) \end{bmatrix} u \tag{5.31}$$

$$y = [\bar{C}_0(\varepsilon) \quad \bar{C}_1(\varepsilon) \quad \bar{C}_2(\varepsilon)]x$$

with all $\bar{A}_i(\varepsilon)$, $\bar{B}_i(\varepsilon)$, $\bar{C}_i(\varepsilon)$ of order 1. The conditions (5.11) are satisfied at time scale t and t/ε and are not satisfied at t/ε^2 as evidenced by the fact that limit as $\varepsilon \downarrow 0$ of

$$\frac{1}{\varepsilon^2} \bar{C}_2(\varepsilon) \exp \bar{A}_2(\varepsilon)t \; \bar{B}_2(\varepsilon)$$

for $t \in [\delta,T]$ does not exist. However, limit as $\varepsilon \downarrow 0$ of

$$\frac{1}{\varepsilon} \bar{C}_2(\varepsilon)\exp\{\bar{A}_2(\varepsilon)\varepsilon t\}\bar{B}_2(\varepsilon)$$

also does not exist for $t \in [\delta,T]$. Consequently the system (5.31) does not have well defined time scale behavior at time scale t/ε. It has well defined time scale behavior only at the fastest time scale t. □

From Example 5.6, we conclude the following:

$$\bar{C}_j(\varepsilon)\bar{B}_j(\varepsilon) \sim O(\varepsilon^j)$$

is only one of the sufficient conditions (5.11) for well defined behavior at time scale t/ε^j. However, if

$$\bar{C}_j(\varepsilon)\bar{B}_j(\varepsilon) \sim O(\varepsilon^q)$$

for $q < j$, the system (5.1) is not well defined at time scales $t/\varepsilon^{q+1},\ldots,$ t/ε^{j-1}, since

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^k} \bar{C}_j(\varepsilon)\exp\{\bar{A}_j(\varepsilon)\varepsilon^{j-k}t\}\bar{B}_j(\varepsilon)$$

$$= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^k} \bar{C}_j(\varepsilon)\bar{B}_j(\varepsilon) = \infty \qquad \text{for} \qquad q < k < j$$

The details of the exact time scale structure of the system (5.1) when conditions (5.11) are not met is currently under investigation.

5.2. Positive Realness of Multiple Time Scale Systems

Consider the multiple time scale linear system

$$\begin{aligned} \dot{x} &= A_0(\varepsilon)x + B_0(\varepsilon)u \\ y &= C_0(\varepsilon)x + D_0u \end{aligned} \tag{5.32}$$

Here $A_0(\varepsilon) \in \mathbb{R}^{nxn}$ satisfies the MSST conditions; the number of inputs is equal to the number of outputs and D_0 is a constant matrix. We consider in this section assumptions on transfer functions of (5.32) valid at different time scales in order to guarantee that (5.32) is SPR for $\varepsilon \in [0,\varepsilon^*]$. In Section 4, we found that in the case of a two-time scale system, the reduced system $\hat{H}_r(s)$ SPR and the fast system $\hat{H}_f(s)$ SPR implied that the augmented system $\hat{H}_\varepsilon(s)$ was SPR for ε small enough.

In the case of (5.32), we will assume that it is well defined in the sense of Definition 5.1 at time scales t, t/ε, ..., t/ε^m (for instance, this is implied by conditions (5.11)). In that case we define the transfer functions at the various time scales using the form (5.9):

$$\hat{H}_0(s) := \lim_{\varepsilon \downarrow 0} D_0 + \bar{C}_0(\varepsilon)(sI-\bar{A}_0(\varepsilon))^{-1}\bar{B}_0(\varepsilon)$$

$$\hat{H}_i(s) := \hat{H}_{i-1}(0) + \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^i} \bar{C}_i(\varepsilon)(sI-\bar{A}_i(\varepsilon))^{-1}\bar{B}_i(\varepsilon) \tag{5.33}$$

Theorem 5.7

The input-output system (5.32) is SPR for $\varepsilon \in [0,\varepsilon^*]$ if $\hat{H}_i(s)$ is SPR for i = 0, ..., m

Proof: Is by induction. Define

$$\tilde{H}_0(\varepsilon,s) := D_0 + \bar{C}_0(\varepsilon)(sI-\bar{A}_0(\varepsilon))^{-1}\bar{B}_0(\varepsilon) \tag{5.34}$$

and for i = 1, ..., m

$$\tilde{H}_i(\varepsilon,s) = \tilde{H}_{i-1}(\varepsilon,s) + \bar{C}_i(\varepsilon)(sI-\varepsilon^i\bar{A}_i(\varepsilon))^{-1}\bar{B}_i(\varepsilon) \tag{5.35}$$

From the definitions (5.34), (5.35) it follows that the transfer function of (5.32) is $\tilde{H}_m(\varepsilon,s)$. By Theorem (4.1), $\tilde{H}_0(\varepsilon,s)$ is SPR for ε small enough if $\hat{H}_0(s)$ is SPR and further

$$\tilde{H}_0(\varepsilon,j\omega) + \tilde{H}_0^*(\varepsilon,j\omega) > \frac{\bar{\nu}_0 I}{1+\omega^2} \tag{5.36}$$

Further, exactly as in the proof of Theorem (4.3), $\tilde{H}_{i-1}(\varepsilon,s)$ SPR and

$$\tilde{H}_{i-1}(\varepsilon,j\omega) + \tilde{H}^*_{i-1}(\varepsilon,j\omega) > \frac{\nu_{i-1}I}{1+(\varepsilon^{i-1}\omega)^2} \tag{5.37}$$

implies that the inequality (5.37) holds for $\tilde{H}_i(\varepsilon,j\omega)$. Combining this with the fact that $\lim_{\varepsilon\downarrow 0} \tilde{H}_i(\varepsilon,\varepsilon^i s) = \hat{H}_i(s)$, with $\hat{H}_i(s)$ SPR, we obtain as in the proof of Theorem 4.3 that $\tilde{H}_i(\varepsilon,s)$ is SPR. Finally, at $i = m$, we get the transfer function of the system (5.32) to be SPR.

Section 6. Concluding Remarks

We have extended results on the time-scales decomposition of autonomous systems to that of input-output systems. We have used these results to study conditions under which positive realness of a transfer function are preserved under singular perturbation.

The computations associated with obtaining the time scales decomposition of Section 5 are straightforward, but are involved and hence omitted from the discussion. We have carried out these computations in detail for systems of the form

$$\begin{aligned} \dot{x} &= (A_0+\varepsilon A_1)x+(B_0+\varepsilon B_1)u \\ y &= (C_0+\varepsilon C_1)x. \end{aligned} \tag{6.1}$$

with three time scales. Roughly speaking, the most involved part of the computations is obtaining the projection matrices $P_0(\varepsilon)$ of (2.22) up to $O(\varepsilon^3)$, $P_1(\varepsilon)$ up to $O(\varepsilon^2)$ and so on using the formulae (2.13)-(2.15). Once these matrices and consequently the diagonalizing transformation $T(\varepsilon)$ of (5.5) are obtained the conditions for the existence of multiple time scales input-output behavior is easily verified. Details of these calculations are available with the authors.

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