ROBUST STABILITY UNDER ADDITIVE PERTURBATIONS

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Robust Stability Under Additive Perturbations

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Abstract

We consider a MIMO linear time-invariant feedback system [1]$S(P,C)$ which is assumed to be $\mathcal{U}$-stable. The plant P is subjected to an additive perturbation ΔP which is proper but not necessarily stable. We prove that the perturbed system is $\mathcal{U}$-stable if and only if $\Delta P[I+Q\cdot\Delta P]^{-1}$ is $\mathcal{U}$-stable. (Here $Q := C(I+PC)^{-1}$.)

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I. Introduction

One of the main purposes of feedback is to reduce the sensitivity of the closed-loop system to changes in the plant, and it is very important to determine whether a feedback system remains stable after being subjected to changes in the plant. There is an abundant literature on this subject with various restrictions imposed on the nature of i) the plant (linear lumped [Des. 1], [Åst. 1], [Fra. 1] [Doy. 1]; linear distributed [Chen 1], [Chen 2]; nonlinear and time-varying [Zam. 1], [San. 1]), ii) the perturbation (stable [Åst. 1], [Fra. 1], [Cru. 1] [Pos. 1], [Zam. 2]) - all giving only sufficient conditions; a class of possibly unstable perturbations [Doy. 1], [Chen 1] with necessary and sufficient conditions (n.a.s.c.); fractional perturbations [Chen 2] which gave n.a.s.c.

In this note we consider exclusively MIMO linear time-invariant systems, we state and give a simple algebraic proof of a necessary and sufficient condition for $\mathcal{U}$-stability ($\mathcal{U}$ refers to an undesirable symmetric region of the complex plane ($\supseteq \mathbb{C}_+$) of the feedback system ${}^1S(P,C)$ (Fig. 1, solid lines) under arbitrary perturbations ΔP (i.e. ΔP is not required to be $\mathcal{U}$-stable). In Section II we formalize the following intuitive argument: a) the addition of ΔP to ${}^1S(P,C)$ (as shown by dotted lines in Fig. 1) creates a new loop; b) the "gain seen by ΔP," through ${}^1S(P,C)$ is equal to $-Q := -C(I+PC)^{-1}$; c) since the nominal system ${}^1S(P,C)$ is $\mathcal{U}$-stable, Q is $\mathcal{U}$-stable, d) view the new loop as ${}^1S(Q,\Delta P)$ as shown in Fig. 2: it is $\mathcal{U}$-stable iff $\Delta P\cdot(I+Q\cdot\Delta P)^{-1} := Q_{{}^1S(Q,\Delta P)}$ is $\mathcal{U}$-stable by the Q-parametrization theorem [Zam. 2], [Des. 2]. If, in addition, ΔP is $\mathcal{U}$-stable, the new loop is stable iff $\det[I+Q\cdot\Delta P](s) \neq 0,\ \forall s \in \mathcal{U}$; e) $\tilde{S}(P,\Delta P,C)$ is $\mathcal{U}$-stable iff the new loop is $\mathcal{U}$-stable. These statements are intuitively appealing, however it is not clear whether some particular restrictions on P, C, ΔP are required to make them true. We prove that they hold even if P, C and ΔP are unstable!

II. Statement and Proof of Theorem

Definition: $\tilde{S}(P,\Delta P,C)$ (defined in Fig. 1) is $\mathcal{U}$-stable iff $H_{yu}:(u_1,u_2,u_3)\mapsto(y_1,y_2,y_3)$ is $\mathcal{U}$-stable.

Assumptions:

A1. $P(s)\in\mathbb{R}_p(s)^{n_o\times n_i}$, $C(s)\in\mathbb{R}_p(s)^{n_i\times n_o}$, $\det[I+PC](\infty)\neq 0$.

A2. All hidden modes of P and C are $\mathcal{U}$-stable.

A3. $\Delta P\in\mathbb{R}_p(s)^{n_o\times n_i}$ and $\det[I+(P+\Delta P)C](\infty)\neq 0$.

Comment: Note that P, C, ΔP are only required to be proper but may be unstable. Of course the nominal and perturbed systems are required to be well-posed (see A1 and A3).

Theorem: Let A1, A2 and A3 hold. If $^1S(P,C)$ is $\mathcal{U}$-stable, then

a) $\tilde{S}(P,\Delta P,C)$ is $\mathcal{U}$-stable $\Leftrightarrow$ $\Delta P\cdot(I+Q\cdot\Delta P)^{-1}$ is $\mathcal{U}$-stable;

b) $\tilde{S}(P,\Delta P,C)$ is $\mathcal{U}$-stable $\Leftrightarrow$ $^1\tilde{S}(Q,\Delta P)$ is $\mathcal{U}$-stable.

If, in addition, ΔP is $\mathcal{U}$-stable, then

c) $\tilde{S}(P,\Delta P,C)$ is $\mathcal{U}$-stable $\Leftrightarrow$ $\det[I+Q\cdot\Delta P](s)\neq 0$, $\forall\, s\in\mathcal{U}$.

Comments: a) Define $H_{eu}:(u_1,u_2,u_3)\mapsto(e_1,e_2,e_3)$. By writing the relation between the e_i's, u_i's and y_i's it is easy to check that H_{yu} is $\mathcal{U}$-stable implies that H_{eu} is $\mathcal{U}$-stable. We will prove H_{yu} is $\mathcal{U}$-stable.

b) Suppose $\Delta P = R/(s-p)$ where $R\in\mathbb{C}^{n\times n}$ and p may be in $\mathcal{U}$. It is easy to check that $\Delta P(I+Q\Delta P)^{-1} = R[(s-p)I+QR]^{-1}$. Since the expression in brackets is analytic in $\mathcal{U}$, by part c) of the theorem, we have:

$\tilde{S}(P,\Delta P,C)$ is $\mathcal{U}$-stable $\Leftrightarrow$ $\det[(s-p)I+Q(s)R] \neq 0,\ \forall s \in \mathcal{U}$

In some applications, [Bha. 1], R turns out to be a dyad, say cb^T (with $b, c \in \mathbb{C}^n$): hence, the n.a.s.c. condition for $\mathcal{U}$-stability reduces to a <u>scalar</u> condition:

$$(s-p) + b^T Q(s) c \neq 0 \quad \forall s \in \mathcal{U}.$$

<u>Proof</u>: The summing node equations for $\tilde{S}(P,\Delta P,C)$ are:

$$e_2 - Ce_1 = u_2 \tag{1}$$

$$Pe_2 + e_1 + \Delta P e_3 = u_1 \tag{2}$$

$$- Ce_1 + e_3 = u_3 \tag{3}$$

Apply the following block elementary row operations $\rho_2 \leftarrow \rho_2 - P\rho_1$ and then $\rho_3 \leftarrow \rho_3 + Q\rho_2$ and note that w.l.o.g. we can set $u_1 = u_2 = 0$, thus obtaining:

$$\begin{bmatrix} I & -C & 0 \\ 0 & I+PC & \Delta P \\ 0 & 0 & I+Q\Delta P \end{bmatrix} \begin{bmatrix} e_2 \\ e_1 \\ e_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ u_3 \end{bmatrix}$$

Using back-substitution and Fig. 1, we obtain:

$$y_3 = \Delta P \cdot e_3 = \Delta P (I+Q\Delta P)^{-1} u_3 \tag{4}$$

$$y_1 = Ce_1 = - C(I+PC)^{-1} \Delta P e_3 = - Qy_3 \tag{5}$$

$$y_2 = Pe_2 = PCe_1 \qquad = - PQy_3 \tag{6}$$

<u>Pf. of a</u>) ($\Rightarrow$) By assumption $\tilde{S}(P,\Delta P,C)$ is $\mathcal{U}$-stable, hence by (4) $\Delta P(I+Q\Delta P)^{-1}$ is $\mathcal{U}$-stable.

($\Leftarrow$) By assumption, ${}^{1}S(P,C)$ is $\mathcal{U}$-stable, hence Q and PQ are $\mathcal{U}$-stable. Also, by assumption, $\Delta P(I+Q\Delta P)^{-1}$ is $\mathcal{U}$-stable. Hence (4)-(6) show that H_{yu} is $\mathcal{U}$-stable.

Pf. of b). Note that because Q is known to be $\mathcal{U}$-stable, ${}^{1}S(Q,\Delta P)$ is $\mathcal{U}$-stable iff $\Delta P(I+Q\Delta P)^{-1}$ is $\mathcal{U}$-stable by the Q-parametrization theorem [Zam. 2], [Des. 2].

Pf. of c). By assumption ${}^{1}S(P,C)$ is $\mathcal{U}$-stable; Q and ΔP are also $\mathcal{U}$-stable. Since $\mathcal{U}$-stable matrices form a ring, $(I+Q\Delta P)^{-1}$ is $\mathcal{U}$-stable $\Leftrightarrow$ $\det[I+Q\Delta P](s) \neq 0$, $\forall\, s \in \mathcal{U}$. □

Comments: Since the proof is purely algebraic, Assumption A1 is not strictly necessary - the theorem holds for linear distributed plants either continuous-time or discrete-time by working in the appropriate algebra (see, for example [Des. 3]).

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Figure Captions

Fig. 1. The figure shows the system $\tilde{S}(P,\Delta P,C)$ with inputs u_1, u_2, u_3 and outputs y_1, y_2, y_3. If the dotted part of the diagram is removed, we are left with ${}^1S(P,C)$ whose inputs are u_1, u_2 and outputs y_1, y_2.

Fig. 2. ${}^1S(Q,\Delta P)$ - obtained from Fig. 1. The "gain seen by ΔP," going from point a to point b through ${}^1S(P,C)$, is equal to $-Q$.

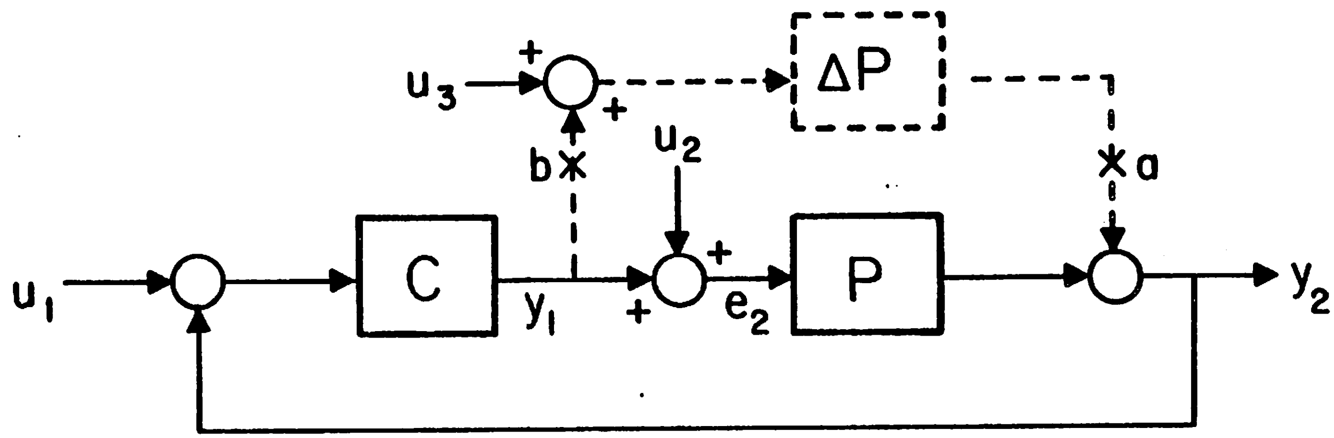

Fig. 1

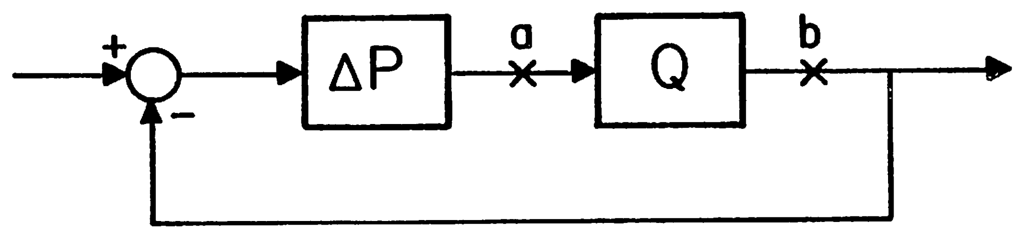

Fig. 2