NECESSARY AND SUFFICIENT CONDITIONS FOR PARAMETER CONVERGENCE IN ADAPTIVE CONTROL

by

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ABSTRACT

Using *Generalized Harmonic Analysis,* we give a complete description of parameter convergence in Model Reference Adaptive Control (MRAC) in terms of the *spectrum* of the exogenous reference input signal. Roughly speaking, if the reference signal "contains enough frequencies" then the parameter vector converges to its correct value. If not, it converges to an easily characterizable subspace in parameter space.

1. Problem Statement

In recent work [1,2,6] on continuous time model reference adaptive control, it has been shown that under a suitable adaptive control law the output y_P of the plant asymptotically tracks the output y_M of a stable reference model, despite the fact that the parameter error vector may not converge to zero (indeed, it may not converge at all). Results that have appeared in the literature on parameter error convergence [3,4,5,8] have established the exponential stability of adaptive schemes under a certain *persistent excitation* (PE) condition. As is widely recognized (e.g. in [9]) the drawback to this condition is that it applies to a certain vector of signals $w(t)$ appearing inside the *nonlinear* feedback loop around the unknown plant.

In earlier work [11] we remedied this shortfall by showing that the persistent excitation condition can be moved from w to w_M, a vector of signals analogous to w but appearing in the linear, time invariant (LTI) model loop. Unlike w, w_M is simply *the output of a LTI system driven by the reference signal* r, and it is thus much easier to determine whether or not it is persistently exciting.

In [11] we gave one simple condition which ensures that w_M is PE:

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If the reference input $r(t)$ contains as many spectral lines as there are unknown parameters, then w_M is PE and consequently the model-plant output error and the parameter error converge exponentially to 0.

Note that a *real* reference signal with a spectral line at frequency ν also has a spectral line at $-\nu$. Thus, for example, a reference signal with a (nonzero) average (DC) value and at least one other spectral line will guarantee exponential convergence of the parameter error vector to zero in a three parameter MRAC. Related results for the scheme of [6] have appeared in [13].

These results made precise the following intuitive argument: assuming the parameter vector *does* converge (but perhaps to the wrong value) the plant loop is "asymptotically time-invariant". If the reference input r has spectral lines at frequencies $\nu_1, \ldots, \nu_k$, we expect y_P will also; since $y_P \to y_M$, we "conclude" that the asymptotic closed loop plant transfer function matches the model transfer function at $s = j\nu_1, \ldots, j\nu_k$. If k is large enough, this implies that the asymptotic closed loop transfer function *is precisely* the model transfer function so that the parameter error converges to *zero*.

In this paper we pursue further this idea that the reference signal must be "rich enough", that is, "contain enough frequencies" for the parameter error to converge to zero. We derive simple *necessary and sufficient* conditions on the reference input r for the parameter error to converge to zero. Roughly speaking, the condition is:

A reference input $r(t)$ results in parameter error convergence to zero *unless* its spectrum is concentrated on $k < N$ lines, where N is the number of unknown parameters in the adaptive scheme.

We will say precisely what we mean by spectrum in the sequel. The results have been announced without proof in [12].

In §2 we briefly describe the MRAC system when the plant has relative degree one. In §3 we review the basic notions of Generalized Harmonic Analysis: *autocovariance* and *spectral measure*. In §4 we state and prove our main result on necessary and sufficient conditions for parameter convergence.

In §5 we discuss partial convergence, i.e. behavior of the parameter error vector when w is *not* PE. This will be the case when the reference signal has its spectrum concentrated on $k < N$ lines, where N is the number of unknown parameters: then the parameter vector can be shown to converge to an affine subspace of dimension $N-k$. The Partial Convergence Theorem of §5 also implies the results of [3,4], but gives a greatly simplified proof.

In §6 we consider higher relative degree cases and show that the results of the previous sections hold, despite the more complicated control strategies. The appendix contains proofs of the theorems of Generalized Harmonic Analysis used in the paper. Although some of these theorems are analogous to results from the theory of wide-sense stationary stochastic processes, these proofs are not, to our knowledge, in the literature.

2. The Model Reference Adaptive System

To fix notation, we review the model reference adaptive system of Narendra, Valavani, et al. [1,2]. The single input single output plant is assumed to be represented by a transfer function

$$\hat{W}_P(s) = k_P \frac{\hat{n}_P(s)}{\hat{d}_P(s)} \tag{2.1}$$

where $\hat{n}_P$, $\hat{d}_P$ are relatively prime monic polynomials of degree $n-1$, n respectively and k_P is a scalar. The following are assumed known about the plant transfer function:

(A1) The degree of the polynomial $\hat{d}_P$, i.e. n, is known.

(A2) The sign of k_P is known (say, + without loss of generality).

(A3) The transfer function $\hat{W}_P$ is assumed to be minimum phase, i.e. $\hat{n}_P$ is Hurwitz.

The objective is to build a compensator so that the plant output asymptotically matches that of a stable reference model $\hat{W}_M(s)$ with input $r(t)$ and output $y_M(t)$ and transfer function

$$\hat{W}_M(s) = k_M \frac{\hat{n}_M(s)}{\hat{d}_M(s)} \tag{2.2}$$

where $k_M > 0$ and $\hat{n}_M$, $\hat{d}_M$ are monic polynomials of degree $n-1$ and n, respectively ($\hat{n}_M$ and $\hat{d}_M$ need not be relatively prime). If we denote the input and output of the plant $u(t)$ and $y_P(t)$, respectively, the objective may be stated: find $u(t)$ so that $y_P(t) - y_M(t) \to 0$ as $t \to \infty$. By suitable prefiltering, if necessary, we may assume that the model $\hat{W}_M(s)$ is strictly positive real.

The scheme proposed by Narendra et al is shown in Figure 1. The dynamic compensator blocks F_1 and F_2 are identical one input, $(n-1)$ output systems, each with transfer function

$$(sI-\Lambda)^{-1}b; \quad \Lambda \in R^{n-1 \times n-1}, \quad b \in R^{n-1}$$

where Λ is chosen so that its eigenvalues are the zeros of $\hat{n}_M$. We assume that the pair (Λ, b) is in controllable canonical form so that

$$(sI-\Lambda)^{-1}b = \frac{1}{\hat{n}_M(s)} \begin{bmatrix} 1 \\ s \\ \cdot \\ \cdot \\ s^{n-2} \end{bmatrix} \tag{2.3}$$

The parameters $c \in R^{n-1}$ in the precompensator block serve to tune the closed loop plant zeros, $d \in R^{n-1}$, $d_0 \in R$ in the feedback compensator assign the closed loop plant poles. The parameter c_0 adjusts the overall gain of the closed loop plant. Thus, the vector of $2n$ adjustable parameters denoted θ is

$$\theta^T = [c_0, c^T, d_0, d^T]$$

If the signal vector $w \in R^{2n}$ is defined by

$$w^T = [r\,, v^{(1)T}, y_P, v^{(2)T}] \tag{2.4}$$

we see that the input to the plant is given by

$$u = \theta^T w \tag{2.5}$$

It may be verified that there is a unique constant $\theta^* \in R^{2n}$ such that when $\theta = \theta^*$, the transfer function of the plant plus controller equals $\hat{W}_M(s)$.† If $r(t)$ is bounded (an assumption we henceforth make) it can be shown that under the parameter update law

$$\dot{\theta} = -e_1 w = -(y_P - y_M)w \tag{2.6}$$

all signals in the loop, i.e. u, $v^{(1)}$, $v^{(2)}$, y_P, y_M are bounded, and in addition $\lim_{t\to\infty} e_1(t) = 0$, that is, the plant output matches the model output and thus our overall objective has been achieved. However the convergence need not be exponential.

Despite the fact that $e_1(t) \to 0$, the parameter vector θ does not necessarily converge to θ^* (it may not converge at all). Various authors [3-5] have established that $e_1(t) \to 0$ and $\theta(t) \to \theta^*$ (i.e. the parameter error converges to 0) *exponentially* iff the signal vector $w(t)$ is *persistently exciting*, (PE) that is, there are $\delta, \alpha > 0$ such that for all $s \geq 0$

$$\int_s^{s+\delta} w w^T dt \geq \alpha I \tag{2.7}$$

Since $w(t)$ contains the signals $v^{(1)}(t)$, $v^{(2)}(t)$, $y_P(t)$ generated inside the *nonlinear plant loop*, translating the PE condition (2.7) on w into an equivalent condition on the exogenous reference input $r(t)$ would seem difficult if even possible. This is precisely what we will now do. Amazingly enough, the condition is very simple when expressed in the frequency domain.

3. Review of Generalized Harmonic Analysis

The integral (2.7) appearing in the definition of PE reminds one of an autocovariance.

Definition 3.1 (Autocovariance): A function $u: R_+ \to R^n$ is said to have *autocovariance* $R_u(\tau) \in R^{n \times n}$ iff

$$\lim_{T\to\infty} \frac{1}{T} \int_s^{s+T} u(t) u(t+\tau)^T dt = R_u(\tau) \tag{3.1}$$

† Indeed θ^* consists of k_M / k_P and the coefficients of the polynomials $\hat{n}_P - \hat{n}_M$ and $\hat{d}_P - \hat{d}_M$.

with the limit uniform in s.

This concept is well known in the theory of time series analysis. There is a strong analogy between (3.1) and $R_u^{stoch}(\tau) = Eu(t)u(t+\tau)$ for u a wide sense stationary stochastic process. Indeed, for a wide sense stationary *ergodic* process $u(t,\omega)$, $R_u(\tau,\omega)$ exists and is $R_u^{stoch}(\tau)$ for almost all ω. But we emphasize that an autocovariance is a completely deterministic notion. Its relation to the notion of PE is simple:

Lemma 3.2 (PE lemma): Suppose w has autocovariance $R_w(\tau)$. Then w is PE iff $R_w(0) > 0$.

Proof: The "if" part is clear. Suppose now that w has an autocovariance R_w and is PE. Let $c \in R^n$, $c \neq 0$. From (3.1) we have for all n

$$\frac{1}{n\delta} \int_s^{s+n\delta} (w^T c)^2 dt \geq \frac{\alpha}{\delta} ||c||^2$$

Hence

$$\overline{\lim_{T \to \infty}} \frac{1}{T} \int_s^{s+T} (w^T c)^2 dt \geq \frac{\alpha}{\delta} ||c||^2 \tag{3.2}$$

Because w has an autocovariance,

$$\lim_{T \to \infty} \frac{1}{T} \int_s^{s+T} (w^T c)^2 dt = c^T R_w(0) c \tag{3.3}$$

From (3.2) and (3.3) we conclude $c^T R_w(0) c \geq \alpha/\delta ||c||^2$, thus $R_w(0) \geq \alpha/\delta > 0$.

We will need a few more simple lemmas concerning autocovariances. The proofs and a more detailed discussion of Generalized Harmonic Analysis appear in the Appendix.

Lemma 3.3: $R_u(\tau)$ is a positive semidefinite function.

Thus, R_u has a Bochner representation:

$$R_u(\tau) = \int e^{i\nu\tau} S_u(d\nu)$$

where S_u is a positive semi-definite matrix of bounded measures, which is called the *spectral measure* of u. If u is scalar valued, then S_u is just a positive bounded measure; $2\,S_u([\omega_0, \omega_1])$ can then be interpreted as the *average energy* contained in u in the frequency band $[\omega_0, \omega_1]$. Thus, for example, if a scalar valued u has a spectral line of amplitude a_ν at ν, then S_u has a point mass at ν of size $|a_\nu|^2$.

Lemma 3.4 (Linear Filter lemma): Suppose $u: R_+ \to R^n$ has autocovariance $R_u(\tau)$, its spectral measure S_u, and h is an $m \times n$ matrix of bounded measures. Then $y = h * u$ has an autocovariance R_y. Its spectral measure is given by:

$$S_y(d\nu) = H(j\nu) S_u(d\nu) H(j\nu)^* \tag{3.8}$$

In particular,

$$R_y(0) = \int H(j\nu) S_u(d\nu) H(j\nu)^* \tag{3.9}$$

where $H(j\nu)$ is the Fourier transform of h.

The reader should note that these formulas are identical to those from the theory of stochastic processes.

Lemma 3.5: If $u-v \in L^2$ and u has an autocovariance R_u, then v has autocovariance R_u.

Thus transients of finite energy do not affect the autocovariance of a signal.

We are now ready to prove our main result.

4. Necessary and Sufficient Conditions for Parameter Convergence

As in Boyd and Sastry [11], redraw Figure 1 as Figure 2 with the model represented in nonminimal form as the plant with compensator and $\theta = \theta^*$. The signal $w_M \in R^{2n}$ in the model loop is given by

$$w_M^T = [r, v_M^{(1)T}, y_M, v_M^{(2)T}]$$

It is shown in [1,2] that $w - w_M \in L_2$

Note that w_M is the output of a stable LTI system driven by $r(t)$, its transfer function

$$\hat{Q}(s) = \begin{bmatrix} 1 \\ \hat{W}_M \hat{W}_P^{-1}(sI-\Lambda)^{-1}b \\ \hat{W}_M \\ \hat{W}_M(sI-\Lambda)^{-1}b \end{bmatrix}$$

The only property of $\hat{Q}$ which we will need is that there is a *constant* invertible matrix M such that

$$\hat{Q}^T(s)M = \frac{1}{\hat{n}_P(s)\hat{d}_M(s)}[\hat{d}_P(s), \ldots, \hat{d}_P(s)s^{n-2}, \hat{n}_P(s), \ldots, \hat{n}_P(s)s^n] \tag{4.1}$$

(This is shown in [11])

We now make the following assumption: r *has an autocovariance.* † Let the spectral measure of r be denoted S_r. We will now derive an explicit formula for $R_w(0)$.

By lemmas 3.4 and 3.5 and the discussion above, w_M has an autocovariance, with spectral measure

$$S_{w_M}(d\nu) = \hat{Q}(j\nu) S_r(d\nu) \hat{Q}(j\nu)^*$$

† Not all r's have autocovariances (e.g. $r(t) = \cos\log(1+t)$) but reasonable ones, whose general characteristics do not change drastically over time, do.

Since $w - w_M \in L^2$, another application of lemma 3.5 shows that w has an autocovariance, its spectral measure also given by

$$S_w(d\nu) = \hat{Q}(j\nu)S_r(d\nu)\hat{Q}(j\nu)^*$$

and autocovariance at 0 given by

$$R_w(0) = \int \hat{Q}(j\nu)S_r(d\nu)\hat{Q}(j\nu)^* \tag{4.2}$$

By the PE lemma, then, we have:

$$w \text{ is PE} \quad \text{iff} \quad R_w(0) = \int \hat{Q}(j\nu)S_r(d\nu)\hat{Q}(j\nu)^* > 0 \tag{4.3}$$

Main Theorem: w is PE iff the spectral measure of r is *not* concentrated on $k < 2n$ points.

Proof: Suppose first that S_r is concentrated at $\nu_1, \ldots, \nu_k$, where $k < 2n$. Then

$$R_w(0) = \int \hat{Q}(j\nu)S_r(d\nu)\hat{Q}(j\nu)^* = \sum_{m=1}^{k} \hat{Q}(j\nu_m)S_r(\{\nu_m\})\hat{Q}(j\nu_m)^*$$

Being the sum of $k < 2n$ dyads, $R_w(0)$ is singular so by (4.3) w is not PE.

Suppose now that w is not PE. Then by the PE lemma there is a nonzero $c \in R^{2n}$ such that

$$0 = c^T R_w(0) c = \int |\hat{Q}(j\nu)^* c|^2 S_r(d\nu) \tag{4.4}$$

Since $|\hat{Q}(j\nu)^* c|^2$ is continuous in ν, (4.4) implies that $\hat{Q}(j\nu)^* c$ vanishes for all ν in $Supt(S_r)$, the support of S_r. Thus for all $\nu \in Supt(S_r)$,

$$0 = \hat{Q}(j\nu)^* c = \hat{Q}(j\nu)^* M(M^{-1}c) \tag{4.5}$$

where M is the constant nonsingular matrix referred to in (4.1). If we let $\tilde{c} = M^{-1}c$ and note that $\hat{d}_P(j\nu)\hat{n}_M(j\nu) \neq 0$, (4.5) says for all $\nu \in Supt(S_r)$,

$$0 = \hat{Q}(j\nu)^* M\tilde{c} = a(j\nu)\hat{d}_P(j\nu) + b(j\nu)\hat{n}_P(j\nu) \tag{4.6}$$

where we define the polynomials $a(s)$ and $b(s)$ by

$$a(s) = \sum_{m=1}^{n-1} \tilde{c}_m s^{m-1} \qquad b(s) = \sum_{m=n}^{2n} \tilde{c}_m s^{m-n} \tag{4.7}$$

Now if $Supt(S_r)$ contains $2n$ or more points, (4.6) vanishes identically since its right hand side is a polynomial of degree $< 2n$, that is

$$a\hat{d}_P + b\hat{n}_P = 0 \tag{4.8}$$

But this contradicts coprimeness of $\hat{d}_P$ and $\hat{n}_P$, since (4.8) implies $\hat{n}_P / \hat{d}_P = -a/b$ and $\partial a \leq n-2 < \partial \hat{n}_P$. So $Supt(S_r)$ must contain $k < 2n$ points, and the Main theorem is proved.

4.1. Discussion

We have proved the following:

Suppose the reference input $r(t)$ to the MRAC system of §2 has an autocovariance. Then the model-plant mismatch error $y_P - y_M$ and the parameter error $\theta - \theta^*$ tend to 0 exponentially *if and only if* the spectral measure of r is not supported on $k < 2n$ points.

Thus in general, one has parameter convergence: only for very special reference signals (which unfortunately sometimes include analytical favorites such as $1(t)$, $\cos(\omega t)$) do we not have $\theta \to \theta^*$.

It is instructive to see how our previous [11] *sufficient* conditions on $r(t)$ fit into the theory above. If r has an autocovariance and has $2n$ spectral lines, then its spectral measure S_r has point masses at the $2n$ frequencies. Thus

$$R_w(0) = \int \hat{Q}(j\nu)S_r(d\nu)\hat{Q}(j\nu)^* \geq \sum_{l=1}^{2n} \hat{Q}(j\nu_l)S_r(\{\nu_l\})\hat{Q}(j\nu_l)^* > 0$$

since the vectors $\hat{Q}(j\nu_l)$ are linearly independent by the argument above.

The terms *sufficiently rich* (SR) and *persistently exciting* (PE) have been used somewhat interchangeably in the literature. We propose that PE refer to property (2.7) for a vector of signals, and that *sufficient richness* be a property of the *reference signal* (scalar valued). A vector of signals is thus PE or not, but whether or not a reference signal is SR depends on the MRAC being studied. More specifically it depends only on the number of unknown parameters in the system, so we propose that a reference signal which results in a PE w in an N-parameter MRAC be referred to as *sufficiently rich of order N*. We then have the following characterization:

If r has an autocovariance, then it is SR of order N iff the support of its spectral measure S_r contains at least N points.

Thus, for example, if r has any *continuous spectrum* (see Wiener [14] for examples of such r's) then r is SR of all orders.

5. Partial Convergence

If w is *not* PE, then the parameter error need not converge to zero (it may not converge at all). In this case S_r is concentrated on $k < 2n$ frequencies $\nu_1, \ldots, \nu_k$. Intuition suggests that although θ need not converge to θ^*, it should converge to the set of θ's for which the *closed-loop plant matches the model* at the frequencies $s = j\nu_1, \ldots, j\nu_k$. This is indeed the case.

Before stating the theorem, we discuss this idea more formally. Suppose that the parameter vector θ is *constant*. Then the plant loop of the MRAC system is LTI: w is in this case Qr. Since the input to the plant is $u = \theta^T w$, the overall closed loop plant transfer function is

$\hat{W}_P(s)\theta^T\hat{Q}(s)$. This transfer function matches $\hat{W}_M$ at $s = j\nu_1, \ldots, j\nu_k$ iff

$$\begin{bmatrix} \hat{W}_P(j\nu_1)\hat{Q}(j\nu_1)^T \\ \vdots \\ \hat{W}_P(j\nu_k)\hat{Q}(j\nu_k)^T \end{bmatrix} \theta = \begin{bmatrix} \hat{W}_M(j\nu_1) \\ \vdots \\ \hat{W}_M(j\nu_k) \end{bmatrix} \tag{5.1}$$

Let us call the set of θ's for which (5.1) holds Θ. Since $\theta^* \in \Theta$, we have

$$\Theta = \theta^* + \text{Nullspace} \begin{bmatrix} \hat{W}_P(j\nu_1)\hat{Q}(j\nu_1)^T \\ \vdots \\ \hat{W}_P(j\nu_k)\hat{Q}(j\nu_k)^T \end{bmatrix} \tag{5.2}$$

Θ thus has dimension $2n-k$. In terms of the parameter error vector $\phi = \theta - \theta^*$, Θ has the simple description

$$\theta \in \Theta \quad \text{iff} \quad R_v(0)\phi = 0 \tag{5.3}$$

We leave the verification of this to the reader; recall that here

$$R_v(0) = \sum_{m=1}^{k} S_r(\{\nu_m\})\hat{Q}(j\nu_m)\hat{Q}(j\nu_m)^*$$

With this discussion in mind, we give

Partial Convergence Theorem:

Suppose that $\dot{r}$ is bounded. Then

$$\lim_{t\to\infty} R_v(0)\phi(t) = 0 \tag{5.4}$$

Remark: If $R_v(0) > 0$, then this theorem tells us nothing more than theorem 1: $\phi \to 0$. But if w is not PE, the conclusion (5.4) can be interpreted as:

$$\theta(t) \to \Theta \quad \text{as} \quad t \to \infty$$

by which we mean $dist(\theta(t), \Theta) \to 0$, *not* $\theta(t) \to \theta(\infty)$ for some $\theta(\infty) \in \Theta$. In particular, θ need not converge to any point as $t \to \infty$.

Proof: Since ϕ and w are bounded, find K such that $||\phi(t)||, ||w(t)|| \leq K$.

Let $\epsilon > 0$ be given. We will find T_0 such that for $t > T_0$, $||R_v(0)\phi(t)|| \leq \epsilon$.

First choose T_1 large enough that for all s,

$$\left|\left| R_v(0) - \frac{1}{T_1}\int_s^{s+T_1} w(t)w(t)^T dt \right|\right| \leq \frac{\epsilon}{3K^2} \tag{5.5}$$

Thus for all t

$$\left| \phi^T(t) R_w(0) \phi(t) - \phi(t)^T \frac{1}{T_1} \int_t^{t+T_1} w(\tau) w(\tau)^T d\tau \phi(t) \right| \leq \frac{\epsilon}{3} \tag{5.6}$$

From our update law $\dot{\phi} = \dot{\theta} = -we_1$; since $e_1 \to 0$, we conclude $\dot{\phi}(t) \to 0$ as $t \to \infty$. The hypothesis $\dot{r}$ bounded implies that $\phi(t)^T w(t) \to 0$ (see [1]). Now find T_0 so that for $t \geq T_0$,

$$(\phi(t)^T w(t))^2 \leq \frac{\epsilon}{3} \tag{5.7a}$$

and

$$\|\dot{\phi}(t)\| \leq \frac{\epsilon}{3K^3 T_1} \tag{5.7b}$$

Then for $t \geq T_0$,

$$\left| \phi(t)^T \frac{1}{T_1} \int_t^{t+T_1} w(\tau) w(\tau)^T d\tau \phi(t) - \frac{1}{T_1} \int_t^{t+T_1} \phi(\tau)^T w(\tau) w(\tau)^T \phi(\tau) d\tau \right| \tag{5.8a}$$

$$= \left| \frac{1}{T_1} \int_t^{t+T_1} w(\tau)^T (\phi(t) - \phi(\tau)) w(\tau)^T (\phi(t) + \phi(\tau)) d\tau \right| \leq \frac{\epsilon}{3} \tag{5.8b}$$

using (5.7b). From (5.7a) we conclude that for $t \geq T_0$,

$$\left| \frac{1}{T_1} \int_t^{t+T_1} \phi(\tau)^T w(\tau) w(\tau)^T \phi(\tau) d\tau \right| \leq \frac{\epsilon}{3} \tag{5.9}$$

From (5.6), (5.8), and (5.9) we have for $t \geq T_0$

$$|\phi(t)^T R_w(0) \phi(t)| \leq \epsilon$$

which completes the proof of the Partial Convergence Theorem.

Remark 1: The proof relies only on the assumptions (5.7), which state, roughly speaking, that *the parameter error eventually becomes orthogonal to* w and that *the updating slows down.* These are nearly universal properties of adaptive systems, so this theorem actually applies quite generally, not just to Narendra's scheme. For example, it applies to all of the schemes described in Goodwin et al [10].

Remark 2: While the $2n-k$ dimensional set Θ to which $\theta(t)$ converges depends only on the frequencies $\nu_1, \ldots, \nu_k$ and not on the average powers $S_r(\{\nu_1\}), \ldots, S_r(\{\nu_1\})$ contained in the reference signal at those frequencies, the *rate of convergence* of θ to Θ depends on both.

Remark 3: As mentioned above, if w is PE then $R_w(0) > 0$ and consequently this theorem yields the original parameter convergence results of [3,4]: uniform, asymptotic convergence of ϕ to zero (and consequently exponential convergence). This proof, however, is considerably simpler than the original proofs.

6. Plant Relative Degree ≥ 2

The scheme of §2 needs to be modified [1] when the relative degree of the plant to be controlled is ≥ 2, i.e. the plant has the transfer function (2.1) with $\hat{n}_P$, $\hat{d}_P$ relatively prime monic polynomials of degree m, n respectively. In addition to the assumptions (A1)-(A3) we add the new assumption (A4):

(A4) The relative degree of the plant, i.e. $(n-m)$, is known.

The model has the form (2.2) with the difference that $\hat{n}_m$ has degree m. The objective of the adaptive control is as before: to get $e_1 = y_P - y_M$ to converge to zero as $t \rightarrow \infty$.

Although the control scheme in this case is considerably more complicated, we will show that the necessary and sufficient conditions for exponential parameter error convergence to zero are *identical* to those given in §4 for the relative degree one case: namely, that $Supt(S_r)$ contain at least $2n$ points.

6.1. The Relative Degree 2 Case

Consider first the scheme of Figure 1 with the difference that Λ is chosen exponentially stable so that its eigenvalues (there are n-1 of them) include the zeros of $\hat{n}_M$ (there are m of them). It may again be verified that there is a unique constant $\theta^* \in R^{2n}$ such that when $\theta = \theta^*$ the transfer function of the plant plus controller equals $\hat{W}_M(s)$. The relationship between θ^* and the coefficients of $\hat{n}_P$ and $\hat{d}_P$ is more complex in this case than in §2. In this case since $\hat{W}_M$ has relative degree 2 it cannot be chosen positive real; however, we may assume (using suitable prefiltering, if necessary) that there is $\hat{L}(s) = (s+\delta)$ with $\delta > O$ such that $\hat{W}_M \hat{L}$ is strictly positive real.

Now, modify the scheme of Figure 1 by replacing each of the gains θ_i, i.e. c_o, d_o, c, d, with the gains $\hat{L}\theta_i\hat{L}^{-1}$ which in turn are given by

$$\hat{L}\theta_i\hat{L}^{-1} = \theta_i + \dot{\theta}_i\hat{L}^{-1} \qquad i = 1,...,2n \tag{6.1}$$

We now define the signal vector

$$\zeta^T(t) \triangleq [\hat{L}^{-1}r,\ \hat{L}^{-1}v^{(1)},\ \hat{L}^{-1}y_P,\ \hat{L}^{-1}v^{(2)}] \tag{6.2}$$

Then

$$\dot{\theta} = -e_1\zeta$$

yields that $e_1(t) \rightarrow 0$ as $t \rightarrow \infty$ provided $r(t)$ is bounded. The persistent excitation condition for exponential parameter and error convergence is on the signal vector $\zeta(t)$ of (6.2): there are

$\alpha,\ \delta > 0$ such that for all $s \geq 0$,

$$\int_{s}^{s+\delta} \varsigma(t)\varsigma(t)^T dt \geq \alpha I \tag{6.3}$$

Now, define the analogous signal vector for the model

$$\varsigma_M{}^T = [\hat{L}^{-1}r,\ \hat{L}^{-1}v_M{}^{(1)T},\ \hat{L}^{-1}y_M,\ \hat{L}^{-1}v_M^{(2)T}]$$

i.e. ς_M is obtained by filtering each component of w_M through the stable system with transfer function $\hat{L}^{-1}$.

Suppose now that r has an autocovariance. ς_M is the output of a LTI filter driven by r, so it has an autocovariance; since $\varsigma - \varsigma_M \in L^2$ (see [1]), ς *has* an autocovariance identical to that of ς_M. In fact

$$R_\varsigma(0) = \int |\hat{L}^{-1}(j\nu)|^2 \hat{Q}(j\nu) S_r(d\nu) \hat{Q}(j\nu)^*$$

Thus $R_\varsigma(0) > 0$ if and only if $R_v(0) > 0$ and hence the necessary and sufficient conditions on r for exponential parameter convergence are exactly the same as in the relative degree one case.

6.2. Relative Degree ≥ 3

As in §6.1, pick a Hurwitz polynominal $\hat{L}$ so that $\hat{L}\hat{W}_M$ is strictly positive real. The trick used in §6.1, namely, to replace each θ_i by $\hat{L}\theta_i\hat{L}^{-1}$, is no longer possible since $\hat{L}\theta_i\hat{L}^{-1}$ depends on second (and possibly higher) derivatives of θ_i. To obtain a positive real error equation we retain the configuration of Figure 1 and attempt to augment the model output by

$$\frac{k_p}{k_M} \hat{W}_M \hat{L} [\theta^T \hat{L}^{-1} - \hat{L}^{-1}\theta^T] w \tag{6.4}$$

The difficulty in implementing (6.4) arises from the fact that k_p is unknown. Consequently the model output is augmented, not by (6.4) but by

$$\hat{W}_M \hat{L}\, \theta_{2n+1}(t) [\theta^T \hat{L}^{-1} - \hat{L}^{-1}\theta^T] w \tag{6.5}$$

with θ_{2n+1} being a new adaptive parameter expected to converge to $\frac{k_p}{k_M}$. To obtain $\phi \in L^2$ and prove stability of the augmented scheme we also add an additional quadratic term as shown in Figure 3 to (6.5) to get

$$\hat{W}_M \hat{L}\, \theta_{2n+1}(t) \left\{ (\theta^T \hat{L}^{-1} - \hat{L}^{-1}\theta^T) w + \alpha \varsigma^T \varsigma e_1 \right\} \tag{6.6}$$

where $\alpha > 0$ and ς is as defined in (6.2). If ξ is defined to be $(\theta^T \hat{L}^{-1} - \hat{L}^{-1}\theta^T) w$ then the update law

$$\dot{\theta} = -e_1 \varsigma$$

$$\dot{\theta}_{2n+1} = e_1\xi$$

yields that as $t \to \infty$, $e_1(t) \to 0$, that is $y_M \to y_P$.

For the scheme of Figure 3, there are $2n+1$ parameters to be considered and the sufficient richness condition for parameter convergence reads: there are α, $\delta > 0$ such that for all $s \geq 0$,

$$\int_s^{s+\delta} \begin{bmatrix} \varsigma \\ \xi \end{bmatrix} [\varsigma^T \xi] dt \geq \alpha I \tag{6.7}$$

where $\xi \triangleq (\theta^T \hat{L}^{-1} - \hat{L}^{-1}\theta^T)w$. However, condition (6.7) can never be satisfied since $\xi \to 0$ as $t \to \infty$ as pointed out by Anderson et al [13]. From the preceding discussion, it follows that the addition of the new parameter θ_{2n+1} in the augmented output signal is what causes this difficulty. If k_p is known, of course, θ_{2n+1}, ξ are unnecessary and the parameter convergence condition (6.7) reduces to (6.3) which is satisfied if $r(t)$ is sufficiently rich of order $\geq 2n$.

When k_P is unknown, and when $r(t)$ is SR of order $\geq 2n$ it follows that the autocovariance at zero of the signal vector $[\varsigma^T, \xi]^T$ is given by

$$\begin{bmatrix} R_\varsigma(0) & 0 \\ 0 & 0 \end{bmatrix} \in R^{2n+1 \times 2n+1} \tag{6.8}$$

with $R_\varsigma(0) > 0$. By the Partial Convergence Theorem of §5, it follows that the parameter error converges to the null space of the matrix in (6.8). Thus *all but the* $(2n+1)$th parameter errors converge to zero. But the $(2n+1)$th parameter is inconsequential since it is the gain parameter associated with the augmented model output y_a.

7. Concluding Remarks

We have shown that a *complete description of parameter convergence* can be given in terms of the *spectrum of the reference input signal.*

Specifically, regardless of the relative degree,

[1] The parameter error ϕ converges exponentially to zero if and only if $Supt(S_r)$ contains at least $2n$ points.

[2] If $Supt(S_r)$ contains only $k < 2n$ points, then ϕ need not converge to zero. Instead it converges to a subspace of dimension $2n-k$. This subspace corresponds *precisely to the set of parameter values for which the closed loop plant matches the model at the frequencies contained in* $Supt(S_r)$.

Appendix: Generalized Harmonic Analysis

The first careful treatment of the notion of autocovariance was Wiener's *Generalized Harmonic Analysis* [14]. The idea is well known in the theory of time-series analysis (see e.g. Koopmans [15]), and is usually presented in the context of stochastic processes. We have been unable to find a clear modern discussion of autocovariances which does not make use of the connection with wide sense stationary stochastic processes. Since the proofs of the various lemmas we used are neither difficult nor long, we give them here.

We should mention that the analogy between *autocovariance* and *stochastic autocovariance* mentioned in §3 is not complete- for example the limit in the definition of R_u makes the proof of the linear filter lemma trickier than the proof of its stochastic analog (which is little more than interchanging integrals and expectation via the Fubini theorem), and there is no stochastic analog of lemma 3.5.

For the remainder of this section we assume that $u:R_+ \to R^n$ has autocovariance R_u. Note that the integral (3.1) in the definition of autocovariance makes sense if and only if u is locally square integrable, i.e. $u \in L^2_{loc}$.

Lemma 3.3: R_u is a positive semi-definite function.

Proof: Suppose $\tau_1, \ldots, \tau_K \in R$, $c_1, \ldots, c_K \in C^n$. We must show

$$\sum_{i,j} c_i^* R_u(\tau_j - \tau_i) c_j \geq 0 \tag{A1}$$

Define the scalar valued function v by:

$$v(t) \triangleq \sum_{k=1}^{K} c_k^* u(t+\tau_k)$$

Then for all $T > 0$

$$0 \leq \frac{1}{T}\int_0^T |v(t)|^2 dt \tag{A2}$$

$$= \sum_{i,j} c_i^* \left[\frac{1}{T}\int_0^T u(t+\tau_i)u(t+\tau_j)^* dt\right] c_j = \sum_{i,j} c_i^* \left[\frac{1}{T}\int_{\tau_i}^{\tau_i+T} u(t)u(t+\tau_j-\tau_i)^* dt\right] c_j \tag{A3}$$

Since u has an autocovariance, as $T \to \infty$ (A3) converges to

$$\sum_{i,j} c_i^* R_u(\tau_j - \tau_i) c_j$$

From (A2) we see that (A3) is nonnegative, so (A1) follows.

Proposition A1 implies that R_u is the transform of a positive semi-definite matrix S_r of bounded measures, that is

$$R_u(\tau) = \int e^{i\nu\tau} S_r(d\nu) \tag{A4}$$

(This is the matrix analog of Bochner's theorem). S_r is symmetric, both in ν and as a matrix, since $R_u(\tau)$ is a real symmetric matrix.

Lemma 3.4 (Linear Filter Lemma): Suppose that $y = h*u$, where h is an $m \times n$ matrix of bounded measures. Then y has an autocovariance R_y given by

$$R_y(\tau) = \int\int h(d\tau_1) R_u(\tau + \tau_1 - \tau_2) h(d\tau_2)^T \tag{A5}$$

and spectral measure S_y given by

$$S_y(d\nu) = H(j\nu) S_u(d\nu) H(j\nu)^* \tag{A6}$$

Proof: We first establish that y has an autocovariance:

$$\frac{1}{T} \int_s^{s+T} y(t) y(t+\tau)^T dt \tag{A7}$$

$$= \frac{1}{T} \int_s^{s+T} \left[h(d\tau_1) u(t-\tau_1)\right] \left[u(t+\tau-\tau_2)^T h(d\tau_2)^T\right] dt \tag{A8}$$

For each T, the integrals in (A8) exist absolutely so we may change the order of integration:

$$= \int\int h(d\tau_1) \left[\frac{1}{T} \int_{s-\tau_1}^{s-\tau_1+T} u(t) u(t+\tau+\tau_1-\tau_2)^T dt\right] h(d\tau_2)^T \tag{A9}$$

The bracketed expression in (A9) converges to $R_u(\tau+\tau_1-\tau_2)$ as $T \to \infty$, uniformly in s. Furthermore the bracketed expression in (A9) is *bounded* as a function of T, s, τ_1, and τ_2, for $T \geq 1$, since by Cauchy-Schwarz*

$$\left| \frac{1}{T} \int_{s-\tau_1}^{s-\tau_1+T} u(t) u(t+\tau+\tau_1-\tau_2)^T dt \right| \leq \sup_{s,\, T\geq 1} \frac{1}{T} \int_s^{s+T} \|u(t)\|^2 dt < \infty \tag{A10}$$

So by dominated convergence (A9) converges, uniformly in s, as $T \to \infty$, to

$$\int\int h(d\tau_1) R_u(\tau+\tau_1-\tau_2) h(d\tau_2)^T \tag{A11}$$

y thus has an autocovariance, given by (A11). This establishes (A5); to finish the proof, we substitute the Bochner integral for R_u in (A11):

$$R_y(\tau) = \int\int h(d\tau_1) \int e^{i(\tau+\tau_1-\tau_2)\nu} S_u(d\nu) h(d\tau_2)^T \tag{A12}$$

*The restriction $T \geq 1$ is required if u is not bounded but only in L^2_{loc}.

$$= \int e^{i\nu\tau}\left[\int e^{-i\tau_1\nu}h(d\tau_1)\right]S_u(d\nu)\left[\int e^{-i\tau_2\nu}h(d\tau_2)\right]^* \tag{A13}$$

(since all the measures are finite)

$$= \int e^{i\nu\tau}H(j\nu)S_u(d\nu)H(j\nu)^* \tag{A14}$$

(A14) is the Bochner representation of R_y, so

$$S_y(d\nu) = H(j\nu)S_u(d\nu)H(j\nu)^* \tag{A15}$$

establishing the linear filter lemma.

Lemma 3.5 (Transient Lemma): Suppose $e(t) = u(t) - v(t) \in L^2$ (and u has autocovariance R_u). Then v also has autocovariance R_u.

Proof:

$$\left| \frac{1}{T}\int_s^{s+T} u(t)u(t+\tau)^T dt - \frac{1}{T}\int_s^{s+T} v(t)v(t+\tau)^T dt \right| \tag{A16}$$

$$= \left| \frac{1}{T}\int_s^{s+T} e(t)u(t+\tau)^T dt + \frac{1}{T}\int_s^{s+T} u(t)e(t+\tau)^T dt + \frac{1}{T}\int_s^{s+T} e(t)e(t+\tau)^T dt \right|$$

$$\leq \frac{1}{\sqrt{T}}\|e\|_2\left[\frac{1}{T}\int_s^{s+T}\|u(t+\tau)\|^2\right]^{1/2} + \frac{1}{\sqrt{T}}\|e\|_2\left[\frac{1}{T}\int_s^{s+T}\|u(t)\|^2\right]^{1/2} + \frac{1}{T}\|e\|_2^2 \tag{A17}$$

using the Cauchy-Schwarz inequality. The two bracketed expressions in (A17) converge uniformly in s as $T \to \infty$ to Trace$R_u(0)$, so we conclude that the entire expression (A17), and thus (A16), converges to zero, uniformly in s, as $T \to \infty$. Thus

$$\frac{1}{T}\int_s^{s+T} v(t)v(t+\tau)^T dt \to R_u(\tau) \quad \text{as} \quad T \to \infty$$

uniformly in s, and lemma 3.5 is proved. *Remark:* Actually the hypothesis can be weakened to $R_e = 0$, that is, e has *zero average energy.*

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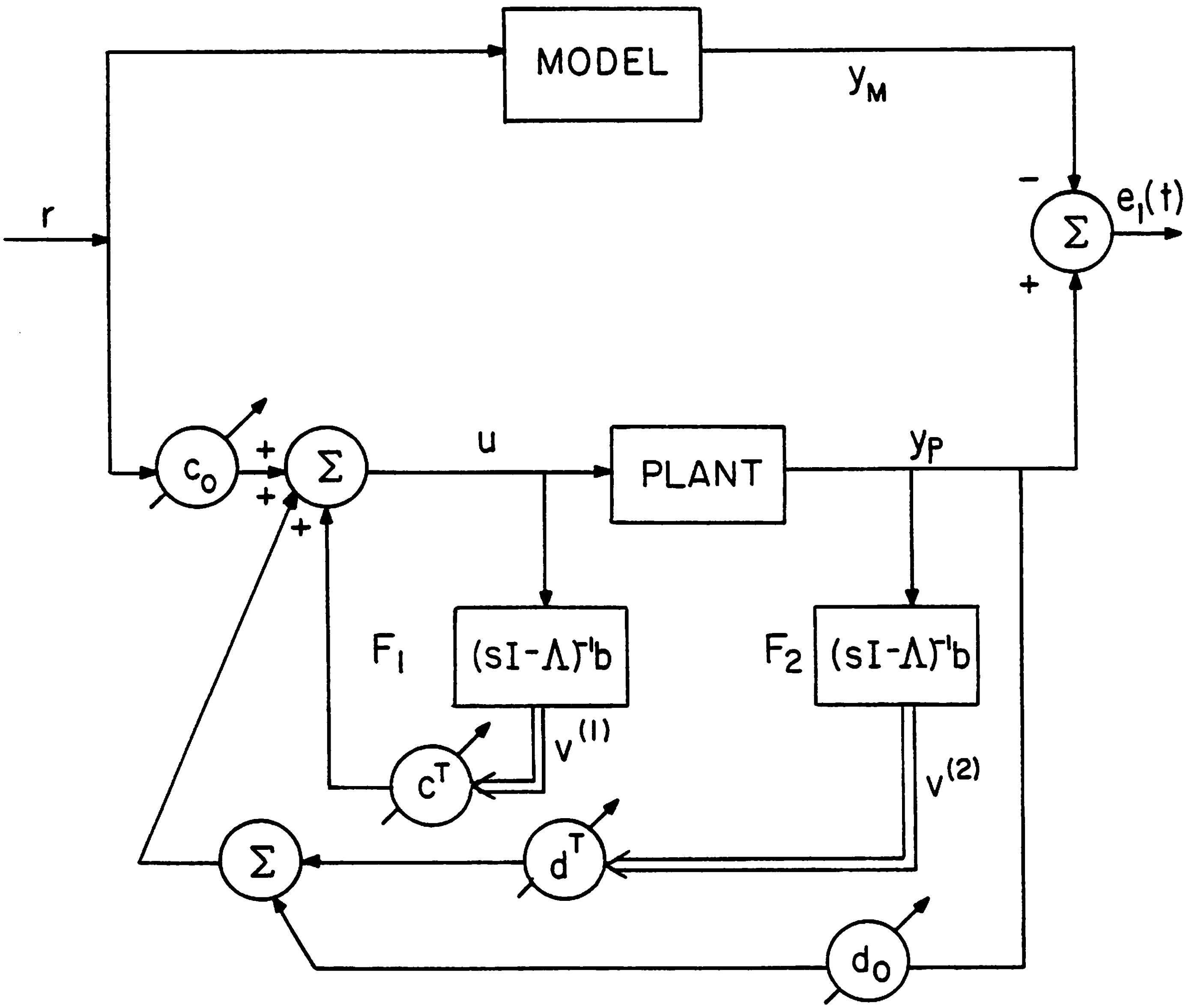
Figure Captions

Figure 1: The adaptive system for the relative degree one case.

Figure 2: The adaptive system of Figure 1 with a new representation for the model.

Figure 3: Modification of the adaptive scheme when the relative degree ≥ 3.


MODEL
y_M
$-$
$e_1(t)$
Σ
$+$
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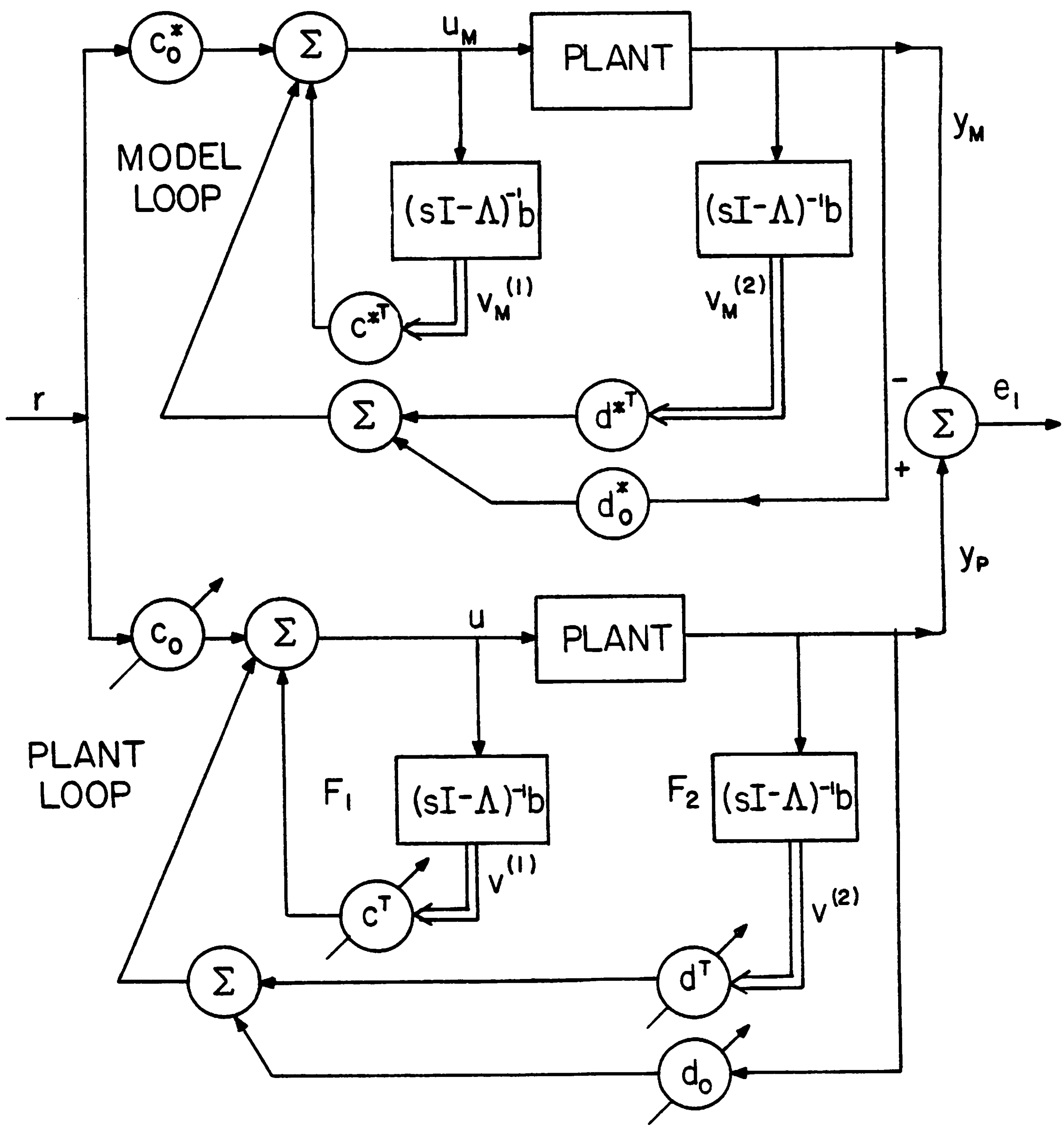
c_0
$+$
$+$
$+$
Σ
u
PLANT
y_P
F_1
$(sI-\Lambda)^{-1}b$
F_2
$(sI-\Lambda)^{-1}b$
$v^{(1)}$
c^T
$v^{(2)}$
Σ
d^T
d_0

Fig. 1



c_0^*
Σ
u_M
PLANT
y_M
MODEL
LOOP
$(sI-\Lambda)^{-1}b$
$(sI-\Lambda)^{-1}b$
c^{*T}
$v_M^{(1)}$
$v_M^{(2)}$
r
Σ
d^{*T}
-
e_1
Σ
+
d_0^*
y_P
c_0
Σ
u
PLANT
PLANT
LOOP
F_1
$(sI-\Lambda)^{-1}b$
F_2
$(sI-\Lambda)^{-1}b$
$v^{(1)}$
c^T
$v^{(2)}$
Σ
d^T
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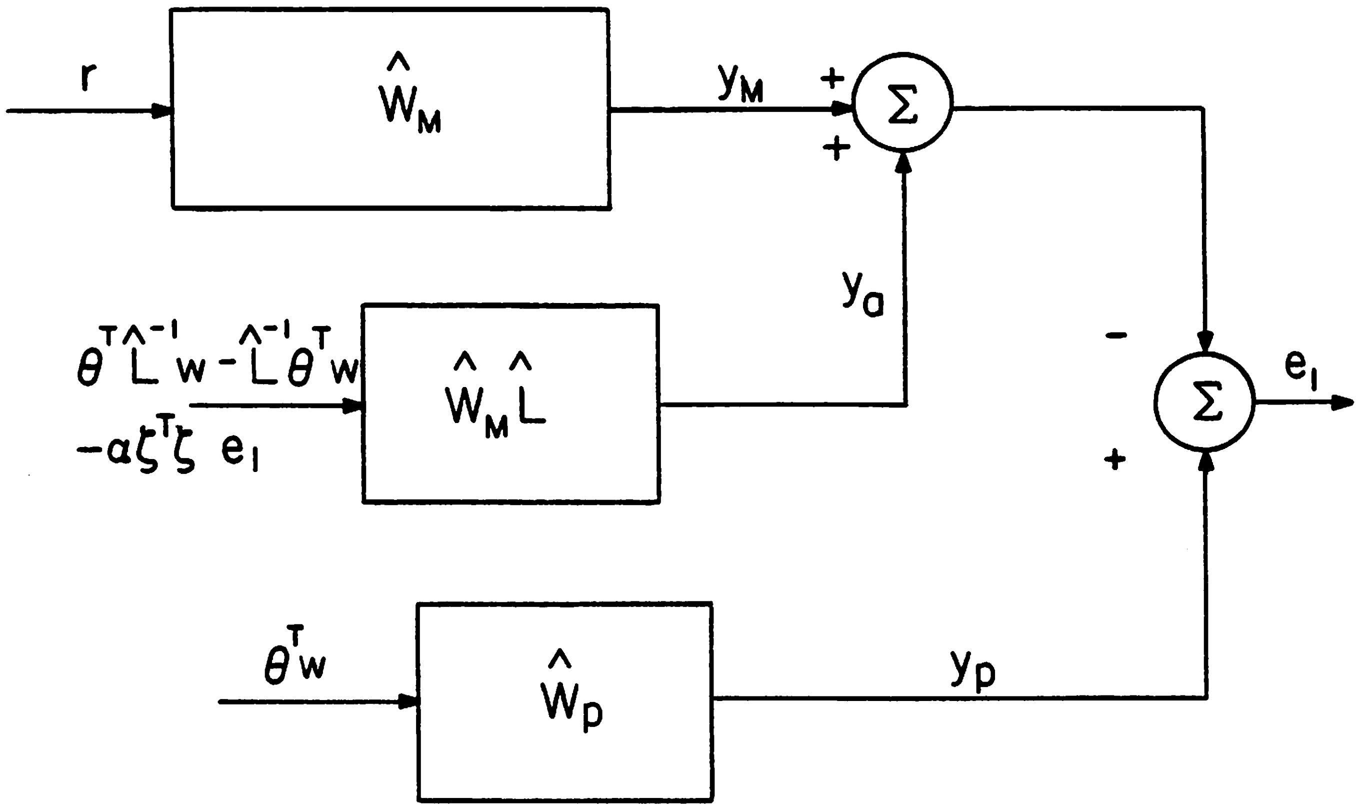
Fig. 2


r
$\hat{W}_M$
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$\hat{W}_M \hat{L}$
-
Σ
e_1
+
$\theta^T w$
$\hat{W}_p$
y_p

Fig. 3