MULTIPLE TIME SCALES FOR NONLINEAR SYSTEMS

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Multiple Time Scales for Nonlinear Systems

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Abstract

In this paper we extend the results on the multiple time-scale structure for linear autonomous systems of the form

$$\dot{x} = A_0(\varepsilon)x$$

(c.f. Coderch *et al.*[1]) to nonlinear autonomous systems. Our main result is in obtaining conditions under which the linearized system and the nonlinear system around an equilibrium point have the same time-scale structure.

Keywords: multiple time-scales, singular perturbation, center manifold, nonlinear

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Section 1 Introduction

Recently Coderch *et al.*[1] carried out a detailed analysis of the multiple time-scale behavior of singularly perturbed linear systems of the form

$$\dot{x} = A_0(\varepsilon)x \tag{1.1}$$

where $A(\varepsilon)$ is analytic in the small parameter $\varepsilon \in [0,\ \varepsilon_0]$. Here we extend there results to the nonlinear case

$$\dot{z} = q(z,\varepsilon) \tag{1.2}$$

where $q \in C^r$, r some large integer. Our contribution is in showing under what conditions the nonlinear system has the same multiple time-scale behavior as the linearized system locally around some equilibrium point.

In the following we briefly review the results of Coderch *et al.*[1]. Then we state some results from center manifold theory and some results obtained by Fenichel for nonlinear singularly perturbed systems with two time-scales. Armed with these results we then consider a procedure which uncovers the multiple time-scale structure of a nonlinear system in a step by step manner reminiscent to that developed by Coderch *et al.*[1].

In the following we will decompose (1.2) to the form

$$\dot{x} = \varepsilon f(x,y,\varepsilon) \tag{1.3a}$$

$$y = g(x,y,\varepsilon) \tag{1.3b}$$

When we consider the above system at the slow time-scale $\tau = t/\varepsilon$ and set $\varepsilon = 0$ we obtain

$$\dot{x} = f(x,y,0)$$

$$0 = g(x,y,0)$$

which is in the semistate form (see Newcomb[7]). Thus, we see that the approximate evolution of the system (1.3a,b) on the time scale t/ε is described by a semistate equation. More generally, the system (1.2) will evolve at several time-scales t, t/ε, t/ε^2, ... The approximate model of (1.2) at time-scales t/ε, t/ε^2, ... etc. will be in the semistate form. Thus, the systems (1.2) we study are related to semistate systems.

Section 2 Linear Case-Review and Preliminaries

Coderch *et al.*[1] considered the linear autonomous system

$$\dot{x} = A_0(\varepsilon)\, x(t) \tag{2.1}$$

where $A_0(\varepsilon)$ is analytic in ε and for all $\varepsilon \in [0, \varepsilon_0]$ $\sigma(A_0(\varepsilon) \subset \mathbb{C}_-$ (the closed left half plane). The above system is said to have **well defined behavior at time-scale** $t/\alpha(\varepsilon)$ where $\alpha(\cdot)$ is an order function (α:$[0,\varepsilon_0] \to \mathbb{R}_+$; $\alpha(0) = 0$, and $\alpha(\cdot)$ continuous and monotone increasing), if there exists a continuous matrix $Y(t)$ such that, for and $\delta>0$, $T<\infty$

$$\lim_{\varepsilon \downarrow 0} \sup_{t \in [\delta, T]} || \exp\{A_0(\varepsilon)t / \alpha(\varepsilon)\} - Y(t) || = 0 \tag{2.2}$$

The system (2.1) is said to be **regularly perturbed** if

$$\lim_{\varepsilon \downarrow 0} \sup_{t \geq 0} || \exp\{A_0(\varepsilon)t\} - \exp\{A_0(0)t\} || = 0 \tag{2.3}$$

singularly perturbed otherwise. In the case where (2.1) is regularly perturbed then $Y(t)$ can be chosen to be the constant matrix P_0, the zero eigenvalue projection of $A_0(0)$, for any time-scale. Thus for regularly perturbed systems no interesting behavior occurs at different time-scales. Coderch *et al.* [1] show that (2.1) is singularly perturbed if and only if $A_0(\varepsilon)$ drops rank when ε=0. In the following we assume this is the case and hence we are dealing with a singularly perturbed system.

A matrix A is said to have **semisimple null structure** (SSNS) if its zero eigenvalue is semisimple. Equivalently A has SSNS if there exists a change of basis such that it is of the form

$$\begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \tag{2.4}$$

where A_{11} is nonsingular. The extension of the above definition to $A_0(\varepsilon)$ is called the **multiple semisimple null structure** (MSSNS) condition. We show in [2],[3] that $A_0(\varepsilon)$ satisfies the MSSNS condition if and only if there exists an analytic change of basis $T(\varepsilon)$ such that $A_0(\varepsilon)$ can be transformed to:

$$\begin{bmatrix} \bar{A}_0(\varepsilon) & & & & 0 \\ \cdot & \cdot & & & \cdot \\ \cdot & & \varepsilon^i \bar{A}_i(\varepsilon) & & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & & \cdot & & \varepsilon^m \bar{A}_m(\varepsilon) \end{bmatrix} \tag{2.5}$$

where $\bar{A}_i(0)$ are nonsingular for $i=0, \ldots, m$. Here for simplicity we assume that $A_0(\varepsilon)$ has full normal rank. $A_0(\varepsilon)$ is said to have the **multiple semistability** (MSST) condition if $\sigma(\bar{A}_i(0)) \subset \mathring{\mathbb{C}}_-$, for $i=0, \ldots, m$ in (2.5). Coderch *et al.*[1] show that only under the MSST condition does (2.1) have well defined time-scale behavior at all time-scales. The following claim also holds (assume for simplicity that $A_0(\varepsilon)$ is in the form (2.5))

Claim 2.1 If $A_0(\varepsilon)$ satisfies the MSST condition then for any $\delta>0$ and $T<\infty$

$$i)\ \lim_{\varepsilon \downarrow 0} \sup_{t \in [0,T]} \left\| \exp\{A_0(\varepsilon)t\} - \begin{bmatrix} \exp\{\bar{A}_0(0)t\} & \cdot & 0 \\ \cdot & I & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & \cdot & I \end{bmatrix} \right\| \tag{2.6a}$$

$$ii)\ \lim_{\varepsilon \downarrow 0} \sup_{t \in [\delta,T]} \left\| \exp\{A_0(\varepsilon)t / \varepsilon^i\} - \begin{bmatrix} 0 & \cdot & & \cdot & 0 \\ \cdot & 0 & & \cdot & \cdot \\ \cdot & \cdot & \exp\{\bar{A}_i(0)\}t & & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & \cdot & & \cdot & I \end{bmatrix} \right\| \tag{2.6b}$$

for $i=1, \ldots, m-1$ and

$$iii)\ \lim_{\varepsilon\downarrow 0}\ \sup_{t\in[\delta,\infty]}\left|\left|\exp\{A_0(\varepsilon)t/\varepsilon^m\}-\begin{bmatrix}0 & \cdot & & 0\\ \cdot & 0 & & \cdot\\ \cdot & \cdot & & \cdot\\ \cdot & \cdot & & \cdot\\ 0 & \cdot & & \exp\{\bar{A}_m(0)t\}\end{bmatrix}\right|\right| \tag{2.6c}$$

From the above claim it follows that the reduced ordered subsystems (obtained via (2.5))

$$\dot{x}_i = \bar{A}_i(0)x_i \tag{2.7}$$

approximate the nontrivial behavior of the overall system at the time-scale t/ε^i. Moreover any solution $x^\varepsilon(t)$ of (2.1) is of the form

$$\sum_{i=0}^{i=m} E_i x_i(\varepsilon^i t) + o(1) \tag{2.8}$$

where $x_i(\cdot)$ are solutions of (2.7) with the appropriate initial conditions and

$$E_{ii} = \begin{bmatrix}0\\ \cdot\\ \cdot\\ I_{ii}\\ \cdot\\ 0\end{bmatrix}$$

Following we extend the above results to the case where $A_0(\varepsilon)$ is no longer analytic in ε but continuous.

Consider the autonomous linear system

$$\dot{x} = A(\varepsilon)x \tag{2.9}$$

where $A(\varepsilon)$ is C^r (instead of analytic) in ε, $\varepsilon \in [0,\varepsilon_0]$. $A(\varepsilon)$ has a Taylor series expansion around $\varepsilon=0$ of the form:

$$A(\varepsilon) = A_0 + \varepsilon A_1 + \cdots + A_m + \varepsilon^{m+1}l(\varepsilon) \tag{2.10}$$

where $m+1<r$ and $l(\cdot)$ continuous. Consider applying our results for the analytic case to the truncated series

$$\tilde{A}(\varepsilon) = A_0 + \varepsilon A_1 + \cdots + A_m\ . \tag{2.11}$$

Assume $\tilde{A}(\varepsilon)$ has MSST and has *full normal rank.* This implies there exists an analytic change of basis $T(\varepsilon)$ which diagonalizes $\tilde{A}(\varepsilon)$. The original linear operator $A(\varepsilon)$ in the new basis is of the form:

$$T(\varepsilon)A(\varepsilon)T^{-1}(\varepsilon) = \begin{bmatrix} \bar{A}_0(\varepsilon) & & & 0 \\ & \varepsilon\bar{A}_1(\varepsilon) & & \\ & & \ddots & \\ 0 & & & \varepsilon^{\bar{m}}\bar{A}_{\bar{m}}(\varepsilon) \end{bmatrix} + 0(\varepsilon^{m+1}) \tag{2.12}$$

where all the $\bar{A}_i(0)$ are exponentially stable for $i = 0,\ldots\bar{m}$. If we assume that $\bar{m} \leq m$ then the time-scale structure of $A(\varepsilon)$ is apparent in (2.12) {i.e., the system is sufficiently smooth to uncover the time-scale structure in its truncated Taylor expansion}. Following we assume this will always be the case and define the MSST condition for an operator if it has an expansion as in (2.12). Note this is a generalization of the MSST condition for the class of analytic operators with full normal rank.

Section 3 Nonlinear Systems

In this section we consider nonlinear systems of the form

$$\dot{z} = q(z,\varepsilon) \tag{3.1}$$

with $q \in C^r$. Fenichel [4] gives a complete analysis for the behavior of these systems in the two time-scale case. We will use his local results around an equilibrium point. Our contribution consists of obtaining conditions under which the multiple time-scale structure for local solutions around an equilibrium point is identical to that of the linearized equation. The main tool we will use for studying singularly perturbed nonlinear systems is center manifold theory. In the following subsection we first state a local version of center manifold theory as found in Carr [5] and then state the results from Fenichel [4] that we shall need. In the previous section we relaxed the analyticity requirement. This was a necessary preliminary step because there is no guarantee that our system will remain analytic in ε once we restrict it to it's center manifold.

Section 3.1 Center Manifolds and Geometric Singular Perturbation Theory

Fenichel [4] shows that under certain conditions the study of singularly perturbed nonlinear systems can be simplified by studying their behavior on the center manifold. In the following we first describe the local theory for center manifolds. We use Carr's approach (c.f. [5]) which is well suited for our purposes. The proofs can be found in Carr [5].

Consider the system

$$\dot{x} = Ax + f(x,y) \tag{3.2a}$$

$$\dot{y} = By + g(x,y) \tag{3.2b}$$

where $f,g \in C^r$, $f(0,0)=0$, $g(0,0)=0$, $f'(0,0)=0$, $g'(0,0)=0$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and A and B are constant matrices such that all the eigenvalues of A have zero real parts while all the eigenvalues of B have negative real parts (Here prime denotes the Jacobian with respect to both arguments).

Definition 3.1

The graph of the function $y = h(x)$ is a **local center manifold** for (3.2) if it is an invariant manifold and if $h(0)=0$, $h'(0)=0$.

Theorem 3.2

There exists a center manifold for (3.2), graph of the function $y = h(x)$, $|x| < \delta$ (some $\delta > 0$) where h is C^{r-1} smooth .

The flow on the center manifold is governed by the n-dimensional system

$$\dot{u} = Au + f(u,h(u)) \tag{3.3}$$

The following lemma and theorem link the trajectories on the center manifold to those with initial conditions in a neighborhood of the equilibrium point.

Lemma 3.3

Let $(x(t),y(t))$ be a solution of (3.2) with $|(x(0),y(0))|$ sufficiently small. Then there exists positive C_1 and μ such that

$$|y(t)-h(x(t))|\leq C_1 e^{-\mu t}|y(0)-h(x(0))| \tag{3.4}$$

for all $t\geq 0$.

Theorem 3.4

Assume that the zero solution of (3.2) is stable. Let $(x(t),y(t))$ be a solution of (3.2) with $(x(0),y(0))$ sufficiently small. Then there exists a solution $u(t)$ of (3.3) such that as $t\to\infty$

$$x(t)=u(t)+0(e^{-\gamma t}) \tag{3.5a}$$

$$y(t)=h(u(t))+0(e^{-\gamma t}) \tag{3.5b}$$

where $\gamma>0$ is a constant dependent only on the matrix $B\in\mathbb{R}^{m\times m}$.

As an example of the application of center manifold theory consider the singularly perturbed system

$$\dot{x}=f(x,y,\varepsilon) \tag{3.6a}$$

$$\varepsilon\dot{y}=By+g(x,y,\varepsilon) \tag{3.6b}$$

where $f(0,0,0)=0$ $g(0,0,0)=0$, $g'(0,0,0)=0$ and $\sigma(B)\subset\mathring{\mathbb{C}}_-$. If we now consider the faster time-scale $\tau=t/\varepsilon$ and augment the state space by ε (suspension technique) we have

$$x'=\varepsilon f(x,y,\varepsilon) \tag{3.7a}$$

$$\varepsilon'=0 \tag{3.7b}$$

$$y'=By+g(x,y,\varepsilon) \tag{3.7c}$$

Theorem 3.2 can now be applied to the above system. Note that the Jacobian for (3.7) is of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B \end{bmatrix}$$

hence there exists a function $y=h(x,\varepsilon)$ whose graph is a center manifold for (3.7). On the center manifold the dynamical system is governed by the equation

$$x' = \varepsilon f(x,h(x,\varepsilon),\varepsilon) := \varepsilon\tilde{f}(x,\varepsilon) \tag{3.8}$$

Studying the system on the center manifold reduces the dimensionality of the problem and the dynamics occur at a slower time-scale than for the overall system. For this reason the center manifold is sometimes referred to as the **slow** manifold. Lemma 3.7 implies any arbitrary solution starting sufficiently close to the equilibrium point tends to the center manifold at an exponential rate at the fast time-scale. For the autonomous case $\dot{x}=A_0(\varepsilon)x$ an analogous step was performed when obtaining the operator $A_1(\varepsilon)$ (see Coderch *et al.*[1]). In that case we restricted the study to the range space of the 0-group eigenvalue projection $P_0(\varepsilon)$. This yielded

$$P_0(\varepsilon)\dot{x} = P_0(\varepsilon)A_0(\varepsilon)x = \varepsilon A_1(\varepsilon)P_0(\varepsilon)x \tag{3.9}$$

In fact for the "nonlinear system"

$$\dot{x} = A_0(\varepsilon)x \tag{3.10a}$$

$$\dot{\varepsilon} = 0 \tag{3.10b}$$

the center manifold is $\{\ \mathbb{R}(P_0(\varepsilon))\times(-\varepsilon_0,\varepsilon_0)\ \}$. (Here $\mathbb{R}(P_0(\varepsilon))$ is the range space of $P_0(\varepsilon)$).

Example (3.6) is a special case of a singularly perturbed system- the separation of slow and fast variables is explicit. Fenichel considers the more general case

$$\dot{z} = q(z,\varepsilon) \tag{3.11}$$

where $q \in C^r$, $z \in M$ an open subset of $\mathbb{R}^n$. Under certain conditions similar to the SSNS condition for the linear case (3.11) can be changed to the form (3.7) after a coordinate change. Fenichel [4] defines (3.11) to be a singularly perturbed problem if for $\varepsilon \neq 0$ the equilibrium points of (3.11) are isolated while at $\varepsilon = 0$ a subset of the equilibrium points forms a ν-dimensional manifold Λ ($\nu < n$). Coderch *et al.* [1] obtain an analogous definition for the linear autonomous system $\dot{x} = A_0(\varepsilon)x$ by showing it to be a singularly perturbed problem iff $A_0(\varepsilon)$ drops rank at $\varepsilon = 0$ (if we assume $A_0(\varepsilon)$ has full normal rank then the definitions are equivalent).

Consider now the case when (3.11) is singularly perturbed. Linearizing (3.11) around some $z_0 \in \Lambda$ yields

$$\delta \dot{z} = D_1 h(z_0, 0)\delta z \tag{3.12}$$

The square matrix $D_1 q(z_0, 0)$ has ν zero eigenvalues whose eigenvectors span the tangent space of Λ at z_0. These are called **trivial** with the remaining eigenvalues **nontrivial**. Fenichel shows that under the assumption that the nontrivial eigenvalues all have non-zero real parts then the system equations on the center manifold can be studied at the slower time-scale (cf.[4] Theorem 9.1 p75). In analogy to the linear case we have the following definitions.

Definition 3.5 (Nonlinear semi-simple null structure)

The system (3.1) is said to have nonlinear semisimple null structure (NLSSNS) at the point z_0 if all its nontrivial eigenvalues have non-zero real part.

Note that in particular $D_1 q(z_0, 0)$ has SSNS.

Definition 3.6 (Nonlinear semistable)

The system (3.1) is said to have nonlinear semistability (NLSST) at the point z_0 if all its nontrivial eigenvalues have negative real parts.

Under the NLSST condition we have the following lemma.

Lemma 3.7

If the system (3.1) has nonlinear semistability at z_0 then there exists a local change of coordinates $\varphi(z,\varepsilon)=(x,y)$, $x \in \mathbb{R}^{\nu}$, $y \in \mathbb{R}^{\mu}$, $\varphi \in C^{r-1}$, $\varphi(z_0,0)=(0,0)$ such that (3.11) is locally of the form

$$\dot{x} = \varepsilon f(x,y,\varepsilon) \tag{3.13a}$$

$$\dot{y} = g(x,y,\varepsilon) \tag{3.13b}$$

with $f(0,0,0)=0$, $g(0,0,0)=0$ and $D_2 g(0,0,0)$ exponentially stable.

Proof

The proof follows directly from Lemma 5.1 and Theorem 11.1 in Fenichel [4].

Remarks

i) The SST condition in the linear case suffices to obtain a linear change of basis $T(\varepsilon)$ so that the system becomes:

$$\dot{x} = \begin{bmatrix} A_0(\varepsilon) & 0 \\ 0 & \varepsilon A_1(\varepsilon) \end{bmatrix} x \tag{3.14}$$

similar to (3.13).

ii) The lemma above yields the decoupling of the state variables into the 'slow' and 'fast' variables.

iii) In the definition of NLSSNS and NLSST the system is assumed to be singularly perturbed (as defined by Fenichel) if it has trivial eigenvalues. This assumption is independent of the structure of the linearized system (i.e., SSNS or SST).

In the following section we generalize the MSST condition to the non-linear case. In particular we consider the connection between the multiple time-scale structure of the linearized system and the time-scale structure of the nonlinear system locally around some equilibrium point.

3.2. Nonlinear Multiple Time-Scale Structure

Assume (3.1) has NLSST at the point z_0. Then by Lemma 3.3 under a suitable change of coordinates it is of the form (3.13). By NLSST, it follows that the Jacobian

$$\begin{bmatrix} \varepsilon D_1 f(0,0,\varepsilon) & \varepsilon D_2 f(0,0,\varepsilon) \\ D_1 g(0,0,\varepsilon) & D_2 g(0,0,\varepsilon) \end{bmatrix} \tag{3.15}$$

satisfies the SST condition. This implies that there exists an analytic change of basis $(x,y) \to T(\varepsilon)(x,y)$ which diagonalizes (3.15) up to order $0(\varepsilon^m)$. Here m is assumed sufficiently large such that the time-scale structure of (3.15) is determined by smaller powers of ε as described in §2. The fact that $D_2 g(0,0,0)$ is exponentially stable implies that

$$T(0) = \begin{bmatrix} I & 0 \\ F & I \end{bmatrix}$$

where $F = D_2 g(0,0,0)^{-1} D_1 g(0,0,0)$. Hence the system equations in the new coordinates remain in the form (3.13). Thus assume we have already performed the change of coordinates and that

$$D_2 f(0,0,\varepsilon) \sim 0(\varepsilon^m) \tag{3.16a}$$

$$D_1 g(0,0,\varepsilon) \sim 0(\varepsilon^m) \tag{3.16b}$$

in (3.15). Applying the suspension technique to (3.13) and expanding in a Taylor series around (0,0) yields

$$\dot{x} = \varepsilon f(x,y,\varepsilon) \tag{3.17a}$$

$$\dot{\varepsilon} = 0 \tag{3.17b}$$

$$\dot{y} = D_2 g(0,0,0)y + D_3 g(0,0,0)\varepsilon + \tilde{g}(x,y,\varepsilon) \tag{3.17c}$$

This is almost in the form (3.2) used in Theorem 3.2 . In order to achieve this we remove the linear term in ε in (3.17c) by performing another change of basis $(x,y,\varepsilon)\to N(x,y,\varepsilon)$ where N is of the form

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & F \\ 0 & 0 & I \end{bmatrix} \tag{3.18}$$

with $F=\{(D_1 g(0,0,0))^{-1} D_3 g(0,0,0)\}$. Again assume this has been done and set $D_3 g(0,0,0)=0$ in (3.17c). Now we can apply Theorem 3.2 and deduce the existence of a function $y=h(x,\varepsilon)$ whose graph is a center manifold for (3.7). The differential equation on the center manifold can be reduced to:

$$\dot{u} = \varepsilon f(u,h(u,\varepsilon),\varepsilon) := \varepsilon \tilde{f}(u,\varepsilon) \tag{3.19}$$

Having obtained the above preliminary results by assuming the equilibrium point z_0 satisfies the NLSST condition we shall now consider first the case where the linearized system (3.12) also satisfies the MSST condition for exactly the two time-scales t and t/ε (i.e. in (3.12) $D_1 q(z_0,0)$ has eigenvalues of order O(1) and O(ε) but none of order O(ε^2). This implies that in the new coordinates (x,y), $D_1 f(0,0,0)$ in (3.15) is exponentially stable. Following we describe the time-scale behavior of the nonlinear system by obtaining reduced-order models approximating the overall system at these two time-scales. Consider the reduced order equation (3.19) we have

$$D_1\tilde{f}(0,0) = D_1 f(0,h(0,0),0) + D_2 f(0,h(0,0),0) D_1 h(0,0)$$

Since $h(0,0)=0$ and $D_1 f(0,h(0,0),0)$ is exponentially stable it follows that

$D_1\tilde{f}(0,0)$ is exponentially stable ($D_2 f(0,0,0)=0$ by (3.16)). Thus our reduced system is regularly perturbed and locally around $u=0$ the solutions on the center manifold behave like $0(e^{(-\gamma t\varepsilon)})$ for some $\gamma>0$. At the fast time-scale t we have for any $T<\infty$

$$\lim_{\varepsilon\downarrow 0}\ \sup_{t\in[0,T]} ||(x(t),y(t))-(x(0),Y(t))||=0 \tag{3.20}$$

where $Y(\cdot)$ solves the differential equation

$$\dot{Y}(t)=D_2 g(0,0,0)Y(t)+\tilde{g}(x(0),Y(t),0) \tag{3.21}$$

with initial condition $Y(0)=y(0)$. The above follows from the continuity of the solution of the ordinary differential equation (3.17) with respect to the parameter ε on a compact interval. At the slow time-scale we have the following approximation. For any $\delta>0$

$$\lim_{\varepsilon\downarrow 0}\ \sup_{t\in[\delta,\infty]} ||(x(t/\varepsilon),y(t/\varepsilon))-(X(t),h(X(t),0))||=0 \tag{3.22}$$

where $X(\cdot)$ solves

$$\dot{X}(t)=f(X(t),h(X(t),0),0)=\tilde{f}(X(t),0),\quad X(0)=x(0) \tag{3.23}$$

To prove this requires a little more work. By the MSST condition it follows that the zero solution of (3.17) is stable (i.e. $||(x(t),y(t)||=0(||(x(0),y(0),\varepsilon)||)$ for $t\geq 0$ and sufficiently small initial conditions). Therefore we can apply Theorem 3.4 to deduce that $x(t)=u(t)+0(e^{-\gamma t})$ and $y(t)=h(u(t),\varepsilon)+0(e^{-\gamma t})$ for some $\gamma>0$ and where $u(t)$ solve (3.19) with $u(0)=x(0)$. Thus for any $\delta<0$ we have

$$\lim_{\varepsilon\downarrow 0}\ \sup_{t\in[\delta,\infty]} ||(x(t/\varepsilon),y(t/\varepsilon))-(u(t/\varepsilon),h(u(t/\varepsilon),\varepsilon))||=0 \tag{3.24}$$

For any $T<\infty$ we have

$$\lim_{\varepsilon\downarrow 0}\ \sup_{t\in[\delta,T]} ||u(t/\varepsilon)-X(t)||=0 \tag{3.25}$$

by continuity with respect to the parameter ε. The reduced equation being regularly perturbed implies $\forall\varepsilon\in(0,\varepsilon_0]$, $\exists K>0$ and $\gamma>0$ such that $\forall$ $T<\infty$ and $t>T$

$$||u(t/\varepsilon)||<Ke^{-\gamma T} \tag{3.26a}$$

$$||X(t)||<Ke^{-\gamma' T} \tag{3.26b}$$

Thus

$$\lim_{\varepsilon\downarrow 0}\sup_{t\in[\delta,\infty]}||u(t/\varepsilon)-X(t)||\le\lim_{\varepsilon\downarrow 0}\sup_{t\in[\delta,T]}||u(t/\varepsilon)-X(t)|| + 2Ke^{-\gamma T}$$

for any $T<\infty$. Hence

$$\lim_{\varepsilon\downarrow 0}\sup_{t\in[\delta,\infty]}||u(t/\varepsilon)-X(t)||=0 \tag{3.27}$$

which together with (3.24) implies (3.22). Locally around (x,y)=(0,0) $(x(t),y(t))=0(e^{-\gamma t})+0(e^{-\gamma t/\varepsilon})$ thus having nontrivial behavior only at time-scales t and t/ε.

Proposition 3.8

Assume (3.1) has nonlinear semistability (NLSST) at z_0 and $D_1q(z_0,\varepsilon)$ satisfies the MSST condition for the two time-scales t and t/ε then the solutions of (3.1) starting in a neighborhood of z_0 have two time-scale with corresponding reduced-order approximating equations.

The extension of the above proposition to the multiple time-scale case appears obvious but as we show in the following example care must be taken when more than the two time-scales t and t/ε are involved.

Example 3.9

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{bmatrix}-\varepsilon^2x-\varepsilon xy\\ -y-5\varepsilon\end{bmatrix}=f(x,y,\varepsilon) \tag{3.28}$$

For this example we have

$$Df(0,0,\varepsilon)=\begin{bmatrix}-\varepsilon^2 & 0\\ 0 & -1\end{bmatrix}$$

Hence the NLSST condition is satisfied. Nonetheless asymptotically as $t\to\infty$ $x(t)$ approaches the solutions of

$$\dot{u} = 4\varepsilon^2 u\,,\ \ u(0) = x(0) \tag{3.29}$$

This can be seen by inspection. It also follows from the fact that the center manifold for (3.28) is $y = -5\varepsilon$ and its reduced equation is $\dot{u} = 4\varepsilon^2 u$. (Note that the above center manifold differs from that of Theorem 3.2 because (3.28) is not in the form (3.2)). Thus our system (3.28) is unstable around ($x = 0, y = 0$).

The previous theorem cannot be generalized, as is, to the multiple time-scale case because even though the linearization around the equilibrium point satisfies the MSST condition when we consider the linearization of the reduced system on the center manifold it need not satisfy the MSST condition as it occurred in the previous example. For the two time-scale case the property of the center manifold $h(0)=0$ implies that the Jacobian of the reduced equation (3.19) is exponentially stable thus eliminating this possibility from occurring. In order to extend our results to the multiple time-scale case we would like the reduced equation (3.19) to retain the same MSST condition for its linearization around the equilibrium point as that for the original system except that the fastest time-scale is eliminated. In this way a step by step procedure from one reduced equation to another will result in a complete time-scale decomposition for the original system. In order to accomplish this we add the following assumption.

Assumption 3.10 Persistence of the Equilibrium Point z_0

$$0 = q(z_0, \varepsilon) \ \text{ for } \ \varepsilon \ \text{ small}$$

Suppose $D_1 q(z_0, \varepsilon)$ satisfies the MSST condition then (3.15) also satisfies the MSST condition in the same manner. If we assume (as discussed previously) that the off diagonal terms are of some order $0(\varepsilon^m)$ for large enough m (i.e., we assume the slowest time-scale is $t/\varepsilon^{\overline{m}}$, $\overline{m} < m$) then $D_1 f(0,0,\varepsilon)$ must also satisfy the MSST condition with one less time-scale than the original system.

The Jacobian of the reduced equation (3.19) satisfies

$$D_1\tilde{f}(0,\varepsilon) = D_1 f(0,h(0,\varepsilon),\varepsilon) + D_2 f(0,h(0,\varepsilon),\varepsilon) D_1 h(0,\varepsilon)$$

Now the persistence of the equilibrium point z_0 implies that $(x,y,\varepsilon) = (0,0,\varepsilon)$ is an equilibrium point for (3.17). This implies that $(0,0,\varepsilon)$ is on the center manifold. Therefore $h(0,\varepsilon) \equiv 0$. This fact plus the fact that $D_2 f(0,0,\varepsilon) \sim O(\varepsilon^m)$ implies that $D_1\tilde{f}(0,\varepsilon)$ is equal $D_1 f(0,0,\varepsilon)$ up to order $O(\varepsilon^m)$ so that they satisfy the same MSST condition. Hence we are now in a position to extend the definition of the MSST condition to the nonlinear case. If we assume that z_0 is a persistent equilibrium point and $D_1 q(z_0,\varepsilon)$ satisfies the MSST condition then the linearization of the reduced equation

$$\dot{u} = \varepsilon \tilde{f}(u,\varepsilon)$$

$\varepsilon D_1\tilde{f}(0,\varepsilon)$ satisfies the same MSST condition as the original system except that the fastest time-scale is eliminated. If we assume the reduced equation (3.19) satisfies the NLSST we can repeat the whole procedure and obtain another reduced equation whose linearization has two time-scales less than the original system. Thus if we continue in this fashion the process ends at the step where we are at the slowest time-scale and the reduced equation is regularly perturbed. We make the following definition:

Definition 3.11 (Nonlinear Multiple Semistability)

The system is said to satisfy the nonlinear semistability assumption at z_0 if z_0 is a persistent equilibrium point, $D_1 q(z_0,\varepsilon)$ satisfies the MSST condition and each subsequent reduced equation satisfies the NLSST condition .

As was done in (3.19) we must choose at each step a center manifold to obtain the reduced system. As these manifolds are not unique the question arises as to whether the choice of manifolds will effect the NLSST condition for the reduced equation. We show below that this is not the case. If we consider the set of equations (3.17) since $D_2g(0,0,0)$ is exponentially stable it follows by the implicit function theorem that there exists a unique function $y=p(x,\varepsilon)$, with $p(0,0)=0$ and $p(\cdot,\cdot)$ continuous in a small neighborhood around $(x,\varepsilon)=(0,0)$ such that

$$0 = D_2g(0,0,0)p(x,\varepsilon) + \tilde{g}(x,p(x,\varepsilon),\varepsilon) \tag{3.32}$$

The equilibrium points for the reduced system (3.19) are the solutions of

$$0 = f(x,p(x,0),0) \tag{3.34}$$

Thus the reduced system (3.19) is singularly perturbed if (3.34) has a manifold of solutions in $\mathbb{R}^\nu$ regardless of the center manifold chosen. Similarly for the subsequent reduced equations where we now obtain the equilibrium points by studying the reduced equation

$$\dot{u} = f(u,p(u,\varepsilon),\varepsilon)$$

which is independent of the choice of center manifolds.

Proposition 3.12

If (3.1) satisfies the NLMSST condition around the point z_0 then the solutions of (3.1) in a neighborhood of z_0 have the same time-scale structure as the linearized system. Furthermore reduced-order models approximating the overall system at specific time-scales t/ε^i can be obtained.

proof

Consider the series of reduced equations constructed from the NLMSST condition. If a reduced equation derived from (3.17) is stable around the equilibrium point (0,0) it follows by Lemma 3.3 that the general equation is also stable around the equilibrium point (0,0,0). At the slowest time-scale the Jacobian of the reduced equation is exponentially stable implying stability. Thus working backwards it follows that all the reduced equations are stable around the equilibrium point and thus the overall equation (3.1) is stable around z_0. We can thus apply Theorem 3.4 at each step of the process to show that any solution of (3.1) is of the form

$$z(t) = \sum_{i=0}^{\overline{m}} 0(e^{-\gamma_i \varepsilon^i t}) \tag{3.35}$$

where $\gamma_i > 0$. Thus excluding nontrivial behavior at all slower time-scales t/ε^k for $k > \overline{m}$. As in the two time-scale case reduced-order differential equations can be constructed which approximate the solution of the overall system at specific time-scales. For the time-scale t (3.20) holds where $Y(\cdot)$ solves (3.21). We also have

$$\lim_{\varepsilon \downarrow 0} \sup_{t \in [\delta, T]} ||(x(t/\varepsilon) - y(t/\varepsilon)) - (X(t), h(X(t), 0)|| = 0 \tag{3.36}$$

where $X(\cdot)$ solves (3.23). If we consider the reduced equation (3.19) at the slower time-scale t/ε: $\dot{u} = \tilde{f}(u, \varepsilon)$, by another change of basis $\tilde{\varphi}$ it can be transformed to:

$$\dot{u}_1 = \varepsilon f_1(u_1, u_2, \varepsilon) \tag{3.37a}$$

$$\dot{u}_2 = g_1(u_1, u_2, \varepsilon) \tag{3.37b}$$

similar to (3.17). We then have

$$\lim_{\varepsilon \downarrow 0} \sup_{t \in [\delta, T]} ||(x(t/\varepsilon), y(t/\varepsilon)) - (\tilde{\varphi}^{-1}(u_1(0), U_2(t)), h(\tilde{\varphi}^{-1}(u_1(0), U_2(t)), 0)|| = 0 \tag{3.38}$$

where U_2 solves

$$\dot{U}_2 = g_1(u_1(0), U_2, 0) \quad , \quad U_2(0) = u_2(0) \tag{3.39}$$

and

$$\lim_{\varepsilon \downarrow 0} \sup_{t \in [\delta, T]} ||(x(t/\varepsilon^2), (y(t/\varepsilon^2)) - \tilde{\varphi}^{-1}(U_1(t), h_1(U_1(t), 0)), h(\tilde{\varphi}^{-1}(U_1(t), h_1(U_1(t), 0)), 0)|| = 0$$

where $h_1(\cdot,\cdot)$ is the center manifold for (3.37) and U_1 solves

$$\dot{U}_1 = f_1(U_1, h_1(U_1, 0), 0) \tag{3.41}$$

and so on. The proof of the above are similar to the two time-scale case. At the slowest time-scale the uniform approximation occurs on the infinite time interval $[\delta,\infty]$.

Consider the following examples.

Example 3.13 (Regularly perturbed)

$$\dot{x} = -\varepsilon x - x^3 + xy \tag{3.42a}$$

$$\dot{y} = -y + y^2 - x^2 \tag{3.42b}$$

Its center manifold is the graph of $y=h(x,\varepsilon)=x^2+$ cubic terms. Now if we set $\varepsilon=0$ the above system has three equilibrium points. Hence if we considered any action at the slower time-scale it would be simply to lie on one of these equilibrium points. The point $(0,\varepsilon)$ is an equilibrium point and the linearized system around 0 is.

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}-\varepsilon & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} \tag{3.43}$$

Thus the linearized version has a two time-scale structure.

Example 3.14 (Two time-scale)

$$\dot{x} = -y - \varepsilon(x + x^3) \tag{3.44a}$$

$$\dot{y} = -y \tag{3.44b}$$

Satisfies the NLSST condition. It's center manifold is $h(x,\varepsilon)\equiv 0$. The reduced order fast system is

$$\dot{Y}(t) = -Y(t)\,, \quad Y(0) = y(0) \tag{3.45a}$$

$$\dot{X}(t) = -Y(t)\,, \quad X(0) = x(0) \tag{3.45b}$$

The slow system is

$$\dot{X}(t) = -(X(t) + X(t)^3)\,, \quad X(0)=x(0) \tag{3.46a}$$

$$y \equiv 0 \tag{3.46b}$$

Example 3.15 (Three time-scale)

$$\begin{bmatrix}\dot{x}_0\\ \dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}A_0 & 0 & 0\\ 0 & \varepsilon A_1 & 0\\ 0 & 0 & \varepsilon^2 A_2\end{bmatrix}+\begin{bmatrix}\varepsilon f_0(x_0,x_1,x_2)\\ \varepsilon^2 f_1(x_0,x_1,x_2)\\ \varepsilon^3 f_2(x_0,x_1,x_2)\end{bmatrix} \tag{3.47}$$

where $\sigma(A_i)\subset\mathbb{C}_-^{\circ}$. The fastest time-scale reduced system is:

$$\dot{X}_0=A_0(0)X_0\,,\ X(0)=x_0(0)\ \ X_1=x_1(0)\ \ ,\ X_2=x_2(0) \tag{3.48a}$$

The approximating reduced order system for the time-scale t/ε is:

$$\dot{X}_1=A_1X_1,\ X_1(0)=x_1(0)\ \ ,X_0\equiv 0,\ \ X_2=x_2(0) \tag{3.48b}$$

and that for the slowest time-scale t/ε^2:

$$\dot{X}_2=A_2X_2\,,\ X_2(0)=x_2(0)\ \ ,\ X_0\equiv 0,\ \text{and}\ \ X_1\equiv 0 \tag{3.48c}$$

As illustrated by the previous examples the multiple time-scale structure for nonlinear systems around an equilibrium point does not follow directly from that of the linearized equations. More assumptions are needed to guarantee the multiple time-scale structure persists for the nonlinear system. In practice ε will be some fixed number hence in the region where the linearized equations are a good approximation for the overall system the behavior of the nonlinear system can be predicted from the linear system. For example consider

$$\dot{x} = -\varepsilon x + x^3 \tag{3.49a}$$

$$\dot{y} = -y \tag{3.49b}$$

for a fixed ε. If in our domain of interest ε and $||x||$ are comparable then the above system has only one time-scale. If on the other hand $\varepsilon \gg ||x||$ in the domain of interest then the above system can be approximated by the two time-scale system

$$\dot{x} = -\varepsilon x \tag{3.50a}$$

$$\dot{y} = -y \tag{3.50b}$$

Remark

Dynamical systems which exhibit jump behavior (see Sastry and Desoer[6]) also have dynamics which occur at different time-scales. The possibility of this occurring in our case is excluded by the stability assumptions we place on the equilibrium point.

Section 4 Conclusion

We have extended Coderch *et al.*[1] results to the nonlinear case by applying Fenichel's results in a step by step procedure similar to that done for the linear case. Analytically the time-scale structure for the nonlinear case does not follow directly from the linearized equations around the equilibrium point. In practice the linearized equations predict the correct time-scale structure if we restrict our system to the domain where the linearized equations approximate the nonlinear system.

In the nonlinear case the computational difficulties involved in calculating the center manifolds and the change of coordinates are much greater than for the linear case perhaps making it unfeasible in general to calculate reduced order models. More research needs to be done in this area.

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