UNIQUENESS OF A BASIC NONLINEAR STRUCTURE

by

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Uniqueness of a Basic Nonlinear Structure*

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ABSTRACT

In this paper we show that systems consisting of a memoryless nonlinearity sandwiched between two linear time invariant operators are *unique* modulo scaling and delays. We mention a few corollaries and applications of general circuit and system theoretic interest.

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1. Introduction

In nonlinear systems theory two types of operators are especially important: *linear time invariant* (LTI) operators and *memoryless* or *static* nonlinear operators. Many important and well-known results pertain to systems which are interconnections of these operators, for example the Popov criterion for the Lur'e structure. Indeed if multi-input multi-output (MIMO) operators are considered, all dynamical systems are included.

In this paper we consider what is perhaps the simplest interconnection of these operators, shown in fig. 1, and ask the question: in what sense are such systems unique, that is, under what conditions could two such systems have the same input/output (I/O) map? Some conditions are easy to think of, for example we can rescale the operators or distribute any delay in A and C arbitrarily between them ($\tilde{A}=\alpha\exp(-sT)A$, $\tilde{C}=\gamma\exp(sT)C$, $\tilde{B}(x)=\alpha^{-1}B(\gamma^{-1}x)$) . We show that these are the *only* ways these systems fail to be unique.

W. J. Rugh and others [1-5] have shown that certain systems containing lumped LTI operators and memoryless power nonlinearities or multipliers are unique in a certain sense, and this paper is inspired by their work. Our emphasis, however, is slightly different: we consider memoryless nonlinearities as opposed to multipliers and pure power nonlinearities, and general as opposed to lumped LTI operators.

2. Notation

We consider operators with a Volterra series:

$$Nu(t) = \sum_{n=1}^{\infty} y_n(t)$$

$$y_n(t)=\int \cdots \int h_n(\tau_1,\tau_2,\ldots\tau_n)u(t-\tau_1)u(t-\tau_2) \cdots u(t-\tau_n)d\tau_1 d\tau_2 \cdots d\tau_n$$

where h_n is a symmetric real tempered distribution supported on $(R^+)^n$, and the inputs u belong to some subset of $C^\infty(R^+)$ which ensures y_n summable.* We

*This formulation includes operators such as differentiation and has a correspondingly restricted signal space. If you like, the h_n can be bounded measures, the signal space the open ball in L^∞ with radius $R^{-1} = \overline{\lim}\,||h_n||^{1/n}$.

refer to h_n as the nth time domain Volterra kernel of N; we will work with their Laplace transforms, called the (frequency domain) Volterra kernels or nonlinear transfer functions:

$$H_n(s_1,s_2,\ldots s_n)=\int \cdots \int h_n(t_1,t_2,\ldots t_n)\exp(-\sum_{i=1}^{n} s_i t_i)\,dt_1 dt_2 \cdots dt_n$$

defined and analytic at least in $\left\{ s \,\middle|\, \mathrm{Re}\, s_k > 0,\ k=1\ldots n \right\}$, henceforth denoted $(C^+)^n$. For more details, see [6-11].

A LTI operator has all kernels above the first vanishing; a memoryless operator is one with all kernels constant, and a positive radius of convergence. To keep the notation simple, we will use the same symbol for a LTI operator and its first kernel, and similarly for a memoryless operator and its associated function from R to R. Juxtaposition of operators will denote composition, equality of operators will mean that they have the same I/O map.

We should mention that the Volterra kernels are completely determined by the operator N, i.e. by its I/O map. Indeed for $\omega_k \neq 0$,

$$H_n(j\omega_1,\ldots j\omega_n) = \frac{\partial^n}{\partial\alpha_1 \cdots \partial\alpha_n}\left[\widehat{N(2\sum_{k=1}^{n}\alpha_k \cos\omega_k t)}\right](\sum_{k=1}^{n} j\omega_k)\ \Big|_{\alpha=0}$$

where the right hand side refers only to the *operator* N, and not to any particular representation of it. This means that the kernels can be measured [12].

3. Statement and Proof of Theorem

Theorem 1: Suppose A, $\tilde{A}$, C, $\tilde{C}$ are nonzero LTI operators, B and $\tilde{B}$ are memoryless operators, at least one of which is not linear. If $ABC=\tilde{A}\tilde{B}\tilde{C}$, then there are real constants α, γ, T such that:

$$\tilde{A}(s)=\alpha\exp(-sT)A(s) \qquad \tilde{C}(s)=\gamma\exp(sT)C(s)$$
$$\tilde{B}(x)=\alpha^{-1}B(\gamma^{-1}x)$$

That is, systems of the form (1) which are not linear have a unique representation of the form (1), modulo scaling and delays.

Proof: Under the hypotheses of theorem 1, the two systems have the same kernels $H_n =$

$$=A(s_1+\cdots+s_n)B_n C(s_1)C(s_2)\cdots C(s_n)= \tag{1}$$
$$=\tilde{A}(s_1+\cdots+s_n)\tilde{B}_n \tilde{C}(s_1)\tilde{C}(s_2)\cdots \tilde{C}(s_n) \tag{2}$$

Consider now any $n>1$ for which H_n is not identically zero (and there is at least one such n). Find an open ball D in $(C^+)^n$ on which $H_n \neq 0$. Indeed $\left\{ s \varepsilon (C^+)^n \;\middle|\; H_n(s) \neq 0 \right\}$ is open and connected in $(C^+)^n$. On D define $Q=$

$$=\ln\left[B_n (C/\tilde{C})(s_1)\cdots(C/\tilde{C})(s_n)\right]= \tag{3}$$
$$=\ln\left[\tilde{B}_n (\tilde{A}/A)(s_1+\cdots+s_n)\right] \tag{4}$$

Any branch of ln will do. Then on D,

$$\frac{\partial^2 Q}{\partial s_1 \partial s_2} = 0 \tag{5}$$

when calculated from (3) and

$$\frac{\partial^2 Q}{\partial s_1 \partial s_2} = \left[\ln(\tilde{A}/A)\right]''(s_1+\cdots+s_n) \tag{6}$$

when calculated from (4). Note that $n>1$ is *crucial;* this is where the requirement of *strict* nonlinearity enters. From (5) and (6) we conclude for some η and T,

$$\ln(\tilde{A}/A)(s_1+\cdots+s_n) = \eta - T(s_1+\cdots+s_n) \tag{7}$$

on D and hence *everywhere* in $(C^+)^n$. Thus

$$\tilde{A}(s)=\alpha\exp(-sT)A(s) \tag{8}$$

for $s \varepsilon C^+$, where $\alpha=\exp\eta$. From $A(\bar{s})=\overline{A(s)}$ we conclude α and T are real. Substituting (8) into (1) and (2) yields

$$\tilde{C}(s)=\gamma\exp(sT)C(s) \tag{9}$$

where $\gamma^n = B_n \tilde{B}_n^{-1}\alpha^{-1}$ and as above γ real. Thus we have $\tilde{B}_n = \alpha^{-1}B_n\gamma^{-n}$, which remains true for those n for which $\tilde{B}_n = B_n = 0$, hence

$$\tilde{B}(x)=\alpha^{-1}B(\gamma^{-1}x) \tag{10}$$

and the theorem is proved.

Corollary 1: Systems of the form HN are *completely disjoint* from systems of the form NH, where H is LTI *nonconstant* and N is memoryless *strictly* nonlinear. (See fig. 2).

Corollary 2: Given any operator N with at least two nonzero kernels, the only LTI operators which commute with N are delays (or delays and negation, if N is odd).

Corollary 3: Chua [13] has defined *algebraic* circuit elements as those with constitutive relations of the form $\Phi(v^{(\alpha)}, i^{(\beta)})=0$ (where $f^{(\alpha)}$ is the αth derivative, or integral if $\alpha<0$, of f). Nonlinear resistors, capacitors, and inductors are examples. Under weak conditions theorem 1 shows that if such an element is *strictly nonlinear* its order (α, β) and its characteristic curve $\Phi(x,y)=0$ are *unique*, that is, such elements have only one description as algebraic elements.

Application: Consider a communications system consisting of N cable-repeater sections, each with frequency response $R(s)$. Suppose the output stage of the kth repeater drifts off bias and starts distorting slightly. The faulted system I/O operator is then $R^{N-k}f(.)R^k$, where $f(.)$ represents the nonlinear output stage: see fig. 3. Theorem 1 tells us that *from I/O measurements alone* (of the whole system) we can locate the faulty repeater.* This should be compared to a linear fault: suppose an element in the kth repeater amplifier drifts in such a way as to, say, halve the bandwidth of the repeater. The kth repeater is still linear, but with frequency response $\tilde{R}(s)$. I/O measurements alone *cannot locate* this fault, since the system's linear (and only) frequency response is $R(s)^{N-1}\tilde{R}(s)$ no matter where the fault is.

4. Comments and Generalizations

The theorem remains true under a wide variety of generalizations. It is true for discrete time systems, with the obvious modification of the conclusion $\tilde{A}(z)=\alpha z^{-d}A(z)$ and $\tilde{C}(z)=\gamma z^{d}C(z)$, d an integer. It holds for multidimensional systems as well, for example for two-dimensional systems we get

* One might suspect that this is possible. The advantage of our machinery is that it can tell us exactly which distortion products to look at.

$$H_n(s_1, \cdots s_n; p_1, \cdots p_n) = \\ = A(s_1 + \cdots + s_n; p_1 + \cdots + p_n) B_n C(s_1, p_1) \cdots C(s_n, p_n)$$

and a proof analogous to the one above establishes

$$\tilde{A}(s,p) = \alpha \exp(-sX - pY) A(s,p) \quad \tilde{C}(s,p) = \gamma \exp(sX + pY) C(s,p) \\ \tilde{B}(x) = \alpha^{-1} B(\gamma^{-1} x)$$

The theorem is also true for most noncausal A and C. For example when their impulse responses fall off exponentially $A(s)$ and $C(s)$ are analytic in a strip $-\varepsilon < \mathrm{Re}s < \varepsilon$ and the proof above applies directly. And under weaker conditions it is usually true as a consequence of the fact that the functional equation $f(x+y) = \alpha g(x) g(y)$ only has exponential solutions under quite general conditions, e.g. f and g measurable and nonzero.* But there are pathological cases in which the theorem fails, for example consider

$$A(j\omega) = \begin{cases} 1 & |\omega| < 1 \\ 0 & |\omega| \geq 1 \end{cases} \qquad B(x) = x^2 \qquad C(j\omega) = \begin{cases} 1 & |\omega| < 3 \\ 0 & |\omega| \geq 3 \end{cases}$$

Then $ABC = ABI$, where $I(s) = 1$.

From these comments we may conclude, for example, that the theorem holds for image processing operators of the form (1). Other generalizations, however, are not straightforward. We do not know under what conditions the theorem holds in the MIMO case. We suspect but cannot prove that the theorem holds for any measurable nonlinearity, and not just the analytic ones considered here.

5. A Stable Decomposition Method

Our proof, which relies on partial derivatives and analytic continuation, might suggest that the decomposition of H_n into $A(s)$, B_n, and $C(s)$ is quite sensitive. The main purpose of this section is to show that this is not so. We now give a sketch of the simplest case: discrete time, minimum phase exponentially decaying A and C. We decompose the second kernel since the higher order kernels decompose similarly. We assume that H_2 has been measured: there are

*See e.g. Shapiro [14]

simpler methods to estimate A and C based on partial knowledge of H_2 (e.g. from $H_2(e^{j\Omega},e^{-j\Omega})=A(0)B_2|C(e^{j\Omega})|^2$; cf. [2,3]) but measuring the kernels allows us to *verify* that the system has the form (1), as well as estimate A and C. It will be convenient to normalize $A(0)=C(0)=1$. Then $\ln H_2$ is analytic in $\left\{(z_1,z_2) \,\middle|\, |z_1|\le 1,\ |z_2|\le 1\right\}$ and

$$\ln H_2(e^{j\Omega_1},e^{j\Omega_2}) = \ln A(e^{j(\Omega_1+\Omega_2)}) + \ln B_2 + \ln C(e^{j\Omega_1}) + \ln C(e^{j\Omega_2})$$

The assumptions imply that the terms above containing A, B_2, and C, when considered elements of $L_2(T\times T)$,* are contained in the mutually orthogonal subspaces S_1, S_2, and S_3, where

$$S_1 = \left\{ g(\Theta_1+\Theta_2) \,\middle|\, g \,\varepsilon\, L_2(T),\ \int g = 0 \right\}$$

$$S_3 = \left\{ f(\Theta_1)+f(\Theta_2) \,\middle|\, f \,\varepsilon\, L_2(T),\ \int f = 0 \right\}$$

and S_2 is the constants. A natural method to estimate $\ln A$, $\ln B_2$, and $\ln C$ is to project $\ln H_2$ on these subspaces, i.e.

$$\ln B = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\ln H_2(e^{j\Omega_1},e^{j\Omega_2})\,d\Omega_1 d\Omega_2$$

$$\ln A(e^{j\Omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\ln H_2(e^{j(\Omega-\Omega_1)},e^{j\Omega_1})\,d\Omega_1 - \ln B$$

$$\ln C(e^{j\Omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\ln H_2(e^{j\Omega},e^{j\Omega_1})\,d\Omega_1 - \ln B$$

In fact these formulas can be used to estimate $\ln|A|$, $\ln|C|$, and B_2 when A or C is not minimum phase,** but the method must be modified to yield the correct phases. The point is that A, B_2, and C can be estimated in a stable way, without taking partial derivatives.

6. Conclusion

The theorem has the interpretation that from I/O measurements alone, we can in principle extract information about the *internal structure* of a system of the form (1). We believe that this is an instance of a general property of non-

*T is the unit circle with normalized measure.

**With the modified normalization for the transfer functions $A(0) = \prod_k |\alpha_k|$, where the α_k are the zeros of A in the unit disk.

linear systems: the same complexity which makes nonlinear systems difficult to represent, analyze, and design (e.g. noncommutativity, nondistributivity...) also allows much more information about internal structure to be extracted from I/O measurements.

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8. Figure Captions

fig1: The system considered: A and C are LTI with frequency responses $A(s)$ and $C(s)$, B is memoryless with characteristic function $B(x)$.

fig2: Two simple nonlinear structures: (a) is of the form HN, and (b) is of the form NH. Except in the trivial cases H constant or N linear, the two types are exclusive.

fig3: (a) cable-repeater section; (b) communications system; (c) system with kth repeater nonlinearly faulted at output; (d) system with kth repeater linearly faulted.

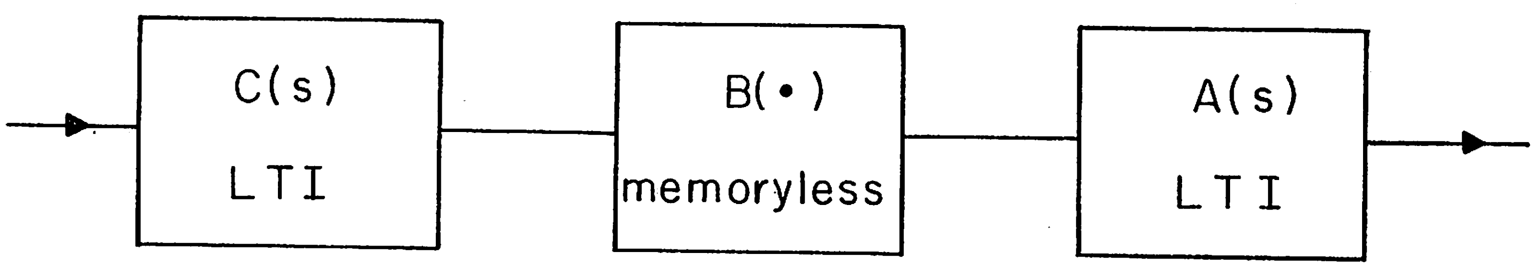

Fig. 1

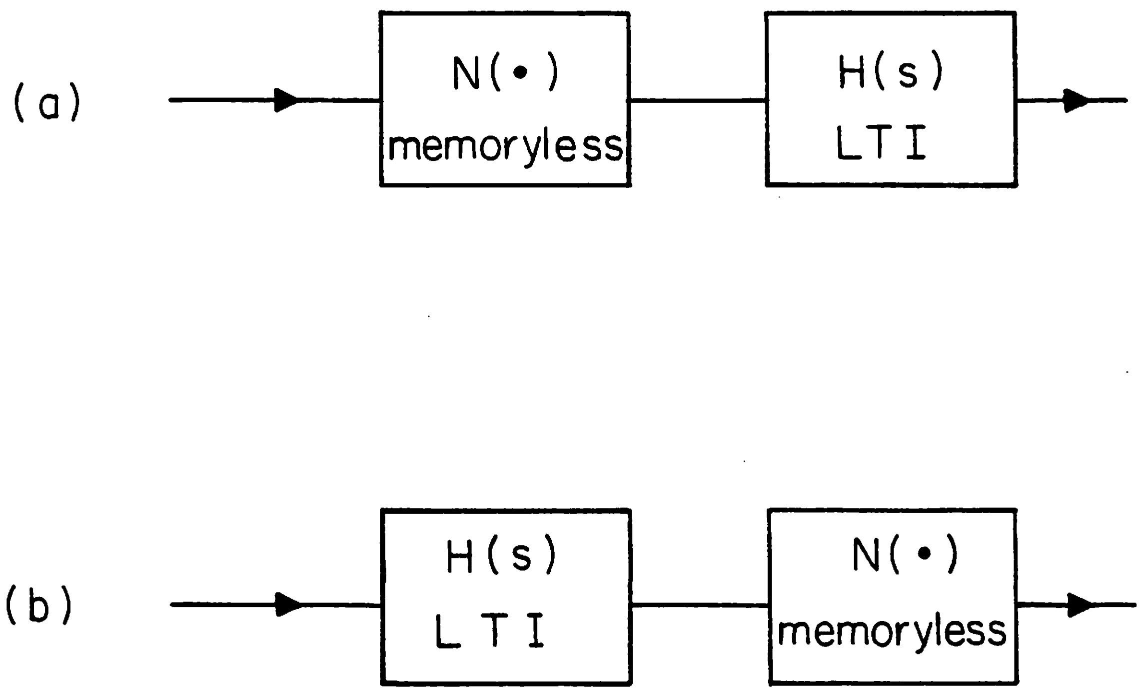

Fig. 2

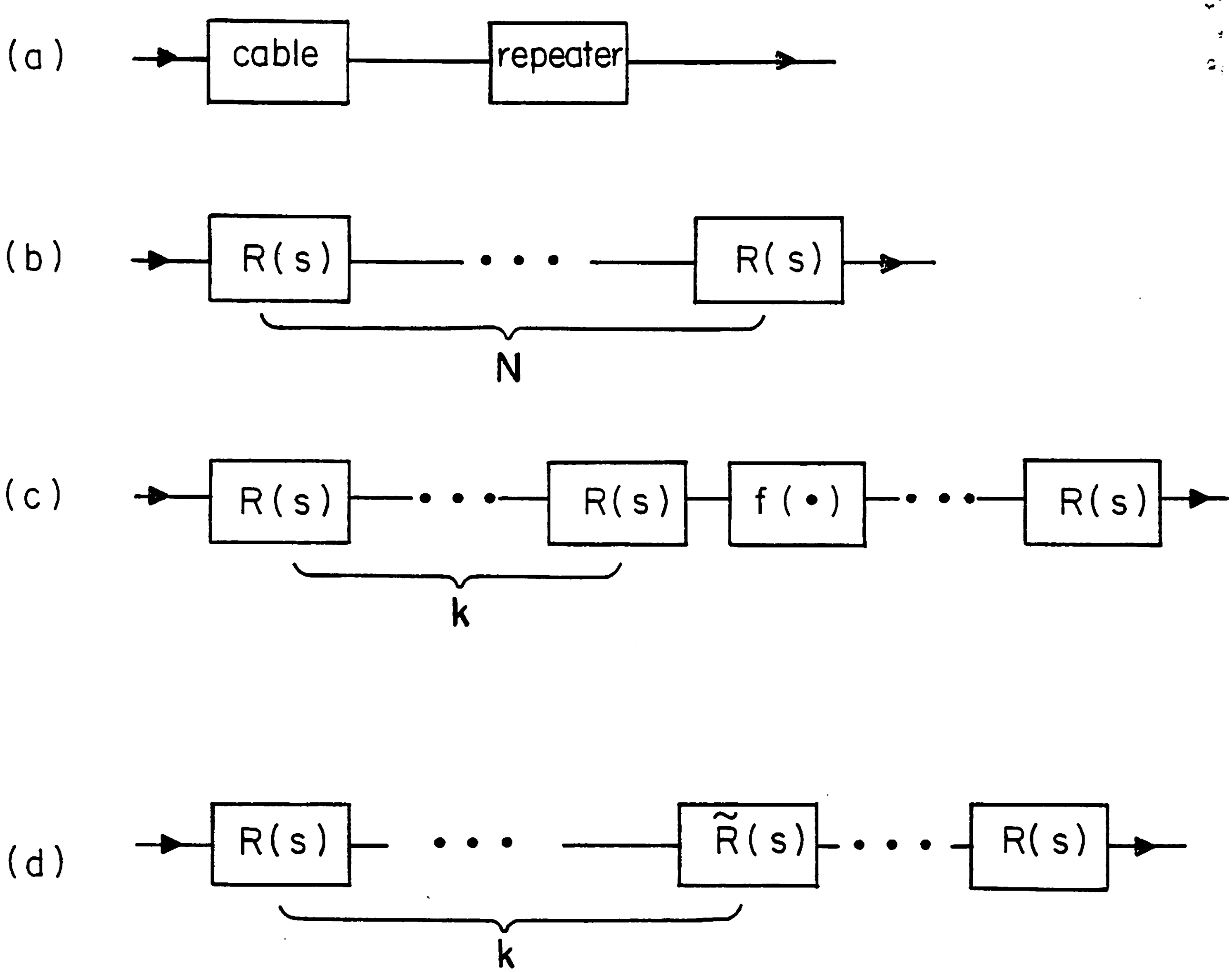

Fig. 3