GEOMETRIC INTERPRETATION OF MANIPULATOR SINGULARITIES

by

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ABSTRACT

The study of singular configurations, i.e. those configurations where the Jacobian of the forward kinematic solution of a manipulator become singular, is important in understanding branching of solutions to the inverse kinematic equations, path planning, and trajectory generation problems. We present a geometric derivation of the cross product form of the Jacobian for manipulators. Using this cross product form of the Jacobian we classify some common singularities. In addition, we investigate the special case of manipulators having three consecutive revolute joints with intersecting axes.

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1. Introduction.

It is well known [1] that differential motions of a manipulator's gripper may be related through the Jacobian of the forward kinematic solution to differential motions of the joints. An interesting subject of study has been those configurations of six degree-of-freedom manipulators where the Jacobian is singular. Understanding these singular configurations is important for the following reasons.

(1) At singularities, bounded gripper velocities may produce unbounded joint velocities.

(2) Points on the boundary[1] of the manipulator's reachable space correspond to singular configurations.

(3) A technique commonly used [2] to plan bounded error, straight line paths in Cartesian space generates more knot points near singularities. This is due to the fact that the distance between knot points in *joint space* determines bounds on error in Cartesian space [3]. Schemes presented in [2] and [3] reduce distances between knot points in *Cartesian space* to improve tolerances - this is effective everywhere except near singularities where small distances in Cartesian space do not necessarily correspond to small distances in joint space.

(4) Points in the manipulator's workspace which are reachable only when the manipulator is in a singular configuration may become unreachable under perturbation of link parameters. Since the position and orientation of a manipulator's gripper are given by a continuously differentiable function of the joint variables and the link parameters, the implicit function theorem guarantees solutions to the inverse kinematic problem under perturbation of link

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[1] Here we restrict "boundary" to mean those points on the boundary of the geometrical model's reachable space where mechanical limits are not an issue.

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The study of singular configurations, i.e. those configurations where the Jacobian of the forward kinematic solution of a manipulator become singular, is important in understanding branching of solutions to the inverse kinematic equations, path planning, and trajectory generation problems. We present a geometric derivation of the cross product form of the Jacobian for manipulators. Using this cross product form of the Jacobian we classify some common singularities. In addition, we investigate the special case of manipulators having three consecutive revolute joints with intersecting axes.

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parameters only where the Jacobian is nonsingular.

Manipulator singularities have received considerable attention. Whitney [1] presents the Jacobian in a cross product form similar to the one we use here. In [4] singularities of robot wrists are analyzed and working regions away from these singularities are defined. Screw calculus has been used in [5] to describe singularities and Luh [6] has used screw calculus to analyze redundant manipulators; in particular, he presents a method for avoiding singularities by taking advantage of redündancy. Since singularities occur at boundary points, they can been used to define the reachable workspace [7]. Finally, Litvin and Castelli have found singular configurations for a Cincinnati Milacron manipulator and the Unimation Puma manipulator in [8]. Additional related work appears in [9] and [10].

Our work focuses on the geometric interpretation of singularities for manipulators with six degrees-of-freedom. Several properties of a manipulator's configuration are independent of the coordinate system used to express them and depend only on the angles and distances between links and joints. The singularity of the Jacobian is such a property and this fact will be exploited to describe singularities geometrically.

The layout of this paper is as follows: In section 2 we outline the notation used for kinematics. Section 3 contains a derivation of the cross product form of the Jacobian, and section 4 gives some examples of configurations where the Jacobian is singular. Section 5 describes the decoupling of singularities which occurs in manipulators with three consecutive revolute joints with intersecting axes. Our conclusions are summarized in section 6.

2. Kinematic Preliminaries.

Consider the standard approach to manipulator kinematics using homogeneous transformations [9]. In this approach a coordinate system $(\tilde{o}_j, \underline{C}_j)$ consisting of a point[2] or origin $\tilde{o}_j$ and a coordinate frame $\underline{C}_j$ is attached to link j, $j \geq 0$, with the motion of the jth joint[3] being on or

[2] Our notational conventions are the following: Objects which are described relative to a fixed world coordinate system (typically the base of the manipulator) are marked - coordinate frames and vectors are underbarred and points have a tilde. Coordinate frames and matrices are represented by uppercase characters whereas points and vectors are represented by lowercase characters. Axes are represented by a point on the axis and a unit vector in the axis direction. Also, $\hat{x} \equiv [1\,0\,0]^T, \hat{y} \equiv [0\,1\,0]^T, \hat{z} \equiv [0\,0\,1]^T$.

about the axis $(\tilde{o}_j, \underline{C}_j \hat{z})$. The coordinate system $(\tilde{o}_{j+1}, \underline{C}_{j+1})$ may be expressed in terms of $(\tilde{o}_j, \underline{C}_j)$ by a homogeneous transformation as follows.

$$\begin{bmatrix} \underline{C}_{j+1} & \tilde{o}_{j+1} \\ 000 & 1 \end{bmatrix} = \begin{bmatrix} \underline{C}_j & \tilde{o}_j \\ 000 & 1 \end{bmatrix} \begin{bmatrix} Q_j & p_j \\ 000 & 1 \end{bmatrix} \tag{2.1}$$

where Q_j is a 3×3 orthogonal matrix with determinant equal to one, and p_j is a column vector. For convenience we decompose (2.1) into the recursions

$$\underline{C}_{j+1} = \underline{C}_j Q_j \tag{2.2}$$

$$\tilde{o}_{j+1} = \tilde{o}_j + \underline{C}_j p_j .$$

Now Q_j and p_j are functions of the joint variable θ_j, so define

$$R_j \equiv Q_j |_{\theta_j = 0} \tag{2.3}$$

$$\alpha_j \equiv p_j |_{\theta_j = 0}$$

Then for the case when joint j is revolute

$$Q_j = e^{\theta_j S(\hat{z})} R_j \tag{2.4}$$

$$p_j = e^{\theta_j S(\hat{z})} \alpha_j .$$

Here

$$S(a) \equiv \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{2.5}$$

so that $e^{\theta S(\hat{a})}$ is a rotation[4] about the unit vector $\hat{a}$ by theta. Also $S(a)b = a \times b$.

For the case when joint j is prismatic

$$Q_j = R_j \tag{2.6}$$

$$p_j = \alpha_j + \theta_j \hat{z}.$$

[3] Note that our numbering begins with 0 rather that 1 for joints and columns of the Jacobian.

[4] See[11] for a discussion of matrix exponentials applied to manipulators.

Thus, the recursions (2.2) become:

For the case when joint j is revolute

$$\underline{C}_{j+1} = \underline{C}_j e^{\theta_j S(\hat{z})} R_j \tag{2.7a}$$

$$\tilde{o}_{j+1} = \tilde{o}_j + \underline{C}_j e^{\theta_j S(\hat{z})} \alpha_j .$$

For the case when joint j is prismatic

$$\underline{C}_{j+1} = \underline{C}_j R_j \tag{2.7b}$$

$$\tilde{o}_{j+1} = \tilde{o}_j + \underline{C}_j (\alpha_j + \theta_j \hat{z}).$$

Example. Figure 1. depicts the Stanford Manipulator with all revolute joints set to their zero positions. The extension of the prismatic joint[5] is nonzero, however, and equal to d_2. Our notation for these schematic diagrams is the following: Revolute joints are represented by cylinders - line segments drawn to the side of a cylinder represent rigid connections and line segments drawn to an end represent connections with one revolute degree of freedom about the axis of the line segment. Prismatic joints are represented by rectangular blocks - line segments drawn to one of the larger faces represent rigid connections and line segments drawn to one of the smaller faces represent connections with one translational degree of freedom along the axis of the line segment. The base of a manipulator is marked by a series of short line segments. We abbreviate the joint axis directions by $\underline{z}_j \equiv \underline{C}_j \hat{z}$.

From the figure, (2.2), and (2.3), we have

$$R_0 = e^{-\frac{\pi}{2} S(\hat{z})} \qquad \alpha_0 = 0 \tag{2.8}$$

$$R_1 = e^{\frac{\pi}{2} S(\hat{z})} \qquad \alpha_1 = d_1 \hat{z}$$

$$R_2 = I \qquad \alpha_2 = 0$$

[5] Here we represent the extension of joint 2 by d_2 rather than θ_2.

$$R_3 = e^{-\frac{\pi}{2}S(\hat{x})} \qquad \alpha_3 = 0$$

$$R_4 = e^{\frac{\pi}{2}S(\hat{x})} \qquad \alpha_4 = 0$$

$$R_5 = I \qquad \alpha_5 = 0.$$

From (2.7) we have

$$\underline{C}_6 = \underline{C}_0 e^{\theta_0 S(\hat{z})} R_0 e^{\theta_1 S(\hat{z})} R_1 e^{\theta_3 S(\hat{z})} R_3 e^{\theta_4 S(\hat{z})} R_4 \tag{2.9}$$

$$\tilde{o}_6 = \tilde{o}_0 + \underline{C}_0 e^{\theta_0 S(\hat{z})} R_0 (e^{\theta_1 S(\hat{z})} (d_1 \hat{z} + R_1 d_2 \hat{z})).$$

The advantages of using the exponential form for rotations is that the angle and axis of rotation are explicit and that differentiation is trivial. We will exploit these to derive the cross product form of the Jacobian.

3. The Manipulator Jacobian.

The position and orientation of the gripper, for a six degree-of-freedom manipulator, is determined by the coordinate system $(\tilde{o}_6, \underline{C}_6)$ attached to the last link. The Jacobian of the manipulator is given by

$$\underline{J} \equiv \begin{bmatrix} \frac{\partial \tilde{o}_6}{\partial \theta_0} & \frac{\partial \tilde{o}_6}{\partial \theta_1} & \cdots & \frac{\partial \tilde{o}_6}{\partial \theta_5} \\ \frac{\partial \underline{C}_6}{\partial \theta_0} & \frac{\partial \underline{C}_6}{\partial \theta_1} & \cdots & \frac{\partial \underline{C}_6}{\partial \theta_5} \end{bmatrix}. \tag{3.1}$$

$\underline{J}$ is not a matrix, but the singularity of $\underline{J}$ is defined naturally in terms of the linear dependence of its "columns". Since $\underline{C}_6$ is an orthogonal matrix for all joint values the derivatives $\frac{\partial \underline{C}_6}{\partial \theta_i}$ are products of skew symmetric matrices and $\underline{C}_6$. The Jacobian may be represented by replacing these products by a vector of the off diagonal terms of the skew symmetric matrices. This will be done after the following proposition.

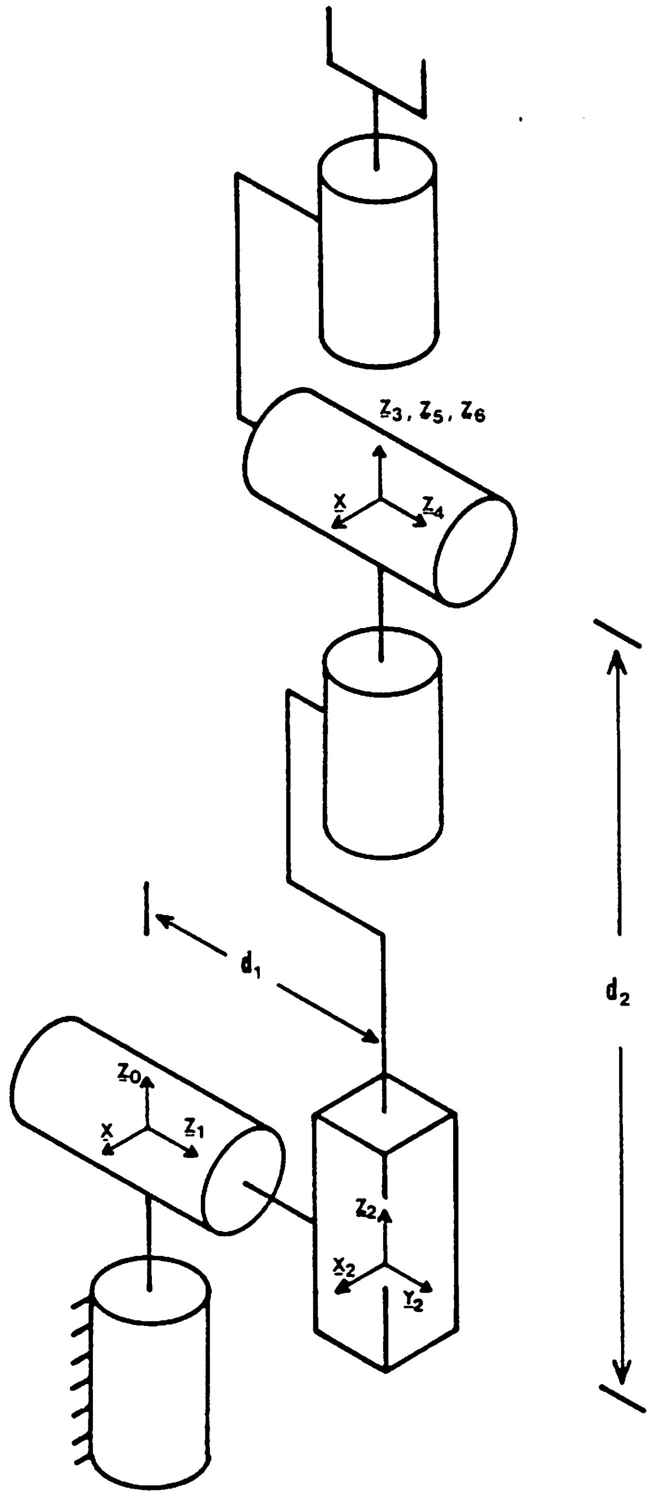

Figure 1. Stanford Manipulator.

Proposition. For a rigid link manipulator with n revolute or prismatic joints the following differential relationships hold.

For $0 \leq j \leq i < n$

$$\frac{\partial \tilde{o}_j}{\partial \theta_i} = 0$$

$$\frac{\partial \underline{C}_j}{\partial \theta_i} = 0. \tag{3.2}$$

For $0 \leq i < j \leq n$ and joint i revolute

$$\frac{\partial \tilde{o}_j}{\partial \theta_i} = \underline{z}_i \times (\tilde{o}_j - \tilde{o}_i)^*$$

$$\frac{\partial \underline{C}_j}{\partial \theta_i} = \underline{z}_i \times \underline{C}_j . \ddagger \tag{3.3}$$

For $0 \leq i < j \leq n$ and joint i prismatic

$$\frac{\partial \tilde{o}_j}{\partial \theta_i} = \underline{z}_i$$

$$\frac{\partial \underline{C}_j}{\partial \theta_i} = 0. \tag{3.4}$$

Proof:

We illustrate the computation for the case when joint i is revolute and $0 \leq i < j \leq n$. By induction:

Set $j = i+1$, then by (2.7a) we obtain

$$\frac{\partial \tilde{o}_{i+1}}{\partial \theta_i} = \underline{C}_i S(\hat{z}) e^{\theta_i S(\hat{z})} \alpha_i \tag{3.5}$$

* Recall $\underline{z}_i \equiv \underline{C}_i \hat{z}$.

‡ The cross product of a vector and a matrix, by notation, is a matrix whose columns are the cross product of the vector and the columns of the original matrix.

$$= [\underline{C}_i S(\hat{z}) \underline{C}_i^T] \underline{C}_i e^{\theta_i S(\hat{z})} \alpha_i .$$

Now by (2.7a) and the fact that $\underline{C}_i$ is orthogonal with determinant equal to one we have

$$\frac{\partial \widetilde{o}_{i+1}}{\partial \theta_i} = S(\underline{C}_i \hat{z})(\widetilde{o}_{i-1} - \widetilde{o}_i) . \tag{3.6}$$

$$= \underline{z}_i \times (\widetilde{o}_{i+1} - \widetilde{o}_i).$$

and

$$\frac{\partial \underline{C}_{i+1}}{\partial \theta_i} = \underline{C}_i S(\hat{z}) e^{\theta_i S(\hat{z})} R_i \tag{3.7}$$

$$= [\underline{C}_i S(\hat{z}) \underline{C}_i^T] \underline{C}_i e^{\theta_i S(\hat{z})} R_i$$

$$= \underline{z}_i \times \underline{C}_{i+1}.$$

So the result holds for $j = i+1$. For the induction step, assume that the formula is valid for $n > j > i \geq 0$. We need to show that the result holds for $j+1$ with joint j prismatic or revolute.

Case 1 Joint j revolute.

$$\frac{\partial \widetilde{o}_{j+1}}{\partial \theta_i} = \frac{\partial \widetilde{o}_j}{\partial \theta_i} + \frac{\partial \underline{C}_j}{\partial \theta_i} e^{\theta_j S(\hat{z})} \alpha_j \tag{3.8}$$

$$= \underline{z}_i \times (\widetilde{o}_j - \widetilde{o}_i) + \underline{z}_i \times \underline{C}_j e^{\theta_j S(\hat{z})} \alpha_j$$

$$= \underline{z}_i \times (\widetilde{o}_j - \widetilde{o}_i) + \underline{z}_i \times (\widetilde{o}_{j+1} - \widetilde{o}_j)$$

$$= \underline{z}_i \times (\widetilde{o}_{j+1} - \widetilde{o}_i).$$

$$\frac{\partial \underline{C}_{j+1}}{\partial \theta_i} = \frac{\partial \underline{C}_j}{\partial \theta_i} e^{\theta_j S(\hat{z})} R_j \tag{3.9}$$

$$= \underline{z}_i \times \underline{C}_j e^{\theta_j S(\hat{z})} R_j$$

$$= \underline{z}_i \times \underline{C}_{j+1}.$$

Case 2 Joint j prismatic.

$$\frac{\partial \widetilde{o}_{j+1}}{\partial \theta_i} = \frac{\partial \widetilde{o}_j}{\partial \theta_i} + \frac{\partial \underline{C}_j}{\partial \theta_i}(\alpha_j + \theta_j \hat{z}) \tag{3.10}$$

$$= \underline{z}_i \times (\widetilde{o}_j - \widetilde{o}_i) + \underline{z}_i \times \underline{C}_j(\alpha_j + \theta_j \hat{z})$$

$$= \underline{z}_i \times (\widetilde{o}_j - \widetilde{o}_i) + \underline{z}_i \times (\widetilde{o}_{j+1} - \widetilde{o}_j)$$

$$= \underline{z}_i \times (\widetilde{o}_{j+1} - \widetilde{o}_i).$$

$$\frac{\partial \underline{C}_{j+1}}{\partial \theta_i} = \frac{\partial \underline{C}_j}{\partial \theta_i} R_j \tag{3.11}$$

$$= \underline{z}_i \times \underline{C}_j R_j$$

$$= \underline{z}_i \times \underline{C}_{j+1}.$$

So we have, with joint i revolute and $0 \leq i < j \leq n$, that the result holds. The cases $i \geq j$, and joint i prismatic are procedural.

Let the ith column, $i \in \{0,1,...,5\}$, of the Jacobian $\underline{J}$ be denoted $\underline{J}_i$. Then from the proposition and the definition of $\underline{J}$ we have:

For joint i revolute

$$\underline{J}_i = \begin{bmatrix} \underline{z}_i \times (\widetilde{o}_6 - \widetilde{o}_i) \\ \underline{z}_i \times \underline{C}_6 \end{bmatrix}. \tag{3.12}$$

For joint i prismatic

$$\underline{J}_i = \begin{bmatrix} \underline{z}_i \\ \mathbf{0} \end{bmatrix}.$$

Now the map $\underline{z}_i \rightarrow \underline{z}_i \times \underline{C}_6$ is linear and invertible so the linear dependence of the columns of $\underline{J}$ may be determined by testing the singularity of the modified Jacobian J defined as follows:

For joint i revolute

$$J_i = \begin{bmatrix} \underline{z}_i \times (\tilde{o}_6 - \tilde{o}_i) \\ \underline{z}_i \end{bmatrix}. \tag{3.13}$$

For joint i prismatic

$$J_i = \begin{bmatrix} \underline{z}_i \\ 0 \end{bmatrix}.$$

This corresponds to the form of the Jacobian presented in [1] .

Example. The features of the Stanford manipulator (Figure 1.) which determine the form of the its Jacobian are

(a) Joint 2 is prismatic.

(b) $\tilde{o}_3 = \tilde{o}_4 = \tilde{o}_5 = \tilde{o}_6$.

The modified Jacobian[6] is therefore

$$J = \begin{bmatrix} \underline{z}_0 \times (\tilde{o}_6 - \tilde{o}_0) & \underline{z}_1 \times (\tilde{o}_6 - \tilde{o}_1) & \underline{z}_2 & 0 & 0 & 0 \\ \underline{z}_0 & \underline{z}_1 & 0 & \underline{z}_3 & \underline{z}_4 & \underline{z}_5 \end{bmatrix}. \tag{3.14}$$

4. Singular Configurations.

This section contains some examples of manipulator configurations where the Jacobian in singular. These examples have particularly simple geometric descriptions, but it is important to remember that, in general, singular configurations have no simple description. The fact that common manipulators are simple geometrically may be the reason for their easily described singularities.

[6] The modified Jacobian will be referred to as simply the Jacobian for the rest of the paper as this is standard.

Example 1. Two Collinear Revolute Joints Axes.

Without loss of generality[7], take the joints to be 0 and 1. Then

(a) Their axes, $(\tilde{o}_0, \underline{z}_0)$ and $(\tilde{o}_1, \underline{z}_1)$, have parallel directions: $\underline{z}_0 = \pm \underline{z}_1$

(b) The vector $(\tilde{o}_0 - \tilde{o}_1)$ is parallel to $\underline{z}_0$ and $\underline{z}_1 : \underline{z}_i \times (\tilde{o}_0 - \tilde{o}_1)$ for $i \in \{0,1\}$, and

$$J = \begin{bmatrix} \underline{z}_0 \times (\tilde{o}_6 - \tilde{o}_0) & \underline{z}_1 \times (\tilde{o}_6 - \tilde{o}_1) & \cdots \\ \underline{z}_0 & \underline{z}_1 & \end{bmatrix} \in \mathbf{R}^{6\times 6}. \tag{4.1}$$

By the elementary row operation[8] row 1 $\leftarrow$ row 1 $+ (\tilde{o}_6 - \tilde{o}_0) \times$ row 2 we have[9]

$$J \sim \begin{bmatrix} 0 & \underline{z}_1 \times (\tilde{o}_0 - \tilde{o}_1) & \cdots \\ \underline{z}_0 & \underline{z}_1 & \end{bmatrix}. \tag{4.2}$$

By (b) it follows that

$$J \sim \begin{bmatrix} 0 & 0 & \cdots \\ \underline{z}_0 & \underline{z}_1 & \end{bmatrix}. \tag{4.3}$$

It is now clear by (a) that J is singular. The Stanford manipulator (Figure 1.) exhibits this singularity when joints 3 and 5 line up. When two revolute joints are collinear there are a continuum of solutions since the links between the two revolute joints may be rotated without affecting the position of the gripper.

Example 2. Three Parallel, Coplanar Revolute Joint Axes.

Without loss of generality take the three joints to be 0, 1, and 2, with axes $(\tilde{o}_i, \underline{z}_i)$, $i \in \{0,1,2\}$. The condition that the joints are parallel is then

(a) $\underline{z}_i = \pm \underline{z}_j = \quad i,j \in \{0,1,2\}$

and the condition that the three joints are coplanar is

(b) There exists a plane containing the axes with unit normal $\hat{\underline{n}}$ such that $\hat{\underline{n}}^T \underline{z}_i = 0$ and $\hat{\underline{n}}^T(\tilde{o}_i - \tilde{o}_j) = 0 \quad i,j \in \{0,1,2\}$.

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[7] Elementary column operations allow us to obtain the form of (4.1) regardless of the of the joint numbering.

[8] By row1 we mean the first row of vectors and similarly for row 2.

[9] $A \sim B$ means that there exists nonsingular C,D such that $A = CBD$.

Since joints 0,1, and 2 are revolute the Jacobian has the form

$$J = \begin{bmatrix} \underline{z}_0 \times (\tilde{o}_6 - \tilde{o}_0) & \underline{z}_1 \times (\tilde{o}_6 - \tilde{o}_1) & \underline{z}_2 \times (\tilde{o}_6 - \tilde{o}_2) & \cdots \\ \underline{z}_0 & \underline{z}_1 & \underline{z}_2 & \end{bmatrix}. \quad (4.4)$$

By the elementary row operation row 1 ← row 1 + $(\tilde{o}_6 - \tilde{o}_0) \times$ row 2, we obtain

$$J \sim \begin{bmatrix} 0 & \underline{z}_1 \times (\tilde{o}_0 - \tilde{o}_1) & \underline{z}_2 \times (\tilde{o}_0 - \tilde{o}_2) & \cdots \\ \underline{z}_0 & \underline{z}_1 & \underline{z}_2 & \end{bmatrix}. \quad (4.5)$$

By (a) there exists elementary column operations to yield

$$J \sim \begin{bmatrix} 0 & \underline{z}_1 \times (\tilde{o}_0 - \tilde{o}_1) & \underline{z}_2 \times (\tilde{o}_0 - \tilde{o}_2) & \cdots \\ \underline{z}_0 & 0 & 0 & \end{bmatrix}. \quad (4.6)$$

By (b) columns[3] 1 and 2 of (4.6) are in the range of $[\hat{\underline{n}}, 0]^T$ and are therefore linearly dependent. Thus, J is singular. The elbow manipulator in Figure 2. has this singularity when the elbow is fully extended as shown. In this configuration the manipulator is at the boundary of its reachable space.

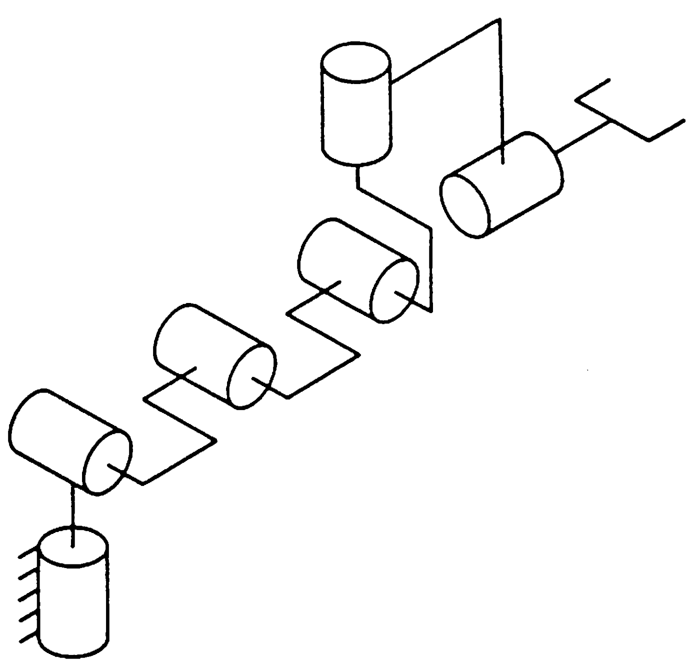

Figure 2. Elbow Manipulator.

Example 3. Four Intersecting Revolute Joint Axes.

When four axes, say $(\tilde{o}_i, \underline{z}_i)$, $i \in \{0,1,2,3\}$, intersect at a point $\tilde{o}$, the point $\tilde{o}$ satisfies

(a) $\underline{z}_i \times (\tilde{o}_i - \tilde{o}) = 0, \quad i \in \{0,1,2,3\}$.

Now

$$J = \begin{bmatrix} \underline{z}_0 \times (\tilde{o}_6 - \tilde{o}_0) & \underline{z}_1 \times (\tilde{o}_6 - \tilde{o}_1) & \underline{z}_2 \times (\tilde{o}_6 - \tilde{o}_2) & \underline{z}_3 \times (\tilde{o}_6 - \tilde{o}_3) & \cdots \\ \underline{z}_0 & \underline{z}_1 & \underline{z}_2 & \underline{z}_3 & \end{bmatrix}. \tag{4.7}$$

By the elementary row operation row 1 $\leftarrow$ row 1 + $(\tilde{o}_6 - \tilde{o}) \times$ row 2 and (a) yield

$$J \sim \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ \underline{z}_0 & \underline{z}_1 & \underline{z}_2 & \underline{z}_3 & \end{bmatrix} \tag{4.8}$$

which is clearly singular since the first four columns are contained in a 3 dimensional subspace of $\mathbf{R}^6$. The Intelledex 605 robot, diagramed in Figure 3 has three intersecting axes at its shoulder. This type of singularity occurs when the final joint axis intersects the shoulder adding a fourth axis as shown.

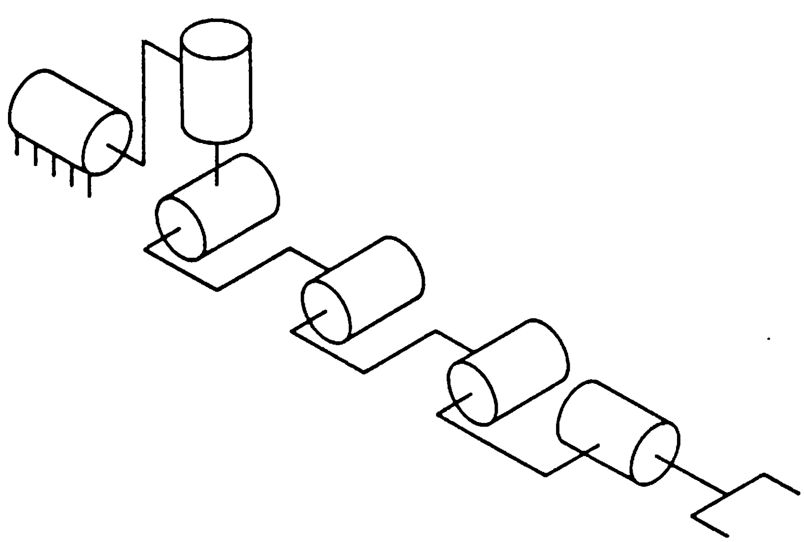

Figure 3. Intelledex 605 Robot.

Example 4. Four Parallel Revolute Joint Axes.

If the joint axes $(\tilde{o}_i, \underline{z}_i)$, $i \in \{0,1,2,3\}$, are parallel then

(a) $\underline{z}_i = \pm \underline{z}_j$, $i \in \{0,1,2,3\}$.

$$J = \begin{bmatrix} \underline{z}_0 \times (\tilde{o}_6 - \tilde{o}_0) & \underline{z}_1 \times (\tilde{o}_6 - \tilde{o}_1) & \underline{z}_2 \times (\tilde{o}_6 - \tilde{o}_2) & \underline{z}_3 \times (\tilde{o}_6 - \tilde{o}_3) & \cdots \\ \underline{z}_0 & \underline{z}_1 & \underline{z}_2 & \underline{z}_3 & \cdots \end{bmatrix}. \quad (4.9)$$

By the elementary row operation row 1 $\leftarrow$ row 1 $+ (\tilde{o}_6 - \tilde{o}_0) \times$ row 2, we have

$$J \sim \begin{bmatrix} 0 & \underline{z}_1 \times (\tilde{o}_0 - \tilde{o}_1) & \underline{z}_2 \times (\tilde{o}_0 - \tilde{o}_2) & \underline{z}_3 \times (\tilde{o}_0 - \tilde{o}_3) & \cdots \\ \underline{z}_0 & \underline{z}_1 & \underline{z}_2 & \underline{z}_3 & \cdots \end{bmatrix}. \quad (4.10)$$

Using (a) and elementary column operations we obtain

$$J \sim \begin{bmatrix} 0 & \underline{z}_1 \times (\tilde{o}_0 - \tilde{o}_1) & \underline{z}_2 \times (\tilde{o}_0 - \tilde{o}_2) & \underline{z}_3 \times (\tilde{o}_0 - \tilde{o}_3) & \cdots \\ \underline{z}_0 & 0 & 0 & 0 & \cdots \end{bmatrix}. \quad (4.11)$$

Now columns[3] 1,2, and 3, in (4.11), are in the null space of $\begin{bmatrix} \underline{z}_0^T & 000 \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$ which has dimension 2. It follows that J is singular.

Example 5. Four Coplanar Revolute Joint Axes.

Let $\hat{\underline{n}}$ be the unit normal to the plane containing the four joint axes. These axes $(\tilde{o}_i, \underline{z}_i)$, $i \in \{0,1,2,3\}$, then satisfy

(a) Each axis direction is orthogonal to $\hat{\underline{n}}$; $\hat{\underline{n}}^T \underline{z}_i = 0$, $i \in \{0,1,2,3\}$.

(b) The vector from $\tilde{o}_i$ to $\tilde{o}_j$ is orthogonal to $\hat{\underline{n}}$; $\hat{\underline{n}}^T(\tilde{o}_i - \tilde{o}_j) = 0$, $i \in \{0,1,2,3\}$.

Now

$$J = \begin{bmatrix} \underline{z}_0 \times (\tilde{o}_6 - \tilde{o}_0) & \underline{z}_1 \times (\tilde{o}_6 - \tilde{o}_1) & \underline{z}_2 \times (\tilde{o}_6 - \tilde{o}_2) & \underline{z}_3 \times (\tilde{o}_6 - \tilde{o}_3) & \cdots \\ \underline{z}_0 & \underline{z}_1 & \underline{z}_2 & \underline{z}_3 & \cdots \end{bmatrix}. \quad (4.12)$$

By the elementary row operation row 1 $\leftarrow$ row 1 $+ (\tilde{o}_6 - \tilde{o}_0) \times$ row 2, we obtain

$$J \sim \begin{bmatrix} 0 & \underline{z}_1 \times (\tilde{o}_0 - \tilde{o}_1) & \underline{z}_2 \times (\tilde{o}_0 - \tilde{o}_2) & \underline{z}_3 \times (\tilde{o}_0 - \tilde{o}_3) & \cdots \\ \underline{z}_0 & \underline{z}_1 & \underline{z}_2 & \underline{z}_3 & \cdots \end{bmatrix}. \quad (4.13)$$

Then elementary column operations yield

$$J = \begin{bmatrix} 0 & \underline{z}_1 \times (\tilde{o}_0 - \tilde{o}_1) & \underline{z}_2 \times (\tilde{o}_0 - \tilde{o}_2) & \underline{z}_3 \times (\tilde{o}_0 - \tilde{o}_3) & \\ \underline{z}_0 & \underline{z}_1 - \underline{z}_0\underline{z}_0^T\underline{z}_1 & \underline{z}_2 - \underline{z}_0\underline{z}_0^T\underline{z}_2 & \underline{z}_3 - \underline{z}_0\underline{z}_0^T\underline{z}_3 & \cdots \end{bmatrix}. \tag{4.14}$$

Columns 1,2, and 3, are in the range of the rank 2 matrix $\begin{bmatrix} \hat{\underline{n}} & 0 \\ 0 & \hat{\underline{n}} \times \underline{z}_0 \end{bmatrix}$ by (a) and (b) above and are therefore linearly dependent.

The Stanford manipulator reaches this configuration when joints 0,1,3, and 4, are coplanar as shown in Figure 1.

Example 6. Six revolute Joint Axes Intersecting a Line.

This configuration occurs in a six degree-of-freedom manipulator with all revolute joints joints when the manipulator is at full reach. For this reason, this configuration is useful for describing the reachable space of a manipulator. Let the line which the six revolute axes intersect be represented by the axis $(\tilde{o}, \hat{\underline{b}})$. Each axis $(\tilde{o}_i, \underline{z}_i)$, $i \in \{0,1,\ldots,5\}$, has a point in common with the axis $(\tilde{o}, \hat{\underline{b}})$ so there exists γ_i, β_i , $i \in \{0,1,\ldots,5\}$, such that

(a) $\tilde{o}_i + \gamma_i \underline{z}_i = \tilde{o} + \beta_i \hat{\underline{b}}$.

For a manipulator with all revolute joints, the columns of J are

$$J_i = \begin{bmatrix} \underline{z}_i \times (\tilde{o}_6 - \tilde{o}_i) \\ \underline{z}_i \end{bmatrix}, \qquad i \in \{0,1,\ldots,5\}. \tag{4.15}$$

From (a) we have

$$\tilde{o}_i = \tilde{o} + \beta_i \hat{\underline{b}} - \gamma_i \underline{z}_i \tag{4.16}$$

Using (4.15), (4.16), and the fact that the cross product of a vector with itself is zero yields

$$J_i \sim \begin{bmatrix} \underline{z}_i \times (\tilde{o}_6 - \tilde{o} - \beta_i \hat{\underline{b}}) \\ \underline{z}_i \end{bmatrix}. \tag{4.17}$$

Applying the elementary row operation row 1 $\leftarrow$ row 1 + $(\tilde{o}_6 - \tilde{o}) \times$ row 2, we obtain

$$J_i \sim \begin{bmatrix} -\beta_i \underline{z}_i \times \hat{\underline{b}} \\ \underline{z}_i \end{bmatrix}. \tag{4.18}$$

It follows that J is singular since $[\hat{\underline{b}}^T, 0^T]$ is in the left nullspace of J. The elbow manipulator in Figure 2. is at full reach and exhibits this singularity. This singularity occurs at other configurations besides those of maximum reach as well.

Example 7. Prismatic Joint Axis Normal to a Plane Containing Two Parallel Revolute Axes.

Label the two revolute joints 0 and 1, and the prismatic joint 2. The revolute axes are therefore $(\tilde{o}_0, \underline{z}_0)$ and $(\tilde{o}_1, \underline{z}_1)$, and the prismatic joint axis is $(\tilde{o}_2, \underline{z}_2)$. The condition that $(\tilde{o}_0, \underline{z}_0)$ and $(\tilde{o}_1, \underline{z}_1)$ are in a plane orthogonal to the prismatic joint axis is

(a) $\underline{z}_2^T \underline{z}_i = 0 \quad i \in \{0,1\}$

$\underline{z}_2^T (\tilde{o}_0 - \tilde{o}_1) = 0.$

The condition that the two revolute axes are parallel is

(b) $\underline{z}_0 = \pm \underline{z}_1.$

From (3.13)

$$J = \begin{bmatrix} \underline{z}_0 \times (\tilde{o}_6 - \tilde{o}_0) & \underline{z}_1 \times (\tilde{o}_6 - \tilde{o}_1) & \underline{z}_2 & \cdots \\ \underline{z}_0 & \underline{z}_1 & 0 & \end{bmatrix}. \tag{4.19}$$

By the elementary operation row 1 $\leftarrow$ row 1 $+ (\tilde{o}_6 - \tilde{o}_0) \times$ row 2 we have

$$J \sim \begin{bmatrix} o & \underline{z}_1 \times (\tilde{o}_0 - \tilde{o}_1) & \underline{z}_2 & \cdots \\ \underline{z}_0 & \underline{z}_1 & 0 & \end{bmatrix}. \tag{4.20}$$

Using the fact that the revolute axes are parallel, (b), together with an elementary column operation yields

$$J \sim \begin{bmatrix} o & \underline{z}_1 \times (\tilde{o}_0 - \tilde{o}_1) & \underline{z}_2 & \cdots \\ \underline{z}_0 & 0 & 0 & \end{bmatrix}. \tag{4.21}$$

Now by (a), both $\underline{z}_1$ and $(\tilde{o}_0 - \tilde{o}_1)$ are orthogonal to $\underline{z}_2$ so $\underline{z}_1 \times (\tilde{o}_0 - \tilde{o}_1)$ is in the range of $\underline{z}_2$. It follows that columns 1 and 2 of (4.21) are linearly dependent and that J is singular. A schematic diagram of the Rhino robot is shown in Figure 4. It reaches this singular configuration when joints 1 and 5 are parallel and in a plane perpendicular to the sliding motion of joint 0.

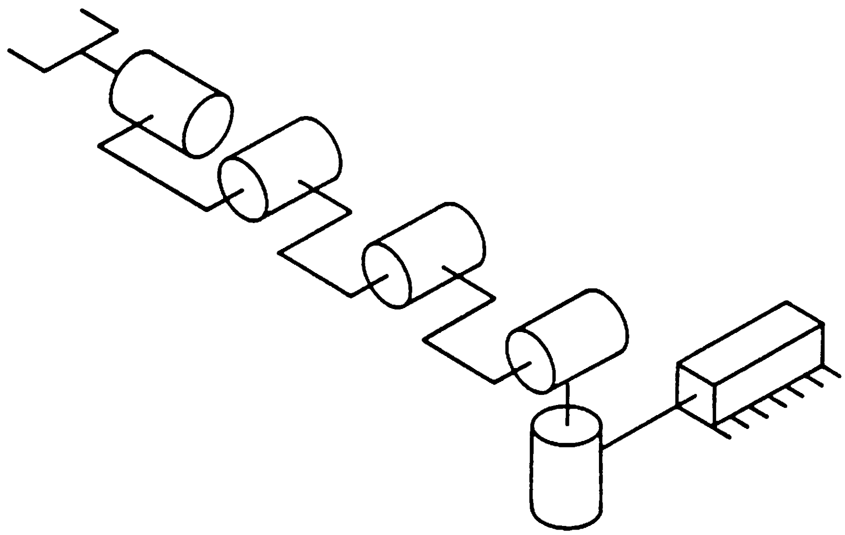

Figure 4. Rhino Robot.

5. Decoupled Singularities.

In this section we demonstrate the decoupling of singularities which occurs in manipulators having three consecutive revolute joints with intersecting axes. For these manipulators, singular configurations are easily recognized. We may assign coordinate systems to a manipulator in this class such that three link coordinate systems share a common origin at the intersection of the of the three axes. For concreteness we discuss six degree-of-freedom manipulators with revolute joints only. By renumbering the joints, the origins to coordinate systems 3,4, and 5 may be chosen to coincide with the intersection of the three axes. The Jacobian is then

$$J = \begin{bmatrix} \underline{z}_0 \times (\tilde{o}_6 - \tilde{o}_0) & \underline{z}_1 \times (\tilde{o}_6 - \tilde{o}_1) & \underline{z}_2 \times (\tilde{o}_6 - \tilde{o}_2) & \underline{z}_3 \times (\tilde{o}_6 - \tilde{o}_3) & \underline{z}_4 \times (\tilde{o}_6 - \tilde{o}_3) & \underline{z}_5 \times (\tilde{o}_6 - \tilde{o}_3) \\ \underline{z}_0 & \underline{z}_1 & \underline{z}_2 & \underline{z}_3 & \underline{z}_4 & \underline{z}_5 \end{bmatrix}. \tag{5.1}$$

Note that the last three joints share the same origin labeled $\tilde{o}_3$. By the elementary row operation row 1 $\leftarrow$ row 1 + $(\tilde{o}_6 - \tilde{o}_3) \times$ row 2 , we have

$$J \sim \begin{bmatrix} \underline{z}_0 \times (\tilde{o}_3 - \tilde{o}_0) & \underline{z}_1 \times (\tilde{o}_3 - \tilde{o}_1) & \underline{z}_2 \times (\tilde{o}_3 - \tilde{o}_2) & 0 & 0 & 0 \\ \underline{z}_0 & \underline{z}_1 & \underline{z}_2 & \underline{z}_3 & \underline{z}_4 & \underline{z}_5 \end{bmatrix}. \tag{5.2}$$

Therefore J is singular if and only if either

(a) $\underline{z}_3, \underline{z}_4$, and $\underline{z}_5$ are coplanar.

or

(b) $\underline{z}_0 \times (\tilde{o}_3 - \tilde{o}_0), \underline{z}_1 \times (\tilde{o}_3 - \tilde{o}_1)$, and $\underline{z}_2 \times (\tilde{o}_3 - \tilde{o}_2)$ are coplanar.

For the elbow manipulator in Figure 2 we may choose $\tilde{o}_3$ to coincide with the intersection of the three wrist axes, and the joint numbering in (5.1) is the natural numbering from base to gripper. Looking at the first and last three joints separately we may determine the singularities of this manipulator by inspection using (a) and (b) above. First, $\underline{z}_3, \underline{z}_4$, and $\underline{z}_5$ are coplanar if and only if joints 3 and 5 are collinear. This is the only singularity contributed by the wrist. Second, the three vectors in (b) are coplanar in the following two cases.

(i) The elbow is fully extended, or is 180 degrees from full extension, so that $\underline{z}_1 \times (\tilde{o}_3 - \tilde{o}_1)$ and $\underline{z}_2 \times (\tilde{o}_3 - \tilde{o}_2)$ are linearly dependent.

(ii) $\tilde{o}_3$ is directly above the base on the axis $(\tilde{o}_0, \underline{z}_0)$ so that $\underline{z}_0 \times (\tilde{o}_3 - \tilde{o}_0) = 0$.

These singularities may be interpreted in terms of the examples as well as (a) and (b) above.

6. Conclusion.

By using the manipulator Jacobian in cross product form, we have described several singular configurations geometrically. The descriptions are manipulator *independent* and therefore apply to any six degree-of-freedom manipulator which can attain the singular configurations. These simple descriptions allow the evaluation of singular configurations without explicitly computing the determinant of the Jacobian. We have also shown that for manipulators with three consecutive intersecting joint axes the evaluation of singularities is particularly simple. In future work we will study branching, or bifurcation, in the solutions of the inverse kinematic solutions. From the inverse function theorem, it follows that branching can only occur at singular configurations. Various bifurcations such as the *fold* consisting of two solution branches annihilating each other, the *pitchfork* consisting of three solution branches merging to one, could occur. The study of exactly which bifurcation occurs at a singularity requires second and higher order derivatives of the forward kinematic solution - a substantial task which we will undertake in future work.

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