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E. Polak*, D. Q. Mayne** and D. M. Stimler*

ABSTRACT

This paper presents a survey of the basic aspects involved in the design of linear multi-variable control systems via semi-infinite optimization. Specific topics treated are (i) data-base and simulation requirements, (ii) techniques for the transcription of design specifications into semi-infinite inequalites, and (iii) semi-infinite optimization algorithms for control system design.

1. INTRODUCTION.

Parametric optimization is a powerful tool for the selection of favorable values for design variables. At the present time, its use in many areas of engineering design is expanding rapidly (see e.g., [Pol.5]). As design specifications became more complex and computing tools more advanced, attempts to use parametric optimization in control system design, became inevitable. Early interpretations of control system design as an optimization problem, can be found in [Dav.1, Kar.1, Pol.1, Zak.1]. In fact, even the linear-quadratic regulator problem [Ath.1, Kwa.1] can be viewed as a parametric optimization problem: the optimal gain matrix for the linear-quadratic regulator problem is a solution to an unconstrained parametric optimization problem of the form

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$$\min_{K \in \mathbb{R}^{n \times m}} f(K), \tag{1.1}$$

where $f(K)$ is the largest eigenvalue of a symmetric matrix of the form

$$\int_0^\infty exp[t(A+BK)]^T (Q+K^T RK) exp[t(A+BK)] dt \tag{1.2}$$

with Q symmetric and positive semi-definite and R symmetric and positive definite.

The parametric optimization algorithms of the sixties and early seventies (see, e.g., [Lue.1], [Pol.4]) were only able to deal with optimization problems of the form

$$\min\{f(x) \mid g^j(x) \leq 0,\ j = 1,2,\ldots,m;\ h^k(x) = 0,\ k = 1,2,\ldots,l\} \tag{1.3}$$

in which the cost function $f: \mathbb{R}^n \to \mathbb{R}$, and the constraint functions $g^j, h^k: \mathbb{R}^n \to \mathbb{R}$, are continuously differentiable and in which there are only a *finite number* of equality and inequality constraints. Now, in the design of single-input single-output (SISO) control systems, specifications on quantities such as phase and gain margins, in the frequency domain, and step response rise time, settling time and overshoot, in the time domain, are commonplace (see, e.g., [Hor.1]). With the development of modern multi-input multi-output (MIMO) control system design theory, these SISO frequency domain specifications have been generalized in terms of norm (e.g., largest singular value) bounds on various transfer function matrices (see, e.g., [Doy.1]). In addition to these "obvious" extensions of SISO requirements, in the frequency domain, a number of multivariable control system specific requirements, such as decoupling and integrity [Ros.1], have also been introduced. It takes very little time to come to the conclusion that all these requirements do not lead to a classical, differentiable, finitely constrained optimization problem of the form (1.3), but rather to a *nondifferentiable*, *infinitely constrained* optimization

problem in the *finite dimensional* design parameter x which represents the free compensator coefficients. Problems of this kind are referred to as semi-infinite optimization problems (or semi-infinite optimization programs, i.e., SIP's). Thus, the appropriate canonical optimization problem for which codes are written and into which control system design problems must be transcribed has the form

$$\min\{f(x) \mid g^j(x) \leq 0,\ j = 1,2,\ldots,m;\ \varphi^k(x,y_k) \leq 0,\ y_k \in Y_k,\ k = 1,2,\ldots,l\}, \quad (1.4)$$

where the Y_k are finite time or frequency intervals.

The realization that existing optimization algorithms were inadequate for solving many engineering design problems has stimulated much research. In [Pol.2, Gon.1, Gon.2] we find described a new generation of optimization algorithms which were developed specifically for the solution of optimization problems of the form (1.4) under assumptions commonly true in engineering situations, while in [Pol.3] we find an algorithm for the specific case of (1.4) that corresponds to control system design. To date the design experience using these algorithms is still very limited, though quite promising. One can expect that a steady sequence of small improvements and innovations will make the use of semi-infinite optimization in control system design progressively more effective.

This paper is a survey of techniques involved in control system design via semi-infinite optimization. The paper is organized as follows.

In Section 2 we establish a data base for control system specification and response evaluation, to be used in optimization-based design. In Section 3 we will show how a typical control system design problem is transcribed into a semi-infinite optimization problem. Finally, in Section 4, we present an exposition of semi-infinite optimization algorithms for computer-aided control system design.

2. CONTROL SYSTEM SPECIFICATION AND RESPONSE EVALUATION.

We assume that the designer has selected a control system configuration, e.g., as in Fig. 1, and that semi-infinite optimization is to be used to compute compensator parameter values so as to satisfy design specifications and to minimize some cost function. Semi-infinite optimization can be used both for computing a *nominal* design and a *worst case* design. In the case of a nominal design, a mathematical model of *a single plant* is used and the mathematical model of the overall control system is adjusted so that the design specifications are met. In the case of worst case design, the mathematical model of the plant contains a mechanism for expressing *modelling uncertainty* and the overall control system is adjusted so that the design specifications are met over the entire plant uncertainty range. In either case, the mathematical models used for the plant and compensators must be such as to facilitate the computations required by the optimization algorithms as well as being compatible with various methodological and system theoretical considerations.

We begin with the simpler case of nominal design. In this case, we assume that the control system to be designed consists of interconnected subsystems. The free, or designable, parameters of the subsystems form the finite dimensional *design vector* $x \in \mathbb{R}^n$. The manner in which subsystems are represented depends partly on whether both time and frequency domain specifications are to be met and partly on the method of parametrization used for the compensators.

When the subsystems are represented in state space form, as

$$S_i \quad \begin{cases} \dot{z}_i = A_i(x)z_i + B_i(x)u_i \\ y_i = C_i(x)z_i + D_i(x)u_i \end{cases} \tag{2.1}$$

they define (assuming there are N subsystems) an *assembly* of subsystems S of the form

$$S \begin{cases} \dot{z} = A(x)z + B(x)u \\ y = C(x)z + D(x)u \end{cases} \tag{2.2}$$

where $A(x) = diag(A_1(x), A_2(x), \ldots A_N(x))$, $B(x) = diag(B_1(x), B_2(x), \ldots B_N(x))$, $C(x) = diag(C_1(x), C_2(x), \ldots, C_N(x))$, $D(x) = diag(D_1(x), D_2(x), \ldots, D_N(x))$. Note that although all matrices were shown to depend on the design parameter x, some of them may, in fact be constant, as in the case of the ones in the plant representation. The interconnections between the subsystems are expressed algebraically:

$$u = Ey + Jr \tag{2.3}$$

where r is a vector of external inputs and E and J are matrices whose elements are zeros and ones.

In control system design, the design vector x can have very large dimension, a fact that leads to a considerable computational burden in the evaluation of gradients with respect to x. The dimension of the design vector x can be somewhat reduced by using minimal forms for the designable system matrices. For example, the system matrices A_i corresponding to compensator blocks may be specified in block diagonal form

$$A_i = \text{diag}(A_{1i}, A_{2i}, \ldots, A_{ki}, \lambda_{(2k+1)i}, \ldots, \lambda_{(N_i)i}) \tag{2.4}$$

where the λ_{ji} are real (some may be frozen at zero for integral action), while

$$A_{ji} = \begin{bmatrix} 0 & 1 \\ a_{1ji} & a_{0ji} \end{bmatrix}. \tag{2.5}$$

This allows design of both real and complex compensator poles. Some structural simplification of the B matrix is also possible. The interconnection equation (2.3) can be eliminated to produce a reduced description of the form

$$S \begin{cases} \dot{z} = A_c(x)z + B_c(x)r \\ y = C_c(x)z + D_{c(x)}r \end{cases} \tag{2.6}$$

where $A_c(x) = A(x) + B(x)[I - ED(x)]^{-1}EC(x)$, $B_c(x) = B(x)J + B(x)[I - ED(x)]^{-1}ED(x)$, $C_c(x) = [I-ED(x)]^{-1}C(x)$, and $D_c(x) = D(x)[I - ED(x)]^{-1}J$, in terms of the matrices in (2.2) and (2.3). Since the closed loop system (2.6) always has distinct eigenvalues (at least with probability 1), the computation of responses can be considerably simplified by diagonalization (more robust techniques, based on Schur decomposition, are also being contemplated). Thus, rewriting (2.6) with the parameters made explicit, we get

$$\dot{z}(t,x) = A_c(x)z(t,x) + B_c(x)r(t) \tag{2.7a}$$

$$y(t,x) = C_c(x)z(t,x) + D_c(x)r(t)\,. \tag{2.7b}$$

We begin with the time responses to inputs $r(t)$. With $W(x)$ a matrix of eigenvectors of $A_c(x)$, we obtain,

$$z(t,x) = W(x)e^{\Lambda(x)t}W(x)^{-1}x(0) + \int_0^t W(x)e^{\Lambda(x)(t-s)}W(x)^{-1}r(s)ds \tag{2.8}$$

where $\Lambda(x)$ is a diagonal matrix of eigenvalues of $A_c(x)$ The output $y(t,x)$ is then computed according to (2.6). When the input $r(t)$ is a polynomial, as is often the case, the integral in (2.8) can be and is evaluated analytically [Bec.1]. Next, optimization algorithms require derivatives of responses with respect to design parameters. The derivatives of the time responses with respect to x of $z(t,x)$ and $y(t,x)$ in (2.6) can be computed by solving

$$\begin{aligned}\frac{d}{dt}(\partial z(t,x)/\partial x) &= A_c(x)(\partial z(t,x)/\partial x)\\ &+ (\partial A_c(x)/\partial x)z(t,x) + (\partial B_c(x)/\partial x)r(t)\end{aligned} \tag{2.9}$$

$$\begin{aligned}\partial y(t,x)/\partial x = C_c(x)(\partial z(x,t)/\partial x) &+ (\partial C_c(x)/\partial x)z(t,x)\\ &+ (\partial D_c(x)/\partial x)r(t)\,.\end{aligned} \tag{2.10}$$

The diagonalization matrix $W(x)$ can be used again to produce fairly simple formulas for the derivatives $\partial z(t,x)/\partial x$ and $\partial y(t,x)/\partial x$. Next we turn to the frequency response of the interconnected system. The input-output transfer func-

tion of the interconnected system is given by

$$G(j\omega,x) = C_c(x)[j\omega I - A_c(x)]^{-1}B_c(x) + D_c(x)\,. \tag{2.11}$$

Since the derivative of G with respect to x is not a matrix, it is easiest to obtain componentwise expressions for it, viz.,

$$\partial G(j\omega,x)/\partial x^i = (\partial C_c(x)/\partial x^i)[j\omega I - A_c(x)]^{-1}\,B_c(x)$$

$$+\, C_c(x)[j\omega I - A_c(x)]^{-1}(\partial A_c(x)/\partial x^i)[j\omega I - A_c(x)]^{-1}$$

$$+\, C_c(x)[j\omega I - A_c(x)]^{-1}(\partial B_c(x)/\partial x^i) + (\partial D_c(x)/\partial x^i) \tag{2.12}$$

Assuming that the time response derivatives are computed first, the only major computation left in the evaluation of the frequency responses and their derivatives as specified by (2.11),(2.12) is the evaluation of the matrix $[j\omega I - A_c(x)]^{-1}$. Since a diagonalization for $A_c(x)$ is already available, this computation can be considerably simplified by making use of the formula

$$[j\omega I - A_c(x)]^{-1} = W(x)[j\omega I - \Lambda(x)^{-1}]W(x)^{-1}\,. \tag{2.13}$$

A recent approach to two degrees of freedom control system design [Des.1, Per.1, Per.2], see Fig. 2, uses polynomial matrix descriptions for the plant and controllers, with the controllers expressed in terms of parametrized transfer function matrices $M(x,s)$ and $Y(x,s)$. This requires that the plant be factored into left and right factorizations which are co-prime relative to a prespecified stability region S which is symmetric about the real axis in the complex plane. In this case, denoting the plant transfer function by $P(s)$, we have

$$P(s) = D_l(s)^{-1}N_l(s) = N_r(s)D_r(s)^{-1} \tag{2.14}$$

with $D_l(s)$, $N_l(s)$, $D_r(s)$, $N_r(s)$ rational matrices which have no poles in S. In the process, two rational matrices $U(s), V(s)$ are constructed which have no poles in S and which satisfy the relation

$$U(s)N_r(s) + V(s)D_r(s) = I. \tag{2.15}$$

It was shown in [Per.1, Per.2, Vid.1] that all the S-stable closed loop transfer functions that can be realized by our two degree of freedom system have the form

$$H_{yu}(s) = N_r(s)M(x,s), \tag{2.16}$$

$$H_{yd}(s) = I - N_r(s)[U(s) + Y(x,s)D_l(s)], \tag{2.17}$$

where $M(x,s)$ and $Y(x,s)$ are arbitrary, proper, rational matrices with S-stable elements. The free coefficients of the elements of M and Y make up the design vector x. The closed loop transfer functions in (2.16), (2.17) are realized by the controller configuration

$$C_n(x,s) = M(x,s) \tag{2.18a}$$

$$C_s(x,s) = V_r(s) - Y(x,s)N_l(s) \tag{2.18b}$$

$$C_f(x,s) = U_r(s) + Y(x,s)D_l(s) \tag{2.18c}$$

As we shall see later, the parametrization (2.18) has the advantage of allowing closed loop stability to be easily ensured and of allowing easy modification of the transient behavior. However, it gives the designer very little control over the complexity of the controller.

Once uncertainty is introduced into the plant model, worst case design becomes essentially restricted to the frequency domain either because the specification of uncertainty is such as to provide inadequate information for time domain response evaluations, or because the computaional complexity of the totality of time domain evaluations needed for worst case design appears to make such designs practically impossible. Hence, uncertainty is usually specified in terms of transfer functions. There are basically two approaches. The first is to assume a nominal plant transfer function $P(s)$, whose elements are proper rational functions and then to consider in the design all transfer

functions of the form

$$P(s)[I + L(s)], \tag{2.19}$$

where $L(\cdot) \in \mathbf{L}$, a family of complex valued functions which is assumed to have the following properties:

Assumption 2.1: (a) There is a continous real valued function $b(\omega)$ such that

$$||L(j\omega)||_2 \leq b(\omega) \ \forall \ \omega >= 0 . \tag{2.20}$$

(Usually, the function $b(\omega)$ grows as ω goes to ∞)

(b) The transfer function $P(s)[I + L(s)]$ has the same number of unstable ($\mathbb{C}_+$) poles (counting multiplicities) as $P(s)$. ▪

Plant model uncertainty which is specified only to the extent assumed in Assumption 2.1 is called *unstructured uncertainty*. It is particularly useful for modelling high frequency discrepancies between the behavior predicted by the nominal plant and that of the actual plant. Typically, bounds such as (2.20) are used to ensure that the unstructured uncertainty does not destabilize the closed loop system. When the unstructured uncertainty is significant at frequencies in the bandwidth of the closed loop system, it may be necessary to take unstructured uncertainty into account in formulating other design requirements as well.

Once one admits that there is uncertainty in the plant model, one must also admit that there is also uncertainty in the specification of the transfer function of the nominal plant which suggests that one should also introduce *parametric* (i.e., structured) uncertainty into the coefficients of the nominal plant. The effect of this is to tremendously increase the computational complexity of the resulting semi-infinite optimization problem. So far, there are effective tech-

niques for overcoming this difficulty for single-input, single-output plants specified in the form

$$P(s,\alpha) = \frac{K\Pi_i(s + z^i)}{\Pi_j(s + p^j)} \tag{2.22}$$

where α denotes a vector whose components are the gain K, the zeros z^i and the poles p^j. The gain K, the zeros z^i and the poles p^j are assumed to lie in confidence intervals, i.e.,

$$K\in[\underline{K},\overline{K}],\ z^i\in[\underline{z}^i,\overline{z}^i]\ ,\ p^i\in[\underline{p}^i,\overline{p}^i]\ . \tag{2.23}$$

For the complex zeros and poles, which must occur in complex conjugate pairs, we assume that that the "intervals" of uncertainty are, in fact rectangles in $\mathbb{C}$ defined by

$$[\underline{z}^i,\overline{z}^i] \triangleq \{(u,v)\in\mathbb{R}^2 \mid Re(\underline{z}^i)\le u\le Re(\overline{z}^i),\ \ Im(\underline{z}^i)\le v\le Im(\overline{z}^i)\ \} \tag{2.24}$$

and similarly for the poles. The product of all the uncertainty intervals in (2.23) will be denoted by $\mathbf{A}$ and is a compact subset of $\mathbb{R}^{n_\alpha}$. For the design problem to be well posed, we shall need the following hypothesis to hold.

Assumption 2.2. (i) The plant transfer function $P(\cdot,\alpha)$ has the same number of poles (counting multiplicities) in $(int\ S)^c$, the complement of the interior of S, for all $\alpha\in\mathbf{A}$. (ii) No $(int\ S)^c$ pole of $P(\cdot,\alpha)$ is cancelled by a zero of $P(\cdot,\alpha)$ for any $\alpha\in\mathbf{A}$ ▪

Next we shall turn to the generation of semi-infinite inequalities from control system design requirements.

3. TRANSCRIPTION OF SPECIFICATIONS INTO SEMI-INFINITE INEQUALITIES.

We shall illustrate the process of transcribing control system design specifications into semi-infinite inequalities by considering a typical example:

the design of a two degrees of freedom control system of the form shown in Fig. 2. We assume that the plant is linear and finite dimensional, with a *nominal*, $m \times m$ proper rational transfer function matrix $P(s)$. We assume that we are required to design three compensator transfer function matrices $C_\pi(x,s)$, $C_s(x,s)$, and $C_f(x,s)$, with elements that are rational functions. The free coefficients of the compensator transfer function matrices make up the design vector $x \in \mathbb{R}^n$. We shall consider time and frequency domain specifications both on the nominal closed loop system as well as on the family of closed loop systems resulting from multiplicative perturbations of the plant transfer function, i.e., for plant transfer functions of the form $P(s)(I + L(s))$, introduced in the preceeding section (see (2.19). Towards the end of this section we shall consider the special case of design in the presence of both structured and unstructured uncertainty for SISO plants.

In formulating design specifications as semi-infinite inequalities one must take care to ensure that the resulting functions satisfy certain minimum continuity requirements and that the implied computations are reasonably well conditioned. In particular, in order to develop rigorously algorithms for optimization problems involving nondifferentiable functions, it is necessary that these functions be at least locally Lipschitz continuous [Cla.1]. In this respect, control system design presents particular difficulties which stem from the fact that transfer functions are discontinuous at their poles and hence any functions defined as functions of transfer functions are also likely to be discontinuous. We shall comment on ways of coping with this difficulty as we progress.

We break up the presentation to follow into two parts. The first one deals with nominal and worst case design in the presence of unstructured uncertainty. The second part deals with worst case design in the presence of both structured and unstructured uncertainty and is restricted to SISO systems.

3.1. Design in the Presence of Unstructured Uncertainty.

We begin with a basic nominal design requirement.

(i) S-Stability for the Nominal System.

We begin with the requirement of confining the closed loop poles to a prespecified region $S \subset \mathbb{C}$. We shall say that the closed loop system is *S-stable* if all the roots of its characteristic polynomial are in S. For example, one may wish to design a control system with a minimum bandwith, while at the same time ensuring that all closed loop complex poles have a minimum damping ratio. Crudely, this can be accomplished by letting the region S be as shown in Fig. 3, i.e., to the left of the boundary defined by

$$Re(s) = -\kappa|Im(s)| - \gamma, \tag{3.1}$$

where Re denotes the real part, Im denotes the imaginary part and $\kappa, \gamma > 0$. We must consider two possibilities: (i) the compensators are defined as in (2.18) and (ii) the compensators are defined as in (2.4) and (2.5).

First, let us consider the easier case when the compensator structure is as in (2.18a) - (2.18c). In this case, the plant transfer function matrix must be factored into left and right co-prime factors which have no poles in S. Hence, referring to [Des.1], we see that the closed loop system is S-stable if and only if the matrices $M(x,s)$, and $Y(x,s)$ are S-stable. Hence, if $M(x,s)$ is specified, componentwise, as

$$M_{ij}(x,s) = \frac{K_{ij}\Pi_k(s + z_{ij}{}^k)}{\Pi_l(s + p_{ij}{}^l)\Pi_q(s^2 + 2\zeta_{ij}{}^q\omega_{ij}{}^q s + (\omega_{ij}{}^q)^2)} \tag{3.2}$$

with similar expressions for $Y(x,s)$, then S-stability for the *nominal* closed loop system is ensured by the following *finite* set of inequalities:

$$p_{ij}{}^{l} > \gamma > 0 \tag{3.3a}$$

$$\zeta_{ij}{}^{q}\omega_{ij}{}^{q} >= \kappa[1-(\zeta_{ij}{}^{q})^{2}]^{\frac{1}{2}}\omega_{ij}{}^{q} + \gamma, \quad \text{if } (\zeta_{ij})^{q\,2} \leq 1\,, \tag{3.3b}$$

$$\zeta_{ij}{}^{q}\omega_{ij}{}^{q} >= \omega_{ij}[(\zeta_{ij}{}^{q})^{2}-1]^{\frac{1}{2}} + \gamma, \quad \text{if } (\zeta_{ij}{}^{q})^{2} > 1 \tag{3.3c}$$

with similar inequalities imposed on the elements of $Y(x,s)$. The inequalities (3.3a) and (3.3b) can also be used to ensure minimum bandwidth.

Now, referring to Fig. 2, suppose that the plant and compensators are specified in state space form:

$$\frac{d}{dt}z_p = A_p z_p + B_p u_p$$

$$y_p = C_p z_p + D_p u_p \tag{3.4a}$$

$$\frac{d}{dt}z_\pi = A_\pi z_\pi + B_\pi u_\pi$$

$$y_\pi = C_\pi z_\pi + D_\pi u_\pi \tag{3.4b}$$

$$\frac{d}{dt}z_s = A_s z_s + B_s u_s$$

$$y_s = C_s z_s + D_s u_s \tag{3.4c}$$

$$\frac{d}{dt}z_f = A_f z_f + B_f u_f$$

$$y_f = C_f z_f + D_f u_f \tag{3.4d}$$

with the interconnection specified by

$$u_\pi = r, u_s = y_\pi - y_f, u_p = y_s, u_f = d + y_p\ . \tag{3.4e}$$

As can be easily verified, the system matrix for the closed loop of the system (not including the precompensator) is given by

$$A(x) = \begin{bmatrix} A_p - B_p(I + D_C D_p)^{-1} D_c C_p & -B_p(I+D_c D_p)^{-1} C_c \\ B_c(I + D_p D_c)^{-1} C_p & A_c - B_c(I + D_P D_c)^{-1} D_p C_c \end{bmatrix}. \tag{3.5}$$

Let the eigenvalues of $A(x)$ be denoted by $\lambda^{j}[A(x)]$. Then the simplest way to ensure S-stability for the example under consideration is to require that these eigenvalues satisfy the system of inequalities

$$\mathrm{Im}[\lambda^j[A(x)]] + \kappa[\mathrm{Re}\lambda^j[A(x)]] + \gamma \leq 0 \quad \text{for } j = 1,2,\ldots,N_c\ , \tag{3.6}$$

(where N_c is the dimension of the state vector) with similar inequalities imposed on the eigenvalues of A_π. Note that in (3.6) we are exploiting the fact that complex eigenvalues must come in conjugate pairs.

There are two objections to this approach. The first is that eigenvalues can be extremely sensitive to parameter variations, resulting in severe computational ill conditioning. The second is that eigenvalues are differentiable only when they are distinct. Now, while one can generally expect closed loop eigenvalues to be distinct, it is possible in the course of an optimization computation to approach a point of eigenvalue multiplicity with dire effects on the behavior of an optimization algorithm.

An alternative approach is to use a modification of the Nyquist stability criterion described in [Pol.6], which leads to a well conditioned semi-infinite inequality. First we shall explain the modified Nyquist criterion for the simplest case, when the nominal plant transfer functions $P(s)$ is scalar valued. Let us write the loop gain in the form of a quotient of two polynomials, as follows:

$$C_s(x,s)C_f(x,s)P(s) = n(x,s)/d(x,s)\ . \tag{3.7}$$

The nominal closed loop system will be S-stable if $C_\pi(x,s)$ is S-stable and all the zeros of the characteristic polynomial

$$c(x,s) = n(x,s) + d(x,s) \tag{3.8}$$

are in S. The simplest extension of the Nyquist stability criterion for determining whether or not the zeros of $c(x,s)$ are in S, with S defined as in (3.1), consists of plotting

$$c(x,s)/d(x,s) \tag{3.9}$$

for $s = -\kappa|\omega| - \gamma + j\omega$, with $\omega \in (-\infty,\infty)$ and observe whether the number of coun-

terclockwise encirclements of the origin is equal to the number of zeros of $d(x,s)$ which are not in S. Thus, instead of using the usual Nyquist contour, one uses one which replaces the $j\omega$-axis by the boundary of S. Intrinsically, this approach implies that an integer valued function is used in the stability determination. Hence such a function is not continuous in the design vector x, since a small change in x can cause an integer change in the number of encirclements.

Our first observation is that a major source of our difficulty is attributable to the denominator polynomial, $d(x,s)$, which may have roots outside of S. Now the main function of this polynomial in the stability test is to provide scaling, so that the graph one is drawing remains confined to one's sheet of paper. However, because it may have roots outside of S, the use of $d(x,s)$ imposes the need to count encirclements. This suggests that one way of eliminating the need of counting encirclements is to replace the polynomial $d(x,s)$ by one that does not have roots outside of S. Let $\tilde{d}(s)$ be a polynomial of the same degree as $d(x,s)$ and let all the zeros of $\tilde{d}(s)$ be in the interior of S. Let

$$t(x,s) \triangleq c(x,s)/\tilde{d}(s) . \tag{3.10a}$$

S-stability can now be ensured by requiring that the plot of $t(x,-\kappa|\omega| - \gamma + j\omega)$, for $-\infty < \omega < \infty$, does not encircle the origin. The easiest way to express this fact as a semi-infinite inequality is to encase the origin in a parabolic region, with boundary defined by $Im(s) = k_1(Re(s))^2 - k_2$, with $k_1, k_2 > 0$, see Fig. 4, and require that the locus of $t(x,-\kappa|\omega| - \gamma + j\omega)$ stay out of this region. This leads to the following sufficient condition of stability:

$$Im[t(x,-\kappa|\omega| - \gamma + j\omega)] - k_1\{Re[t(x,-\kappa|\omega| - \gamma + j\omega)]\}^2 + k_2 \leq 0 ,$$
$$\forall\ \omega \in [\omega',\omega''] \tag{3.10b}$$

where $[\omega',\omega'']$ is a critical interval of frequencies. Note that (3.10b) is only a sufficient and not a necessary and sufficient condition of bibo stability.

Referring to the Nyquist stability criterion for the multivariable case, as stated in [Che.1] we see that, for the system in Fig. 2, with $C_\pi(s) \equiv I$, $C_f(s) \equiv I$, the appropriate definition of the rational function $t(x,s)$ becomes

$$t(x,s) = \frac{det(sI - A_s(x))det(sI - A_p)det(I + G_p(s)G_c(x,s)C_s(x,s))}{\tilde{d}(s)} \quad (3.11)$$

where $C_s(x,s)$ and $G_p(s)$ are the plant and compensator transfer functions and $\tilde{d}(s)$ is a polynomial with roots in the interior of S, of the same degree as the characteristic polynomial of the loop.

(ii) Stability Robustness.

Next, we turn to the problem of ensuring closed loop system stability in the presence of unstructured plant uncertainty, as is required in worst case design. Referring to [Che.2, Doy.1, San.1], we see that if the nominal design, for the plant model $P(s)$, is bounded -input -bounded -output (bibo) stable, then the closed loop system will remain bibo stable for all plant transfer functions of the form $P(s)(I + L(s))$, with $L(s)$ as defined in (2.20), if the nominal feedback system satisfies

$$\bar{\sigma}[H_{yu}(x,j\omega)] \leq 1/b(\omega), \quad \forall\ \omega \in [\omega',\omega''], \quad (3.12)$$

where $[\omega',\omega'']$ is a critical range of frequencies.

(iii) Disturbance Suppression.

The effect of the output disturbance on the output of the nominal closed loop system can be suppressed by making the closed loop system output-to-disturbance transfer function $H_{yd}(x,j\omega)$ "small" in a critical frequency range. This is usually accomplished by imposing bounds on the norm of the transfer function $H_{yd}(x,j\omega)$. When the induced L_2 norm of a matrix is used, the L_2 norm can be computed by making use of the fact that it is equal to the *maximum singular value* $\bar{\sigma}[H_{yd}(x,j\omega)]$. Typically, these bounds have the form:

$$\bar{\sigma}[H_{yd}(x,j\omega)] \leq b_d(\omega), \quad \forall\ \omega \in [\omega', \omega''] . \tag{3.13}$$

(iv) Plant Saturation Avoidance.

To avoid plant saturation by, say, output disturbances, it may be necessary to impose constraints on the norm (maximum singular value $\bar{\sigma}[H_{u_p d}(x,j\omega)]$) of the transfer function $H_{u_p d}$ over a range of frequencies. This can be done conveniently only for the nominal design, as follows:

$$\bar{\sigma}[H_{u_p d}(x,j\omega)]) \ < \ b_s(\omega) \ \forall \ \ \omega \in [\omega', \omega''] . \tag{3.14}$$

(v) Input Signal Tracking.

Satisfactory input signal tracking, for a class of inputs $\{u_j\}_{j \in J}$, can be ensured only for the nominal design. Tracking is specified by semi-inifnite inequalities of the form (see Fig. 5)

$$\underline{b}^{jk}(t) \leq y^k(x,t,u_j) \leq \bar{b}^{jk}(t) \forall \ \ t \in [0,T], \ \forall \ \ j \in J, \ \forall \ \ k = 1,2,...,m . \ ((3.15))$$

Above, y^j denotes the j-th component of the output y.

3.2. Design in the Presence of Structured and Unstructured Uncertainty.

As we have already mentioned, semi-infinite inequalities which arise from worst case design of multi-input multi-output systems, with both structured and unstructured uncertainty are not tractable at the present time. Hence, we restrict ourselves to SISO systems in this subsection. In particular, we shall show how one can transcribe into semi-infinite inequalities the design requirements for a SISO feedback system in which the plant model contains both structured and unstructured uncertainty. In this case, the plant transfer function to be considered in a worst case design has the form

$$P_w(s,\alpha,L(s)) \triangleq P(s,\alpha)(1 + L(s)) \quad \alpha \in \mathbf{A} \tag{3.16}$$

where $P(s,\alpha)$ and $\mathbf{A}$ are defined by (2.22) and (2.23), (2.24), respectively, and the

function $L : \mathbb{C} \to \mathbb{C}$ satisfies (2.20). The structured uncertainty is introduced to take account of variations in plant parameters (such as those arising when components are manufactured to tolerances) or to account for errors in the model-fitting process. When both amplitude and phase bounds are available for the unstructured uncertainty, it is possible to remove the conservatism inherent in the approach taken in the preceeding section, where worst case stability was ensured by means of two inequalities, one guaranteeing the stability of the nominal system (3.10a) and (3.10b) and the second one ensuring robustness under expected multiplicative perturbations (3.12), by combining both of these requirements into a single semi-infinite inequality.

Assumption 3.1: There exist continuously differentiable functions, $\underline{l}_M, \bar{l}_M, \underline{l}_A, \bar{l}_A : \mathbb{R}_+ \to \mathbb{R}$ such that the multiplicative perturbations in (3.16) are bounded as follows:

$$0 < \underline{l}_M(\omega) \leq |1 + L(j\omega)| \leq \bar{l}_M(\omega) \quad \forall\, \omega \in \mathbb{R}_+ \tag{3.17a}$$

$$\underline{l}_A(\omega) \leq \arg(1 + L(j\omega)) \leq \bar{l}_A(\omega) \quad \forall\, \omega \in \mathbb{R}_+ \tag{3.17b}$$

and these bound functions satisfy

$$0 < \underline{l}_M(\omega) \leq 1 \leq \bar{l}_M(\omega) \quad \forall\, \omega \in \mathbb{R}_+ \tag{3.18a}$$

$$\underline{l}_A(\omega) \leq 0 \leq \bar{l}_A(\omega) \quad \forall\, \omega \in \mathbb{R}_+. \quad \blacksquare \tag{3.18b}$$

Definition 3.2: We denote by $\tilde{\mathbf{L}}$ the set of functions, $L(\cdot)$, which satisfy (3.17a) and (3.17b). $\blacksquare$

In what follows, we shall consider some of the requirements described in Subsection 3.1.

(i) Worst Case Stability

We shall consider only the case in which the closed loop poles are to lie in $\mathbb{C}_-^o$ (*i.e.*, $S = \mathbb{C}_-^o$). Next, we must add to Assumption 2.2, the following hypothesis.

Assumption 3.2: For all $L \in \tilde{\mathbf{L}}$, the transfer function $P_w(\cdot,\alpha,L(\cdot))$ has the same number of $\mathbb{C}_+$ poles (counting multiplicities) as $P(\cdot,\alpha)$ where, $P(\cdot,\alpha)$ satisfies Assumption 2.2. ▪

Proceeding analogously to (3.7) and (3.10), we decompose the loop gain as follows

$$C_s(x,s)C_f(x,s)P_w(s,\alpha,L(s)) = n(x,s,\alpha)(1 + L(s))/\, d(x,s,\alpha)\,. \qquad (3.19)$$

Stability can be ensured by making use of the locus of the function

$$t(x,s\alpha,L(s)) \triangleq [n(x,s,\alpha)(1 + L(s)) + d(x,s,\alpha)]/\, \tilde{d}(s) \qquad (3.20)$$

where $\tilde{d}(\cdot)$ is a polynomial of degree equal to the maximum degree† of $d(x,s,\alpha)$ over A. The zeros of $\tilde{d}(\cdot)$ are all in $\mathbb{C}_-^o$. Again, using a parabolic exclusion region, as in (3.11), a sufficient condition for stability is given by

$$\mathrm{Im}[t(x,j\omega,\alpha,L(j\omega))] - k_1\{\mathrm{Re}[t(x,j\omega,\alpha,L(j\omega))]\}^2 + k_2 \le 0\,,$$
$$\forall\ \omega\in[\omega',\omega''],\quad \forall\ \alpha\in\mathbf{A} \text{ and } \forall\ L\in\tilde{\mathbf{L}}\,, \qquad (3.21)$$

where $[\omega',\omega'']$ is the critical interval of frequencies. Observe that (3.21) differs from (3.11) in that the former inequality is parametrized not only by an interval of frequencies but also by the uncertainty in the plant model. To determine whether a given design vector x satisfies the worst case stability test, one must evaluate

$$\max_{\substack{\omega\in[\omega',\omega''] \\ \alpha\in\mathbf{A} \\ L\in\tilde{\mathbf{L}}}} \left\{ \mathrm{Im}[t(x,s,j\omega,\alpha,L(j\omega))] - k_1\{\mathrm{Re}[t(x,j\omega,\alpha,L(j\omega))]\}^2 + k_2 \right\} \qquad (3.22)$$

which is an extremely difficult maximization problem since, typically, $\mathbf{A}$ is a multidimensional hyperectangle and the objective function in (3.22) is non-convex and so appears to allow no simplification of the problem. Since an evaluation of

† Since there may be pole-zero cancellations for some $\alpha\in A$ in forming the ratio $n(x,s,\alpha)/\, d(x,s,\alpha)$, we preserve the scaling feature of the formulation (3.10) by using the maximum degree of $d(x,s,\alpha)$ over $\alpha\in\mathbf{A}$

the max in (3.22) would be required at least once during each iteration of a semi-infinite optimization algorithm, some modification of this maximization problem seems essential for our optimization-based design procedure to remain computationally tractable. A procedure for systematically modifying inequalities such as (3.21) into more tractable, though somewhat more conservative forms, will be presented in Section 3.3.

(ii) Bounds on Transfer Function Magnitudes

We shall consider only one of the possible design requirements of this form since other constraints on transfer function magnitudes may be formulated analogously.

By analogy to the the formulation of disturbance suppression for a nominal plant model in Subsection 3.1 (iii), one obtains the worst case disturbance suppresion requirement (c.f. (3.11))

$$|H_{yd}(x,j\omega,\alpha,L(j\omega))| \leq b_{d(}\omega) \quad \forall\ \omega\in[\omega',\omega''],\ \forall\ \alpha\in A \text{ and } \forall\ L\in\tilde{\mathbf{L}}. \qquad (3.23)$$

As in worst case stability, c.f. (3.21), verification of worst case disturbance suppression amounts to an evaluation of

$$\max_{\substack{\omega\in[\omega',\omega''] \\ \alpha\in A \\ L\in\tilde{\mathbf{L}}}} \left\{ |H_{yd}(x,j\omega,\alpha,L(j\omega))| \right\} \qquad (3.24)$$

which is extremely difficult because of the complexity of the plant uncertainty.

3.3 Complexity Reduction via Majorization

We now present a set of majorization techniques [Pol.9, Pol.10]. These techniques yield computationally tractable inequalities as a replacement for intractable design inequalities, such as (3.21) and (3.23), which arise when the plant model includes structured uncertainty. The computational simplification is achieved at the expense of an added amount of conservatism expressed by the

inequality. We begin by defining what we mean by majorization.

Definition 3.2: Let $\rho,\bar{\rho}:\mathbb{R}^n\times\mathbb{C}\to\mathbb{R}$ and $b:\mathbb{C}\to\mathbb{R}$ be given functions and let B be a subset of $\mathbb{C}$. If

$$\rho(x,s)\leq\bar{\rho}(x,s) \quad \forall\, x\in\mathbb{R}^n \text{ and } \forall\, s\in B\,, \tag{3.25a}$$

then we say that the inequality

$$\bar{\rho}(x,s) - b(s)\leq 0, \quad \forall\, s\in B \tag{3.25b}$$

majorizes the inequality

$$\rho(x,s) - b(s)\leq 0, \quad \forall\, s\in B\,. \quad \blacksquare \tag{3.25c}$$

Design constraints for a SISO system such as (3.21) and (3.23) are of the form,

$$\xi(x,j\omega,\alpha,L(s)) \leq 0 \quad \forall\, \omega\in\Omega, \forall\, \alpha\in\mathbf{A} \text{ and } \forall\, L\in\tilde{\mathbf{L}} \tag{3.26}$$

where $\Omega\subset\mathbb{R}$ is compact. Since the uncertainty vector, $(\alpha,L(\cdot))$, enters (3.26) only as an argument of the plant transfer function, we may show that there exist $\eta:\ \mathbb{C}\times\mathbb{R}^{n_\alpha}\times\tilde{\mathbf{L}}\to\mathbb{R}^{2k}$, with $k = 1$ or 2, and $\zeta:\mathbb{R}^n\times\mathbb{C}\times\mathbb{R}^{2k}\to\mathbb{R}$ such that

$$\xi(x,s,\alpha,L(s)) = \zeta(x,s,\eta(s,\alpha,L(s)))\,. \tag{3.27}$$

For example, if

$$\xi(x,s,\alpha,L(s)) = H_{yd}(x,s,\alpha,L(s)) \tag{3.28a}$$

we may take

$$\zeta(x,s,\eta) = |1+\eta C(x,s)|^{-1} \tag{3.28b}$$

and

$$\eta(s,\alpha,L(s)) = P_w(s,\alpha,L(s)))\,. \tag{3.28c}$$

Let

$$\eta(s,\mathbf{A},\tilde{\mathbf{L}}) \triangleq \{\eta(s,\alpha,L(s)) \mid \alpha\in\mathbf{A}, L\in\tilde{\mathbf{L}}\}. \tag{3.29}$$

Our majorization techniques are based on the following results [Pol.9, Pol.10].

Theorem 3.3: Suppose that ξ,ζ and η satisfy (3.27). Let $N:\mathbb{C}\to 2^{\mathbb{R}^{2k}}$ be a set-valued map such that $\mathbf{N(s)}$ is a closed subset of $\mathbb{R}^{2k}$ with $k = 1$ or 2 for all $s \in \mathbb{C}$ and

$$\eta(s,\mathbf{A},\tilde{\mathbf{L}}) \subset \mathbf{N(s)} \tag{3.30}$$

Then

$$\sup_{\substack{\alpha\in\mathbf{A}\\ L\in\tilde{\mathbf{L}}}}\xi(x,s,\alpha,L(s)) \le \sup_{\eta\in\mathbf{N}(s)}\zeta(x,s,\eta) \quad \forall\, s \in \mathbb{C}. \quad \blacksquare \tag{3.31}$$

Corollary 3.4: Let $B \subset \mathbb{C}$, $b: \mathbb{C}\to\mathbb{R}$ and

$$\varphi(x,s) \triangleq \sup_{\substack{\alpha\in\mathbf{A}\\ L\in\tilde{\mathbf{L}}}}\xi(x,s,\alpha,L(s)) \tag{3.32a}$$

$$\overline{\varphi}(x,s) \triangleq \sup_{\eta\in\mathbf{N}(s)}\zeta(x,s,\eta)\,. \tag{3.32b}$$

Then the inequality,

$$\overline{\varphi}(x,s) - b(s) \le 0 \quad \forall\, s \in B \tag{3.33a}$$

majorizes the inequality,

$$\varphi(x,s) - b(s) \le 0 \quad \forall\, s \in B. \quad \blacksquare \tag{3.33b}$$

The advantage of specifying a design requirement in the form (3.33a) rather than (3.33b) is that $\overline{\varphi}(x,s)$ is potentially far simpler to evaluate because $\mathbf{N(s)} \subset \mathbb{R}^{2k}$ with $k = 1$ or 2 while $\mathbf{A} \subset \mathbb{R}^{n_\alpha}$, with, typically, $n_\alpha \gg 1$. The disadvantage of this majorization is that the set of $x \in \mathbb{R}^n$ which satisfies (3.33a) is, in general, smaller than the one which satisfies (3.33b). To keep the resulting conservatism small, the set $\mathbf{N(s)}$ satisfying (3.30) should be a "tight" approximation to $\eta(s,\mathbf{A},\tilde{\mathbf{L}})$. To further simplify the evaluation of (3.32b), the set $\mathbf{N(s)}$ is constructed to be a polyhedral convex subset† of $\mathbb{R}^2$ (see [Pol.9]).

In constructing majorizing inequalities for constraints defined in terms of closed loop transfer functions, we shall make use of a polyhedral convex set,

† A polyhedral convex set is one which is the intersection of finitely many closed half spaces.

$\mathbf{R}_H(j\omega)$, which contains the set (c.f. (3.29)),

$$\mathbf{P}(j\omega,\mathbf{A},\tilde{\mathbf{L}}) \triangleq \{(m,\varphi)\in\mathbb{R}^2 \mid m = m_P(j\omega,\alpha,L(j\omega)),\ \varphi = \varphi_P(j\omega,\alpha,L(j\omega)),\ \alpha\in\mathbf{A}, L\in\tilde{\mathbf{L}}\} \tag{3.34}$$

where m_P and φ_P are defined by

$$P_w(j\omega,\alpha,L(j\omega)) = m_P(j\omega,\alpha,L(j\omega))e^{j\varphi_P(j\omega,\alpha,L(j\omega))}. \tag{3.35}$$

Clearly, $\mathbf{P}(j\omega,\mathbf{A},\tilde{\mathbf{L}})$ is the set of gain-phase variations of all plants described by the structured and unstructured uncertainty, (3.16). The simplest polyhedral convex set $\mathbf{R}_H(j\omega)$, containing $\mathbf{P}(j\omega,\mathbf{A},\tilde{\mathbf{L}})$ that we can construct is a rectangle in $\mathbb{R}^2$. It has been shown in [Pol.9, Pol.10] that for any $x\in\mathbb{R}^n$ and for all $\omega\in\mathbb{R}_+$, the following maxima and minima may be readily computed by making use of mathematical programming decomposition results:

$$\hat{M}(j\omega) \triangleq \max_{\substack{\alpha\in\mathbf{A}\\ L\in\tilde{\mathbf{L}}}} m_P(j\omega,\alpha,L(j\omega)) \tag{3.36a}$$

$$\check{M}(j\omega) \triangleq \min_{\substack{\alpha\in\mathbf{A}\\ L\in\tilde{\mathbf{L}}}} m_P(j\omega,\alpha L(j\omega)) \tag{3.36b}$$

$$\hat{\Phi}(j\omega) \triangleq \max_{\substack{\alpha\in\mathbf{A}\\ L\in\tilde{\mathbf{L}}}} \varphi_P(j\omega,\alpha,L(j\omega)) \tag{3.36c}$$

$$\check{\Phi}(j\omega) \triangleq \min_{\substack{\alpha\in\mathbf{A}\\ L\in\tilde{\mathbf{L}}}} \varphi_P(j\omega,\alpha L(j\omega)). \tag{3.36d}$$

For each $\omega\in\mathbb{R}_+$, we define the rectangle,

$$\mathbf{R}_H(j\omega) \triangleq [\check{M}(j\omega),\hat{M}(j\omega)]\times[\check{\Phi}(j\omega),\hat{\Phi}(j\omega)]. \tag{3.37}$$

It then follows that for all $\omega\in\mathbb{R}_+$,

$$\mathbf{P}(j\omega,\mathbf{A},\tilde{\mathbf{L}}) \subset \mathbf{R}_H(j\omega) \subset \mathbb{R}^2 \tag{3.38}$$

and that each side of the rectangle touches $\mathbf{P}(j\omega,\mathbf{A},\tilde{\mathbf{L}})$ (see Fig. 6). Hence the disturbance rejection constraint (3.23), which may be written as

$$\max_{\substack{\alpha\in\mathbf{A}\\ L\in\mathbf{L}}} |\, H_{yd}(x,j\omega,\alpha,L(j\omega))\,| - b_d(\omega) \le 0 \quad \forall\ \omega\in[\omega',\omega''], \tag{3.39a}$$

is majorized by the constraint,

$$\max_{(m,\varphi)\in\mathbf{R}_H(j\omega)} \zeta_{yd}(x,j\omega,m,\varphi) - b_d(\omega) \le 0 \quad \forall\ \omega\in[\omega',\omega''] \tag{3.39b}$$

and ζ_{yd} is obtained from H_{yd} by expressing the plant transfer function in polar coordinates.

It has been shown in [Pol.9] that, provided ζ_{yd} is not a constant, $\zeta_{yd}(x,j\omega,\cdot,\cdot)$ can have no stationary points in the interior of $\mathbf{R}_H(j\omega)$ for each $\omega\in\mathbb{R}_+$. This makes the evaluation of the max in (3.39b) particularly straightforward since the maximization need only be performed over the boundary of $\mathbf{R}_H(j\omega)$ which consists of four straight line segments.

The majorization of the stability constraint (3.21) which employs the "Extended Nyquist Criterion" is less straightforward. It is shown in [Pol.9, Pol.10] that (3.21) is majorized by an inequality of the form

$$\max_{(\varphi_1,\varphi_2)\in\mathbf{R}_s(j\omega)} \zeta_s(x,j\omega,\varphi_1,\varphi_2) \le 0 \quad \forall\ \omega\in\mathbb{R}_+, \tag{3.40a}$$

where

$$\mathbf{R}_s(j\omega) \triangleq [\overset{\mathbf{v}}{\Phi}_1(j\omega),\hat{\Phi}_1,(j\omega)]\times[\overset{\mathbf{v}}{\Phi}_2(j\omega),\hat{\Phi}_2(j\omega)] \tag{3.40b}$$

is an easily computed rectangle of possible phase values. To reduce the conservatism induced by the use of sets $\mathbf{R}_H$ and $\mathbf{R}_s$ which are rectangles, more complicated polyhedral convex sets may be used, though their construction is computationally not as straightforward as the construction of a rectangle.

3.4 Differentiability Properties of the Majorizing Functions

Before we can exploit the computational simplicity of the majorizing inequalities that have been constructed in Subsection 3.3, we must verify that the

majorizing functions are locally Lipschitz continuous, and that their generalized gradients can be computed easily (see Section 4 for definitions) as required by the semi-infinite optimization algorithms to be presented in Section 4. In the forms stated in Subsection 3.3, this is not true, because the rectangles $\mathbf{R}$ used in the majorizing inequalities are not constant, compact sets, and because the rectangles $\mathbf{R}_H$ may be unbounded.

These difficulties can be removed by various transformations. First, to make the rectangles $\mathbf{R}_H$ always bounded, see [Pol.9, Pol.10], we introduce a change of variables in (3.39b):

$$m' = \frac{m}{m+1} \tag{3.41a}$$

and replace m by m' according to the rule

$$m = \frac{m'}{1-m'}. \tag{3.41b}$$

This also necessitates the definition of approriate maxima and minima $\hat{M}'(j\omega)$ and $\overset{\mathbf{v}}{M}'(j\omega)$ defined by replacing m_P in (3.36a) and (3.36b) my m'_P. The resulting $\hat{M}'(j\omega)$ and $\overset{\mathbf{v}}{M}'(j\omega)$ are bounded and continuous and hence so are the rectangles $R'_H(jw)$ which they define (by substitution in (3.37)). To distinguish the resulting new inequalities from the old ones (for example (3.39b)) we replace ζ by ζ' to obtain

$$\max_{\nu \in \mathbf{R}_H(j\omega)} \zeta'_{yd}(x, j\omega, \nu) - b_d(\omega) \leq 0\,, \quad \forall\, \omega \in [\omega', \omega''] \tag{3.42a}$$

where $\nu \triangleq (m', \varphi)$ and

$$\mathbf{R}'_H(j\omega) \triangleq co\{v_{H,i}(j\omega) \mid i \in \{1,2,3,4\}\} \tag{3.42b}$$

and the $v_{H,i}: \mathbb{R}_+ \to \mathbb{R}^2, i \in \{1,2,3,4\}$ are differentiable, except on a set of finite cardinality, on which $\mathbf{R}'_H(\cdot)$ is not even continuous. For the stability constraint we do not need a tranformation for m. It is shown in [Pol.10] that the points of discon-

tinuity of the vertices of $\mathbf{R}'_H$ and $\mathbf{R}_s$, $\{\omega_l \in \mathbb{R}_+ | \, l \in \underline{k}\}$, correspond to certain $j\omega$-axis poles and zeros of the extremizers phase of $P(s,\alpha,L(s))$ at which the phase of the corresponding rational function is discontinuous.

Because the sets $\mathbf{R}'_H(j\omega)$ and $\mathbf{R}_s(j\omega)$ are polyhedral convex for each $\omega \in \mathbb{R}_+$, any $v \in \mathbf{R}'_H(j\omega)$ can be represented in terms of the vertices of $\mathbf{R}'_H(j\omega)$, i.e.,

$$v = \sum_{i=1}^{4} \mu^i v_{H,i}(j\omega) \tag{3.43}$$

where $\sum_{i=1}^{4} \mu^i = 1 \, , \mu^i >= 0 \quad \forall \, i \in \{1,2,3,4\}$, and similarly for $v \in \mathbf{R}_s(j\omega)$. Hence we obtain that

$$\max_{v \in \mathbf{R}'_H(j\omega)} \zeta'_H(x,j\omega,v) = \max_{\mu \in \Sigma} \xi_H(x,j\omega,\mu) \tag{3.44a}$$

where

$$\xi_H(x,j\omega,\mu) \triangleq \zeta'_H(x,j\omega,\sum_{i=1}^{4} \mu^i v_{H,i}(j\omega)) \tag{3.44b}$$

and $\Sigma \subset \mathbb{R}^4$ is a simplex defined by

$$\Sigma \triangleq \{\mu \in \mathbb{R}^4 | \sum_{i=1}^{4} \mu^i = 1, \mu^i >= 0 \quad \forall \, i \in \{1,2,3,4\}\} \, . \tag{3.44c}$$

Similarly, we define

$$\xi_s(x,j\omega,\mu) \triangleq \zeta_s(x,j\omega,\sum_{i=1}^{4} \mu^i v_{s,i}(j\omega)). \tag{3.44d}$$

Now let

$$\psi_H(x) \triangleq \max_{\substack{\omega \in \Omega_H \\ \mu \in \Sigma}} \xi_H(x,j\omega,\mu) \tag{3.45a}$$

and

$$\psi_s(x) \triangleq \max_{\substack{\omega \in \Omega_s \\ \mu \in \Sigma}} \xi_s(x,j\omega,\mu) \tag{3.45b}$$

where Ω_H, Ω_s are the critical ranges of frequency.

If we assume, to simplify the exposition, that the system has a unity precompensator, C_π, in Fig. 2, we find that stability is ensured if

$$\psi_s(x) \leq 0 . \tag{3.46}$$

Let

$$\mathbf{X} \triangleq \{x \in \mathbb{R}^n \mid \psi_s(x) \leq 0\} . \tag{3.47}$$

It was shown in [Pol.10] that $\xi_s : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^4 \to \mathbb{R}$ and $\xi_H : X \times \mathbb{R}_+ \times \mathbb{R}^4 \to \mathbb{R}$ are continuously differentiable except at finitely many frequencies, at which these functions are bounded but discontinuous.

Because of the discontinuities, we must must replace the max by a sup in our definitions of ψ'_H and ψ_s, i.e., we define

$$\psi'_H(x) \triangleq \sup_{\substack{\omega \in \Omega_H \\ \mu \in \Sigma}} \xi_H(x, j\omega, \mu) \tag{3.49a}$$

$$\psi'_s(x) \triangleq \sup_{\substack{\omega \in \Omega_s \\ \mu \in \Sigma}} \xi_s(x, j\omega, \mu). \tag{3.49b}$$

The algorithm in Section 4 assumes that the functions ξ_s and ξ_H are continuous on the constraint intervals Ω_s and Ω_H. When these functions have discontinuities in Ω_s or Ω_H, respectively, the algorithm in Section 4 can still be used provided that a minor, rather technical modification (see [Pol.11]) is introduced.

We have thus completed the transcription of the constraints on scalar transfer function magnitudes, $|H(j\omega)|$, as well as the requirement of robust stability, in the presence of both structured and unstructured uncertainty, into a form that is compatible with the algorithm to be presented in the next section.

4. SEMI-INFINITE OPTIMIZATION ALGORITHMS FOR CONTROL SYSTEM DESIGN.

We shall now describe a family of algorithms for solving semi-infinite optimization problems of the form

$$\min\{f(x) \mid g^j(x) \leq 0,\ j = 1,2,\ldots,m;\ \varphi^k(x,y_k) \leq 0,\ y_k \in Y_k,$$
$$k = 1,2,\ldots,l;\ x \in X\}\ . \tag{4.1a}$$

These algorithms were developed over the years by Polak and Mayne [Pol.2], Gonzaga, Polak and Trahan [Gon.1], Polak, Trahan and Mayne [Pol.7], Polak and Wardi [Pol.3] and Polak and Stimler [Pol.11]. The algorithms in [Pol.2, Gon.1, Pol.7, Pol.11] were constructed with broad engineering applications in mind, while the Polak-Wardi and Polak-Stimler algorithms [Pol.3, Pol.11] were specifically conceived for control system design.

The above mentioned algorithms can be used whenever the following conditions are satisfied:

Assumption 4.1:

(i) The sets Y_k in (4.1a) are closed and $\max_{\substack{\omega \in \Omega \\ \mu \in \Sigma}} \xi(x,j\omega,\mu)$;

(ii) For every $x \in \mathbb{R}^n$, the functions $\varphi^k(x,\cdot)$ have only a finite number of local maximizers in Y_k;

(iii) The cost function $f(\cdot)$ is continuously differentiable;

(iv) The constraint functions $\varphi^k(\cdot,\cdot)$ are either continuously differentiable on X or the pointwise maxima over a fixed compact set of functions which are continuously differentiable on X (this includes functions of the form (3.49a,b) and functions defined in terms of the square of the maximum singular value of a continuously differentiable (in x) $m \times m$ transfer function matrix $H(x,s)$, whose elements are proper rational functions whose elements are continuously differentiable on X);

(v) The set $X \subset \mathbb{R}^n$ is defined by

$$X \triangleq \{x \in \mathbb{R}^n \mid \psi_s(x) \leq 0\} \tag{4.1b}$$

where $\psi_s: \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\psi_s(x) \triangleq \max \{g_s{}^j(x) \leq 0,\ j = 1,2,\ldots,m';\ \varphi_s{}^k(x,y_k) \leq 0,\ y_k \in Y_{s,k},$$
$$k = 1,2,\ldots,l'\}\,, \tag{4.1c}$$

with all the functions in (4.1c) continuously differentiable, so that the Danskin formula (4.9b) can be used to evaluate the directional derivatives of ψ_s. ▪

The set X is introduced to account for two phenomena. The first is the fact that functions $\varphi^k(x,y_k)$ in (4.1a), defined in terms of transfer functions, may have poles on the $j\omega$ axis ($y = \omega$) for some values of x and hence are not even continuous at such x. The second phenomenon is that the time responses of unstable systems may blow up within the time intervals under consideration. The functions defining $\psi_s(\cdot)$ are those described in the preceeding section under the subtitle *S-stability*.

From now on, without loss of generality, we shall consider only the simplest case of problem (4.1), which includes only one semi-infinite constraint and no ordinary, differentiable constraints, so as to avoid much cumbersome notation. Thus, we shall explain algorithms for solving (4.1) in terms of the problem

$$\min\{f(x) \mid \varphi(x,y) \leq 0,\ y \in Y,\ x \in X\}\,, \tag{4.2}$$

which retains all the important features of (4.1).

A mathematically elegant exposition of algorithms for solving (4.2) requires the use of concepts of nondifferentiable analysis [Cla.1] The interested reader will find such a presentation in [Pol.8]. In this paper we will attempt to give a more elementary exposition which makes use only of the concept of directional derivative and ε-directional derivative of a max functions.

Let

$$\psi(x) \triangleq \max_{y \in Y} \varphi(x,y)\,. \tag{4.3}$$

We can now rephrase (4.2) in the alternative form

$$\min\{f(x) \mid \psi(x) \leq 0,\ x \in X\}\ . \tag{4.4}$$

When $\varphi(x,y)$ is defined by an expression of the form

$$\varphi(x,y) = \bar{\sigma}[H(x,jy)]^2 - b(y)\ , \tag{4.5}$$

where $\bar{\sigma}[H(x,jy)]$ is the maximum singular value of a differentiable (in x) transfer function matrix $H(x,jy)$, $\psi(x)$ can be expressed as

$$\psi(x) = \max\{||H(x,jy)z||^2 \mid ||z|| = 1,\ y \in Y\}\ . \tag{4.6}$$

Hence, whether φ arises from a time domain constraint, a differentiable frequency domain constraint, or from a singular value frequency domain constraint, the function ψ defined by (4.3) is a max function of the type that has directional derivatives for which formulas can be found in [Cla.1, Dan.1]. We shall state these formulas as it becomes necessary.

Since the constraint function in (4.2) may not be continuous outside of the set X, the solution of problem (4.2) must begin with the computation of an initial design $x^0 \in X$. Under fairly weak assumptions such an x^0 can be computed in a finite number of iterations by a subprocedure which we shall call *phase 0* of our algorithm. Once an $x^0 \in X$ has been computed, the simplest approach to solving problem (4.4), is to proceed in two additional *phases* which are analogous to the ones used in linear programming. In *phase I* one computes a sequence $\{x^I{}_k\}$, such that $x^I{}_0 = x^0$ and $\psi_s(x^I{}_k) \leq 0$ for all k. Hopefully, this sequence is finite and terminates in a *feasible* design vector $x^I{}_k = x_0$, i.e., one that satisfies all the constraints in (4.4). In *phase II*, one constructs a sequence $\{x_i\}_{i=0}^{\infty}$, with first element x_0, along which the cost sequence, $\{f(x_i)\}_{i=0}^{\infty}$, decreases monotonically, while the constraints $\psi(x_i) \leq 0, \psi_s(x_i) \leq 0$ are satisfied for all i. Usually, the sequence $\{x_i\}_{i=0}^{\infty}$ converges to a local minimizer x^* of (4.4).

In phase I - phase II algorithms, the two separate operations are combined, with the advantage that the construction of a feasible point is carried out taking

into account the need for eventually decreasing the cost. Normally, phase I - phase II methods construct a feasible point x_k in a finite number of iterations. When the construction of a feasible point $\hat{x}$ takes an infinite number of iterations, the point $\hat{x}$ turns out to be a local minimizer for (4.4).

Phase I - phase II algorithms consist of three blocks: (i) a block which computes the search direction, (ii) a block which computes the step size, and (iii) an update block which combines the results of the preceeding two operations to obtain the new design vector x_{i+1} from the old design vector x_i. The most important of these blocks is the search direction finding block. However, before we can proceed, we must digress to discuss methods for computing directional derivatives of max functions.

4.1. Directional Derivatives of Max Functions.

We recall that the *directional derivative* of a function $\psi:\mathbb{R}^n \to \mathbb{R}$ at a point x, in the direction h is given by

$$d\psi(x;h) \triangleq \lim_{t\to 0+} \frac{\psi(x+th) - \psi(x)}{t}. \tag{4.7}$$

When $\psi(\cdot)$ is continuously differentiable, its directional derivative at x, in the direction h is given by

$$d\psi(x;h) = \langle \nabla\psi(x), h \rangle . \tag{4.8}$$

When

$$\psi(x) = \max_{y\in Y} \varphi(x,y) , \tag{4.9a}$$

it does not have a gradient everywhere and hence fomula (4.8) does not always apply. Nevertheless, it is still possible to obtain useful formulas for the directional derivative $d\psi(x;h)$. We shall consider the two cases that concern us most.

When $\varphi(\cdot,\cdot)$ is continuously differentiable in x, the directional derivative $\psi(\cdot)$ at x, in the direction h is given by the Danskin formula (see [Dan.1])

$$d\psi(x;h) = \max_{y \in Y(x)} \langle \nabla_x \varphi(x,y), h \rangle \tag{4.9b}$$

where

$$Y(x) \triangleq \{y \in Y \mid \varphi(x,y) = \psi(x)\} . \tag{4.9c}$$

It follows from (4.9b) that $\psi(\cdot)$ is differentiable at all x such that $Y(x)$ is a singleton, $\{y(x)\}$, in which case $\nabla\psi(x) = \nabla_x \varphi(x, y(x))$.

The above results can be extended to the case where ψ is defined by

$$\psi(x) \triangleq \max_{i \in I} \psi_i(x) , \tag{4.10a}$$

where I is a finite index set and

$$\psi_i(x) \triangleq \max_{y_i \in Y_i} \varphi^i(x, y_i) \tag{4.10b}$$

with the $\varphi_i(\cdot,\cdot)$ continuously differentiable and the Y_i compact. In this case, the formula for the directional derivative of ψ becomes

$$d\psi(x;h) = \max_{i \in I(x)} d\psi_i(x;h) , \tag{4.10c}$$

where

$$I(x) \triangleq \{i \in I \mid \psi_i(x) = \psi(x)\} . \tag{4.10d}$$

An alternative way of writing (4.9b) is

$$d\psi(x;h) = \max_{\xi \in \partial\psi(x)} \langle \xi, h \rangle , \tag{4.11}$$

where

$$\partial\psi(x) \triangleq co\{\nabla\varphi(x,y) \mid y \in Y(x)\} \tag{4.12}$$

is the *Clarke generalized gradient* of $\psi(\cdot)$ at x (see [Cla.1]) and co denotes the convex hull of the set enclosed in the braces.

The Clarke generalized gradient of $\psi(\cdot)$ at x is equal to $\nabla\psi(x)$ when the gradient exists at x. When the gradient of $\psi(\cdot)$ does not exist at x, it can be shown to satisfy the following relationship to gradients at nearby points x', where the gradient exists:

$$\partial\psi(x) = co\{\lim_{x' \to x} \nabla\psi(x')\} . \tag{4.13a}$$

i.e., it consists of limits of gradients at points x' converging to x.

Formula (4.13a) can be used to define the Clarke generalized gradient for arbitrary locally Lipschitz continuous functions $\psi(\cdot)$ [Cla.1].

Definition 4.1: A function $\psi:\mathbb{R}^n \to \mathbb{R}$ is said to be *locally Lipschitz continuous* at x if there exists an $L\in(0,\infty)$ and a $\rho > 0$ such that

$$|\psi(x') - \psi(x'')| \leq L\|x' - x''\| \quad \forall\, x', x'' \in B(x,\rho) . \tag{4.13b}$$

$\psi(\cdot)$ is locally Lipschitz continuous if it is locally Lipschitz continuous at all $x\in\mathbb{R}^n$. ∎

In general, the expression (4.11), with $\partial\psi(x)$ the Clarke generalized gradient of $\psi(\cdot)$, yields not the directional derivative of $\psi(\cdot)$, but the *generalized* directional derivative [Cla.1], which is defined by

$$d^o\psi(x;h) \triangleq \lim_{\substack{t\to 0+ \\ y\to x}} \frac{\psi(y + th) - \psi(y)}{t} . \tag{4.13c}$$

It can be shown that in general, $d^o\psi(x;h) >= d\psi(x;h)$. However, in the case of the max functions that we are considering, the ordinary and generalized directional derivatives coincide and hence (4.11) can be used to evaluate the directional derivative provided the correct expression for $\partial\psi(x)$ is used.

When $\varphi(x,y) \triangleq \bar{\sigma}[\mathbf{H(x,jy)}]^2 - \mathbf{b(y)}$, with $\bar{\sigma}[H(x,jy)]$ the maximum singular value of a proper, rational transfer function matrix $H(x,jy)$, whose coefficients are continuously differentiable in x, and $b(y)$ is continuous, $\nabla_x\varphi(x,y)$ exists only when $\bar{\sigma}[H(x,jy)]$ is a distinct singular value. In that case, the gradient of $\varphi(\cdot,\cdot)$ is

given by

$$\nabla_x \varphi(x,y)=(\langle v,[dQ(x,y)/\,dx^1]v\rangle,.....,\langle v,[dQ(x,y)/\,dx^n]v\rangle)^T\,, \qquad (4.14)$$

where v is the corresponding unit eigenvector of the matrix $Q(x,y) \triangleq H(x,jy)^*H(x,jy)$. Assuming that (4.14) is valid for all $y \in Y(x)$, formula (4.12) for $\partial\psi(x)$ remains valid for this case also. When the maximum singular value is multiple for some $y \in Y(x)$, which happens over a set of measure 0 in $\mathbb{R}^n$, the Clarke generalized gradient assumes the slightly more complex form

$$\partial\psi(x) = co\{\xi \mid \xi^i = \langle v, dQ(x,y)/\,dx^i]v\rangle,\ i = 1,2,...,n,$$
$$v = U(x,y)z, \|z\| = 1,\ y \in Y(x)\}\,, \qquad (4.15)$$

where $U(x,y)$ is any matrix of orthogonal unit eigenvectors spanning the eigenspace corresponding to the largest eigenvalue $\bar{\sigma}[H(x,jy)]^2$ of $Q(x,y)$.

We are now ready to proceed with an explanation of the search direction computation rules.

4.2. Primitive Search Direction Computations.

Given a design vector x, the search direction for problem (4.4) is computed according to four distinct rules, depending on whether $\psi_s(x) < 0$, $\bar{\psi}(x) > 0$, $\bar{\psi}(x) < 0$ or $\bar{\psi}(x) = 0$, where

$$\bar{\psi}(x) \triangleq \max\{\psi(x), \psi_s(x)\}\,. \qquad (4.16a)$$

The first case corresponds to phase 0, the second case corresponds to phase I, while the last two cases correspond to phase II. We begin by describing a set of *primitive* search direction finding rules. These are simple to explain, but they have the disadvantage that they yield search directions that behave discontinuously near "corners" of the level sets

$$F_\delta \triangleq \{x \in \mathbb{R}^n \mid \bar{\psi}(x) \leq \delta\}\,. \qquad (4.16b)$$

This discontinuous behavior is known to cause algorithm jamming and hence the

primitive search direction rules have to be modified so as to "smear out" this discontinuous behavior by anticipating the possible presence of a "corner" in the vicinity of the current design vector x. We shall present a "corner" anticipation technique after we have explained the primitive search direction finding procedures.

Phase 0 (primitive): When $\psi_s(x) > 0$, $\psi_s(x)$ can be decreased along the steepest descent direction

$$h^I(x) \triangleq \arg\min_{h \in R^n} \{\frac{1}{2}|\,|h|\,|^2 + d\psi_s(x;h)\}\,, \tag{4.17a}$$

provided that $\Theta^0(x) < 0$, where $\Theta^0(x)$ is defined by

$$\Theta^0(x) \triangleq \min_{h \in R^n} \{\frac{1}{2}|\,|h|\,|^2 + d\psi_s(x;h)\}\,. \tag{4.17b}$$

Note that the term $\frac{1}{2}|\,|h|\,|^2$ in (4.17a) serves the purpose of limiting the size of the solution vector $h^I(x)$.

We note that $\Theta^0(x) = 0$ must hold at all local minimizers x of $\psi_s(\cdot)$. Hence, to be certain that our algorithms does not hang up at an infeasible point we will need

Assumption 4.2: For all x such that $\psi(x) > 0$, $\Theta^0(x) < 0$. ▪

When $\psi_s(x) = \max_{y \in Y} \varphi(x,y)$ and $\varphi(\cdot,\cdot)$ is continuously differentiable in x, as is the case when it expresses the stability test (3.22), we can substitute for the directional derivative in (4.17a) from (4.9), (4.10), to obtain

$$h^0(x) = \arg\min_{h \in R^n} \max_{y \in Y(x)} \{\frac{1}{2}\|h\|^2 + \langle \nabla\varphi(x,y), h\rangle\}\,. \tag{4.18}$$

It should now be apparent why the primitive phase 0 search direction $h^0(x)$ can behave discontinuously. The reason is that the cardinality of the set $Y(x)$ (i.e., the number of points it contains) can change abruptly.

Substituting for $d\psi(x;h)$ from (4.12) into (4.17a), we obtain the alternative formula

$$h^0(x) = arg \min_{h \in R^n} \max_{\xi \in \partial \psi_s(x)} \{ \frac{1}{2} | |h| |^2 + \langle \xi, h \rangle \} . \tag{4.19}$$

By making use of the Von Neumann minimax theorem [Ber.1] it can be shown that (4.19) can be converted to

$$h^0(x) = - arg \min_{\xi \in \partial \psi_s(x)} \|\xi\|^2 \tag{4.20}$$

Because by Assumption 4.1 the set $Y(x)$ has finite cardinality, the minimization problem in (4.20) is a finite quadratic programming problem which can be solved in a finite number of iterations by means of special algorithms, see e.g., [Pol.4]. The min max problem in (4.18) can also be transformed into a finite quadratic program, but that program is more difficult to solve than the one in (4.20). Hence (4.20) is the one that is usually implemented in optimization codes.

Phase Ia (primitive): Suppose that $\psi_s(x) < 0$. In this case there is a ball $B(x,\rho)$, centered on x such that for all $x' \in B$, $\psi(x') \leq 0$. Hence the cost can be decreased, without incurring a constraint violation, along the *steepest descent* direction

$$\begin{aligned} h^{Ia}(x) &\triangleq arg \min_{h \in R^n} \{ \frac{1}{2} | |h| |^2 + d\psi(x;h) \} \\ &= arg \min_{h \in R^n} \max_{\xi \in \partial \psi(x)} \{ \frac{1}{2} | |h| |^2 + \langle \xi, h \rangle \} . \end{aligned} \tag{4.21}$$

Again making use of the Von Neumann minimax theorem, we obtain from (4.21) that

$$h^{Ia}(x) = - arg \min_{\xi \in \partial \psi(x)} \|\xi\|^2 . \tag{4.22}$$

We note that when $\varphi(x,y) \triangleq \bar{\sigma}[H(x,jy)]^2 - b(y)$, with $\bar{\sigma}[H(x,jy)]$ the maximum singular value of a proper, rational transfer function matrix $H(x,jy)$, whose coefficients are continuously differentiable in x, and $b(y)$ is continuous, we use (4.15) to define $\partial\psi(x)$. Unfortunately, in this case, (4.21) is no longer a finite quadratic program and hence the evaluation of the search direction by means of formula (4.21) requires the use of a "nearest point" or "proximity" algorithm, see e.g., [PoL4]. Proximity algorithms do not converge finitely and hence a truncation rule must always be included in any algorithm using a proximity algorithm as a subprocedure.

Phase Ib (primitive): Next, suppose that $\psi_s(x) = 0$. In this case, one can decrease the cost, without violating constraints, along the direction

$$h^{Ib}(x) \triangleq \arg\min_{h\in R^n} \{\frac{1}{2}|\,|h\,|\,|^2 + \max\{d\psi(x;h), d\psi_s(x;h)\}\}\,. \tag{4.23}$$

whenever $\Theta^I(x) < 0$, where

$$\Theta^I(x) \triangleq \arg\min_{h\in R^n} \{\frac{1}{2}|\,|h\,|\,|^2 + \max\{d\psi(x;h), d\psi_s(x;h)\}\} \tag{4.24}$$

since in that case both $d\psi(x;h)$ and $d\psi_s(x;h)$ are negative.

Now we must not allow the computation to get hung up at a point x such that $\bar{\psi}(x) > 0$ and hene we impose the following requirement on the problem formulation.

Assumption 4.3: For all x such that $\bar{\psi}(x) >= 0$, $\Theta^I(x) < 0$. square

Phase IIa (primitive): Suppose that $\bar{\psi}(x) < 0$. In this case there is a ball $B(x,\rho)$, centered on x such that for all $x' \in B$, $\bar{\psi}(x') \leq 0$. Hence the cost can be decreased, without incurring a constraint violation, along the *steepest descent* direction

$$h^{IIa}(x) \triangleq \arg\min_{h\in\mathbb{R}^n}\{\frac{1}{2}|\,|h\,|\,|^2 + df\,(x;h)\} \tag{4.25a}$$

where the directional derivative of f at x, in the direction h is given by

$$df\,(x;h) = \langle \nabla f\,(x), h\rangle\ . \tag{4.25b}$$

Making use of (4.25a), (4.25b), we obtain the explicit answer: $h^{IIa}(x) = -\nabla f\,(x)$. Clearly, no first order reduction can be obtained when $\nabla f\,(x) = 0$, since x is a stationary point for problem (4.4) in that case.

Phase IIb (primitive): Next, suppose that $\overline{\psi}(x) = 0$. In this case, one can decrease the cost, without violating constraints, along the direction

$$h^{IIb}(x) \triangleq \arg\min_{h\in\mathbb{R}^n}\{\frac{1}{2}|\,|h\,|\,|^2 + \max\{df\,(x;h), d\overline{\psi}(x;h)\}\}\ . \tag{4.26a}$$

whenever $\Theta_0(x) < 0$, where

$$\Theta_0(x) \triangleq \arg\min_{h\in\mathbb{R}^n}\{\frac{1}{2}|\,|h\,|\,|^2 + \max\{df\,(x;h), d\overline{\psi}(x;h)\}\} \tag{4.26b}$$

since in that case both $df\,(x;h)$ and $d\overline{\psi}(x;h)$ are negative.

This leaves us with the question as to what happens when $\Theta_0(x) = 0$ (since $\Theta_0(x)\leq 0$ must always hold). The answer to this question is given by the following result.

Theorem 4.1: Suppose that $\hat{x}$ is solves problem (4.4). If $\overline{\psi}(\hat{x}) < 0$, then $\nabla f\,(\hat{x}) = 0$. If $\overline{\psi}(\hat{x}) = 0$, then $\Theta_0(\hat{x}) = 0$. ▪

Thus, if $\hat{x}$ is a point such that $\overline{\psi}(\hat{x}) = 0$ and $\Theta_0(\hat{x}) = 0$, then $\hat{x}$ is stationary for the problem (4.4) and it is not possible to dislodge oneself from $\hat{x}$ by means of a first order algorithm.

4.3. Convergent Phase 0, Phase I - Phase II Search direction Computations.

We shall first show how the discontinuous behavior of the search directions defined in the preceeding subsection can be tempered by anticipating corners in the level sets F_δ and how the phase I and phase II computations can be combined into a single one. The anticipation of corners is achieved by ε-smearing (augmentation) of the sets $\partial\psi_s(x)$, $\partial\bar{\psi}(x)$, see Fig. 8. The augmented set correspond to a possible generalized gradient $\partial\psi^s(x')$, or $\partial\bar{\psi}(x')$ at a nearby "corner" of a level set. The smearing parameter $\varepsilon > 0$ is driven to zero adaptively. The combination of the phase I and phase II computations into a single one will be achieved by means of a cross-over mechanism.

Let

$$\psi_s(x)_+ \triangleq \max\{\psi_s(x),0\} \tag{4.27a}$$

$$\psi(x)_+ \triangleq \max\{\psi(x),0\} \tag{4.27b}$$

$$\bar{\psi}(x)_+ \triangleq \max\{\psi_s(x),\psi(x),0\}\,. \tag{4.27c}$$

Then, for $\varepsilon > 0$, let

$$Y_{s,\varepsilon}(x) \triangleq \{y\in Y_s \mid \varphi_s(x,y) >= \psi_s(x)_+ - \varepsilon\,,$$
$$\text{and y is a local maximizer of } \psi(x,\cdot) \text{ in } Y_s\} \tag{4.28a}$$

$$Y_\varepsilon(x) \triangleq \{y\in Y \mid \varphi(x,y) >= \psi(x)_+ - \varepsilon,$$
$$\text{and y is a local maximizer of } \psi(x,\cdot) \text{ in Y}\} \tag{4.28b}$$

Next, we define

$$\partial_\varepsilon\psi_s(x) \triangleq \{\partial\varphi(x,y) \mid y\in Y_{s,\varepsilon}(x)\}\,. \tag{4.29a}$$

When $\varphi(\cdot,\cdot)$ is differentiable in x, we define

$$\partial_\varepsilon\psi(x) \triangleq \{\partial\varphi(x,y) \mid y\in Y_\varepsilon(x)\}\,. \tag{4.29b}$$

When $\varphi(\cdot,\cdot)$ is defined in terms of the maximum singular value $\bar{\sigma}H[(x,jy)]$, we define

$$\partial_\varepsilon\psi(x) \triangleq co\{y \mid y^i = <v,dQ(x,y)/dx^i]v>,\ i = 1,2,\ldots,n,\ v = U_\varepsilon(x,y)z,$$

$$||z|| = 1,\ y \in Y_\varepsilon(x)\}\ . \tag{4.29c}$$

In (4.29c) $U_\varepsilon(x,y)$ is a matrix whose columns are a maximal set of orthonormal eigenvectors corresponding to all the eigenvalues $\lambda_k(x,y)$, $k \in \{1,2,\ldots,m\}$, of the $m \times m$ matrix $Q(x,y) \triangleq H(x,jy)^* H(x,jy)$, such that

$$\bar{\sigma}[H(x,jy)]^2 - \lambda_k(x,y) \leq \varepsilon\ , \tag{4.29d}$$

where these eigenvalues are ordered so that $\bar{\sigma}[H(x,jy)]^2 = \lambda_1(x,y) >= \lambda_2(x,y) >= \ldots. >= \lambda_m(x,y)$.

Finally we define

$$\begin{aligned} \partial_\varepsilon \bar{\psi}(x) &\triangleq co\{\partial\psi_s(x), \partial\psi(x)\} \ \text{ if } |\psi_s(x) - \psi(x)| \leq \varepsilon\ , \\ &= \partial\psi_s(x), \quad \text{if} \psi_s(x) - \psi(x) >= \varepsilon\ , \\ &= \partial\psi(x), \quad \text{if} \psi_s(x) - \psi(x) \leq \varepsilon\ . \end{aligned} \tag{4.30}$$

Next, we define

$$\begin{aligned} d_\varepsilon \psi_s(x;h) &\triangleq \max_{\xi \in \partial_\varepsilon \psi_s(x)} < \xi, h >, \quad \text{when } \psi_s(x) >= -\varepsilon; \\ &= -\infty, \quad \text{otherwise.} \end{aligned} \tag{4.31a}$$

$$\begin{aligned} d_\varepsilon \bar{\psi}(x;h) &\triangleq \max_{\xi \in \partial_\varepsilon \bar{\psi}(x)} < \xi, h >, \quad \text{when } \bar{\psi}(x) >= -\varepsilon; \\ &= -\infty, \quad \text{otherwise.} \end{aligned} \tag{4.31b}$$

We now define

$$\Theta^0_\varepsilon(x) \triangleq \min_{h \in \mathbb{R}^n} \{\frac{1}{2}|\ |h'|\ |^2 + d_\varepsilon \psi_s(x;h')\ \} \tag{4.32a}$$

and with $\gamma >= 0$, we define

$$\Theta_\varepsilon(x) \triangleq \min_{h \in \mathbb{R}^n} \{\frac{1}{2}|\ |h'|\ |\ .^2 + \max\{df\,(x;h') - \gamma\bar{\psi}(x)_+, d_\varepsilon \bar{\psi}(x;h')\}\} \tag{4.32b}$$

Correspondingly, we define

$$h^0{}_\varepsilon(x) = \arg\min_{h\in\mathbb{R}^n}\{\frac{1}{2}|\,|h'\,||^2 + d_\varepsilon\psi_s(x;h')\}\,. \tag{4.32c}$$

$$h_\varepsilon(x) = \arg\min_{h\in\mathbb{R}^n}\{\frac{1}{2}|\,|h'\,||^2 + \max\{df\,(x;h') - \gamma\bar{\psi}(x)_+, d_\varepsilon\bar{\psi}(x;h')\}\}\,. \tag{4.32d}$$

Note that when $\bar{\psi}(x)\leq 0$, the search direction $h_\varepsilon(x)$ is a phase II search direction. When $\bar{\psi}(x) > 0$, the term $\gamma\bar{\psi}(x)_+$, in (4.32b) and (4.32d), effectively suppresses the contribution of the derivative of the cost function when the constraint violation is large. This suppression is reduced in a continuous manner as a feasible design is approached and the algorithm makes an attempt to decrease, or at least not to increase the cost as the the constraint violation is reduced, see Fig. 7.

Next we must define a mechanism for reducing the "smearing" parameter ε as a solution to (4.4) is approached. Let $\delta\in(0,1)$ and let

$$\mathrm{E} \triangleq \{0,1,\delta,\delta^2,\delta^3,\dots\}\,. \tag{4.33}$$

We then define the smearing functions

$$\varepsilon^0(x) \triangleq \max\{\varepsilon\in\mathrm{E}\,|\,\Theta^0{}_\varepsilon(x)\leq -\varepsilon\}\,, \tag{4.34a}$$

$$\varepsilon(x) \triangleq \max\{\varepsilon\in\mathrm{E}\,|\,\Theta_\varepsilon(x)\leq -\varepsilon\}\,. \tag{4.34b}$$

Finally, we define the combined phase I - phase II search direction at x by

$$h(x) = h_{\varepsilon(x)}(x)\,. \tag{4.35}$$

Note that the computation of $h(x)$ involves a loop. This loop can be made quite efficient in an algorithm which computes a sequence $\{x_i\}$, by searching E for $\varepsilon(x_i)$ not starting from 1, but from the preceeding value $\varepsilon(x_{i-1})$.

4.4. Step Size Computation.

The step size computation is also broken up into a phase 0, phase I and a phase II rule. The step size rules that we find most efficent are derived from one

proposed by Armijo [Arm.1] for unconstrained differentiable optimization. This step size rule makes use of three parameters: $\alpha,\beta\in(0,1)$ and $K > 0$.

Phase 0 Step Size: Suppose that $x\in\mathbb{R}^n$ is such that $\psi_s(x) > 0$. It can then be shown that $\varepsilon(x) > 0$ because of Assumption 4.2 and hence $d\psi_s(x;h(x)) < \Theta^0_{\varepsilon(x)}(x) \leq -\varepsilon(x) < 0$. The phase 0 step size $s(x)$ is defined to be the largest number s in $\{K,K\beta,K\beta^2,\cdots\}$ such that, see Fig. 9,

$$\psi_s(x + sh(x)) - \psi_s(x) \leq s\alpha\varepsilon^0(x) . \tag{4.36a}$$

Phase I Step Size: Next, suppose that $x\in\mathbb{R}^n$ is such that $\psi_s(x)\leq 0$, and $\overline{\psi}(x)\leq 0$. It can then be shown that $\varepsilon(x) > 0$ because of Assumption 4.3 and hence $d\overline{\psi}(x;h(x)) < \Theta_{\varepsilon(x)}(x) \leq -\varepsilon(x) < 0$. The phase I step size $s(x)$ is defined to be the largest number s in $\{K,K\beta,K\beta^2,\cdots\}$ such that, see Fig. 9,

$$\overline{\psi}(x + sh(x)) - \overline{\psi}(x) \leq -s\alpha\varepsilon(x) . \tag{4.36b}$$

Phase II Step Size: Now suppose that $x\in\mathbb{R}^n$ is such that $\overline{\psi}(x)\leq 0$ and $\Theta_0(x) < 0$. It can then be shown that $\varepsilon(x) > 0$ and hence that $df(x;h(x)) \leq \Theta_{\varepsilon(x)}(x) \leq -\varepsilon(x) < 0$), and if $\overline{\psi}(x) > -\varepsilon(x)$, then we also have $d\overline{\psi}(x;h(x)) < \Theta_{\varepsilon(x)}(x) \leq -\varepsilon(x) < 0$. The phase II step size $s(x)$ is defined to be the largest number s in $\{K,K\beta,K\beta^2,\cdots\}$ such that

$$f(x + sh(x)) - f(x) \leq -s\alpha\varepsilon(x) , \tag{4.36c}$$

and

$$\psi(x + sh(x)) \leq 0 . \tag{4.36d}$$

4.5 The Algorithm.

We are now ready to collect all of our blocks into a phase 0, phase I phase II algorithm for solving the specific forms of problem (4.2) that we have been discussing.

Algorithm 4.1:

Parameters: $\alpha, \beta, \delta \in (0,1)$, $K, \gamma > 0$.

Data: x_0.

Step 0: Set i = 0.

Step 1: If $\psi_s(x_i) > 0$, compute $\varepsilon^0(x_i)$ and $h_i \triangleq h^0(x_i)$. Else, compute $\varepsilon(x_i)$ and $h_i \triangleq h(x_i)$.

Step 2: If $\psi_s(x_i) > 0$, compute the largest step size $s_i \in \{K, K\beta, K\beta^2, \ldots\}$ such that

$$\psi_s(x_i + x_i h_i) - \psi_s \leq -s_i \alpha \varepsilon^0(x_i) \tag{4.37a}$$

If $\psi_s(x_i) \leq 0$ and $\overline{\psi}(x_i) > 0$, compute the largest step size $s_i \in \{K, K\beta, K\beta^2, \ldots\}$ such that

$$\begin{aligned} \overline{\psi}(x_i + s_i h_i) - \overline{\psi}(x_i) &\leq -s_i \alpha \varepsilon(x_i) \\ \psi_s(x_i + s_i h_i) &\leq 0 \end{aligned} \tag{4.37b}$$

If $\overline{\psi}(x) \leq 0$, compute the largest step size $s_i \in \{K, K\beta, K\beta^2, \ldots\}$ such that

$$\begin{aligned} f(x_i + s_i h_i) - f(x_i) &\leq -s_i \alpha \varepsilon(x_i) \\ \overline{\psi}(x_i + s_i h_i) &\leq 0 \,. \end{aligned} \tag{4.37c}$$

Step 3: Set

$$x_{i+1} = x_i + s_i h_i \,, \tag{4.38}$$

set $i = i + 1$, and go to Step 1. ▪

The convergence properties of the above algorithm can be stated as follows.

Theorem 4.2: If $\{x_i\}_{i \in J}$, with J an infinite subset of $\{0,1,2,3,\ldots\}$, is any infinite subsequence of a sequence $\{x_i\}_i^{\infty}$ constructed by Algorithm 4.1, which converges to a point $\hat{x}$, then $\overline{\psi}(\hat{x}) \leq 0$ and $\Theta_0(\hat{x}) = 0$, i.e., $\hat{x}$ is feasible and satisfies a standard first order optimality condition. ▪

5. CONCLUSION

Since one cannot successfully use semi-infinite optimization algorithms in a batch mode and since, in any event, engineering design involves a great deal of trade off, the implementation of the control system design techniques described in this paper is best carried out in a highly flexible and highly interactive computing environment. DELIGHT.MIMO [Con.1, Nye.1, Nye.2, Pol.12] is an experimental, interactive, optimization-based control system design package which is currently being developed at the University of California, Berkeley and Imperial College, London. Because this system is still undergoing considerable evolution, there is little up-to-date documentation available on this research.

The use of semi-infinite optimization in control system design is still in its infancy. This is partly due to the fact that present day control theory was not developed with an awareness of the full power of semi-infinite optimization, and partly to the lack of easily utilizable, semi-infinite optimization software tools. Hopefully, DELIGHT.MIMO will alleviate this problem.

A great deal of research remains to be done in (i) the construction of control theoretic results which take into account the availability of semi-infinite optimization tools, in particular in the areas of formulation of uncertainty and computational tests for ensuring closed loop system stability; (ii) the development of systematic techniques for transcribing physical requirements into numerically well conditioned semi-infinite inequalities, and (iii) software implementations.

Initial experimentation with semi-infinite optimization in control system design is certainly very encouraging, though, clearly, there remains a great deal to be done. Hence, we would like to encourage our readers to experiment with semi-infinite optimization in control system design, since it is only out of a large collective experience with a design tool that intuition and folk wisdom emerge, which fuel further research efforts.

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Figure Captions

Fig. 1. Two Degrees of Freedom Control System.

Fig. 2. Three Controller Feeback Structure.

Fig. 3. S-stability Region.

Fig. 4. Exclusion Region for Stability Test.

Fig. 5. Envelope Constraint for Time Responses

Fig. 6. Construction of Rectangular Approximation for the Set of Plant Variations.

Fig. 7. Trajectory of Successive Iterates in Phase I-Phase II Algorithm

Fig. 8. Illustrating the Smoothing Effect of ε-Smearing.

Fig. 9. Step Size Computations.

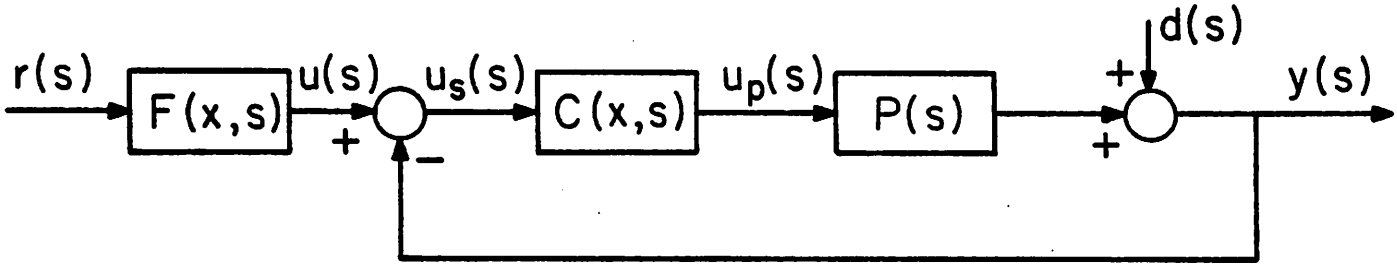
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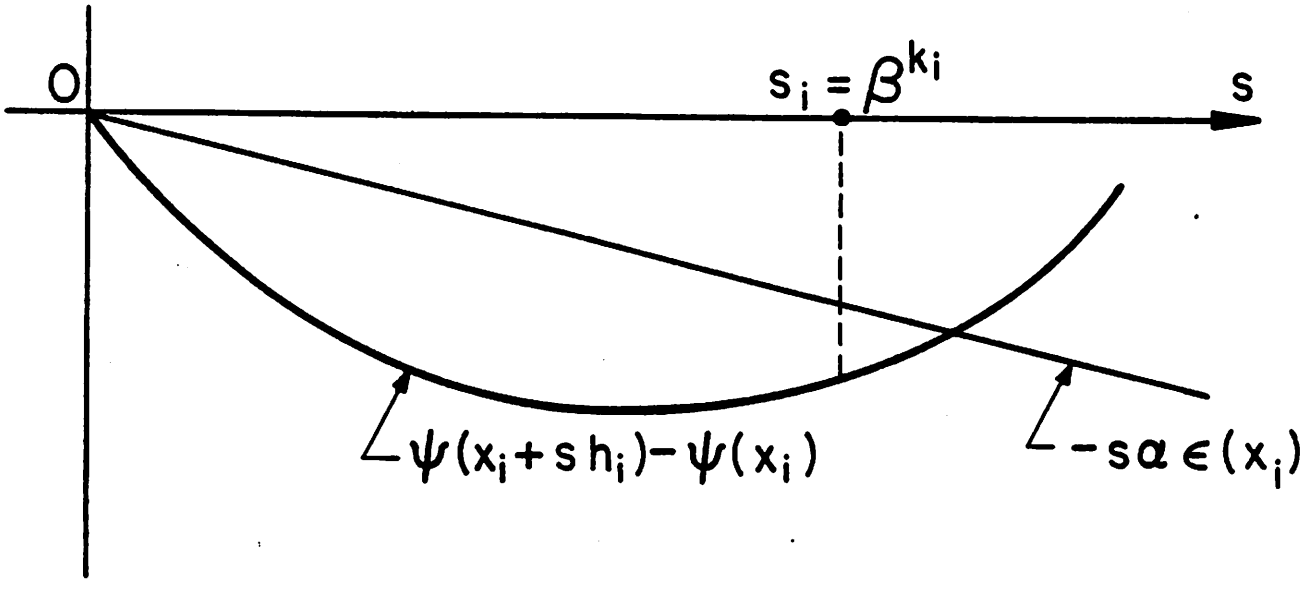
Fig. 1


O
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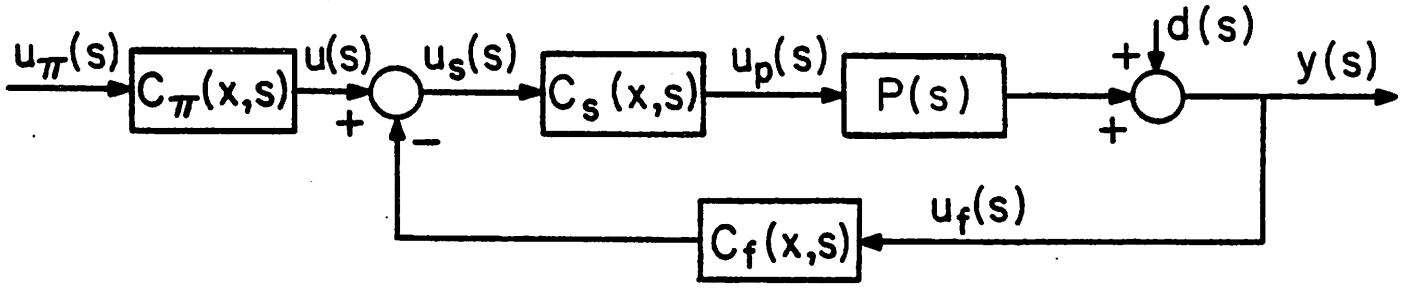
$-s\alpha\,\epsilon(x_i)$

Fig. 9



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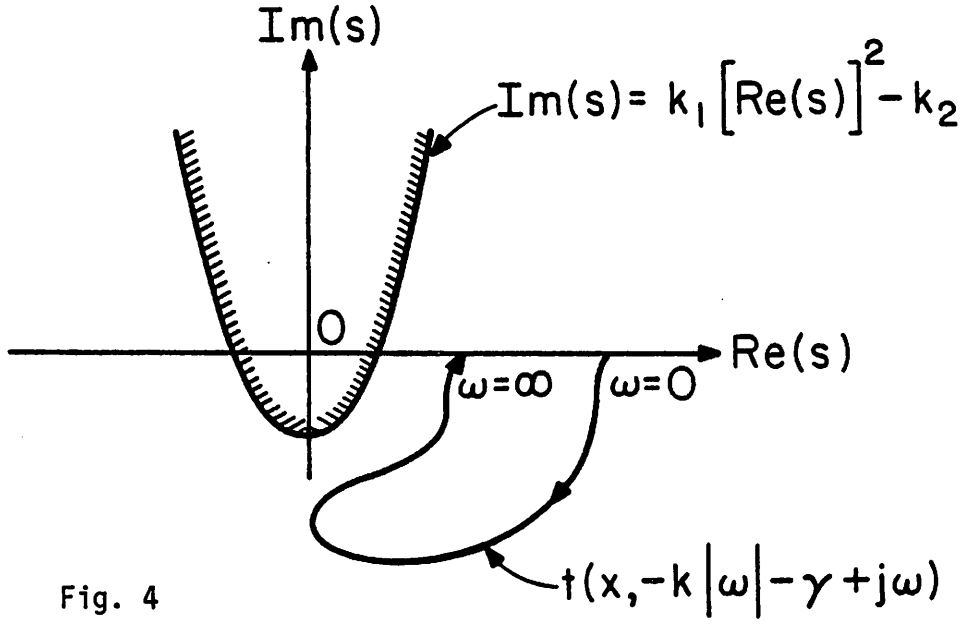

Fig. 4

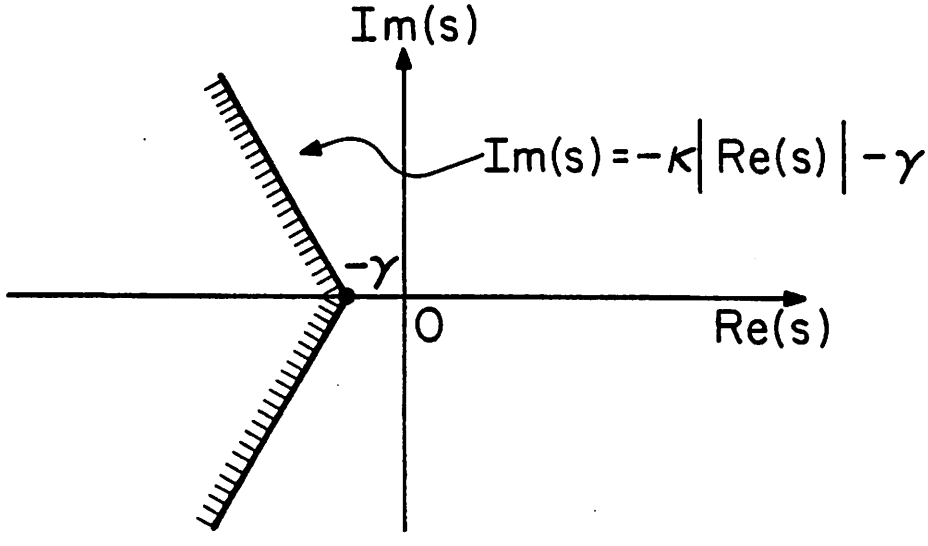

Fig. 3

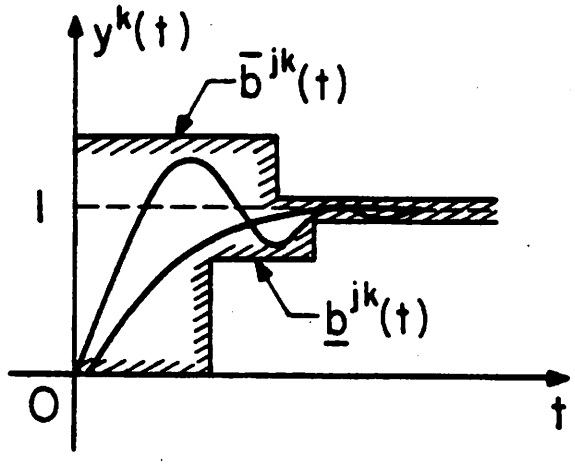
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Fig. 5

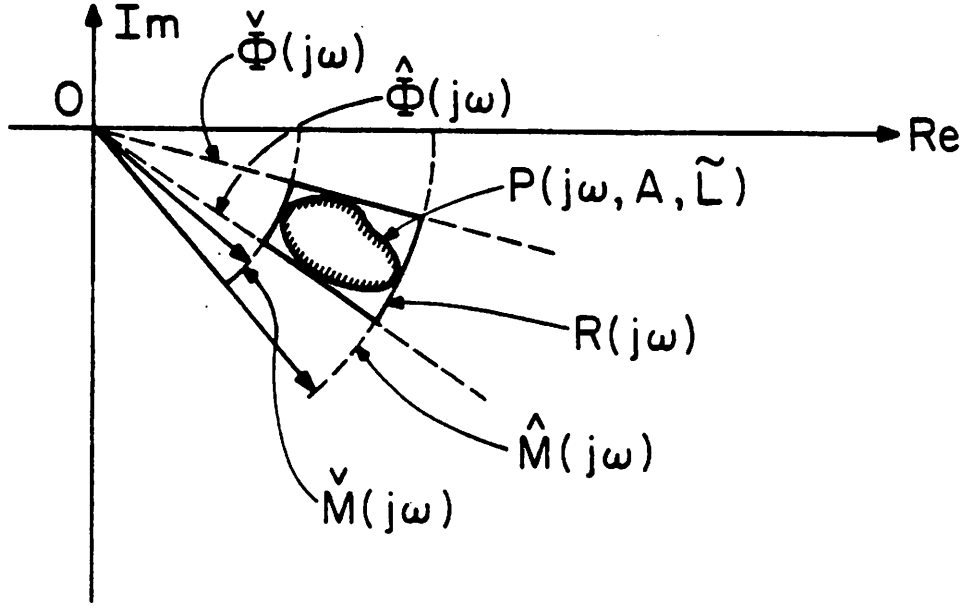
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Fig. 6

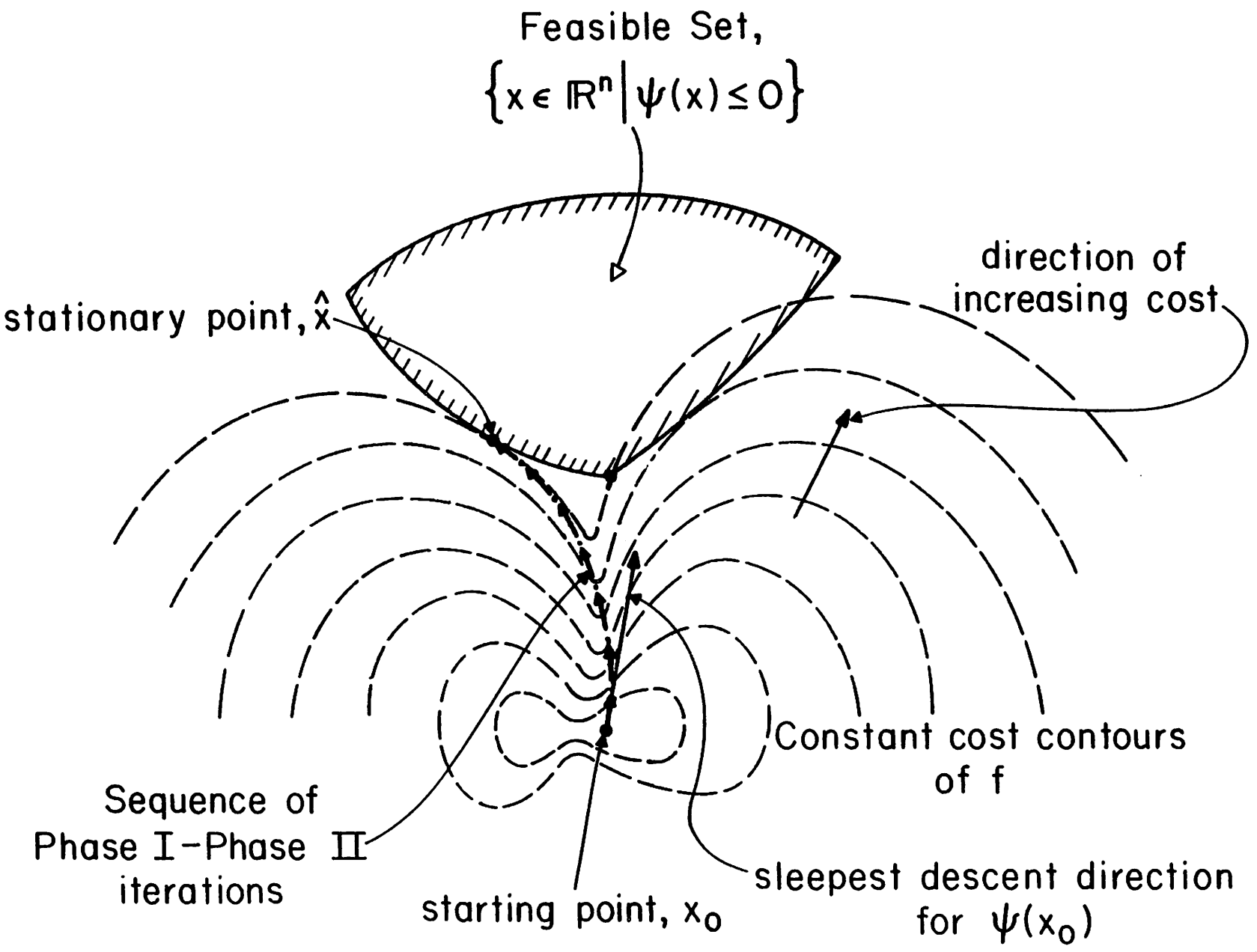
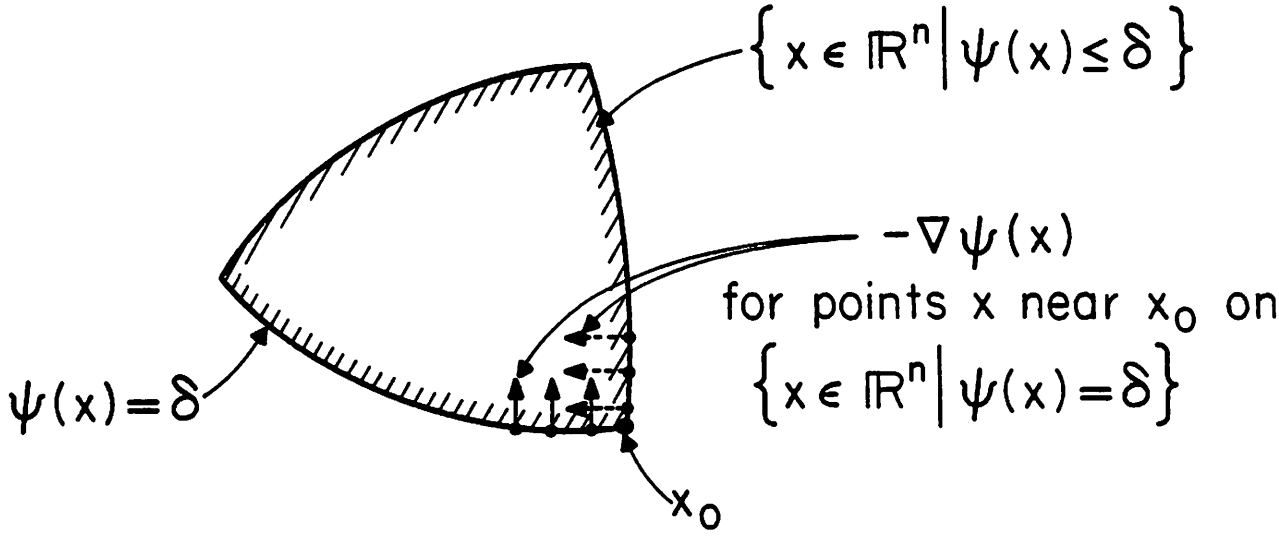

Feasible Set,
$\{x \in \mathbb{R}^n \mid \psi(x) \leq 0\}$
stationary point, $\hat{x}$
direction of increasing cost
Constant cost contours of f
Sequence of Phase I-Phase II iterations
starting point, x_0
sleepest descent direction for $\psi(x_0)$

Fig. 7


$\{x \in \mathbb{R}^n \mid \psi(x) \leq \delta\}$
$-\nabla\psi(x)$
for points x near x_0 on
$\{x \in \mathbb{R}^n \mid \psi(x) = \delta\}$
$\psi(x) = \delta$
x_0

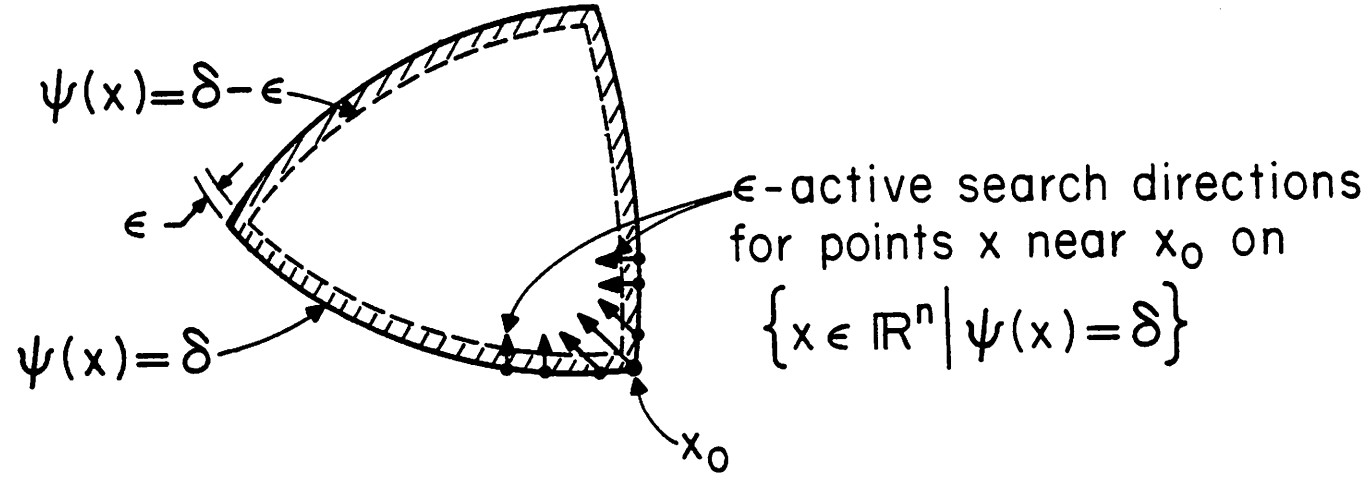

$\psi(x) = \delta - \epsilon$
ϵ
ϵ-active search directions
for points x near x_0 on
$\{x \in \mathbb{R}^n \mid \psi(x) = \delta\}$
$\psi(x) = \delta$
x_0

Fig. 8