SMALL SIGNAL I/O STABILITY OF NONLINEAR CONTROL SYSTEMS: APPLICATION TO THE ROBUSTNESS OF A MRAC SCHEME

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ABSTRACT

This paper presents general results relating the internal exponential stability of nonlinear time varying systems to their external input/output stability. Provided that the undriven system is exponentially stable, we give explicit bounds on the size of the input under which the driven system is stable. Moreover, the deviation from equilibrium of the driven system is at most proportional to the (L_∞) size of the input, and the L_∞ gain depends inversely on the rate of exponential convergence. These results are used to study the robustness properties of a model reference adaptive control scheme to various kinds of disturbances: input disturbances, plant parameter variation, output disturbances, and unmodelled dynamics. In most adaptive algorithms, the exponential convergence follows from a persistent excitation condition, so that this condition appears central to the robustness problem in adaptive control. The paper concludes with some remarks on the interpretation of these results for practical applications.

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1. Introduction

In recent years, there has been a very substantial interest in the robustness properties of adaptive control algorithms. Simulation work by Rohrs (1982), and Rohrs et al (1982), showed that, under certain conditions, current adaptive control algorithms lacked robustness margins relevant to several practical applications. They concluded that these algorithms needed to be modified before their implementation could be considered. While some modifications to adaptive control algorithms have been proposed by various authors (Peterson & Narendra (1982), Kreisselmeier & Narendra (1982), Sastry (1984), Ioannou & Kokotovic (1984)), it has been argued by others that the robustness of adaptive algorithms depends primarily on the persistency of excitation of the controlled system, and, consequently, on the choice of the exogeneous reference input applied to the system. Astrom (1983) related the lack of persistency of excitation to improper identification, and to the drift of the parameters of the control system in the presence of disturbances. Other authors (Kosut et al (1983), Anderson & Johnstone (1983), Chen & Cook (1984)) related the persistency of excitation condition to the uniform asymptotic stability , or to the exponential stability of the adaptive system. It is recognised widely that exponentially stable systems are, in general, robust to disturbances, and, for most adaptive algorithms, exponential stability has been proven, provided that persistency of excitation conditions are satisfied (Morgan & Narendra (1977), Anderson (1977) Anderson & Johnson (1982)).

In this paper, we make precise the relationship between the exponential stability of nonlinear time varying systems, and their input/output stability. The input is considered here to be a disturbance, so that the result is a robustness result. In §2, we take a general approach to study perturbation of

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nonlinear time varying ordinary differential equations by disturbances. We generalize results that were partially known in the literature to obtain quantitative relationships between the size of the disturbances, and the size of the consequent deviation of the system from its equilibrium point. Such results are valuable tools for the study of robustness margins of a large class of continuous time nonlinear control systems, in particular adaptive control systems. This is illustrated in §3 for a well-known adaptive control algorithm. We show that input disturbances, time variation of plant parameters, and output disturbances of the plant can be brought to the general framework of §2. Explicit bounds on the disturbances are obtained such that the stability of the system is preserved in the presence of these disturbances. Moreover, a bound on the gain from the disturbances to the deviations from equilibrium is obtained, and the relationship between the gain and the system parameters is studied. Unmodelled dynamics can be studied in the same framework, and some bounds are also obtained for this case. We conclude in §4 with some general conclusions on the robustness of adaptive systems.

2. Mathematical Preliminaries

2.1. Notations and General Assumptions

We consider the differential equations:

$$\dot{x} = f(x,t,u) \tag{2.1}$$

and:

$$\dot{x} = f(x,t,0) \tag{2.2}$$

where $x \in R^n$, $t \geq 0$, and $u \in R^m$. In the sequel, we will refer to (2.1) as the *perturbed* system, and to (2.2) as the *unperturbed* system. We assume the existence of some closed balls B_h and B_c (in R^n and R^m respectively), in which the following assumptions hold:

(A1) $x = 0$ is a stable equilibrium point of the unperturbed system, i.e. $f(0,t,0) = 0$ for all $t \geq 0$, and there exists $h' > 0$ such that, for all $x_0 \in B_{h'}$, $t_0 \geq 0$, and $t \geq t_0$, $x(t) \in B_h$ along the solutions of (2.2) starting at x_0.

(A2) the function f has continuous and bounded first order partial derivatives in x and u, and is piecewise continuous in t, for all $x \in B_h$, $t \geq 0$, $u \in B_c$. We define:

$$A_{ij}(x,t) := \frac{\partial f_i(x,t,u)}{\partial x_j} \tag{2.3}$$

and:

$$k_h := \sup_{x \in B_h, t \geq 0} \left(\|A(x,t)\|_\infty \right) \tag{2.4}$$

where:

$$\|A\|_\infty := \max_i \left(\sum_{j=1}^{n} |A_{ij}| \right) \qquad (2.5)$$

Assumption (A2) implies that f satisfies a Lipschitz condition, i.e. that there exists $L_{hc}>0$ such that, for all $x,x' \in B_h$, $u,u' \in B_c$, $t \ge 0$:

$$\|f(x,t,u)-f(x',t,u')\| \le L_{hc} \left(\|x-x'\|+\|u-u'\| \right) \qquad (2.6)$$

This also implies the existence, and uniqueness, of the solutions of (2.1) and (2.2), as long as they remain in B_h, B_c. For $x_0 \in B_{h'}$, and $t_0 \ge 0$, the solution of (2.2) exists for all $t \ge t_0$, and we will denote it $\underset{\sim}{x}(t,x_0,t_0)$.

We denote by $\|x(t)\|$ the euclidean norm of the vector x at time t, while:

$$\|x(.)\| := \sup_{t \ge 0} \|x(t)\| \qquad (2.7).$$

2.2. Theorem 1 (Converse Theorem of Lyapunov)

Consider the system (2.2), with assumptions (A1) and (A2). Then, the following statements are equivalent:

(a) $x=0$ is an *exponentially stable* equilibrium point of the unperturbed system, i.e. there exist α, $M>0$, such that, for all $x_0 \in B_{h'}$, $t_0 \ge 0$, $t \ge t_0$:

$$\|x(t)\| \le M \|x_0\| e^{-\alpha(t-t_0)} \qquad (2.8)$$

along the solutions of (2.2).

(b) there exists a function $v(x,t)$, and some constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$, such that, for all $x \in B_{h'}$, $t \ge 0$:

$$\alpha_1 \|x\|^2 \le v(x,t) \le \alpha_2 \|x\|^2 \qquad (2.9)$$

$$\frac{dv(x,t)}{dt} \le -\alpha_3 \|x\|^2 \qquad (2.10)$$

$$\left| \frac{\partial v(x,t)}{\partial x_i} \right| \le \alpha_4 \|x\| . \qquad (2.11)$$

where $i=1,\ldots,n$.

Comments

Theorem 1 can be found in Krasovskii (1963), and Hahn (1967). It is known as one of the converse theorems. The proof of the theorem is constructive: it provides an explicit Lyapunov function $v(x,t)$. The proof presented here is similar to the proof in Hahn (1967), with the difference that we derive explicit values of the constants involved in (2.9)-(2.11) to obtain some interpretation of the results of the theorem in §2.3.

The derivative in (2.10) is a *total* derivative, or derivative taken along the trajectories of (2.2), i.e.:

$$\frac{dv(x,t)}{dt} = \frac{\partial v(x,t)}{\partial t} + \sum_{i=1}^{n} \frac{\partial v(x,t)}{\partial x_i} f_i(x,t,0) \tag{2.12}$$

Proof of Theorem 1

(a) implies (b)

(i) Define:

$$v(x,t) = \int_t^{t+T} \|p(\tau,x,t)\|^2 d\tau \tag{2.13}$$

where $T>0$ will be defined in (ii). From the exponential stability:

$$\|p(\tau,x,t)\| \le M\|x\| e^{-\alpha(\tau-t)} \tag{2.14}$$

The Lipschitz condition implies:

$$\|f(x,t,0)\| \le L_{hc}\|x\| \tag{2.15}$$

so that:

$$\|p(\tau,x,t)\| \ge \|x\| e^{-nL_{hc}(\tau-t)} \tag{2.16}$$

Inequality (2.9) follows with:

$$\alpha_1 := \left(1-e^{-2nL_{hc}T}\right) / \, 2nL_{hc} \tag{2.17}$$

$$\alpha_2 := M^2\left(1-e^{-2\alpha T}\right) / \, 2\alpha \tag{2.18}$$

(ii) Differentiating (2.13) with respect to t, we obtain:

$$\frac{dv(x,t)}{dt} = \|p(t+T,x,t)\|^2 - \|p(t,x,t)\|^2 + \int_t^{t+T} \frac{d}{dt}\left(\|p(\tau,x,t)\|^2\right) d\tau \tag{2.19}$$

Note that $\frac{d}{dt}$ is a total derivative with respect to the *initial* time t, and by definition of the solution p:

$$p(\tau,x(t+\Delta t),t+\Delta t) = p(\tau,x(t),t) \tag{2.20}$$

so that the term in the integral is identically zero over $[t,t+T]$. The second term in the right-hand side of (2.19) is simply $\|x\|^2$, while the first is related to $\|x\|^2$ by the assumption of exponential stability. It follows that:

$$\frac{dv(x,t)}{dt} \le -\left(1-M^2e^{-2\alpha T}\right)\|x\|^2 \tag{2.21}$$

Inequality (2.10) follows provided that::

$$T > \frac{1}{\alpha}\ln M \tag{2.22}$$

and:

$$\alpha_3 := 1 - M^2 e^{-2\alpha T} \tag{2.23}$$

(iii) Differentiating (2.13) with respect to x_i, we have:

$$\frac{\partial v(x,t)}{\partial x_i} = 2 \int_t^{t+T} \sum_{j=1}^{n} p_j(\tau,x,t) \frac{\partial p_j(\tau,x,t)}{\partial x_i} d\tau \tag{2.24}$$

From the exponential stability:

$$|\, p_j(\tau,x,t) \,| \leq M \|x\| e^{-\alpha(\tau-t)} \tag{2.25}$$

while, under the assumptions,

$$\begin{aligned} \frac{d}{d\tau}\left(\frac{\partial p_j(\tau,x,t)}{\partial x_i}\right) &= \frac{\partial}{\partial x_i}\left(\frac{d}{d\tau} p_j(\tau,x,t)\right) \\ &= \frac{\partial}{\partial x_i}\left(f_j(p(\tau,x,t),\tau,0)\right) \\ &= \sum_{k=1}^{n} \left.\frac{\partial f_j}{\partial x_k}\right|_{p(\tau,x,t),\tau,0} \cdot \frac{\partial p_k(\tau,x,t)}{\partial x_i} \end{aligned} \tag{2.26}$$

(except possibly at points of discontinuity of $f(.,\tau,0)$).

Denoting:

$$Q_{ij}(\tau,x,t) = \left(\frac{\partial p_i(\tau,x,t)}{\partial x_j}\right) \tag{2.27}$$

equation (2.26) becomes the following differential equation:

$$\frac{d}{d\tau}(Q(\tau,x,t)) = A(p(\tau,x,t),\tau).\,Q(\tau,x,t) \tag{2.28}$$

This equation defines $Q(\tau,x,t)$, when integrated from $\tau=t$ to $\tau=t+T$, with initial conditions:

$$Q(t,x,t) = I \tag{2.29}$$

In fact, $Q(\tau,x,t)$ is the *transition matrix* associated with the time varying matrix $A(p(\tau,x,t),\tau)$. It follows that (Vidyasagar (1978)):

$$\|Q(\tau,x,t)\|_\infty \leq e^{k_h(\tau-t)} \tag{2.30}$$

and:

$$\left|\frac{\partial v(x,t)}{\partial x_i}\right| \leq 2 \int_t^{t+T} M \|x\| e^{(k_h-\alpha)(\tau-t)} d\tau \tag{2.31}$$

which is (2.11), defining:

$$\alpha_4 = 2M \left(e^{(k_h-\alpha)T} - 1\right) / \, (k_h - \alpha) \tag{2.32}$$

Provided that $x \in B_{h'}$, the trajectories in the steps of the proof remain in B_h, so that all assumptions are valid. Note that one can simply define:

$$h' = h / M \tag{2.33}$$

(b) implies (a)

This direction is straightforward, using (2.9) and (2.10):

$$M := \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{2}} \tag{2.34}$$

$$\alpha := \frac{1}{2}\frac{\alpha_3}{\alpha_2} \tag{2.35}$$

In this case, we can also define:

$$h' = h\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{2}} \tag{2.36}$$

2.3. Theorem 2 (Small Signal I/O Stability)

Consider the perturbed and unperturbed systems (2.1) and (2.2), with assumptions (A1) and (A2). Then:

(a) $x=0$ is an *exponentially stable* equilibrium point of the unperturbed system

implies:

(b) the perturbed system is *small-signal L_∞-stable* , i.e. there exist γ_∞, $c_\infty>0$, such that $\|u(.)\| \le c_\infty$ implies:

$$\|x(t)\| \le \gamma_\infty \|u(.)\| \tag{2.37}$$

for all $t \ge 0$, along the solutions of (2.1) starting at $x_0=0$;

(c) moreover, there exists $h_0>0$ such that, for all $x_0 \in B_{h_0}$, $\|u(.)\| \le c_\infty$ implies that $x(t)$ converges to a B_δ ball of radius $\delta = \gamma_\infty \|u(.)\|$. More precisely, for all $\varepsilon>0$, there exists $T \ge 0$ such that:

$$\|x(t)\| \le (1+\varepsilon)\delta \tag{2.38}$$

for all $t \ge T$, along the solutions of (2.1) starting at x_0.

Comments

Part (b) of theorem 2 is a direct extension of theorem 1 of Vidyasagar & Vannelli (1982) (see also Hill & Moylan (1980)) to the non autonomous case. Part (c) is new and applies to non zero initial conditions.

Theorem 2 relates *internal* exponential stability to *external* input/output stability (the output is here identified with the state). Although lack of exponential stability does not imply input/output instability, it is known that simple stability, and even (non uniform) asymptotic stability are *not* sufficient conditions to guarantee I/O stability (see e.g. Kalman & Bertram (1960) Ex.5 p 379). As a matter of fact, the proof of theorem 2 relies strongly on the assumption of exponential stability of the original system. In conclusion, one would be tempted to consider, for all *practical* purposes, the exponential stability, or at least the uniform asymptotic stability, a necessary condition for input-output stability.

Proof of Theorem 2

The differential equation (2.2) satisfies the conditions of theorem 1, so that there exists a (Lyapunov) function $v(x,t)$ satisfying the inequalities (2.9)-(2.11). If we consider the same function to study the differential equation (2.1), the inequalities (2.9) and (2.11) still hold, while (2.10) is modified, since the derivative is now to be taken along the trajectories of (2.1), instead of (2.2). The two derivatives are, however, related through equation (2.12), and:

$$\left.\frac{dv(x,t)}{dt}\right|_{(2.1)} = \frac{\partial v(x,t)}{\partial t}+\sum_{i=1}^{n}\frac{\partial v(x,t)}{\partial x_i}f_i(x,t,u(t))$$

$$= \left.\frac{dv(x,t)}{dt}\right|_{(2.2)}+\sum_{i=1}^{n}\frac{\partial v(x,t)}{\partial x_i}\Big[f_i(x,t,u(t))-f_i(x,t,0)\Big] \tag{2.39}$$

Using the results of theorem 1, and the Lipschitz condition:

$$\left.\frac{dv(x,t)}{dt}\right|_{(2.1)} \le -\alpha_3\|x\|^2+\alpha_4\|x\|\sqrt{n}L_{hc}\|u(.)\| \tag{2.40}$$

Assume a bounded input $u(.)$, and define:

$$\gamma_\infty := \frac{\alpha_4}{\alpha_3}\sqrt{n}\ L_{hc}\left(\frac{\alpha_2}{\alpha_1}\right)^{1/2} \tag{2.41}$$

$$\delta := \gamma_\infty\|u(.)\| \tag{2.42}$$

$$\delta' := \left(\frac{\alpha_1}{\alpha_2}\right)^{1/2}\delta = \frac{\alpha_4}{\alpha_3}\sqrt{n}L_{hc}\|u(.)\| \tag{2.43}$$

The inequality (2.40) can now be written:

$$\left.\frac{dv(x,t)}{dt}\right|_{(2.1)} \le -\alpha_3\|x\|(\|x\|-\delta') \tag{2.44}$$

This inequality is the basis of the proof:

(a) implies (b)

Consider the situation when $x_0 \in B_{\delta'}$ (this is true in particular if $x_0=0$). We show that this implies that $x \in B_\delta$ for all $t \ge 0$ (note that $\delta' \le \delta$).

Suppose it was not true. Then, by continuity of the solutions, there would exist $T_0, T_1 (T_1 > T_0 \geq 0)$, such that: $\|x(T_0)\| = \delta'$, $\|x(T_1)\| > \delta$, and for all $t \in [T_0, T_1]$: $\|x(t)\| \geq \delta'$. Consequently, inequality (2.44) shows that, in $[T_0, T_1]$, $dv/dt \leq 0$. However, this contradicts the fact that $v(x(T_0), T_0) \leq \alpha_2 \delta'^2 = \alpha_1 \delta^2$, and $V(x(T_1), T_1) > \alpha_1 \delta^2$.

(a) implies (c)

Assume now that $\|x_0\| > \delta'$. We show the result in two steps.

(i) for all $\varepsilon > 0$, there exists $T \geq 0$ such that $\|x(T)\| = \delta'(1+\varepsilon)$.

Suppose it was not true. Then, for all $t \geq 0$, $\|x(t)\| > \delta'(1+\varepsilon)$ and $dv/dt < -\alpha_3 \delta'(1+\varepsilon)\delta'\varepsilon$, which is a strictly negative constant. However, this contradicts the fact $v(x_0, 0) \leq \alpha_2 \|x_0\|^2$, and $v(x(t), t) > \alpha_1 \delta'^2 (1+\varepsilon)^2$ for all $t \geq 0$.

(ii) for all $t \geq T$, $\|x(t)\| \leq \delta(1+\varepsilon)$. This follows directly from (i), using an argument identical to the one used to prove (b).

Finally, recall that the assumptions require that $x(t) \in B_h$, $u(t) \in B_c$. It is easily seen, from the results, that this is guaranteed, provided that $x_0 \in B_{h_0}$ and $\|u(.)\| \leq c_\infty$, where:

$$h_0 = h' \left(\frac{\alpha_1}{\alpha_2} \right)^{1/2} \tag{2.45}$$

and:

$$c_\infty := min(c, h'/\gamma_\infty) \tag{2.46}$$

Note that although the first part of the proof is, in itself, a result for non zero initial conditions, the size of the ball $B_{\delta'}$ involved decreases with the amplitude of the input, while the size of B_{h_0} is independent of it (and, actually $h_0 \geq \delta'$, when $\|u(.)\| \leq c_\infty$).

Additional Comments

a) If assumptions (A1) and (A2) are valid *globally*, then the results are valid globally too. The system remains stable, and has finite I/O gain, independent of the size of the input *. In the example of §3, and for a wide category of nonlinear systems (bilinear systems for example), the Lipschitz condition is not verified globally. Yet, given *any* balls B_h, B_c, the system satisfies a Lipschitz condition with constant L_{hc} depending on the size of the balls (actually increasing with it). The balls B_h, B_c are consequently arbitrary, but the values of γ_∞ (the L_∞ gain) and c_∞ (the stability bound) will vary with them. In general, it can be expected that, for all h, $c \in R_+$, c_∞ will remain bounded, so that, despite the freedom left in the choice of h and c, the I/O stability will only be local.

* This conclusion is not true if the system is only globally uniformly asymptotically stable (see e.g. Desoer et al (1965)).

b) *Explicit* values of γ_∞ and c_∞ can be obtained from parameters of the differential equation, using equations (2.41) and (2.46). Needless to say, the dependence of γ_∞ and c_∞ on these parameters is rather complex. However, it can be verified that, with other parameters remaining identical, the L_∞ gain is decreased, and the stability margin is increased, *when the rate of exponential convergence α is increased, or the "overshoot" M is decreased.*

3. Robustness of Adaptive Algorithms

3.1. Exponential Convergence

For the purpose of illustration, we consider a simple, continuous time, model reference adaptive control algorithm, as found in Narendra and Valavani (1978).

The plant is described by:

$$\dot{x}_p = A_p x_p + b_p u_p \tag{3.1}$$

$$y_p = h_p^T x_p \tag{3.2}$$

where $x_p, b_p, h_p \in R^n, A_p \in R^{n \times n}, u_p, y_p \in R$.

(B1) Assume that the transfer function:

$$W_p(s) = h_p^T (sI - A_p)^{-1} b_p = k_p\, n_p(s) / d_p(s) \tag{3.3}$$

has relative degree 1, and is minimum phase. Assume that $d_p(s)$ and $n_p(s)$ are monic polynomials of degrees n and $n-1$ respectively, and that the sign of k_p is known.

The state of the plant is obtained from the observers:

$$\dot{v}^{(1)} = \Lambda v^{(1)} + b u_p \tag{3.4}$$

$$\dot{v}^{(2)} = \Lambda v^{(2)} + b y_p \tag{3.5}$$

where $v^{(1)}$, $v^{(2)}$, $b \in R^{n-1}$, $\Lambda \in R^{n-1 \times n-1}$, Λ is exponentially stable, and Λ, b is a controllable pair. The controller is a state feedback controller:

$$u_p = c_0 r + c^T v^{(1)} + d_0 y_p + d^T v^{(2)} := \vartheta^T w \tag{3.6}$$

with $c_0, d_0 \in R$, $c, d \in R^{n-1}$, $\vartheta, w \in R^{2n}$. The control objective is to find a control law such that y_p approaches the output y_m of a model reference system, whose transfer function:

$$W_m(s) = k_m n_m(s) / d_m(s) \tag{3.7}$$

is strictly positive real, with monic polynomials $n_m(s)$ and $d_m(s)$ having the same degrees as the corresponding plant polynomials, and with k_m having the same sign as k_p.

From assumption (B1), it can be shown (Narendra & Valavani (1978)) that there exist unique c_0^*, c^*, d_0^*, d^* such that the closed-loop system described by equations (3.1)-(3.6) has a transfer function equal to $W_m(s)$. For *the purpose of the analysis* the model is described by the same equations ((3.1)-(3.6)) with x_m, u_m, y_m, $v_m^{(1)}$, $v_m^{(2)}$, c_0^*, c^*, d_0^*, d^*, ϑ^*, w_m replacing the corresponding plant variables. This is represented in Fig. 1, where the upper block represents the model, and the lower block the plant, together with the controller. Note that Fig. 1 includes disturbances $d_1(t), d_2(t)$ which will be considered in §2, but are assumed zero at present. In practice, the model is simply realized by the transfer function $W_m(s)$, but the representation of Fig. 1 is convenient for the subsequent analysis.

(B2) Assume that the reference input r is bounded, and piecewise continuous.

Consequently, all signals in the model reference block are bounded, and the plant variables can be alternatively described by their differences with the model reference variables, by defining the errors:

$$e := \begin{bmatrix} x_p - x_m \\ v^{(1)} - v_m^{(1)} \\ v^{(2)} - v_m^{(2)} \end{bmatrix} \tag{3.8}$$

$$\varphi := \vartheta - \vartheta^* \tag{3.9}$$

Simple algebraic manipulations lead to:

$$\dot{e} = A_c e + b_c (\varphi^T w) \tag{3.10}$$

$$e_1 := y_p - y_m = h_c^T e \tag{3.11}$$

where:

$$A_c := \begin{bmatrix} A_p + b_p d_0^* h_p^T & b_p c^{*T} & b_p d^{*T} \\ b d_0^* h_p^T & \Lambda + b c^{*T} & b d^{*T} \\ b h_p^T & 0 & \Lambda \end{bmatrix} \tag{3.12}$$

$$b_c := \begin{bmatrix} b_p \\ b \\ 0 \end{bmatrix} \tag{3.13}$$

$$h_c^T := (h_p^T, 0, 0) \tag{3.14}$$

and $e, b_c, h_c \in R^{3n-2}, A_c \in R^{3n-2 \times 3n-2}, \varphi \in R^{2n}$. From the definition of c_0^*, c^*, d_0^*, d^*:

$$h_c^T (sI - A_c)^{-1} b_c = \frac{k_p}{k_m} W_m(s) \tag{3.15}$$

The realization A_c, b_c, h_c is not completely observable. Actually $2n-2$ modes are the modes corresponding to the observer. They are the modes of Λ and of $\Lambda + bc^{*T}$ (these are located at the zeroes of $n_m(s)$). The $2n-2$ modes are controllable, exponentially stable, but not observable. In other words, A_c, b_c, h_c is a non minimal, yet stabilizable and detectable representation of a strictly positive real transfer function. The positive real lemma (Anderson & Vongpanitlerd (1973)) implies that there exist symmetric positive definite matrices $P, Q \in R^{3n-2 \times 3n-2}$ such that:

$$A_c^T P + P A_c = -Q \tag{3.16}$$

$$P b_c = h_c \tag{3.17}$$

From these considerations, the update law is chosen as:

$$\dot{\vartheta} = -e_1 w \tag{3.18}$$

so that the system is described by (3.10), together with:

$$\dot{\varphi} = -h_c^T e \, w \tag{3.19}$$

Choosing a Lyapunov function candidate:

$$V = e^T P e + \varphi^T \varphi \tag{3.20}$$

results in:

$$\dot{V} = -e^T Q e \tag{3.21}$$

This implies that e, φ are bounded functions of time, and, moreover, that $\lim_{t \to \infty} e(t) = 0$. One can show (Morgan & Narendra (1977), Anderson (1977)), that e, φ converge to 0 exponentially fast, provided that w is persistently exciting, i.e. that there exist positive constants $\delta, k_1 > 0$ such that, for all $s \geq 0$:

$$\int_s^{s+\delta} w(\tau) w(\tau)^T d\tau \geq k_1 I \tag{3.22}$$

For (e, φ) in a ball B_h, this was shown by Boyd & Sastry (1984) to be guaranteed provided that the same condition holds for w_m, or, equivalently, provided that the support of the spectrum of r has at least $2n$ points (i.e. r is sufficiently rich).

Important Remarks:

(i) The persistency of excitation condition is not only important in guaranteeing exponential stability, but also in guaranteeing the *parameter convergence* to the nominal values, and the existence of a *unique* equilibrium point $(e,\varphi)=0$ to the differential equation (3.10),(3.19).

(ii) The description of the system (3.10),(3.19) is misleading, since the right-hand side contains the signal w, which depends on the state e. Actually, this makes the system appear to be linear time varying, although it is truly described by nonlinear differential equations. This can lead to some "circular" arguments when studying the robustness of the algorithm (as pointed out in Narendra & Annaswamy (1984)). To avoid this, we replace w by:

$$w = W e + w_m \tag{3.23}$$

where:

$$W := \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ h_p^T & 0 & 0 \\ 0 & 0 & I \end{bmatrix} . \tag{3.24}$$

with $W \in R^{2n \times 3n-2}$ (a constant matrix). The system description becomes:

$$\dot{e}(t) = A_c e(t) + b_c \varphi^T(t) We(t) + b_c \varphi^T(t) w_m(t) \tag{3.25}$$

$$\dot{\varphi}(t) = -h_c^T e(t) We(t) - h_c^T e(t) w_m(t) \tag{3.26}$$

where we indicated explicitely the time dependence. This can be written:

$$\dot{x} = f(x,t) \tag{3.27}$$

where:

$$x := \begin{bmatrix} e \\ \varphi \end{bmatrix} \in R^{5n-2} \tag{3.28}$$

It is a nonlinear ordinary differential equation (actually, it is bilinear). The time variation of the coefficients in (3.25)-(3.26) is *explicit* since it is obtained only from $w_m(t)$ an exogeneous, bounded function of time, obtained from the model.

3.2. Robustness to Disturbances

In this section we consider three types of disturbances, and show that, in each case, the new differential equation describing the system can be written as:

$$\dot{x} = f(x,t) + p_1(t) + P_2(t)x(t) \tag{3.29}$$

where $p_1(t) \in R^{6n-2}$, and $P_2(t) \in R^{6n-2 \times 6n-2}$.

a) Input Disturbances $d_1(t)$

Let $u_p(t)$ be replaced by $u_p(t)+d_1(t)$ (cf Fig. 1). Equation (3.29) follows with:

$$p_1(t) = \begin{bmatrix} b_o d_1(t) \\ 0 \end{bmatrix}, P_2(t) = 0 \tag{3.30}$$

b) Plant Parameter Variation $\dot{\vartheta}^*(t)$

Let ϑ^* be time dependent, with $\dot{\vartheta}^*$ bounded, so that $\dot{\varphi}(t) = \dot{\vartheta}(t) - \dot{\vartheta}^*(t)$. Equation (3.29) follows with:

$$p_1(t) = \begin{bmatrix} 0 \\ -\dot{\vartheta}^*(t) \end{bmatrix}, P_2(t) = 0 \tag{3.31}$$

c) Output Disturbances $d_2(t)$

Output disturbances $d_2(t)$, as in Fig. 1, have two effects:

- first, on the input of the plant: u_p becomes $u_p + q^T \vartheta d_2$, where q is a constant vector defined by:

$$q^T = (0, 0, 1, 0) \tag{3.32}$$

with $q \in R \times R^{n-1} \times R \times R^{n-1} = R^{2n}$.

- second, on the parameter update: $h_c^T e$ becomes $h_c^T e + d_2$ and We becomes $We + qd_2$. Some manipulations lead again to (3.29) with:

$$p_1(t) = \begin{bmatrix} b_c q^T \vartheta^* d_2(t) \\ -q d_2^2(t) - w_m(t) d_2(t) \end{bmatrix} \tag{3.33}$$

$$P_2(t) = \begin{bmatrix} 0 & b_c q^T \\ -W - q h_c^T & 0 \end{bmatrix} d_2(t) \tag{3.34}$$

For $x \in B_h$, and $w_m(t)$ bounded (and piecewise continuous), equation (3.29) (together with the expressions for $p_1(t), P_2(t)$) is a particular form of equation

(2.1) in §2. Moreover, assumptions (A1) and (A2) are verified, so that, *under persistent excitation conditions*, theorem 2 is applicable.

From this, we draw the following conclusions:

a) specific bounds on $\|d_1(.)\|$, $\|\dot{\vartheta}^*(.)\|$, or $\|d_2(.)\|$ can be obtained such that, within these bounds, and provided the initial error is sufficiently small, *the stability of the adaptive system will be preserved.* Note, however, that for different disturbances acting together, the analysis should be slightly modified, since their effect is *not* additive;

b) the deviations from equilibrium are locally *at most proportional* to the disturbances (in terms of L_∞ norms), and their bounds can be made arbitrarily small by reducing the bounds on the disturbances;

c) the L_∞ gain from the disturbances to the deviations from equilibrium can be reduced by *increasing the rate of exponential convergence of the unperturbed system.*

Although equation (3.29) is specific to our example, most adaptive systems will be described by equations which are included in the general framework of §2, and satisfy assumptions (A1) and (A2).

Design Guidelines

The determination of the constants M and α from design parameters is, unfortunately, very difficult. Kreisselmeier and Joos (1982) presented a scheme similar to the one presented here, and obtained specific rates of convergence (around the equilibrium point). It appears, in their work, that the rate of convergence is proportional to the constant k_1 in the persistent excitation condition (3.22), describing the *level* of persistent excitation (the rate is, however, bounded by the slowest time constant of the model reference). Consequently, the robustness can be increased directly by this means.

In the presence of output disturbances, the dependence has to be analyzed more carefully, since the "constant" L_{hc} will vary with $\|w_m(.)\|$, and the total effect will be different. In other words, *arbitrarily fast adaptive control algorithms are not necessarily arbitrarily robust.* We expect this to be true especially when the rate of convergence is increased through an internal adaptation parameter, for example by replacing the adaptation law (3.18) by:

$$\dot{\vartheta} = -\gamma e_1 w \qquad (3.35)$$

with $\gamma>0$. Although the rate of convergence α may be increased to some extent by increasing γ, the parameter L_{hc} will be directly multiplied by γ, resulting in possibly smaller robustness margins.

It appears then, that increasing the level of persistent excitation of the system, is the best way to increase the stability margins of the adaptive system.

Rohrs et al (1982) example (in §4.2) of instability of an adaptive scheme with output disturbances on a non persistently excited system, is an example of instability when the persistent excitation condition of the nominal system is not satisfied.

3.3. Robustness with Respect to Unmodelled Dynamics

The approach we adopt here is similar to that used by Doyle & Stein (1981) to study the robustness of non adaptive control systems. We assume that there exists a nominal plant $W_p(s)$, satisfying the assumptions (B1) in §3.1, and such that:

$$y_p^*(s) = W_p(s)u_p(s) \tag{3.36}$$

The actual output is modelled as the output of the nominal plant, plus some additive uncertainty represented by a bounded operator H_a:

$$y_p(t) = y_p^*(t) + H_a(u_p)(t) \tag{3.37}$$

The operator $H_a(u_p)$ represents the difference between the real plant, and the idealized plant $W_p(s)$. In the terminology of Doyle & Stein (1981), we refer to it as an *unstructured uncertainty*, and it constitutes all the uncertainty, since it is the purpose of the adaptive scheme to reduce to zero the *structured* or *parametric* uncertainty.

From the results of §3.2, it follows that there exists $c_1 \geq 0$ such that, for *any* output disturbance $d_2(t)$ satisfying $\|d_2(.)\| \leq c_1$ the system will remain stable. In particular, all internal signals will remain bounded, and there exists $c_2>0$ such that $\|u_p(.)\| \leq c_2$. Define:

$$\|H_a\|_{c_2} := \frac{1}{c_2} \sup_{\|u_p(.)\| \leq c_2} \left(\|H_a(u_p)(.)\| \right) \tag{3.38}$$

The perturbed system will remain stable provided:

$$\|H_a\|_{c_2} \leq c_1 / c_2 \tag{3.39}$$

Condition (3.39) is very general, since it includes possible nonlinearities, unmodelled dynamics,etc. provided that they can be modelled as additive, bounded-input bounded-output operators.

If the operator $H_a(u_p)$ is linear time-invariant, the condition is a condition on the L_∞ gain of H_a. One can use:

$$||H_a||_{c_2} = \int_0^\infty |h_a(\tau)| \, d\tau \tag{3.40}$$

where $h_a(\tau)$ is the impulse response of H_a (assuming zero initial conditions for the unmodelled dynamics).

Design Guidelines

Although (3.39) provides a specific condition to guarantee the stability of the adaptive system in the presence of unmodelled dynamics, it does not provide much insight into the frequency domain constraints on $H_a(j\omega)$, when the unmodelled dynamics are considered linear time invariant. In that case, a necessary condition for the transfer function $H_a(j\omega)$ to satisfy is:

$$|H_a(j\omega)| \leq c_1 / c_2 \tag{3.41}$$

for all ω>0. However, this condition is not sufficient.

As a precautionary step, however, it is useful to consider the robustness margins of the adaptive system after convergence of the parameters. Then, the adaptive control system becomes a linear time invariant controller, whose robustness margins can be studied with standard methods. Consider, for simplicity, the case of a stable plant $W_p(s)$. For the example used here, it can be shown (Sastry (1984)) that, using a multiplicative representation of the uncertainty (defining $L_m(j\omega)$ by $L_m(j\omega)W_p(j\omega)=L_a(j\omega)$), the *converged* system is guaranteed to remain stable provided that:

$$|L_m(j\omega)| < \left| \frac{d_m(j\omega)}{d_m(j\omega)-d_p(j\omega)} \right| \tag{3.42}$$

for all ω>0.

For the sake of performance enhancement, the zeroes of $d_m(s)$ are often located farther left of the complex plane, than those of $d_p(s)$. The zeroes of $d_m(s)-d_p(s)$ are located between these sets of zeroes (they are on the root-locus joining the zeroes of $d_m(s)$ to those of $d_p(s)$). The transfer function on the right-hand side of (3.42) is consequently usually as represented on Fig. 2. The robustness margins are seriously degraded in the frequency range between the dynamics of the model, and those of the plant. Actually, condition (3.42) is a constraint on the class of *allowable models*, and puts a limit on the bandwith of the reference model chosen.

The importance of the proper choice of the reference model should not be overlooked. In many methods, which are not considered model reference methods, a reference model is still implicitly defined. For example, some discrete time algorithms have the objective to track a reference model whose transfer function is a pure time delay z^{-d}. This may lead to high frequency closed-loop dynamics, and, from our previous discussion, we deduce that such designs will prone to be non robust.

Finally, it should be remembered that, under normal conditions, the output error is small, so that the input of the plant is roughly equal to:

$$u(j\omega) \sim \frac{W_m(j\omega)}{W_p(j\omega)}\, r(j\omega) \tag{3.43}$$

In the range of frequencies where $H_a(j\omega)$ is significant, $W_m(j\omega)/\, W_p(j\omega)$ is roughly constant (cf previous discussion), so that, in these frequencies:

$$u(j\omega) \sim \frac{k_m}{k_p}\, r(j\omega) \tag{3.44}$$

Clearly then, to minimize the output disturbance due to the unmodelled dynamics, *the spectrum of the reference input should be constrained to the bandwith of the model and of the plant*. Rohrs et al (1982) example (in §4.1) with sinusoidal reference inputs having a strong component in the frequency range of $H_a(j\omega)$, and a weak component in the frequency range of $W_m(j\omega)$, $W_p(j\omega)$, is an example of instability when this condition is violated.

In conclusion, some robustness properties are guaranteed, provided the system is persistently excited, at a sufficient level, and provided this condition is satisfied in the range of frequencies of the model and of the nominal plant.

4. Conclusions

We showed, on a simple example, that adaptive algorithms have some robustness margins, provided that they are exponentially stable. These robustness margins can be increased by increasing the rate of exponential convergence. This can usually be done by increasing the level of persistent excitation of the system through the reference input. We pointed out, however, that in the presence of unmodelled dynamics at higher frequencies, this should be achieved with a reference input whose spectrum is concentrated in the bandwith where these dynamics are small.

Although the results presented here give only sufficient conditions for robustness, there is sufficient evidence in the literature to be *suspicious of any scheme that is not exponentially convergent, or of implementations that are not persistently excited.* For example, the schemes presented by Morse (1980), and by Narendra et al (1980), for relative degree greater than 2, and unknown high frequency gain k_p can never be exponentially stable (cf Dasgupta et al (1983), Sastry (1984)). In fact, one should be suspicious of schemes that do not possess a unique equilibrium point (at least locally), a condition which is implied by exponential stability. In such cases, a typical mechanism of instability is the drift, along the equilibrium surface, from a locally stable equilibrium point to a locally unstable point, due to possibly very small disturbances. At this point, either a "burst" phenomenon occurs (cf Anderson (1983)), bringing the system back to a locally stable point, or instability results (Riedle & Kokotovic (1984)).

In practice, parameter convergence is an important condition for robustness. For this reason, persistent excitation should be maintained for as long as the controller parameters need to converge,or as long as the plant parameters vary. After convergence, one can simply turn-off the adaptation law when the excitation is insufficient (cf Astrom (1983)). A deadzone (Peterson & Narendra (1982), Sastry (1984)) is a simple way to indirectly achieve this. However, persistent excitation must be maintained when the parameters of the plant vary significantly. Other modifications of the update law, as proposed by Ioannou & Kokotovic in (1984), we feel do not really solve this problem, since they aim at limiting the drift of the parameters, which is an effect of the instability, and not the cause of it.

In conclusion, the design of robust adaptive control algorithms includes not only the choice of an efficient algorithm, but also the careful consideration of the reference input, and an adequate choice (explicit or implicit) of the reference model.

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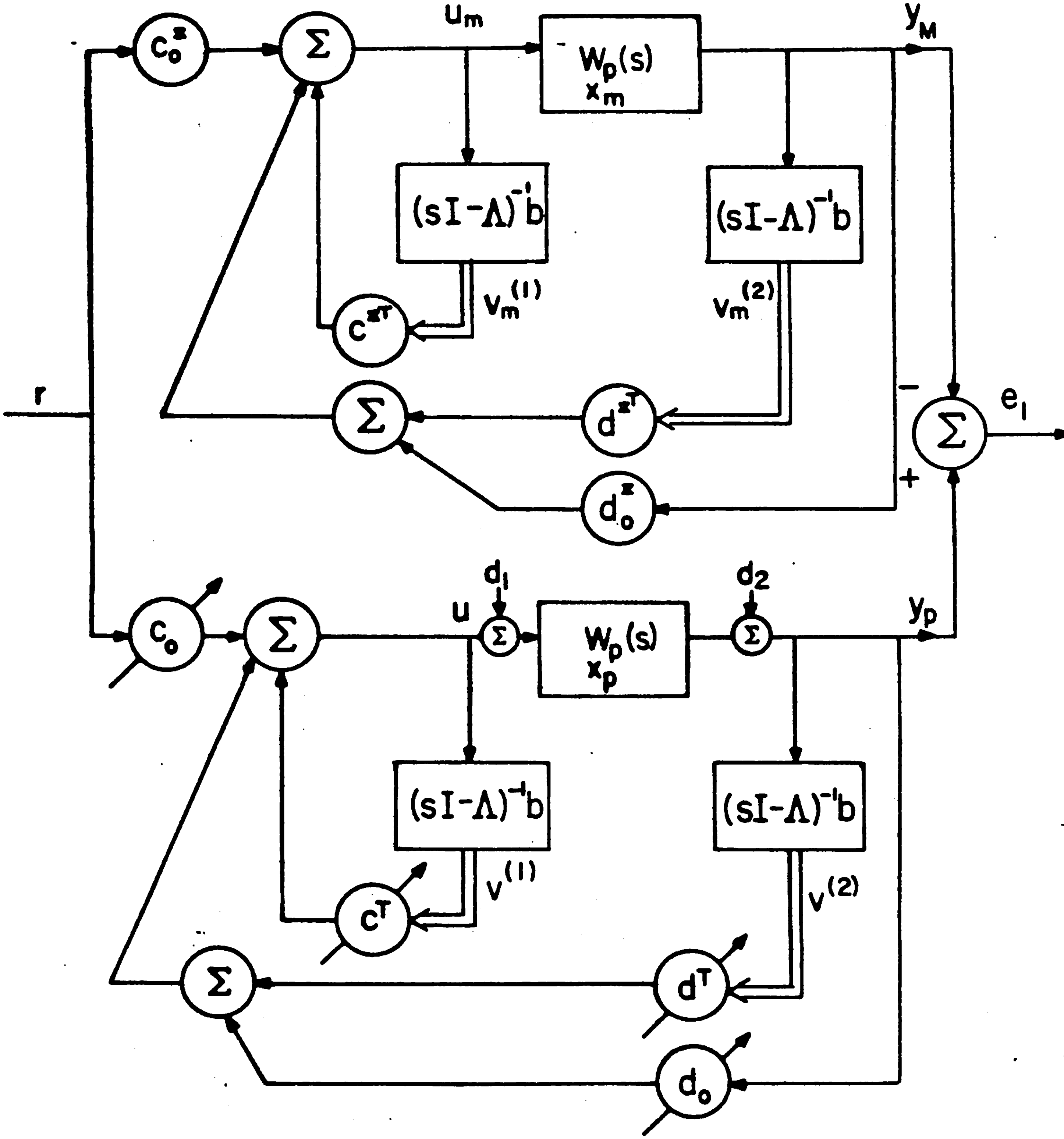
List of Figures



u_m
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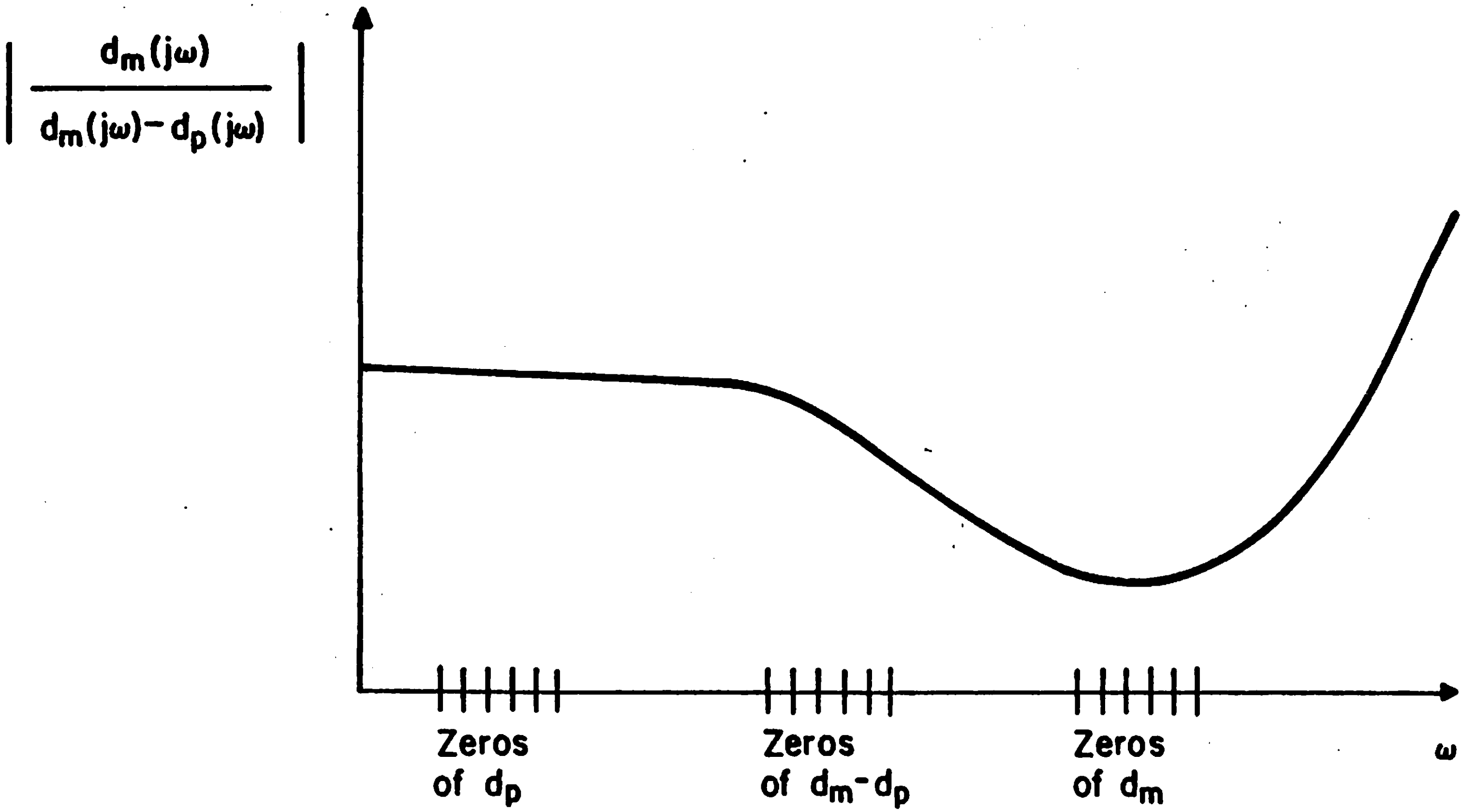

$\left| \frac{d_m(j\omega)}{d_m(j\omega) - d_p(j\omega)} \right|$
Zeros of d_p
Zeros of $d_m - d_p$
Zeros of d_m
ω