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NON-AUTONOMOUS CIRCUIT

T. Matsumoto†, L.O. Chua†† and S. Tanaka†

Abstract A bifurcation diagram is given for an extremely simple non-autonomous circuit. It is a series connection of a linear resistor, linear inductor and a 2-segment piecewise — linear capacitor driven by a sinusoidal voltage source. The diagram resembles, in a surprisingly close manner, experimentally observed ones and numerical results with much more complicated models. A cross section of the chaotic attractor is also given.

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* Research supported in part by the Office of Naval Research under Contract N00014-76-C-0572, and by the National Science Foundation Grants ENG-7722745 and ECS-8313278.

Chaotic behabior have been observed in second order non-autonomous circuits consisting of a linear resistor, a linear inductor and a p-n junction diode driven by a sinusoidal voltage source[1-3]. Numerical results have also been reported with appropriate models for the diode[3-8]. The models, however, are rather complicated and it is not easy to see what cause(s) the chaotic behavior. Our purpose here is to report the simplest chaotic non-autonomous circuit and discuss its relationship with the cause of chaotic behavior of driven R-L-Diode circuits.

Consider the circuit of Fig.1(a) where R and L are linear and the capacitor is characterized by Fig.1(b) ; it is 2-segment piecewise-linear. We fix $R=60\Omega$, $L=100\mu\mathrm{H}$, $C_1=10\,\mathrm{nF}$, $C_2=40\,\mathrm{pF}$, $\omega/2\pi=700\,\mathrm{KHz}$, $E_0=0.1\,\mathrm{V}$, and vary the magnitude E of the driving voltage source from 0 V to 2.0 V. For each value of E we compute the capacitor charge $q(n\tau)$, $\tau=2\pi/\omega$ for $n \geqslant N$ where N is large enough so that the transient has decayed down. The result (bifurcation diagram) is shown in Fig.2. Fig.3 shows a cross section of the chaotic attractor at $E=2.0\,\mathrm{V}$. One can observe period doubling, chaotic band, periodic window etc. that are surprisingly similar to the experimental observations and numerical results with much more complicated models. The circuit of Fig.1 is the simplest because

(i) there is only one nonlinear element and the nonlinearity consists of only two segments each of which is linear (a linear system cannot be chaotic),

(ii) it has only two memory elements (it is well known that a first order non-autonomous system cannot be chaotic, generically), and

(iii) if the resistor is nonlinear, (which is simpler), instead of the capacitor, the circuit would have a unique steady state[10], and hence cannot exhibit chaotic behavior.

Let us now explain how we have come up with the simple circuit of Fig.1. A reasonably accurate model of a p-n junction diode is shown in Fig.4 where

$$i = I_S(\exp(v/v_T) - 1)$$

and

$$v = h(q)\ .$$

The function h is shown in Fig.4(c) and is rather complicated[11]. We have set up an R-L-Diode circuit and observed experimentally (not a numerical simulation) the wave forms of $v(t)$, the capacitor voltage. A typical one in a chaotic band is shown in Fig.5 (The parameters are the same as in the numerical case with $E=2.0\,\mathrm{V}$). We found out that $v(t)<0.2\,\mathrm{V}$ for all t, which means that $v(t)$ always stays in the flat region and never gets into the conduction region. This naturally lets one to remove the resitor. As for the capacitor, observe that the $v-q$ characteristic is still nonlinear in the region $v<0.2\mathrm{V}$. One of the simplest approximations of Fig.4(c) would be Fig.1(b).

The above result shows that the nonlinearity of the capacitor is a <u>definite cause</u> of the chaotic behavior of the circuit of Fig.1. There have been discussions on what is (are) responsible for the chaotic behavior of the R-L-Diode circuits.[3-8] Even though we do not claim that the non-linearity of the capacitor is the cause of chaotic behavior of every driven R-L-Diode circuit (since we have not given rigorous proof), we feel that the above arguments are valid for many R-L-Diode circuits.

Finally, let us mention that E_0 of Fig.1(b) cannot be zero in order for the circuit to be chaotic. A similar observation has been made by Rollins and Hunt[5]. Here we will give a <u>rigorous proof</u>. Let us first note that the piecewise-linear function f of Fig.1(b) can be represented by the following from[12]

$$f(q)=a|q|+bq+E_0$$

where

$$a=\frac{C_2-C_1}{2C_1C_2}, \quad b=\frac{C_1+C_2}{2C_1C_2}$$

and the dynamics of the circuit is given by

$$\begin{cases} \dfrac{dq}{dt}=i \\ L\dfrac{di}{dt}=-Ri-a|q|-bq-E_0+E\sin\omega t. \end{cases} \tag{1}$$

By the change of coordinate system

$$Q=q/E, \quad I=i/E \tag{2}$$

(1) is transformed into

$$\begin{cases} \dfrac{dQ}{dt}=I \\ L\dfrac{dI}{dt}=-RI-a|Q|-bQ-E_0/E+\sin\omega t \end{cases}$$

This means that if $E_0=0$, then for any value of E, (1) is equivalent to the dynamics where the amplitude of the driving voltage source is unity. Therefore, no bifurcations can occurr by changing E. Notice that unless f is piecewise-linear with only two segments, the change of coordinate system (2) would not lead to the desired proof. It would be extremely hard to find the right change of coordinate system even if it is not impossible.

Note also that by the change of coordinate system $Q=q/E_0$, $I=i/E_0$, (1) is transformed into

$$\begin{cases} \dfrac{dQ}{dt}=I \\ L\dfrac{dI}{dt}=-RI-a|Q|-bQ-1+E/E_0\sin\omega t. \end{cases}$$

This means that the bifurcation diagram obtained by changing E_0 while E being fixed is qualitatively the same as the one obtained by changing E while E_0 being fixed.

A more detailed analysis will be reported in a later paper.

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FIGURE CAPTIONS

Fig. 1 The simplest non-autonomous circuit with chaotic behavior ; (a) circuitry, (b) capacitor characteristic.

Fig. 2 The bifurcation diagram.

Fig. 3 Cross section of the chaotic attractor at $E=2.0\,\mathrm{V}$.

Fig. 4 Characterization of a diode ; (a) equivalent circuit, (b) resistor characteristic, (c) capacitor characteristic

Fig. 5 A typical wave form of $v(t)$ in a chaotic band. Vertical scale : 0.17V per division, Horizontal scale : $0.25\,\mu$sec per division.

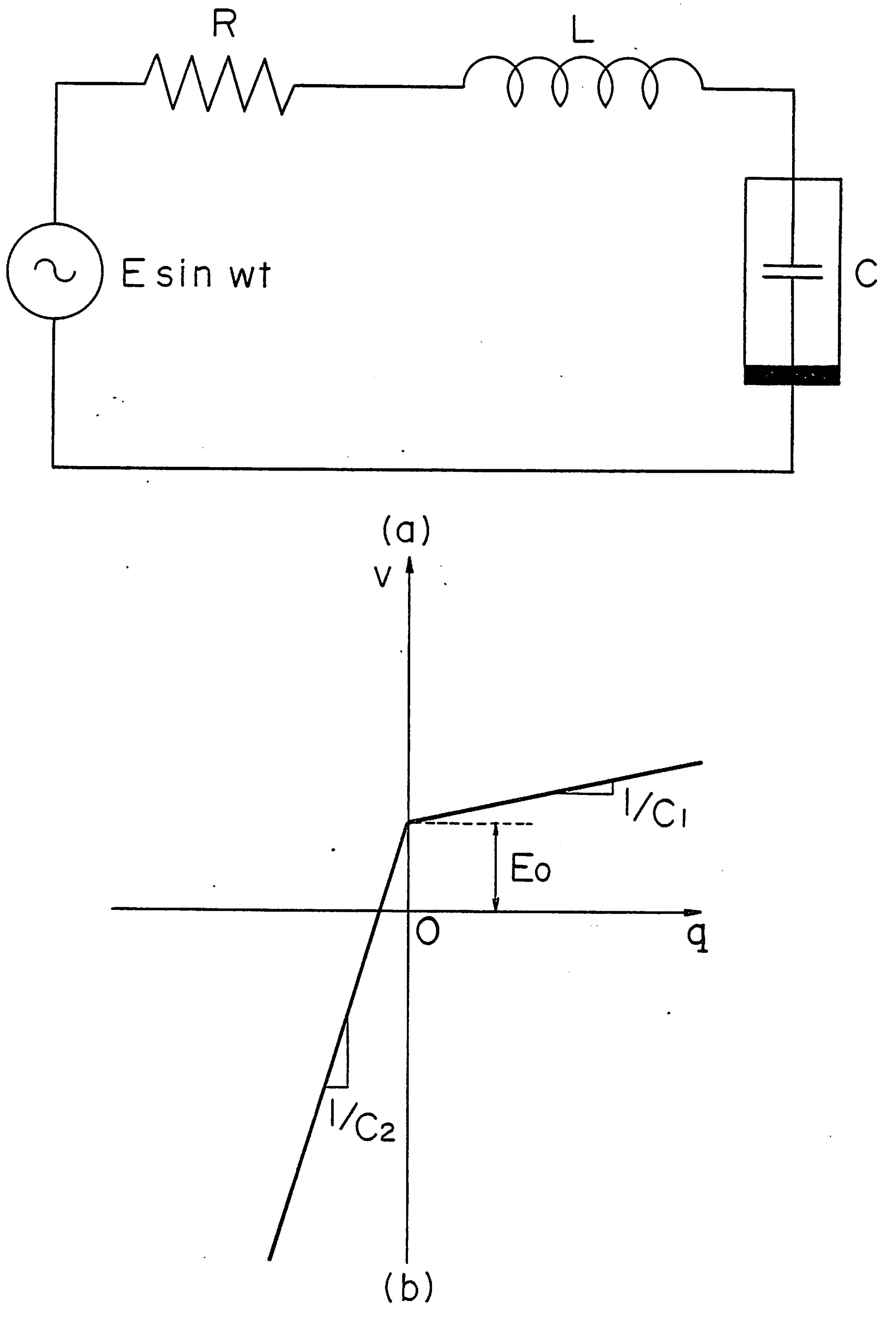
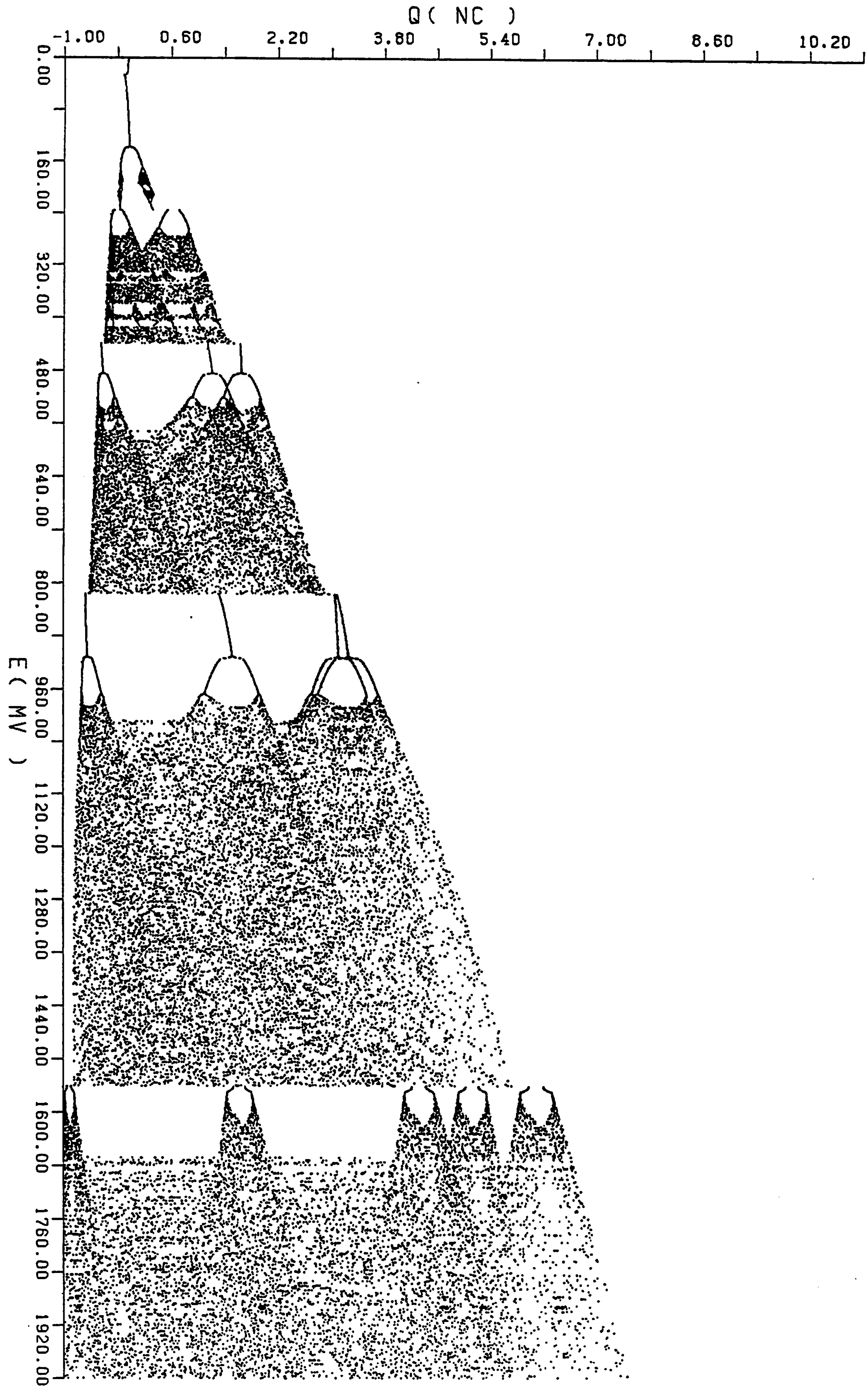


Fig. 1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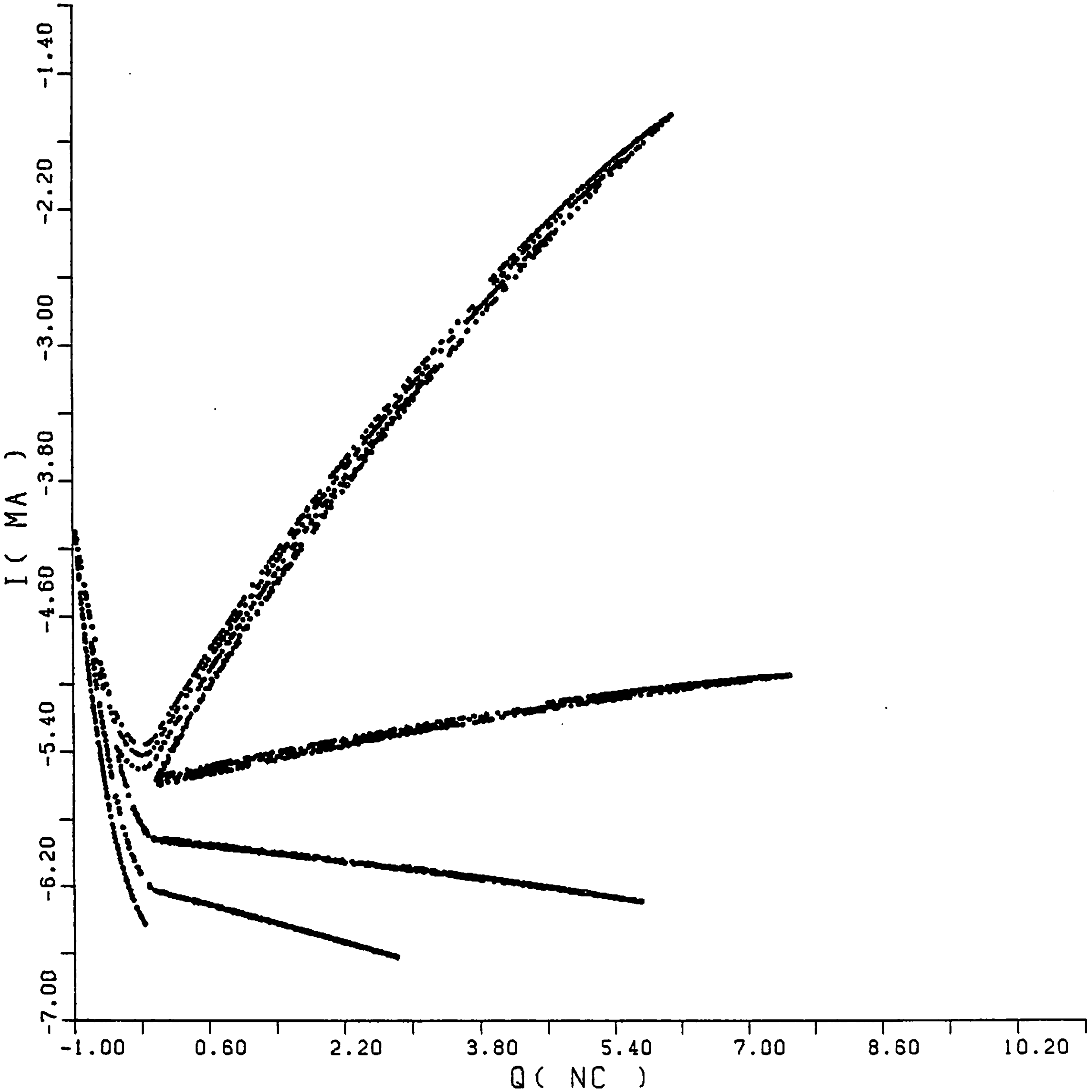
2.20
3.80
5.40
7.00
8.60
10.20
0.00
160.00
320.00
480.00
640.00
800.00
960.00
1120.00
1280.00
1440.00
1600.00
1760.00
1920.00
E(MV)

Fig. 2


I(MA)
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-2.20
-3.00
-3.80
-4.60
-5.40
-6.20
-7.00
-1.00
0.60
2.20
3.80
5.40
7.00
8.60
10.20
Q(NC)

Fig. 3

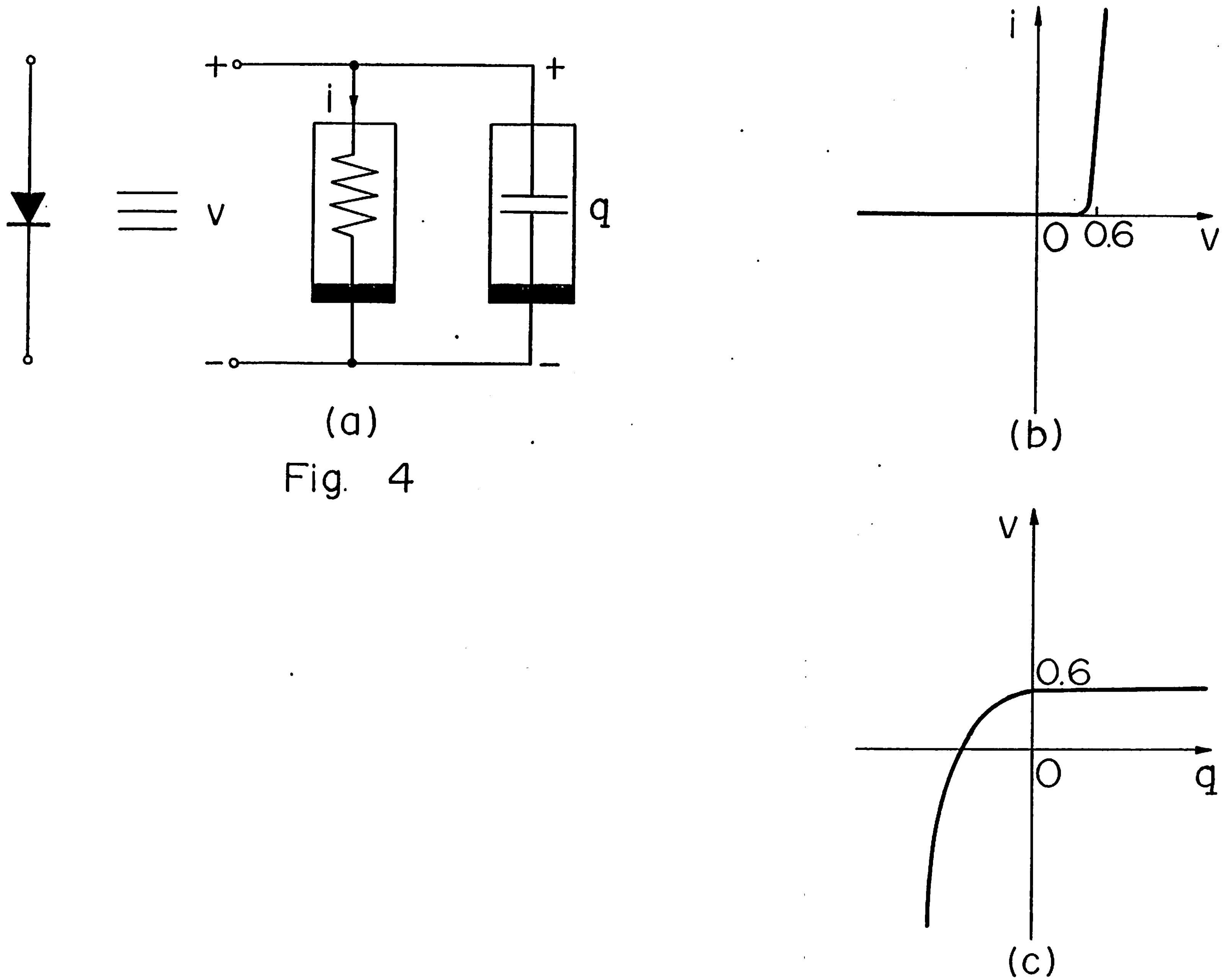

Fig. 4

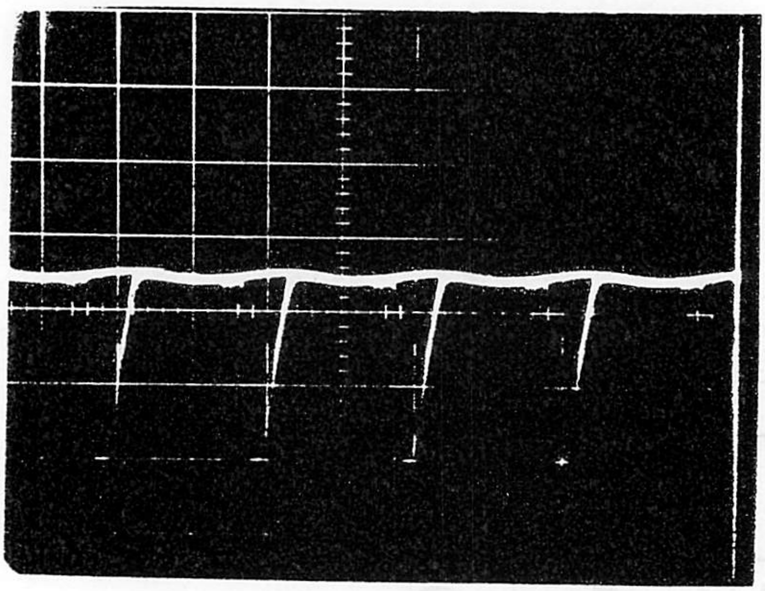

Fig. 5