CONTROLLING PLANTS WITH DELAY

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Title page

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Abstract

This paper studies a generalized form of the MIMO Smith regulator (GS) and one of its variants--the Gray-Hunt-Horowitz (GHH) regulator. The necessary and sufficient conditions for the exponential stability of both systems are determined. The two schemes are analysed and the limitations imposed on system performance by uncertainty in delay, plant saturation etc. are exhibited. The GHH and GS regulators are compared with reference to a fairly general example. Finally, two theorems establish a global parametrization of all stabilizing controllers for the nonlinear GS and GHH regulators.

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1. Introduction

The problem of controlling a plant with "delays" has been studied for a long time. In 1957, O. J. M. Smith proposed an ingenious scheme now called the Smith regulator (e.g. [Åst. 1]) -- with appealing characteristics. Since then the SISO problem has been studied by many authors [Gaw. 1], [Kwo. 1], [Man. 1], [Mar. 1], [Mor. 1], [Pal. 1]; variations have been proposed e.g. [Hor. 1], and some of the theory has been extended to the MIMO case [Ale. 1,2], [Fur. 1], [Ogu. 1], [Pal. 2].

We consider a generalized form of the MIMO Smith regulator which we call the GS regulator (see Fig. 2 below) and one of its variants -- which we call the Gray-Hunt-Horowitz (GHH) regulator (see Fig. 4 below). An analysis of the nominal case for the generalized Smith regulator shows that the I/O map

$$H_{y_2u_1} = N_dPC(I+PC)^{-1} = N_dPQ \tag{1.1}$$

and that the exponential stability of ${}^2S(N_dP,\ (I-N_d)P,C)$ requires that the plant $\mathcal{P}(=N_dP)$ be exponentially stable (see Appendix, Thm. A.2). The formula (1.1) seems to say that the delay N_d can be factored out of the design, and thus that the delay imposes no special restrictions on achievable performance.

In sec. 2 we describe the required factorization assumptions and the assumptions made on the plant. Sections 3 and 4 contain analyses of the GS and GHH regulators respectively, in the nominal and perturbed cases. In section 5 we draw general conclusions on the two regulators and apply results of the analysis to special examples to better understand limitations of the two regulators. Section 6 discusses the extension to the nonlinear case briefly and section 7 compares the two regulators.

Notation:

$a := b$ means a denotes b; $\mathbb{N}^*$ is the set of positive integers; $\mathbb{R}$ is the field of real numbers; $\mathbb{C}$ is the field of complex numbers; $\mathbb{R}_+$ is the set of non-negative real numbers; $\mathbb{C}_+$ ($\mathbb{C}_{\sigma_+}$) is the set of complex numbers such that Re $z \geq 0$ (Re $z \geq \sigma$, resply.). For any set A, $A^{n\times n}$ denotes the set of all $n\times n$ matrices with elements in A. $\mathbb{R}_p(s)$ ($\mathbb{R}_{p,o}(s)$) denotes the set of all proper (resp. strictly proper) rational functions with coefficients in $\mathbb{R}$. For any $A \in \mathbb{C}^{m\times n}$, $\sigma_{max}[A]$ ($\sigma_{min}[A]$) is the maximum (resp. minimum) singular value of A. Given $\sigma \in \mathbb{R}$ (typically $\sigma > 0$), $f \in \mathcal{A}(\sigma)$ iff $f(t) = f_a(t) + \sum_0^\infty f_i\, \delta(t-t_i)$, where $f_a = \mathbb{R} \to \mathbb{R}$ with $f_a(t) = 0$ for $t < 0$ and $t \mapsto \exp(-\sigma t)\, f_a(t) \in L_1$; $t_o = 0$, $t_i > 0$, $\forall i > 0$; $\forall i$, $f_i \in \mathbb{R}$ and $i \mapsto f_i \exp(-\sigma t_i) \in \ell_1$; $f \in \mathcal{A}_-(\sigma)$ iff, for some $\sigma_1 < \sigma$ $f \in \mathcal{A}(\sigma_1)$. $\hat{f}$ denotes the Laplace transform of f. $\hat{\mathcal{A}}(\sigma) := \{\hat{f} : f \in \mathcal{A}(\sigma)\}$; $\hat{\mathcal{A}}_-(\sigma) = \{\hat{f} : f \in \mathcal{A}_-(\sigma)\}$. $\hat{f} \in \hat{\mathcal{A}}_{-,0}(\sigma_0)$ iff $\hat{f} \in \hat{\mathcal{A}}_-(\sigma)$ and $\hat{f}$ goes to zero at infinity in $\mathbb{C}_{\sigma+}$.

2. System description and factorization of $\mathcal{P}$

Let us call the plant transfer function $\mathcal{P}$: it is an $n_o x n_i$ matrix. Roughly speaking, we must assume that $\mathcal{P}$ can be factored into a Blaschke-product-like "all-pass" factor N_d and a factor P, i.e., $\mathcal{P} = N_d P$ (resp. PN_d) depending on whether the plant is more accurately modelled by the "delay" N_d following (resp. preceding) P. Specifically,

Assumption A: The plant $\mathcal{P}$ may be factored as $N_d P$ (resp. PN_d) where

i) for some $\sigma_o < 0$, $N_d \in \hat{\mathcal{A}}_-(\sigma_o)^{n_o x n_o}$ (resp. $\hat{\mathcal{A}}_-(\sigma_o)^{n_i x n_i}$), is unitary on the $j\omega$-axis and $N_d(0) = I^\dagger$.

ii) $P \in \mathbb{R}_{p,o}(s)^{n_o x n_i}$ and is exponentially stable.

Comments: a) Note that, irrespective of the order of factors, $\in \hat{\mathcal{A}}_{-,o}(\sigma_o)^{n_o x n_i}$, i.e., is strictly proper.

† There is no loss of generality in assuming $N_d(0) = I$, for if $N_d(0) = U \neq I$, U unitary, redefine $N_d(s)$ to be $N_d U^{-1}$ (resp. $U^{-1}N_d$) and redefine P as UP (resp. PU) so that $\mathcal{P} = N_d P$ (resp. PN_d) is unchanged.

b) In the appendix we show that, given that N_d has its elements in $\hat{\mathcal{A}}_-(\sigma_0)$, P must be exponentially stable in order for the GHH or GS regulators to be exponentially stable. Thus, we assume exponential stability of P only for convenience and to emphasize that GS or GHH regulators can only be used for exponentially stable plants.

c) For the Smith regulator an explicit knowledge of both factors N_d and P is required in order to construct the regulator. However the plant need not be physically separable into factors N_d and P.

d) We will see below that in order to use the GHH regulator we do not require an explicit knowledge of N_d but we do require that $\mathcal{P}$ factor as PN_d (i.e., "delay" N_d precedes P) and that the plant be physically separable into factors N_d and P with the output of N_d accessible to the designer.

e) Though in assumption A ii) we impose the restriction that P be lumped (i.e., $P \in \mathbb{R}_{p,o}(s)^{n_o x n_i}$) it makes no difference to the methods, theorems or conclusions of this paper if P is distributed (i.e., $P \in \hat{\mathcal{A}}_{-,o}(\sigma_0)^{n_o x n_i}$, for some $\sigma_0 > 0$)

Since, by assumption A, P is exponentially stable we may use the Q-parametrization theorem [Zam. 1], [Des. 1], [Cal. 1]. Throughout this paper, given $\mathcal{P} = N_dP$ or PN_d, we imagine the standard system ${}^1S(P,C)$ (Fig. 1) associated with this P and define $Q := C(I+PC)^{-1}$.† For P exponentially stable, the exponential stability of ${}^1S(P,C)$ is equivalent to the exponential stability of $Q := C(I+PC)^{-1}$; equivalently we can represent C as an exponentially stable Q with a positive feedback of P. This representation of C in Fig. 2 and in Fig. 4 (when $P_1 = P_2 = P$) leads to Fig. 3 and 5 respectively.

†Indeed, the fact that when $\mathcal{P} = N_dP$ the nominal system of Fig. 2 has $H_{y_2u_1} = N_dPQ = N_dPC(I+PC)^{-1}$ led some to think of the Smith regulator as ${}^1S(P,C)$ followed by N_d since this would have the same $H_{y_2u_1}$!

3. Analysis of the generalized Smith regulator (GS)

Let the nominal plant be $\mathcal{P} := N_d P$ and the perturbed plant be $\tilde{\mathcal{P}} := \tilde{N}_d \tilde{P}$. Referring to Fig. 3, we write the summing node equation as:

$$\begin{bmatrix} I-\mathcal{P}Q & \tilde{\mathcal{P}} \\ -Q & I \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{3.1}$$

We shall solve (3.1) below in order to obtain eventually the closed-loop transfer function $H_{yu} : (u_1,u_2) \mapsto (y_1,y_2)$.

In the nominal case ($\tilde{P} = P$, $\tilde{N}_d = N_d$, thus $\tilde{\mathcal{P}} = \mathcal{P} = N_d P$), Eqn. (3.1) gives, for $H^o_{\varepsilon u} : (u_1,u_2) \mapsto (\varepsilon_1,\varepsilon_2)$:

$$H^o_{\varepsilon u} = \begin{bmatrix} I & -\mathcal{P} \\ Q & I-Q\mathcal{P} \end{bmatrix}$$

and the nominal I/O-map

$$H^o_{yu} = \begin{bmatrix} Q & -Q\mathcal{P} \\ \mathcal{P}Q & \mathcal{P}(I-Q\mathcal{P}) \end{bmatrix} = \begin{bmatrix} Q & -QN_d P \\ N_d PQ & (I-N_d PQ)N_d P \end{bmatrix} \tag{3.2}$$

Note that the nominal disturbance to output map is:

$$H^o_{y_2 d_0} = I - \mathcal{P}Q = I - N_d PQ \tag{3.3}$$

Comment: Whereas the "delay" caused the I/O-map $H^o_{y_2 u_1} = N_d PQ$ to be premultiplied by N_d (see eqn. (3.2)), the effect of the "delay" N_d on $H^o_{y_2 d_0}$ (see eqn. (3.3)) is much more drastic.

In the additively perturbed case ($\tilde{\mathcal{P}} = \mathcal{P}+\Delta\mathcal{P}$), we shall establish that:

$$\tilde{H}_{yu} = \begin{bmatrix} Q(I+\Delta\mathcal{P}\cdot Q)^{-1} & -Q\cdot(I+\Delta\mathcal{P}\cdot Q)^{-1}\tilde{\mathcal{P}} \\ \tilde{\mathcal{P}}Q(I+\Delta\mathcal{P}\cdot Q)^{-1} & \tilde{\mathcal{P}}[I-Q\,(I+\Delta\mathcal{P}\cdot Q)^{-1}\cdot\tilde{\mathcal{P}}] \end{bmatrix} \tag{3.4}$$

and that:

$$\tilde{H}_{y_2 d_0} = I - \tilde{\mathcal{P}} Q(I+\Delta\mathcal{P}\cdot Q)^{-1} \tag{3.5}$$

The effect of $\mathcal{P} \leftarrow \tilde{\mathcal{P}}$ is to replace Q by $Q(I+\Delta\mathcal{P}\cdot Q)^{-1}$ and $\mathcal{P}$ by $\tilde{\mathcal{P}}$ in equations (3.2) and (3.3) to get equations (3.4) and (3.5). To prove equations (3.4), (3.5), we solve the summing node equation (3.1) to get:

$$\tilde{H}_{\varepsilon u} = \begin{bmatrix} (I+\Delta\mathcal{P}\cdot Q)^{-1} & -(I+\Delta\mathcal{P}\cdot Q)^{-1}\tilde{\mathcal{P}} \\ Q(I+\Delta\mathcal{P}\cdot Q)^{-1} & I-Q(I+\Delta\mathcal{P}\cdot Q)^{-1}\tilde{\mathcal{P}} \end{bmatrix}$$

and eqn. (3.4) follows since,

$$\tilde{H}_{yu} = \begin{bmatrix} Q & 0 \\ 0 & \tilde{\mathcal{P}} \end{bmatrix} \tilde{H}_{\varepsilon u}$$

Comments:

a) In the multiplicatively perturbed case ($\tilde{\mathcal{P}} = (I+M)\mathcal{P}$) we can use formulae (3.4) and (3.5) above with $\Delta\mathcal{P} = \tilde{\mathcal{P}} - \mathcal{P} = M\mathcal{P}$ and $\tilde{\mathcal{P}} = (I+M)\mathcal{P}$.

b) As special cases, we easily obtain formulae for perturbations in P alone or N_d alone either when $\mathcal{P} = N_d P$ or $\mathcal{P} = PN_d$. For example, if $P = P+\Delta P$, $\tilde{N}_d = N_d$, $\mathcal{P} = N_d P$ then $\tilde{\mathcal{P}} = \tilde{N}_d\tilde{P} = N_d\tilde{P} = N_d(P+\Delta P)$, so $\Delta\mathcal{P} = N_d\cdot\Delta P$.

4. Analysis of the G-H-H regulator

We call Horowitz's modification [Hor. 1] of the Gray-Hunt [Gra. 1] regulator the GHH regulator (Figs. 4,5). In order to be able to implement this feedback scheme it is essential that: first, the "all-pass" subsystem N_d precede P (i.e., $\mathcal{P} = PN_d$ and not $\mathcal{P} = N_d P$ as before) second, the output of N_d be accessible as shown in Figs. 4. and 5.

Consider the nominal GHH regulator, i.e., in Fig. 4 set $P = P_1 = P_2$ and, in Fig. 5, set $\tilde{P} = P_2 = P$; then, by inspection:

$$H^o_{yu} = \begin{bmatrix} Q & 0 \\ PN_dQ & PN_d \end{bmatrix} \tag{4.1}$$

$$H^o_{y_2d_0} = I - PN_dQ \tag{4.2}$$

Suppose now that, in the system of Fig. 5, $\tilde{P} = P+\Delta P$ and $P_2 = P$, then the summing node equations read:

$$\left.\begin{aligned} e_1 &= u_1 - \Delta P\cdot N_d(u_2+Qe_1) \\ e_2 &= u_2 + Qe_1 \end{aligned}\right\} \tag{4.3}$$

Solving equation (4.3) for $\tilde{H}_{eu} : u \mapsto e$ gives:

$$\tilde{H}_{eu} = \begin{bmatrix} (I+\Delta P\cdot N_dQ)^{-1} & -(I+\Delta P\cdot N_dQ)^{-1}\cdot\Delta P\cdot N_d \\ Q(I+\Delta P\cdot N_dQ)^{-1} & (I+Q\cdot\Delta P\cdot N_d)^{-1} \end{bmatrix} \tag{4.4}$$

From equation (4.4),

$$\tilde{H}_{yu} = \left[\begin{array}{c|c} Q(I+\Delta P\cdot N_dQ)^{-1} & -Q\cdot\Delta P\cdot N_d(I+Q\cdot\Delta P\cdot N_d)^{-1} \\ \hline \tilde{P}N_dQ(I+\Delta P\cdot N_dQ)^{-1} & \tilde{P}N_d(I+Q\cdot\Delta P\cdot N_d)^{-1} \end{array}\right] \tag{4.5}$$

Since we can show that $\tilde{H}_{y_2d_0} = I-\tilde{H}_{y_2u_1}$, from equation (4.5):

$$\tilde{H}_{y_2d_0} = I - \tilde{P}N_dQ(I+\Delta P\cdot N_dQ)^{-1} = (I-PN_dQ)(I-\Delta P\cdot N_dQ)^{-1} \tag{4.6}$$

5. Conclusions

5.1. The generalized Smith regulator

In order to avoid vague generalities we apply the analysis developed above to the case when:

i) the plant P is square

ii) $\mathcal{Z}[P] \cap \mathbb{C}_+ = \phi$; (in other words the plant P is such that P^{-1} has no poles in $\mathbb{C}_+$). We choose

$$Q := P^{-1}\left(1 + \frac{s}{\omega_c}\right)^{-m} \tag{5.1}$$

where ω_c is a chosen cutoff frequency and m is chosen large enough such that Q is proper. Then,

$$H^o_{y_2u_1} = N_d PQ = N_d\left(1 + \frac{s}{\omega_c}\right)^{-m} \tag{5.2}$$

$$H^o_{y_2d_0} = I - N_d PQ = I - N_d\left(1 + \frac{s}{\omega_c}\right)^{-m} \tag{5.3}$$

5.1.1. The generalized Smith regulator: nominal case

If we consider exclusively the nominal case ($P = \tilde{P}$, $N_d = \tilde{N}_d$) (and disregard noise and saturation limitations for the moment) then:

i) as Smith pointed out in the SISO case, the feedback design can be carried out as if the "delay" N_d were absent;

ii) eqn. (5.2) suggests that we can "broadband" the plant P arbitrarily by choosing Q as in eqn. (5.1) above with ω_c as large as we please and thus achieve, on paper, an arbitrarily fast rise time in spite of the presence of the "delay" N_d.

We still consider only the <u>nominal case</u> ($P = \tilde{P}$, $N_d = \tilde{N}_d$) but, more realistically, look at saturation and at disturbance rejection constraints:

a) Saturation

The plant input e_2 is given by (see Fig. 3)

$$e_2 = Qu_1 + (I-QN_dP)u_2$$

Thus Q also controls the size of the plant input. Saturation of the plant P puts bounds on the allowed plant input e_2. Hence saturation sets an upper bound on the size of Q, i.e., on $\sigma_{max}[Q]$. From (5.1),

$$\sigma_{max}[Q] = \frac{1}{\sigma_{min}[P]} \quad \left|\left(1 + j\,\frac{\omega}{\omega_c}\right)^{-m}\right| \tag{5.4}$$

From eqn. (5.4) since P is strictly proper it is clear that such a bound on $\sigma_{max}[Q]$ puts a bound on ω_c. Thus it is only on paper that we can "broadband" P arbitrarily, <u>even in the nominal case</u>. These limitations are precisely the same as those one would encounter when $N_d = I$ (i.e., no "delays" in $\mathcal{P}$).

b) Disturbance rejection:

Consider the transfer function $H_{y_2d_0} : d_0 \mapsto y_2$ (eqn. (5.3)): $\sigma_{max}[H_{y_2d_0}(j\omega)]$ is a measure of the disturbance rejection achieved at frequency ω and of the I/O-map sensitivity to plant perturbations. To fix ideas, we consider the special case of "equal delay τ in all channels," i.e.,

$$N_d(j\omega) = e^{-j\omega\tau}\cdot I \tag{5.5}$$

Using (5.5) in (5.3), <u>for</u> a <u>large</u> enough cutoff frequency <u>ω_c</u>

$$\sigma_{max}[H_{y_2d_0}(j\omega)] \simeq \sigma_{max}[I - N_d(j\omega)] = |1 - e^{-j\omega\tau}| = 2\sin(\tfrac{\omega\tau}{2}) \tag{5.6}$$

Equation (5.6) shows that $\sigma_{max}[H_{y_2d_0}]$ starts at 0 for $\omega = 0$ and increases monotonically as ω increases over $[0, \frac{\pi}{\tau}]$; in particular $\sigma_{max}[H_{y_2d_0}]$

reaches 1 at $\omega_d := \frac{\pi}{3\tau}$. Unless τ is very small, we will have $\omega_d \ll \omega_c$. Hence in the nominal case ($\tilde{P} = P$, $\tilde{N}_d = N_d$) we have an appealing I/O-map $H_{y_2u_1}$, but a very unfavorable $H_{y_2d_0}$: at $\omega_d = \frac{\pi}{3\tau}$ rad/s the system ${}^2S(e^{-s\tau}P,\ e^{-s\tau}P,\ Q = P^{-1}(1+\frac{s}{\omega_c})^{-m})$ does not achieve any disturbance rejection (see Fig. 3).

We have just seen that, for the nominal system, the "delay" N_d imposes bounds on the achievable disturbance rejection. This conclusion fits Zames' general theory [Zam. 1]: suppose that we model the "delay" N_d by all-pass sections (i.e., Blaschke products), say, $N_d = \mathrm{diag}(p_1(s), \ldots, p_n(s))$, where $p_k(s) = \prod_j \left(\frac{s-z_{jk}}{s+\bar{z}_{jk}}\right)$, with $\mathrm{Re}(z_{jk}) > 0$ $\forall j, \forall k$, then by [Zam. 1, Thm. 4] for all stabilizing compensators

$$\|w\, H_{y_2d_0}\|_\infty \geq w(z_{jk}) \qquad \forall j, \forall k$$

i.e., the weighted disturbance transmission is bounded below by the value of the weighting function, $w(\cdot)$, at the $\mathbb{C}_+$-zeros of the plant $\mathcal{P}$.

5.1.2. The generalized Smith regulator: the perturbed case

i) For additive perturbations eqn. (3.4) shows that stability of the perturbed GS regulator depends on the Nyquist plot of $\omega \mapsto \det[I+\Delta\mathcal{P}\cdot Q(j\omega$, provided that the perturbed plant $\tilde{\mathcal{P}}$ is stable.

ii) Consider multiplicative perturbations of the form $\tilde{\mathcal{P}} = (I+M)\mathcal{P}$ (or $\tilde{\mathcal{P}} = \mathcal{P}(I+M)$ etc.) with $M \in \mathcal{M}$ where for a given tolerance function $\omega \mapsto \ell_m(\omega)$ which satisfies

a) $\omega \mapsto \ell_m(\omega)$ mapping $\mathbb{R}_+$ into $\mathbb{R}_+\backslash\{0\}$ is continuous

b) $\exists k \in \mathbb{N}^*$ s.t. $\ell_m(\omega)\omega^k > 1$, for all ω sufficiently large

$\mathcal{M} := \{M : M\mathcal{P}$ is exponentially stable and strictly proper;

$$\sigma_{max}[M(j\omega)] < \ell_m(\omega),\ \forall\omega \in \mathbb{R}_+,\ n^+_{\tilde{\mathcal{P}}} = n^+_{\mathcal{P}}\}^{\dagger}$$

Using the results of [Chen. 1] (which generalized to the present case the original results of [Doy. 1]) we write down the necessary and sufficient condition for robust stability of the perturbed system for various locations of the perturbation in Table 1. To see what the necessary and sufficient condition means let us consider a specific row, ii), of Table 1. For a multiplicative $\tilde{\mathcal{P}} = (I+M)\mathcal{P}$ where $M \in \mathcal{M}$, the necessary and sufficient condition for exponential stability of ${}^1S(\tilde{\mathcal{P}},\mathcal{P},Q)$ (GS) is

$$\sigma_{max}[\mathcal{P}Q]\ \sigma_{max}[M] < 1 \Leftrightarrow \sigma_{max}[H_{y_2u_1}]\ \sigma_{max}[M] < 1\ . \tag{5.7}$$

Equation (5.7) shows that the requirement of robust stability puts a constraint on the "size" (as measured by the maximum singular value) of the I/O-map, $H_{y_2u_1}$. The larger the "size" of the perturbation M, the smaller we must choose $\sigma_{max}[H_{y_2u_1}]$ to maintain stability in the presence of perturbations in the class $\mathcal{M}$.

In the example of sec. 5.1, $H_{y_2u_1} = N_d(1 + \frac{s}{\omega_c})^{-m}$ (eqn. (5.2)) and since N_d is unitary, $\sigma_{max}[H_{y_2u_1}] = |1 + j(\frac{\omega}{\omega_c})|^{-m}$; thus a bound $\sigma_{max}[H_{y_2u_1}]$ puts a bound on ω_c, the achievable bandwidth.

iii) The effect of mismatches between plant "delay" and model "delay" can be quite drastic and is best seen from the following example: Consider ${}^2S(e^{-(\tau+\Delta\tau)s}P,\ e^{-\tau s}P,Q)$: i.e., $\tilde{N}_d = e^{-(\tau+\Delta\tau)s}I$ (equal delay, $\tau+\Delta\tau$, in all channels) and $N_d = e^{-\tau s}I$. Note that $(I+(e^{-(\Delta\tau)s}-1)I)e^{-\tau s}P = e^{-(\tau+\Delta\tau)s}P$, thus the mismatch in delay can be modelled as a

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† $n^+_{\tilde{\mathcal{P}}}$ (resp. $n^+_{\mathcal{P}}$) is the number of $\mathbb{C}_+$-poles of $\tilde{\mathcal{P}}$ (resp. $\mathcal{P}$).

multiplicative perturbation of the form (I+M)P and from Table 1:

$$^{2}S(e^{-(\tau+\Delta\tau)s}P,\ e^{-\tau s}P,Q) \text{ is exponentially stable}$$

$$\Leftrightarrow\ \forall\omega \in \mathbb{R},\ \sigma_{max}[e^{-j\omega\tau}P(j\omega)\ Q(j\omega)]\cdot\sigma_{max}[(e^{-j\omega\Delta\tau}-1)I] < 1 \tag{5.8}$$

In particular for $\omega_k = \frac{(2k-1)\pi}{\Delta\tau}$, $k \in \mathbb{N}^*$, stability condition (5.8) gives

$$\forall k \in \mathbb{N}^*,\ \ \sigma_{max}[P(j\omega_k)\ Q(j\omega_k)] < \frac{1}{2} \tag{5.9}$$

Note that the larger the uncertainty in delay, $\Delta\tau$, the smaller is the spacing between the successive ω_k's.

Since $N_d(j\omega)$ is unitary, $\sigma_{max}[H_{y_2u_1}(j\omega)] = \sigma_{max}[P(j\omega)\ Q(j\omega)]$ $\forall\omega \in \mathbb{R}$. Thus delay uncertainty imposes strict (and, in this case, undesirable) conditions on the I/O-map.

In the SISO case, equation (5.9) becomes:

$$\forall\ k \in \mathbb{N}^*,\ |p(j\omega_k)\ q(j\omega_k)| < \frac{1}{2}\ ;\ \text{or equivalently}\ |h^{o}_{y_2u_1}(j\omega_k)| < \frac{1}{2}$$

where $h^{o}_{y_2u_1}(\cdot)$ is the I/O-map for $^{1}S(p,c)$ and this is more explicit than Palmor's result [Pal. 1, Thm. 1] for the case of unity feedback.

iv) Finally we point out that $\Delta\mathcal{P}$ may act in a direction to make $\tilde{H}_{y_2d_0} = I - \tilde{\mathcal{P}}\,Q(I+\Delta\mathcal{P}\cdot Q)^{-1}$ larger than nominal though the perturbed system is still stable. Thus robust disturbance rejection, as defined in [Chen. 2], will impose even more stringent requirements on the system.

5.2. The GHH regulator

5.2.1. The nominal case:

By equations (4.1) and (4.2), the nominal GHH system has the same nominal I/O and disturbance to output maps as the nominal GS system. Hence, using the same example as for the nominal GS system (see sec. 5.1.1), equations (5.2) and (5.3) are valid for the GHH system, so we reach the same conclusions except that the plant input e_2 is now given by $e_2 = Qu_1+u_2$, which makes it even clearer that plant saturation puts a bound on $\sigma_{max}[Q]$ and hence on ω_c (see equation (5.4))

5.2.2. The GHH regulator: the perturbed case

i) In the GHH regulator (Fig. 4) let $\tilde{P} = P$ but $N_d \leftarrow \tilde{N}_d$. Then from equations (4.5) and (4.6) with $\Delta P = 0$ we obtain:

$$\tilde{H}_{yu} = \begin{bmatrix} Q & 0 \\ P\tilde{N}_d Q & P\tilde{N}_d \end{bmatrix} \tag{5.10}$$

$$\tilde{H}_{y_2 d_0} = I - P\tilde{N}_d Q \tag{5.11}$$

Equations (5.10) and (5.11) are merely equations (4.1) and (4.2) with $N_d \leftarrow \tilde{N}_d$, and imply that if $\tilde{N}_d$ is arbitrary but exponentially stable, then the perturbed system is still exponentially stable. (This observation is made intuitively obvious if we note that when $P = \tilde{P}$ there is no feedback, as Fig. 4 shows). This is the precise sense in which one should understand Horowitz's phrase "totally insensitive to the delay τ" [Hor. 1, p. 984].

ii) From a comparison of Figs. 4 and 5, it might appear, at first sight, that use of the Q-parametrization theorem causes us to ignore the effect of differences between P_1 and $\tilde{P}$. This is not true because even if $P_1 \neq \tilde{P}$, but P_1 is exponentially stable, we may use the Q-parametrization theorem with $Q_1 := C(I+P_1C)^{-1}$; and, when $P_2 = \tilde{P}$, this will just change the I/O-map $H_{y_2u_1}$ to $\tilde{P}N_dQ_1$ without affecting the stability of the GHH regulator.

iii) When $P_2 \neq \tilde{P}$ there is feedback, hence there may be instability: Equations (4.5) and (4.6) show that, in order to maintain stability for all expected $\tilde{N}_d$'s and all expected additive perturbations, (ΔP's), $(I+\Delta P\cdot\tilde{N}_d\cdot Q)^{-1}$ must be exponentially stable--which is easily checked by a Nyquist plot.

iv) For multiplicative perturbations in the class $\mathcal{M}$ (defined in sec. (5.1.2), ii) the necessary and sufficient conditions for robust stability for various locations of the perturbation are listed in Table 1 and if we look at row ii) again, we are led to the <u>same</u> conclusions for the GHH regulator, as for the GS regulator in sec. (5.1.2), ii): the larger the "size" [$\sigma_{max}[M]$) of the multiplicative perturbation, M, the smaller we must choose $\sigma_{max}[H_{y_2u_1}]$ ("size" of the I/O-map) to maintain exponential stability.

For two specific locations (see rows 1 and 3b) of Table 1) of perturbations, however, the GHH regulator is <u>stable</u> for <u>all</u> perturbations in class $\mathcal{M}_\infty$ defined by taking $\ell_m(\omega)$ to be arbitrarily large for all ω (i.e., there is no restriction on $\sigma_{max}[M]$ as long as the perturbed plant is still exponentially stable). This phenomenon is explained in i) of the present section (5.2.2)

6. <u>The nonlinear case</u>

The theory of the GS regulator and the GHH regulator extends to the nonlinear case, as is suggested in [Mar. 1, p. 162].

For all <u>nonlinear</u> dynamical systems considered below, we assume that at $t = 0$ they are in a standard state (say, some equilibrium state called the zero-state). Thus we assume that, for any input (say $e_2 : \mathbb{R}_+ \to \mathbb{R}^{n_i}$) the nonlinear causal plant $\underset{\sim}{\mathcal{P}}$ has a unique output $\underset{\sim}{\mathcal{P}}(e_2(\cdot)) : \mathbb{R}_+ \to \mathbb{R}^{n_o}$, which depends <u>only</u> on the input.

6.1. Definitions and Notations

Let $(\mathcal{L}, \|\cdot\|)$ be a normed space of "time functions": $\mathcal{J} \to \mathcal{V}$ where $\mathcal{J}$ is the time set (typically $\mathbb{R}_+$ or $\mathbb{N}$), $\mathcal{V}$ is a normed space typically $\mathbb{R}$, $\mathbb{R}^n$, C^n, $\cdots$) and $\|\cdot\|$ is the chosen norm in $\mathcal{L}$. Let $\mathcal{L}_e$ be the corresponding extended space (see e.g. [Wil. 1], [Des. 4]).

A function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to <u>class K</u> iff ϕ is ϕ <u>continuous</u> and <u>increasing</u>. ϕ is said to belong to <u>class K_0</u> iff $\phi \in K$ and $\phi(0) = 0$. If ϕ_1 and $\phi_2 \in K_0$, then $\phi_1 + \phi_2$ and $\alpha \mapsto \phi_1(\phi_2(\alpha)) \in K_0$.
A nonlinear causal map $\underset{\sim}{H} : \mathcal{L}_e^{n_i} \to \mathcal{L}_e^{n_o}$ is said to be <u>$\mathcal{S}$-stable</u> iff $\exists \phi \in K$ s.t. $\forall x \in \mathcal{L}_e^{n_i}$, $\forall T \in \mathcal{J}$,

$$\|\underset{\sim}{H}x\|_T \leq \phi(\|x\|_T)$$

$\underset{\sim}{H}$ is said to be <u>incrementally $\mathcal{S}$-stable</u> <u>(incr. $\mathcal{S}$-stable)</u> iff (i) $\underset{\sim}{H}$ is $\mathcal{S}$-stable, (ii) $\exists \tilde{\phi} \in K_0$ s.t. $\forall x, x' \in \mathcal{L}_e^{n_i}$, $\forall T \in \mathcal{J}$,

$$\|\underset{\sim}{H}x - \underset{\sim}{H}x'\|_T \leq \tilde{\phi}(\|x - x'\|_T)$$

It can be shown that if the nonlinear causal maps $\underset{\sim}{H}_1$ and $\underset{\sim}{H}_2$ are $\mathcal{S}$-stable, (inc. $\mathcal{S}$-stable), then $\underset{\sim}{H}_1 + \underset{\sim}{H}_2$ and $\underset{\sim}{H}_1 \circ \underset{\sim}{H}_2$ are $\mathcal{S}$-stable, (incr. $\mathcal{S}$-stable, resp.). (For simplicity, in what follows we drop the symbol "$\circ$" denoting the composition of the maps.)

A feedback system is said to be <u>well-posed</u> iff the relation from the exogenous inputs into each subsystem† variable (i.e., subsystem input and subsystem output) is a well-defined nonlinear causal map between the corresponding extended spaces. More precisely, the system ${}^1S(\underset{\sim}{P}, \underset{\sim}{C})$ of Fig. 1, where $\underset{\sim}{P} : \mathcal{L}_e^{n_i} \to \mathcal{L}_e^{n_o}$, $\underset{\sim}{C} : \mathcal{L}_e^{n_o} \to \mathcal{L}_e^{n_i}$ are causal maps, is said to be

†By subsystem we mean any block of the block diagram of the feedback system.

well-posed iff $\underset{\sim}{H} : (u_1,u_2) \mapsto (e_1,e_2,y_1,y_2)$ is well-defined and causal. Note that ${}^1S(P,C)$ is well-posed implies that† $(I+PC)^{-1}$ and $(I+CP)^{-1}$ are well-defined and causal. We say that a well-posed nonlinear feedback system is $\mathcal{S}$-stable (incr. $\mathcal{S}$-stable) iff the map from the exogenous inputs to any subsystem variable is $\mathcal{S}$-stable (incre. $\mathcal{S}$-stable, resp.).

The Q-parametrization theorem [Des. 2], [Des. 3], states that if $\mathcal{P}$ is incrementally $\mathcal{S}$-stable, then the nonlinear feedback system ${}^1S(\underset{\sim}{P},\underset{\sim}{C})$ is $\mathcal{S}$-stable if and only if there is an $\mathcal{S}$-stable $\underset{\sim}{Q}$ such that

$$\underset{\sim}{C} = \underset{\sim}{Q}(\underset{\sim}{I}-\underset{\sim}{P}\underset{\sim}{Q})^{-1} \tag{6.1}$$

or, equivalently, [Des. 2,3]

$$\underset{\sim}{Q} = \underset{\sim}{C}(\underset{\sim}{I}+\underset{\sim}{P}\underset{\sim}{C})^{-1} \quad . \tag{6.2}$$

Note that here $\underset{\sim}{P}\underset{\sim}{Q}$, $\underset{\sim}{P}\underset{\sim}{C}$, etc. denote composition of nonlinear maps.

Consider ${}^2S(\underset{\sim}{N}_d\underset{\sim}{P},(\underset{\sim}{I}-\underset{\sim}{N}_d)\underset{\sim}{P},\underset{\sim}{C})$ shown in Fig. 2, interpreted in the nonlinear context: where $\mathcal{P} = \underset{\sim}{N}_d\underset{\sim}{P}$ with $\underset{\sim}{P}$ nonlinear, causal, incrementally $\mathcal{S}$-stable and the "delay" map $\underset{\sim}{N}_d$ is linear, time-invariant, represented by a transfer function N_d which is proper, analytic in $\mathbb{C}_+$ and unitary on the $j\omega$-axis. Applying the Q-parametrization theorem we can redraw the system of Fig. 1 as shown in Fig. 2, except that we consider the nominal nonlinear case, set P, $\tilde{P}$ to $\underset{\sim}{P}$; set N_d, $\tilde{N}_d$ to $\underset{\sim}{N}_d$. Now for ${}^2S(\underset{\sim}{N}_d\underset{\sim}{P},\underset{\sim}{N}_d\underset{\sim}{P},\underset{\sim}{Q})$ we have the partial I/O-map $\underset{\sim}{H}_{y_2u_1} : (u_1,0,0) \mapsto y_2$ given by $\underset{\sim}{H}_{y_2u_1} = \underset{\sim}{N}_d\underset{\sim}{P}\underset{\sim}{Q}$. This suggests immediately that, since $\underset{\sim}{N}_d$ represents only a linear "delay," we choose $\underset{\sim}{Q}$ to fashion $\underset{\sim}{P}\underset{\sim}{Q}$ to our design requirements. In other words, the design concept that O. J. M. Smith proposed in 1957 can be extended to

†The meaning of $(\underset{\sim}{I}+\underset{\sim}{P}\underset{\sim}{C})^{-1}$ deserves clarification: the map $\underset{\sim}{C}$ is composed with $\underset{\sim}{P}$ then the identity is added, and the resulting map is inverted. Although this formula has the same form as the linear case, it has a completely different interpretation.

the MIMO nonlinear case. Now if, in Fig. 3, we view $\underset{\sim}{Q}$ as $\underset{\sim}{C}$ with a nonlinear (negative) feedback $\underset{\sim}{P}$ (see equation (6.2)) we obtain, for the nonlinear case, the system of Fig. 2 where we set P and $\tilde{P}$ to $\underset{\sim}{P}$; N_d and $\tilde{N}_d$ to $\underset{\sim}{N}_d$ and C to $\underset{\sim}{C}$.

We formalize the above discussion as the following theorem:

Theorem 6.1: Let $\underset{\sim}{P}$ be a nonlinear causal incrementally $\mathcal{S}$-stable map and $\underset{\sim}{N}_d$ a linear time-invariant map represented by a transfer function which is proper, analytic in $\mathbb{C}_+$ and unitary on the $j\omega$-axis. Then the nominal nonlinear GS regulator ${}^2S(\underset{\sim}{N}_d\underset{\sim}{P},(\underset{\sim}{I}-\underset{\sim}{N}_d)\underset{\sim}{P},\underset{\sim}{C})$ is $\mathcal{S}$-stable if and only if $\underset{\sim}{C} = \underset{\sim}{Q}(\underset{\sim}{I}-\underset{\sim}{P}\underset{\sim}{Q})^{-1}$, or equivalently $\underset{\sim}{Q} = \underset{\sim}{C}(\underset{\sim}{I}+\underset{\sim}{P}\underset{\sim}{C})^{-1}$, for some $\mathcal{S}$-stable $\underset{\sim}{Q}$.

Proof: Follows the proof of Thm. 1 a), sec. III of [Des. 3] almost exactly, after defining $\tilde{u}_1 := \underset{\sim}{N}_d\underset{\sim}{P}(\underset{\sim}{C}\varepsilon_1) - \underset{\sim}{N}_d\underset{\sim}{P}(u_2+\underset{\sim}{C}\varepsilon_1)$. ¤

For the nominal nonlinear GHH regulator we state and prove the analog of Theorem 6.1

Theorem 6.2: Let $\underset{\sim}{P}$ be a nonlinear causal $\mathcal{S}$-stable map and $\underset{\sim}{N}_d$ a linear time-invariant map represented by a transfer function which is proper, analytic in $\mathbb{C}_+$ and unitary on the $j\omega$-axis. Then the nominal nonlinear GHH regulator ${}^3S(\underset{\sim}{P}\underset{\sim}{N}_d,\underset{\sim}{P},\underset{\sim}{P},\underset{\sim}{C})$ is $\mathcal{S}$-stable if and only if $\underset{\sim}{C} := \underset{\sim}{Q}(\underset{\sim}{I}-\underset{\sim}{P}\underset{\sim}{Q})^{-1}$, or equivalently $\underset{\sim}{Q} = \underset{\sim}{C}(\underset{\sim}{I}+\underset{\sim}{P}\underset{\sim}{C})^{-1}$ for some $\mathcal{S}$-stable $\underset{\sim}{Q}$.

Proof: ($\Rightarrow$) Set $u_2 = 0$ and $d_0 = 0$: the partial map $\underset{\sim}{H}_{y_1u_1} : (u_1,0,0) \mapsto y_1$ is given by $\underset{\sim}{H}_{y_1u_1} = \underset{\sim}{C}(\underset{\sim}{I}+\underset{\sim}{P}\underset{\sim}{C})^{-1}$ which by assumption is $\mathcal{S}$-stable. Let $\underset{\sim}{Q} := \underset{\sim}{C}(\underset{\sim}{I}+\underset{\sim}{P}\underset{\sim}{C})^{-1}$, then $\underset{\sim}{Q}$ is $\mathcal{S}$-stable, equivalently $\underset{\sim}{C} = \underset{\sim}{Q}(\underset{\sim}{I}-\underset{\sim}{P}\underset{\sim}{Q})^{-1}$

($\Leftarrow$) Refer to Fig. 4; set $d_0 = 0$: the summing node equations read:

$$e_1 = u_1 - \underset{\sim}{P}\underset{\sim}{C}\, e_1 \tag{6.3}$$

$$e_2 = u_2 + \underset{\sim}{C}\, e_1 \tag{6.4}$$

Equation (6.3) is equivalent to:

$$e_1 = (\underset{\sim}{I}+\underset{\sim}{P}\underset{\sim}{C})^{-1}u_1$$

and, from Fig. 4,

$$y_1 = \underset{\sim}{C}\,e_1 = \underset{\sim}{C}(\underset{\sim}{I}+\underset{\sim}{P}\underset{\sim}{C})^{-1}u_1 = \underset{\sim}{Q}\,u_1 \tag{6.5}$$

Define the projection map $\underset{\sim}{\pi}_i : (u_1,u_2) \mapsto u_i$, $i = 1,2$. From equation (6.5), the map $\underset{\sim}{H}_{y_1u} = (u_1,u_2) \mapsto y_1$ is given by

$$\underset{\sim}{H}_{y_1u} = \underset{\sim}{Q}\,\underset{\sim}{\pi}_1$$

Since $\underset{\sim}{\pi}_1$ is $\mathcal{S}$-stable and by assumption $\underset{\sim}{Q}$ is $\mathcal{S}$-stable, the map $\underset{\sim}{H}_{y_1u}$ is $\mathcal{S}$-stable. From Fig. 4, we have

$$y_2 = \underset{\sim}{P}\,\underset{\sim}{N}_d(u_2+y_1)$$

Hence the map $\underset{\sim}{H}_{y_2u} = (u_1,u_2) \mapsto y_2$ is given by

$$\underset{\sim}{H}_{y_2u} = \underset{\sim}{P}\,\underset{\sim}{N}_d(\underset{\sim}{\pi}_2+\underset{\sim}{H}_{y_1u}) \tag{6.6}$$

Now, by assumption, $\underset{\sim}{P}$ is $\mathcal{S}$-stable and $\underset{\sim}{N}_d$ is linear, and $\mathcal{S}$-stable: thus $\underset{\sim}{\pi}_2$, $\underset{\sim}{H}_{y_1u}$ and $\underset{\sim}{P}\underset{\sim}{N}_d$ are all $\mathcal{S}$-stable, and it follows from equation (6.6) that $\underset{\sim}{H}_{y_2u}$ is $\mathcal{S}$-stable. Therefore $\underset{\sim}{H}_{yu} : (u_1,u_2) \mapsto (y_1,y_2)$ is $\mathcal{S}$-stable. ¤

7. A Comparison between the GS and GHH Regulators

i) Refer to equation (3.2) with $\mathcal{P} = PN_d$ and to equation (4.1). The first column of the nominal H^o_{yu} is the same for both regulators as is the first column of the perturbed $\tilde{H}_{yu}$'s (equations (3.4), (4.5)). Consequently:

a) the nominal (resp. perturbed) I/O-maps are identical, each to each, for both configurations;

b) Since $H_{y_2 d_0} = I - H_{y_2 u_1}$ (in the nominal as well as perturbed cases), the nominal (resp. perturbed) disturbance-to-output maps are identical, each to each, for both configurations. We have used a) and b) to abbreviate sec. 5.2.1 drastically.

ii) Table 1 compares the 2 regulator configurations on the basis of necessary and sufficient conditions for robust stability under multiplicative perturbations at various locations. It reiterates an important point that we have discussed and explained in sec. 5.2.2 i), iv):

Given that the assumptions on factorization (see sec. 4) required for the use of the GHH regulator obtain, we may say that provided $P_2 = P$ (which is really the nominal case for the GHH regulator), the GHH regulator has superior stability properties when uncertainty in N_d is potentially large.

In view of our theoretical results (sec. 5.1.2, iii)) on the effects of uncertainty in delay on the GS regulator, the numerical example of [Pal. 1] and simulation studies ([Hor. 1] etc.), this robustness of the GHH regulator with perfect model ($P_2 = P$), in the face of uncertainty in N_d, is significant. However, as rows ii) a) and b) and iii) a) and c) of Table 1 show, the GHH and GS regulators impose identical robust stability constraints as soon as we are in the real world of imperfect ($P_2 \neq P$) models! The importance of a perfect model is seen again in the appendix (see comment c following Theorem A.1, Appendix I).

iii) A comparison of Theorem 6.1 and 6.2 shows that the theory of the GHH regulator extends to the nonlinear case under the assumption that $\underset{\sim}{P}$ is $\mathcal{S}$-stable whereas the extension of the theory of the GS regulator requires the stronger assumption of incremental $\mathcal{S}$-stability of $\underset{\sim}{P}$.

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Notation for Appendices I & II

Referring to the notation of Sec. I, we introduce the following notation. For details, see [Cal. 2].

$\hat{\mathcal{A}}_-^\infty(\sigma)$ and $\hat{\mathcal{A}}_{-,o}(\sigma)$ denote the subsets of $\hat{\mathcal{A}}_-(\sigma)$ consisting of those $\hat{f}$ that are bounded away from zero at infinity in $\mathbb{C}_{\sigma+}$, and those that go to zero at infinity in $\mathbb{C}_{\sigma+}$ respectively.

$$\hat{\mathcal{B}}(\sigma) = [\hat{\mathcal{A}}_{-,o}(\sigma)][\hat{\mathcal{A}}_-^\infty(\sigma)]^{-1}$$

denotes the commutative algebra of fractions $\hat{g} = \hat{n}/\hat{d}$, where $\hat{n} \in \hat{\mathcal{A}}_-(\sigma)$ and $\hat{d} \in \hat{\mathcal{A}}_-^\infty(\sigma)$

$$\hat{\mathcal{B}}_0(\sigma) := [\hat{\mathcal{A}}_{-,o}(\sigma)][\hat{\mathcal{A}}_-^\infty(\sigma)]^{-1}$$

Let $H \in \hat{\mathcal{B}}(\sigma)^{m\times n}$ then:

$N_rD_r^{-1}$ is called a <u>right coprime factorization</u> (r.c.f.) of H if and only if

(i) N_r and D_r have all their elements $\hat{\mathcal{A}}_-(\sigma)$ and

$\det D_r \in \hat{\mathcal{A}}_-^\infty(\sigma)$

(ii) $H = N_rD_r^{-1}$

(iii) N_r, D_r are right coprime (r.c.), i.e., there exists $U_r \in \hat{\mathcal{A}}_-(\sigma)^{n\times m}$ and $V_r \in \hat{\mathcal{A}}_-(\sigma)^{n\times n}$ such that

$$U_rN_r + V_rD_r = I_n$$

$D_\ell^{-1}N_\ell$ is called a <u>left coprime factorization</u> (l.c.f.) of H if and only if

(i') N_ℓ and D_ℓ have all their elements in $\hat{\mathcal{A}}_-(\sigma)$ and

$\det D \in \hat{\mathcal{A}}_-^\infty(\sigma)$

(ii') $H = D_\ell^{-1}N$

(iii') D_ℓ, N_ℓ are left coprime (l.c.), i.e., there exists $U_\ell \in \hat{\mathcal{Q}}_-(\sigma)^{n\times m}$ and $V_\ell \in \hat{\mathcal{Q}}_-(\sigma)^{m\times m}$ such that

$$N_\ell U_\ell + D_\ell V_\ell = I_m$$

For all $H \in \hat{\mathcal{B}}(\sigma)^{m\times n}$, algorithms are available to obtain both r.c.f. and l.c.f. [Cal. 2].

Appendix I: Stability of the GHH Regulator

Since there are proofs available in the literature showing that exponential stability of the plant is necessary for the use of the Smith regulator [Fur. 1], we do an analysis for the GHH regulator and indicate a method for the Smith regulator that does not rely on state-space analysis as in [Fur. 1]. We impose the following 2 assumptions on C, N_d, P, P_1, P_2. For some $\sigma_0 < 0$

$$\left.\begin{array}{l} C \in \hat{\mathcal{B}}(\sigma_0)^{n_i x n_o};\ N_d \in \hat{\mathcal{Q}}_-(\sigma_0)^{n_i x n_i},\ \forall\, \omega \in \mathbb{R}, N_d(j\omega) \text{ is unitary; } N_d(0) = I. \\ P \in \hat{\mathcal{B}}_0(\sigma_0)^{n_o x n_i};\ P_j \in \hat{\mathcal{B}}_0(\sigma_0)^{n_o x n_i},\ j = 1, 2 \end{array}\right\} \quad \text{(A1)}$$

$$\left.\begin{array}{l} D_{c\ell}^{-1} N_{c\ell} \text{ is an l.c.f. of } C;\ N_{m_j r} D_{m_j r}^{-1} \text{ is an r.c.f. of } P_j,\ j=1,2 \\ N_{pr} D_{pr}^{-1} \text{ is an l.c.f. of } P \end{array}\right\} \quad \text{(A2)}$$

Comments: a) Note that, [Cal. 2], the matrices $D_{c\ell}$, D_{pr}, $D_{m_j r}$ may be chosen to be rational; by definition, they are exponentially stable, proper and their determinants become non-zero constants as $|s| \to \infty$.

b) Assumption (A1) says that C, P, N_d and the plant models P_1 and P_2 are not required to be lumped: their transfer functions C, P, P_1, P_2, N_d are analytic in Re $s > \sigma_1$ except for possibly a finite number of poles. The plant P and its models P_1, P_2 are required to be strictly proper.

In terms of ξ_1, ξ_2, ξ_3 defined in Fig. A1, the system ${}^3S(PN_d, P_1, P_2, C)$ is described by

$$D\xi = N_\ell u, \qquad N_r \xi = y \tag{A3}$$

where $\xi := [\xi_1^T \vdots \xi_2^T \vdots \xi_3^T]$, $u := [u_1^T \vdots u_2^T]$, and

$$D := \left[\begin{array}{c|c|c} D_{c\ell}D_{m_1r} + N_{c\ell}N_{m_1r} & -N_{c\ell}N_{m_2r} & N_{c\ell}N_{pr} \\ \hline -N_dD_{m_1r} & D_{m_2r} & 0 \\ \hline 0 & D_{m_2r} & -D_{pr} \end{array}\right] \tag{A4}$$

$$N_\ell := \left[\begin{array}{c|c} N_{c\ell} & 0 \\ \hline 0 & N_d \\ \hline 0 & 0 \end{array}\right] \quad ; \quad N_r := \left[\begin{array}{c|c|c} D_{m_1r} & 0 & 0 \\ \hline 0 & 0 & N_{pr} \\ \hline N_{m_1r} & -N_{m_2r} & N_{pr} \end{array}\right] \tag{A5}$$

From (A3) $H_{yu} : u \mapsto y$ the input-output map of the system ${}^3S(PN_d, P_1, P_2, C)$ is given by

$$H_{yu} = N_rD^{-1}N_\ell \tag{A6}$$

We now define stability.

Definition: We say that a system S specified by a matrix fraction description, as in (A3), is exponentially stable if and only if det[D] has no $\mathbb{C}_+$-zeros.

Comments: a) Since we do not require left (resp. right) coprimeness of D and N_ℓ (resp. D and N_r) in the above definition we cannot use the definition that "S is exponentially stable if and only if $H_{yu} : u \mapsto y$ has all its elements in $\hat{\mathcal{A}}_-(\sigma_0)$", as in [Chen 1]; indeed the description (A3) allows unstable hidden modes.

b) For the system ${}^3S(PN_d,P_1,P_2,C)$, shown in Fig. A.1 it is intuitively obvious that (D,N_r), (D,N_ℓ) are not r.c., l.c. respectively because: i) we are not observing all subsystem outputs (thus there may be output-decoupling zeros); ii) we do not have exogenous inputs to all subsystems (thus there may be input-decoupling zeros) [Cal. 1, Sec. 4.2].

Theorem A.1. Let assumptions (A1) and (A2) hold. The nominal GHH regulator, namely the system ${}^3S(PN_d,P,P,C)$, is exponentially stable if and only if

i) D_{pr} can be taken to be I,

ii) $\forall\ s \in \mathbb{C}_+$, $\det[D_{c\ell}(s) + N_{c\ell}(s)N_{pr}(s)] \neq 0$.

Comment: Note that if we choose $D_{pr} = I$, then $P = N_{pr}$ and by the definition of the r.c.f. ([Cal. 2], [Chen 1]), $N_{pr} \in \hat{\mathcal{A}}_{-,o}(\sigma_o)^{n_o \times n_i}$. Since $N_d \in \hat{\mathcal{A}}_-(\sigma_o)^{n_o \times n_i}$, $\mathcal{P} = PN_d \in \hat{\mathcal{A}}_{-,o}(\sigma_o)^{n_o \times n_i}$ thus $\mathcal{P}$ is exponentially stable, as promised.

Proof: In the nominal case $N_{m_1r} = N_{m_2r} = N_{pr}$ and $D_{m_1r} = D_{m_2r} = D_{pr}$. Making these substitutions in (A4) we use the elementary column operation $\gamma_2 \leftarrow \gamma_2 + \gamma_3$ and the Binet-Cauchy theorem [Gan. 1] to obtain

$$\det[D] = -\ (\det[D_{pr}])^2 \det[D_{c\ell}D_{pr}+N_{c\ell}N_{pr}] \tag{A7}$$

($\Leftarrow$) Clearly if i) and ii) in the statement of Theorem A1 hold, then $\det[D] \neq 0$, $\forall\ s \in \mathbb{C}_+$.

($\Rightarrow$) From equation (A7), we note that, since P is strictly proper and C is proper, for $s \in \mathbb{C}_+$ and $|s| \to \infty$, det [D] tends to a non-zero constant. So we need only consider the possibility of finite $\mathbb{C}_+$-zeros of det[D].

By the assumption of exponential stability, $\det[D(s)] \neq 0 \;\; \forall\, s$ in $\mathrm{Re}(s) > \sigma_0$ where $|\sigma_0|$ is small and $\sigma_1 < \sigma_0 < 0$: thus, by (A7),

$$\forall s : \mathrm{Re}(s) > \sigma_0, \det[D_{pr}(s)] \neq 0 \;\underline{\text{and}}\; \det[D_{c\ell}(s)D_{pr}(s) + N_{c\ell}(s)N_{pr}(s)] \neq 0 \quad \text{(A8)}$$

Now the matrix D_{pr} is analytic in $\mathrm{Re}\, s > \sigma_1$, $\det[D_{pr}(s)] \neq 0$ for $\mathrm{Re}\, s > \sigma_0$, $s \mapsto (D_{pr}(s))^{-1} \in \hat{\mathcal{A}}(\sigma_0)^{n_i \times n_i}$; consequently D_{pr} can be taken to be I. This establishes conclusion i). Conclusion ii) follows by (A8). ¤

<u>Robustness considerations</u>: Since P_1, P_2 are meant to be models of P, they are assumed to be strictly proper. In the nominal case $P_1 = P_2 = P$. If P_1 and P_2 differ slightly from P, we can show that ${}^3S(PN_d, P_1, P_2, C)$ is also exponentially stable.

Let (A1), (A2) hold for the GHH regulator ${}^3S(PN_d, P_1, P_2, C)$. Then D is given by (A4) and by the elementary column operation $(\gamma_2 \leftarrow \gamma_2 + \gamma_3)$ on D we obtain D',

$$D' := \left[\begin{array}{c|c|c} D_{c\ell}D_{m_1r} + N_{c\ell}N_{m_1r} & N_{c\ell}(N_{pr} - N_{m_2r}) & N_{c\ell}N_{pr} \\ \hline -N_d D_{m_1r} & D_{m_2r} & 0 \\ \hline 0 & -(D_{pr} - D_{m_2r}) & -D_{pr} \end{array}\right] \quad \text{(A9)}$$

Recall that N_{pr}, N_{m_2r}, D_{pr}, D_{m_2r} have elements in $\hat{\mathcal{A}}_-$, hence are bounded in $\mathbb{C}_+$. So we <u>assume</u> that P_1 and P_2 are "good" models of P; more precisely, we assume that for j = 1, 2, and for $\varepsilon > 0$ small,

$$(N_{pr} - N_{m_jr}) \text{ and } (D_{pr} - D_{m_jr}) \text{ are } O(\varepsilon) \quad \text{in } \mathbb{C}_{\sigma_0+} \quad \text{(A10)}$$

then, by applying the Binet-Cauchy theorem to D' we obtain:

$$\det[D'] = -\det[D_{c\ell}D_{m_1r} + N_{c\ell}N_{m_1r}]\cdot\det[D_{m_2r}]\cdot\det[D_{pr}] + O(\varepsilon) \qquad \text{(A11)}$$

Then, since $\det[D_{pr}(s)]$ is bounded away from zero in $Re(s) > \sigma_0$, for $\varepsilon > 0$ sufficiently small, (A11) shows that $\det[D']$ will also be bounded away from zero in $Re(s) \geq 0$; equivalently with assumption (A10) ${}^3S(PN_d, P_1, P_2, C)$ is exponentially stable.

Appendix II: Stability of the GS regulator

For the GS regulator (Fig. A.2) the "block" $[I-N_d(s)]N_{mr}(s)$ has a blocking-zero at $s=0$ since $N_d(0) = I$.[†] Thus, at dc block" $[I-N_d]N_{mr}D_{mr}^{-1}$ has an output y_3 which is zero; furthermore, in the nominal case ($\tilde{N}_d = N_d$, so that $\tilde{N}_d(0) = I$), the GS regulator is equivalent to ${}^1S(P,C)$ at dc.

Referring to Fig. A.2, we state the analog of Theorem A.1 for the GS regulator. Let the following assumptions hold:

For some $\sigma_o \leq 0$

$$\left.\begin{array}{l} C \in \hat{\mathcal{B}}(\sigma_o)^{n_o \times n_i},\ \tilde{P} \in \hat{\mathcal{B}}_o(\sigma_o)^{n_o \times n_i};\ P \in \hat{\mathcal{B}}_o(\sigma_o)^{n_o \times n_i};\ \tilde{N}_d, N_d \in \hat{\mathcal{A}}_-(\sigma_o)^{n_o \times n_o} \\ \quad \forall\ \omega \in \mathbb{R},\ N_d(j\omega) \text{ and } \tilde{N}_d(j\omega) \text{ are unitary } \underline{\text{and}}\ N_d(0) = \tilde{N}_d(0) = I. \end{array}\right\} \quad (B.1)$$

$$\left.\begin{array}{l} D_{c\ell}^{-1}N_{c\ell} \text{ is an l.c.f. of } C;\ N_{pr}D_{pr}^{-1} \text{ is an r.c.f. of } \tilde{P}; \\ N_{mr}D_{mr}^{-1} \text{ is an r.c.f. of } P \end{array}\right\} \quad (B.2)$$

Theorem B.1. Let assumptions (B.1) and (B.2) hold. The nominal GS regulator, ${}^2S(N_dP,(I-N_d)P,C)$ is exponentially stable if and only if,

i) D_{pr} can be taken to be I,

ii) $\forall\ s \in \mathbb{C}_+$, $\det[D_{c\ell}(s) + N_{c\ell}(s)N_{pr}(s)] \neq 0$

Proof: As in Appendix I, the elementary column operation $(\gamma_1 \leftarrow \gamma_1 + \gamma_2)$ shows that, in the nominal case, we have

$$\det[D] = \det[D_{c\ell}D_{pr} + N_{c\ell}N_{pr}]\ \det[D_{pr}] \qquad \square$$

[†] The blocking zero shows that there is a windup problem [Åst. 1] if $\det[D_{mr}(0)] = 0$.

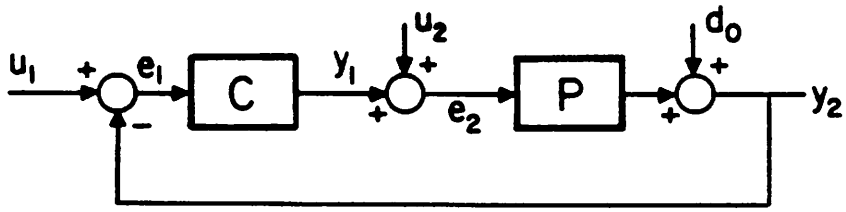


Fig. 1 $^1S(P, C)$

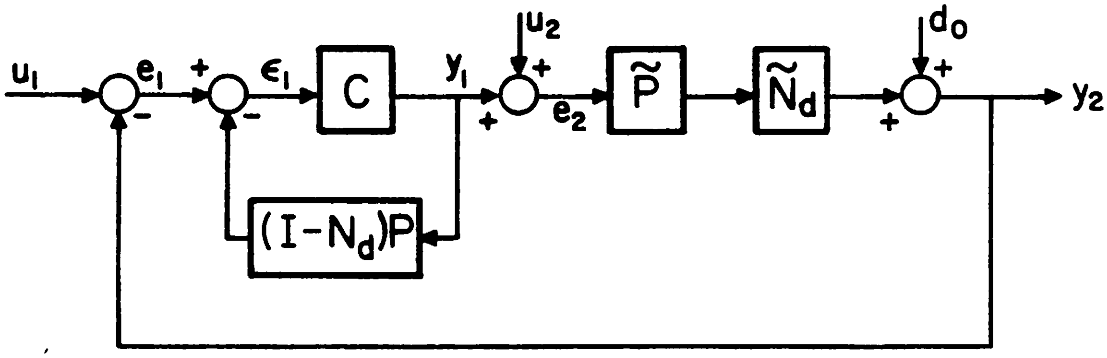


Fig. 2. The GS regulator: $^2S(\tilde{N}_d\tilde{P}, (I-N_d)P, C)$

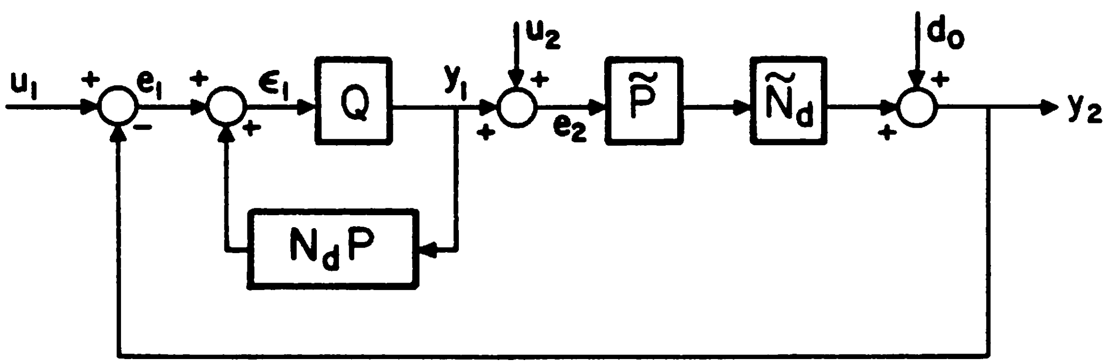


Fig. 3. The GS regulator with Q parametrization:

$$^2S(\tilde{N}_d\tilde{P}, N_dP, Q) \equiv {}^2S(\tilde{N}_d\tilde{P}, (I-N_d)P, C)$$

$$Q := C(I+PC)^{-1}$$

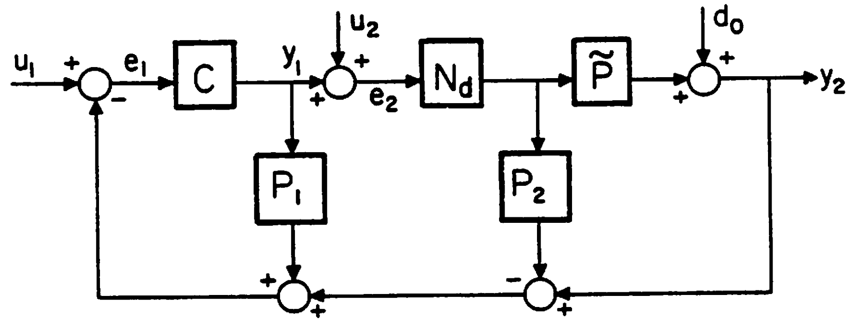


Fig 4. The GHH regulator: ${}^3S(\tilde{P}N_d, P_1, P_2, C)$

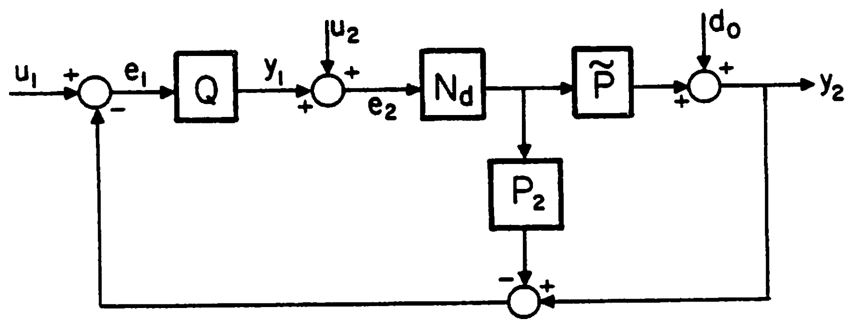
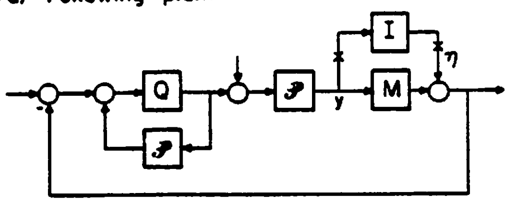
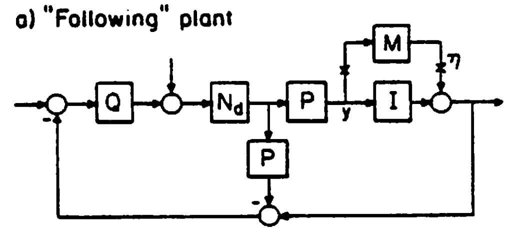
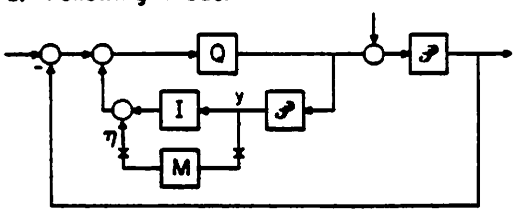
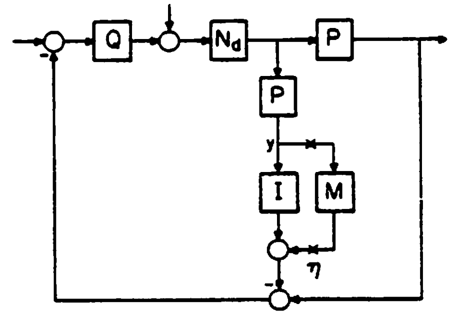
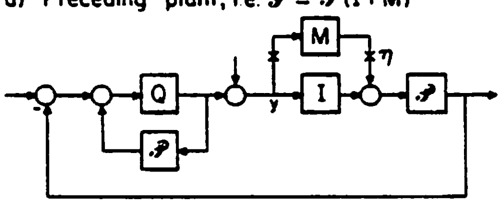
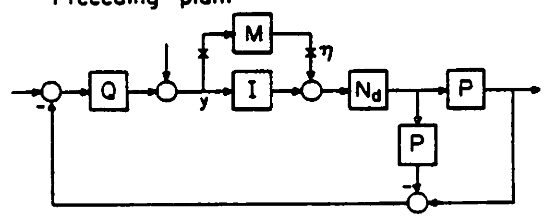
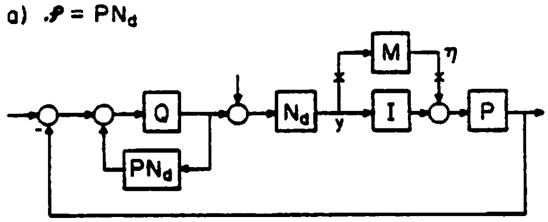
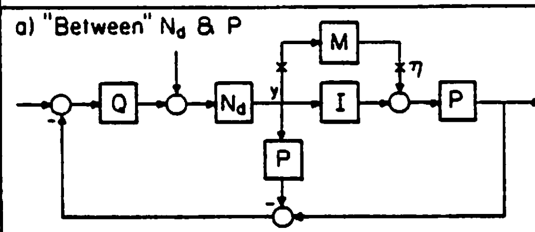
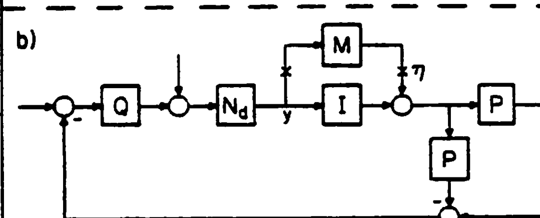
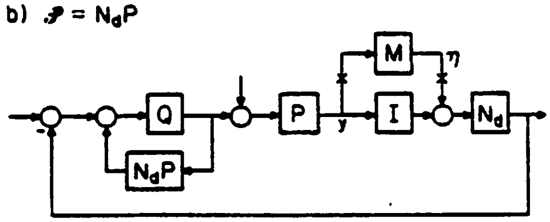
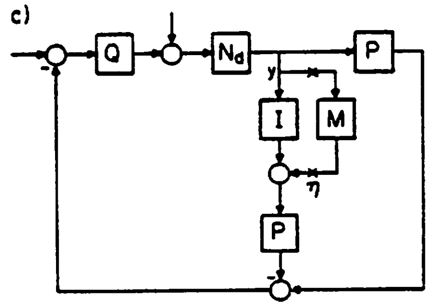


Fig. 5. The GHH regulator with Q-parametrization: ${}^2S(\tilde{P}, P_2, N_d Q)$

TABLE 1

LOCATION OF MULTIPLICATIVE PERTURBATION $M \in \mathcal{M}$ (see Sec. 5.1.2.ii) IN GS REGULATOR	$\frac{GS}{Hy\eta}$	LOCATION OF MULTIPLICATIVE PERTURBATION $M \in \mathcal{M}$ (see Sec. 5.1.2.ii) IN GHH REGULATOR	$\frac{GHH}{Hy\eta}$
i) a) "Following" plant 	$-PQ$ i.e. either $-PN_dQ$ or $-N_dPQ$	a) "Following" plant 	$-PN_dQ$
b) "Following" model 	PQ i.e. either PN_dQ or N_dPQ	b) "Following" model 	PN_dQ
ii) a) "Preceding" plant, i.e. $\tilde{\mathcal{P}} = \mathcal{P}(I+M)$ 	$-QP$ i.e. either $-QN_dP$ or $-QPN_d$	"Preceding" plant 	0
b) "Preceding" model	QP		
iii) "Between" N_d & P a) $\mathcal{P} = PN_d$ 	$-N_dQP$	a) "Between" N_d & P 	$-N_dQP$
		b) 	0
b) $\mathcal{P} = N_dP$ 	$-PQN_d$	c) 	N_dQP

IN ALL CASES THE NECESSARY AND SUFFICIENT CONDITION FOR EXPONENTIAL STABILITY OF THE PERTURBED SYSTEM IS:

$$\sigma_{max}\left[Hy\eta\right] \le \frac{1}{\ell_m(\omega)} \Longleftrightarrow \sigma_{max}\left[Hy\eta\right]\sigma_{max}\left[M\right] < 1$$

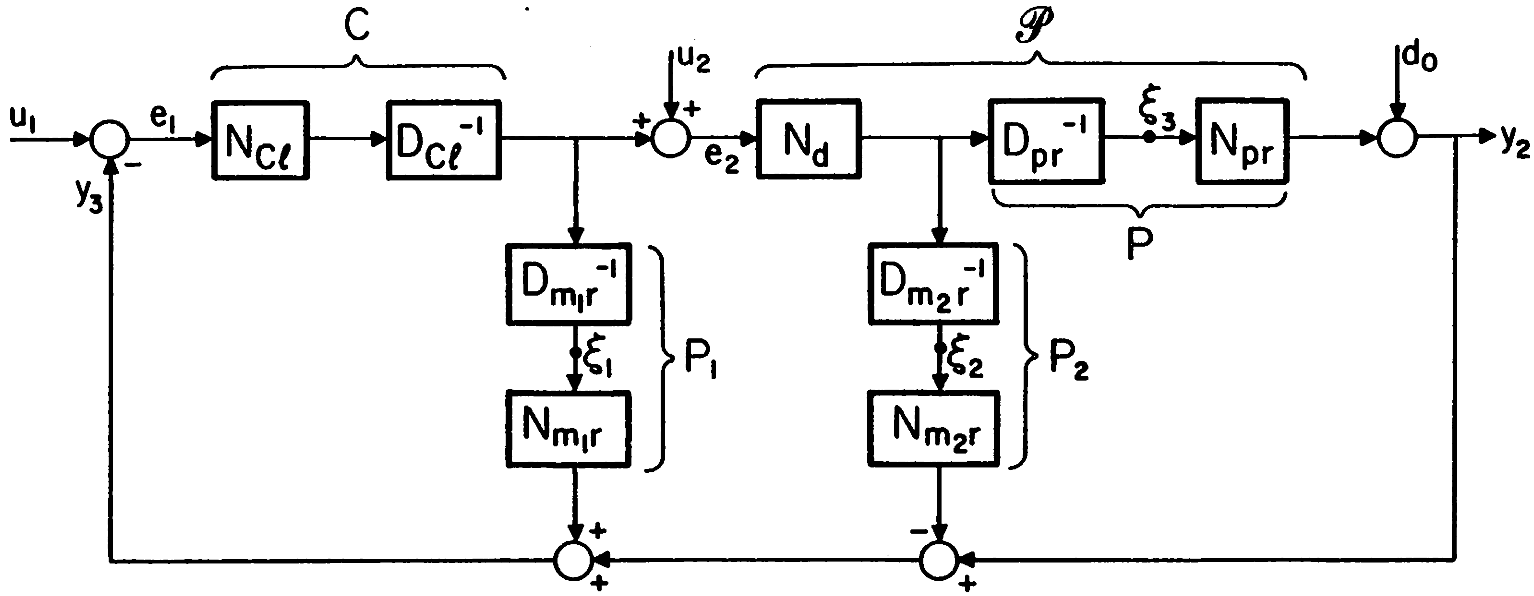


Fig. A.1 The GHH regulator: ${}^3S(PN_d, P_1, P_2, C)$

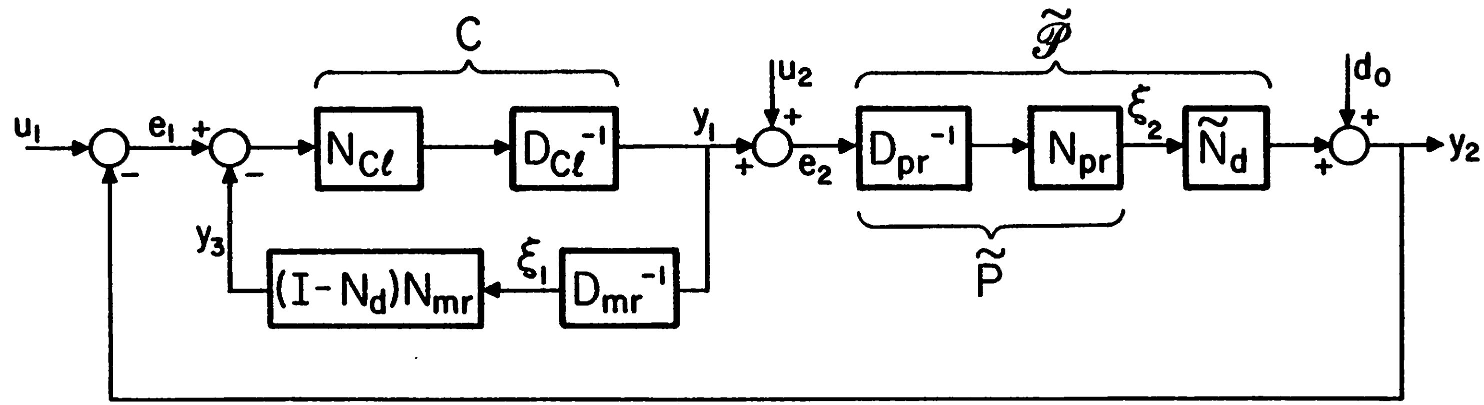


Fig. A.2 The GS regulator: ${}^2S(\tilde{N}_d\tilde{P}, (I-N_d)P, C)$
(Note: $P := N_{mr} D_{mr}^{-1}$)