UNIQUENESS OF SOLUTION FOR NONLINEAR RESISTIVE CIRCUITS CONTAINING CCCSs OR VCVSs WHOSE CONTROLLING COEFFICIENTS ARE FINITE

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Uniqueness of Solution for Nonlinear Resistive Circuits Containing CCCS's or VCVS's Whose Controlling Coefficients are Finite*

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Abstract

Necessary and sufficient conditions are given for the uniqueness of solution of nonlinear resistive circuits made of strictly monotone-increasing nonlinear resistors, dc sources, and k linear current-controlled current sources or linear voltage-controlled voltage sources whose controlling coefficients α_μ are bounded by $0 < \alpha_\mu < \alpha_{\mu max}$, $\mu = 1,2,\ldots,k$. These conditions are cast in explicit topological terms and are therefore easy to check.

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I. Introduction

The Nielsen-Willson theorem [1] gave the topological necessary and sufficient condition for a transistor circuit to have a unique solution for all values of circuit parameters. This theorem assumes the transistors are represented by the Ebers-Moll model consisting of two monotone-increasing nonlinear resistors and two current-controlled current sources (abbreviated as CCCS's) whose current gains lie between 0 and 1. Since the reverse current gain of real transistors is less than 0.5, the following practical question naturally arises: If in the Ebers-Moll model we assume that the reverse current gain α_r is restricted to $0 < \alpha_r < 0.5$ and the forward current gain α_f to $0 < \alpha_f < 1$, does the feedback structure in [1] remain a necessary and sufficient condition for the existence of a unique solution? To answer this question, it is necessary to generalize the above theorem to a more general class of CCCS circuits. The same generalization is also desirable for VCVS (voltage-controlled voltage source) circuits. For, since VCVS circuits can be regarded as a model of op-amp circuits, they are the most important circuits from the practical viewpoint.

Related to this problem, several results have already been given for some classes of circuits [2]-[6]. These circuits may be classified by the "active" elements allowed. They are 1) linear CCCS's whose current gains lie between 0 and ∞ [2], 2) four types of linear controlled sources whose controlling coefficients lie between 0 and ∞ [3], 3) nonlinear op-amps [4], 4) linear op-amps [5], [6], and 5) linear CCCS's whose current gains lie between 0 and 1 [5].

In this paper we consider the most general CCCS (resp., VCVS) circuits composed of dc sources, linear resistors, nonlinear resistors each of which has its own v-i characteristic represented by a strictly-monotone-increasing function mapping R onto R (henceforth referred to as a nonlinear resistor in this paper), and linear CCCS's (resp., VCVS's) whose controlling coefficients α_μ lie between 0 and $\alpha_{\mu\max}$ ($< \infty$). For complete generality, we allow each CCCS (resp., VCVS) to have its own maximum controlling coefficient.† In Theorems 1 and 2, we give the necessary and sufficient conditions for CCCS (resp., VCVS) circuits to have a unique solution for all values of circuit parameters.†† Theorem 1 corresponds

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†A controlling coefficient should be read as a current gain for a CCCS and as a voltage gain for a VCVS.

††That is, <u>arbitrary</u> strictly-monotone-increasing v-i characteristics of nonlinear resistors, <u>arbitrary</u> positive values of linear resistors, <u>arbitrary</u> controlling coefficient, α_μ, of CCCS's or VCVS's satisfying $0 < \alpha_\mu < \alpha_{\mu\max}$, and <u>arbitrary</u> real values of dc sources, which may be connected at any place of a circuit.

to the case where the controlling coefficients of all CCCS's (resp., VCVS's) have the same maximum value, that is, $\alpha_{1max} = \alpha_{2max} = \dots = \alpha_{max}$. Theorem 2, a generalization of Theorem 1, corresponds to the case where each CCCS (resp., VCVS) has its own maximum controlling coefficient.

Due to space limitation, only a few simple examples illustrating Theorems 1 and 2 are given. Many more interesting applications of these theorems, including an alternate proof of the Nielsen-Willson theorem, will be given in future papers.

II. Symbols, Notations, and Assumption

Since our results will be stated in completely topological terms, we first define some graph-theoretic terminologies.

Let N denote a CCCS (resp., VCVS) circuit in which the controlling coefficient α_μ of each CCCS (resp., VCVS) "μ" satisfies $0 < \alpha_\mu < \alpha_{\mu max}$. Let the associated graph G be obtained from N by the following operations.

(i) Short-circuit all dc voltage sources and open-circuit all dc current sources.

(ii) Replace each linear or nonlinear resistor by a nondirected branch (which we call a resistor branch).

(iii) Replace each CCCS (resp., VCVS) "μ" by a pair of branches $(\mu,\hat{\mu})$ as shown in Fig. 1(a) (resp., (b)).† These branches are called CCCS (resp., VCVS) branches.

Note that after applications of operations (i), (ii) and (iii), the resulting graph G contains only resistor and CCCS (resp., VCVS) branches.

Assumption 1. We assume without loss of generality that the associated graph G is connected.

Graph operations $O(\cdot)$, $S(\cdot)$, $O/S(\cdot)$, and $\mathfrak{Z}(\cdot)$ are defined as in the previous paper [3]. That is, operations $O(\mu)$ and $S(\mu)$ mean "open-circuiting the branch μ" and "short-circuiting the branch μ," respectively. Operation $O/S(\mu)$ means $O(\mu)$ or $S(\mu)$. Operation $\mathfrak{Z}(\cdot)$, called zero operation, is applied only to a pair of CCCS or VCVS branches. That is, $\mathfrak{Z}(\mu)$ means " S(input

†Since N contains only one type of controlled sources, it is more convenient to represent each CCCS by two conventional branches. On the other hand, if we adopt the unconventional representation as in [3], a similar result can of course be obtained.

branch μ of the CCCS μ) and $\mathcal{O}$(output branch $\hat{\mu}$ of the same CCCS) or $\mathcal{O}$(input branch μ of the VCVS μ) and $\mathcal{S}$(output branch $\hat{\mu}$ of the same VCVS).

The following operations will be often applied for the reduction of the associated graph G.

(I) Apply $\mathcal{O}/\mathcal{S}(\cdot)$ to each resistor branch of the associated graph G.

(II) Apply $\mathcal{J}(\cdot)$ to some (possibly none) pairs of CCCS or VCVS branches.

Note that after applications of operations (I) and (II), we have a graph consisting exclusively of CCCS or VCVS branches, henceforth called a controlled source graph and denoted by G_0.

Let us consider a controlled source graph G_0 composed exclusively of n pairs of CCCS (resp., VCVS) branches $\{(\mu_1,\hat{\mu}_1), (\mu_2,\hat{\mu}_2),\ldots, (\mu_n,\hat{\mu}_n)\}$. Assume that in G_0 n input branches of CCCS's (resp., VCVS's) (i.e., branches μ_1, μ_2, $\ldots,\mu_n$) form a tree (resp., cotree) of a graph G_0. Then we can obtain a fundamental cutset matrix C_f (resp., loop matrix B_f) with respect to this tree (resp., cotree) as follows:

$$C_f = \begin{array}{c} \\ \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{array}\!\!\begin{array}{c} \begin{array}{cccccccc} \mu_1 & \mu_2 & \cdots & \mu_n & \hat{\mu}_1 & \hat{\mu}_2 & \cdots & \hat{\mu}_n \end{array} \\ \left[\begin{array}{c|c} \\ \quad I \quad & \quad Q \quad \\ \\ \end{array}\right] \end{array} \tag{1a}$$

$$B_f = \begin{array}{c} \\ \hat{\mu}_1 \\ \hat{\mu}_2 \\ \vdots \\ \hat{\mu}_n \end{array}\!\!\begin{array}{c} \begin{array}{cccccccc} \hat{\mu}_1 & \hat{\mu}_2 & \cdots & \hat{\mu}_n & \mu_1 & \mu_2 & \cdots & \mu_n \end{array} \\ \left[\begin{array}{c|c} \\ \quad I \quad & \quad Q \quad \\ \\ \end{array}\right] \end{array} \tag{1b}$$

where I denotes the identity matrix and Q is called the main part of the fundamental cutset matrix (resp., loop matrix).

We define $\Delta^{(1)}(G_0)$ and $\Delta^{(2)}(G_0)$ as follows:

$$\begin{aligned} \Delta^{(1)} &= \Delta^{(1)}(G_0) = \Delta^{(1)}(\alpha,G_0) \\ &\triangleq |I+\alpha Q| \end{aligned} \tag{2}$$

$$\Delta^{(2)} = \Delta^{(2)}(G_0) = \Delta^{(2)}(A,G_0) \triangleq |I+AQ| \tag{3}$$

where $|M|$ denotes the determinant of M, α denotes a scalar, and

$$A \triangleq \mathrm{diag}[\alpha_{\mu_1 max}, \alpha_{\mu_2 max}, \ldots, \alpha_{\mu_n max}].^{\dagger} \tag{4}$$

III. Main Results

Throughout this paper, a CCCS (resp., VCVS) circuit N is a circuit containing only linear positive resistors, strictly monotone-increasing nonlinear resistors, independent voltage and current sources, and linear current-controlled current sources (resp., voltage-controlled voltage sources) having controlling coefficients $0 < \alpha_\mu < \alpha_{\mu max}$, $\mu = 1,2,\ldots,k$, and $\alpha_{\mu max} < \infty$.

If in addition

$$\alpha_{1max} = \alpha_{2max} = \alpha_{kmax} = \alpha_{max} (> 0). \tag{5}$$

we say N has identical maximum controlling coefficient α_{max}.

Theorem 1. A CCCS (resp., VCVS) circuit with identical maximum controlling coefficient α_{max} has a unique solution for all values of circuit parameters if and only if the associated graph G satisfies the following three conditions:

1) G contains no loop (resp., cutset) consisting exclusively of input branches of CCCS (resp., VCVS) branches.
2) G contains no cutset (resp., loop) consisting exclusively of output branches of CCCS (resp., VCVS) branches.
3) By applying operations (I) and (II) defined in Section II to G, we cannot obtain a connected controlled source graph G_0 such that

$$\Delta^{(1)}(\alpha_{max}, G_0) < 0 \tag{6}$$

Remark 1: When we examine whether condition 3) of Theorem 1 is satisfied or not, we need not consider all the controlled source graphs in which input branches form a tree in the case of CCCS circuits or a cotree in the case of VCVS circuits. It

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† Note that $\Delta^{(1)}$ and $\Delta^{(2)}$ are defined only for a controlled source graph G_0 in which the input branches of CCCS's (resp., VCVS's) form a tree (resp., cotree) of G_0. Remember furthermore that $\Delta^{(1)}$ and $\Delta^{(2)}$ are defined by using the fundamental cutset matrix (resp., loop matrix) for CCCS (resp., VCVS) circuits.

suffices to consider only controlled source graphs having neither a self-loop nor a bridge (i.e., a branch which forms a cutset by itself) (see Appendix 1).

Since the above results for CCCS circuits and for VCVS circuits are dual each other, we will explain Theorem 1 mainly for CCCS circuits.

Example 1. Consider the circuit in Fig. 2(a), of which the associated graph G is shown in Fig. 2(b). The graph G satisfies conditions 1) and 2) of Theorem 1. By applying operations (I) and (II) to G, we cannot obtain a controlled source graph G_0 in which the input branch 1 forms a tree of G_0 and which has no self-loop.† Thus condition 3) of Theorem 1 is satisfied by default. Hence, this circuit has a unique solution for arbitrary values of α_{max}. Indeed we can verify the above conclusion by a direct analysis of the circuit.

Example 2. Consider the circuit in Fig. 3(a), of which the associated graph G is shown in Fig. 3(b). The graph G satisfies conditions 1) and 2) of Theorem 1. By applying $\mathcal{S}(R_1)$ we get the controlled source graph G_0 in Fig. 3(c). The fundamental cutset matrix C_f of G_0 is given by

$$C_f = \begin{array}{cc} 1 & \hat{1} \\ \left[1\right. & \left.\vdots\; -1\right] \end{array}$$

Therefore $\Delta^{(1)} = 1 - \alpha_{max}$. If $\alpha_{max} \leqq 1$ then $\Delta^{(1)} \geqq 0$. Therefore we conclude that if $\alpha_{max} \leqq 1$ then the circuit has a unique solution for all values of circuit parameters. If $\alpha_{max} > 1$, then $\Delta^{(1)}(\alpha_{max}) < 0$. Therefore the circuit does not have a unique solution for $\alpha_{max} > 1$ for at least one choice of circuit parameters.

If the direction of the output branch of the CCCS is reversed, then the circuit has a unique solution for any maximum current gain α_{max} (> 0).

Example 3. Consider the circuit in Fig. 4(a), of which the associated graph is shown in Fig. 4(b). Note that G satisfies conditions 1) and 2) of Theorem 1. It remains to investigate condition 3). The first step we have to do is to find a controlled source graph which is obtained from G by applying operations (I) and (II) and in which the input branches of CCCS's form a tree. By inspection we can find a total of four such graphs shown in Figs. 4(c)-(f). For example, the graph in Fig. 4(d) is obtained from G by applying $\mathcal{O}(R_1)$, $\mathcal{S}(R_2)$, $\mathcal{S}(R_3)$, and $\mathfrak{z}(1)$ and the graph in Fig. 4(e) by applying $\mathcal{O}(R_1)$, $\mathcal{S}(R_2)$ and $\mathcal{S}(R_3)$. However, we need not consider the graphs in Figs. 4(d) and (e) since they have

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†Note that if we open-circuit R_2 and R_3, node ② would not be a part of the input branch. If we short-circuit R_2 or R_3, then the output branch $\hat{1}$ would form a self-loop (see Remark 1).

a bridge and/or a self-loop (See Remark 1). The fundamental cutset matrices of the graphs in Figs. 4(c) and (d), denoted by G_c and G_d, are, respectively, given by

$$
C_f^{(c)} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & \hat{1} & \hat{2} \end{array} \\ \left[\begin{array}{cc:cc} 1 & & 0 & -1 \\ & 1 & -1 & 1 \end{array}\right] \end{array}
$$

$$
C_f^{(d)} = \begin{array}{c} \begin{array}{cc} 2 & \hat{2} \end{array} \\ \left[\begin{array}{c:c} 1 & 1 \end{array}\right] \end{array}
$$

Assume that $\alpha_{1max} = \alpha_{2max} = \alpha_{max}$. Therefore $\Delta^{(1)}$ are calculated as follows:

$$
\Delta^{(1)}(G_c) = \begin{vmatrix} 1 & -\alpha_{max} \\ -\alpha_{max} & 1+\alpha_{max} \end{vmatrix} = 1 + \alpha_{max} - \alpha_{max}^2
$$

$$
\Delta^{(1)}(G_d) = 1 + \alpha_{max} > 0
$$

Observe that if $\alpha_{max} \leqq (\sqrt{5}+1)/2$ then $\Delta^{(1)}(G_c) \geqq 0$. Thus we conclude: The circuit in Fig. 4(a) has a unique solution if $0 < \alpha_{max} \leqq (\sqrt{5}+1)/2$. If $\alpha_{max} > (\sqrt{5}+1)/2$ then the circuit does not have a unique solution for some circuit parameters.

If in Fig. 4(a) we reverse the polarity of the output port of the CCCS 2, then the maximum value α_{max} for the circuit to have a unique solution is 1.

Example 4. Consider the VCVS circuit in Fig. 5(a), of which the associated graph G is shown in Fig. 5(b). Since G satisfies conditions 1) and 2) of Theorem 1, we will investigate condition 3). The first step we have to do is to find a controlled source graph G_0 such that (i) in G_0 input branches of VCVS's form a cotree and (ii) G_0 has neither a self-loop nor a bridge.

Suppose first we apply $\mathcal{S}(R_1)$ to G. Since then the branch 1 forms a self-loop, we have to apply $\mathfrak{Z}(1)$ so that we will get a controlled source graph satisfying (i) and (ii) above. From the resulting graph we obtain a controlled source graph shown in Fig. 5(c).

Next suppose we apply $\mathcal{S}(R_3)$ to G. Since in this case the branch 2 forms a self-loop, we have to apply $\mathfrak{Z}(2)$ successively so that we get a graph satisfying (i) and (ii). From the resulting graph we obtain the controlled source graph in Fig. 5(d).

Thirdly we apply $\mathcal{O}(R_1)$ and $\mathcal{O}(R_3)$ and let the resulting graph be G_1. It can easily be seen that if we apply $\mathcal{O}(R_2)$ to G_1, we cannot obtain a controlled source graph satisfying (i) and (ii). So we apply $S(R_1)$ to G_1. From the resulting graph we obtain the graph shown in Fig. 5(e).

The graphs in Figs. 5(c)-(e), denoted by G_c, G_d, and G_e, are the only graphs we need to investigate. The fundamental loop matrices of these graphs are given by

$$B_f^{(c)} = \begin{array}{c c} & \begin{array}{cc} \hat{2} & 2 \end{array} \\ \hat{2} & \left[\begin{array}{c:c} 1 & 1 \end{array}\right] \end{array}$$

$$B_f^{(d)} = \begin{array}{c c} & \begin{array}{cc} \hat{1} & 1 \end{array} \\ \hat{1} & \left[\begin{array}{c:c} 1 & -1 \end{array}\right] \end{array}$$

$$B_f^{(e)} = \begin{array}{c c} & \begin{array}{cccc} \hat{1} & \hat{2} & 1 & 2 \end{array} \\ \begin{array}{c} \hat{1} \\ \hat{2} \end{array} & \left[\begin{array}{cc:cc} 1 & & 0 & -1 \\ & 1 & 1 & 0 \end{array}\right] \end{array}$$

Assume that $\alpha_{1max} = \alpha_{2max} = \alpha_{max}$. Then $\Delta^{(1)}(\cdot)$ are calculated as follows:

$$\Delta^{(1)}(G_c) = 1 + \alpha_{max}$$

$$\Delta^{(1)}(G_d) = 1 - \alpha_{max}$$

$$\Delta^{(1)}(G_e) = 1 + \alpha_{max}^2$$

Therefore we conclude that the circuit in Fig. 5(a) has a unique solution for all values of circuit parameters if and only if $0 < \alpha_{max} \leqq 1$.

To demonstrate the generality of Theorem 1, we will derive a useful corollary, which can be applied by inspection.

Corollary 1.1. Let N be a circuit containing k CCCS's or k VCVS's having identical maximum controlling coefficient α_{max}. If N satisfies conditions 1) and 2) of Theorem 1 and if

$$\alpha_{max} \leqq 1/k \tag{7}$$

then N has a unique solution for all values of circuit parameters.

Proof. See Appendix 2.

Corollary 1.1 is the best result that we can obtain in the sense that if $\alpha_{max} > 1/k$ then there exists a circuit which does not have a unique solution for some choice of circuit parameters (See Appendix 3).

Theorem 2. (General CCCS or VCVS circuit)

Consider a CCCS (resp., VCVS) circuit containing k CCCS's (resp., VCVS) whose controlling coefficients α_μ satisfy $0 < \alpha_\mu < \alpha_{\mu max}$ ($\mu=1,2,\ldots,k$). Then the circuit has a unique solution for all values of circuit parameters if and only if in addition to conditions 1) and 2) of Theorem 1 the following condition holds.

3') By applying operations (I) and (II) defined in Section II to G, we cannot obtain a connected controlled source graph G_0 such that

$$\Delta^{(2)}(G_0) < 0 \tag{8}$$

Remark 2: When we examine whether condition 3') is satisfied or not, it suffices to consider only controlled source graphs in which input branches of CCCS's (resp., VCVS's) form a tree (resp., cotree) and which has neither a self-loop nor a bridge.

Observe that Theorem 1 is a special case of Theorem 2. The proof of Theorem 2 is given in Section V.

Corresponding to Corollary 1.1 we have:

Corollary 2.1. Consider a CCCS (resp., VCVS) circuit containing k CCCS's (resp., k VCVS's) whose controlling coefficients α_μ satisfies $0 < \alpha_\mu < \alpha_{\mu max}$ ($\mu=1,2,\ldots,k$). If the circuit satisfies conditions 1) and 2) of Theorem 1 and if

$$\sum_{\mu=1}^{k} \alpha_{\mu max} \leqq 1 \tag{9}$$

then the circuit has a unique solution for all values of circuit parameters.

Proof. See Appendix 4.

Corollary 2.1 is also the best result possible in the sense that if (9) does not hold then there exists a circuit which does not have a unique solution for some choice of circuit parameters.

Example 5. Consider the flip-flop circuit N_T in Fig. 6(a) where arrows representing the emitters of the transistors T_1 and T_2 are not shown intentionally. Note that N_T is a typical circuit having a feedback structure [1]. By representing two transistors in Fig. 6(a) by the Ebers-Moll model, we have the circuit in Fig. 6(b), of which the associated graph G is shown in Fig. 6(c). Here the CCCS branches $(1,\hat{1})$ and $(2,\hat{2})$ belong to the transistor T_1 and the branches $(3,\hat{3})$ and $(4,\hat{4})$ to the transistor T_2. Let α_μ denote the current gain of the CCCS μ and assume that $\alpha_{\mu max} \leq 1$.

By applying operations (I) and (II) to G, we obtain many controlled source graphs, of which only three graphs, denoted by G_a, G_b and G_c, are shown in Figs. 7(a)-(c). By noting Remark 2, we can easily verify that it suffices to consider only the above three graphs. Now the fundamental cutset matrices of these graphs are given by

$$C_f^{(a)} = \begin{array}{c} \\ 1 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 4 & \hat{1} & \hat{4} \end{array} \\ \left[\begin{array}{cc:cc} 1 & & -1 & 1 \\ & 1 & -1 & 0 \end{array}\right] \end{array}$$

$$C_f^{(b)} = \begin{array}{c} \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 2 & 3 & \hat{2} & \hat{3} \end{array} \\ \left[\begin{array}{cc:cc} 1 & & 0 & -1 \\ & 1 & 1 & -1 \end{array}\right] \end{array}$$

$$C_f^{(c)} \quad \begin{array}{c} \\ 2 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 2 & 4 & \hat{2} & \hat{4} \end{array} \\ \left[\begin{array}{cc:cc} 1 & & -1 & 1 \\ & 1 & 1 & -1 \end{array}\right] \end{array} \quad .$$

Therefore we have

$$\Delta^{(2)}(G_a) = \begin{vmatrix} 1-\alpha_{1max} & \alpha_{1max} \\ -\alpha_{4max} & 1 \end{vmatrix} = 1 - \alpha_{1max} + \alpha_{1max}\,\alpha_{4max}$$

$$\Delta^{(2)}(G_b) = \begin{vmatrix} 1 & -\alpha_{2max} \\ \alpha_{3max} & 1-\alpha_{3max} \end{vmatrix} = 1 - \alpha_{3max} + \alpha_{2max}\,\alpha_{3max}$$

$$\Delta^{(2)}(G_c) = \begin{vmatrix} 1-\alpha_{2max} & \alpha_{2max} \\ \alpha_{4max} & 1-\alpha_{4max} \end{vmatrix} = 1 - \alpha_{2max} - \alpha_{4max}.$$

Since $0 < \alpha_{\mu max} \leq 1$, we have $\Delta^{(2)}(G_a) > 0$ and $\Delta^{(2)}(G_b) > 0$. Therefore, if $\Delta^{(2)}(G_c) > 0$, that is, if

$$\alpha_{2max} + \alpha_{4max} < 1,$$

then N_T has a unique solution for all values of circuit parameters. We therefore conclude that if both α_2 and α_4 are reverse current gains, and if $\alpha_{2max} = \alpha_{4max} < 0.5$, then N_T cannot function as a flip-flop circuit. If, however, either α_2 or α_4 is forward current gain, then we can expect N_T to function as a flip-flop circuit.

Remark 3: This example shows the important fact that in general the Nielsen and Willson theorem is not valid for transistor circuits if the reverse current gains are less than 0.5.

Example 6. Consider the circuit in Fig. 4(a) again. $\Delta^{(2)}(\cdot)$ are calculated for the graphs in Figs. 4(c) and (d) as follows:

$$\Delta^{(2)}(G_c) = 1 + \alpha_{2max} - \alpha_{1max}\,\alpha_{2max}$$

$$\Delta^{(2)}(G_d) = 1 + \alpha_{2max} > 0$$

Therefore if $\alpha_{1max} \leqq 1 + \alpha_{2max}^{-1}$, then the circuit has a unique solution. For example, if $\alpha_{2max} = 0.5$ and $\alpha_{1max} = 3$, then the circuit has a unique solution (cf. Example 3).

If we reverse the polarity of the output port of CCCS 2, we can easily show that the necessary and sufficient condition for a solution to be unique is: $(0 <)\ \alpha_{2max} \leqq 1$ and α_{1max} is an arbitrary positive number.

Example 7. Consider again the circuit in Fig. 5(a). $\Delta^{(2)}(\cdot)$ are calculated for the graphs in Figs. 5(c)-(e) as follows:

$$\Delta^{(2)}(G_c) = 1 + \alpha_{1max}\,\alpha_{2max}$$

$$\Delta^{(2)}(G_d) = 1 - \alpha_{1max}$$

$$\Delta^{(2)}(G_e) = 1 + \alpha_{2max}$$

From this we conclude the circuit has a unique solution if $0 < \alpha_{1max} < 1$ and $0 < \alpha_{2max} < \infty$.

If we reverse the polarity of the output port of VCVS 1, then the condition for a unique solution is $\alpha_{1max}\,\alpha_{2max} \leqq 1$.

Example 8. Consider the VCVS circuit in Fig. 8(a), of which the associated graph G is shown in Fig. 8(b). By applying operations (I) and (II) to G, we find 6 controlled source graphs in Figs. 8(c)-(h) need be considered. Other controlled source graphs obtained by the above operations either

(i) has no cotree composed only of input branches of VCVS's, or

(ii) has a self-loop or a bridge (see Remark 2).

The fundamental loop matrices of the graphs in Figs. 8(c)-(h), denoted by G_c, G_d, ..., G_h, are given by

$$B_f^{(c)} = \begin{array}{cc} & \begin{array}{cccc} \hat{1} & \hat{2} & 1 & 2 \end{array} \\ \begin{array}{c} \hat{1} \\ \hat{2} \end{array} & \left[\begin{array}{cc:cc} 1 & & 0 & -1 \\ & 1 & 1 & 0 \end{array}\right] \end{array}$$

$$B_f^{(d)} = \begin{array}{cc} & \begin{array}{cccc} \hat{1} & \hat{3} & 1 & 3 \end{array} \\ \begin{array}{c} \hat{1} \\ \hat{3} \end{array} & \left[\begin{array}{cc:cc} 1 & & 1 & -1 \\ & 1 & -1 & 1 \end{array}\right] \end{array}$$

$$B_f^{(e)} = \begin{array}{cc} & \begin{array}{cccc} \hat{2} & \hat{3} & 2 & 3 \end{array} \\ \begin{array}{c} \hat{2} \\ \hat{3} \end{array} & \left[\begin{array}{cccc} 1 & & -1 & -1 \\ & 1 & -1 & 0 \end{array}\right] \end{array}$$

$$B_f^{(f)} = \begin{array}{cc} & \begin{array}{cc} \hat{1} & 1 \end{array} \\ \hat{1} & \left[\begin{array}{c:c} 1 & 1 \end{array}\right] \end{array}$$

$$B_f^{(g)} = \begin{array}{cc} & \begin{array}{cc} \hat{2} & 2 \end{array} \\ \hat{2} & \left[\begin{array}{c:c} 1 & -1 \end{array}\right] \end{array}$$

$$B_f^{(h)} = \begin{array}{cc} & \begin{array}{cc} \hat{3} & 3 \end{array} \\ \hat{3} & \left[\begin{array}{c:c} 1 & 1 \end{array}\right] \end{array}$$

Therefore $\Delta^{(2)}(\cdot)$ are calculated as follows:

$$\Delta^{(2)}(G_c) = \begin{vmatrix} 1 & -\alpha_{1max} \\ \alpha_{2max} & 1 \end{vmatrix} = 1 + \alpha_{1max}\,\alpha_{2max}$$

$$\Delta^{(2)}(G_d) = \begin{vmatrix} 1+\alpha_{1max} & -\alpha_{1max} \\ -\alpha_{3max} & 1+\alpha_{3max} \end{vmatrix} = 1 + \alpha_{1max} + \alpha_{3max}$$

$$\Delta^{(2)}(G_e) = \begin{vmatrix} 1-\alpha_{2max} & -\alpha_{2max} \\ -\alpha_{3max} & 1 \end{vmatrix} = 1-\alpha_{2max} - \alpha_{2max}\,\alpha_{3max}$$

$$\Delta^{(2)}(G_f) = 1 + \alpha_{1max}$$

$$\Delta^{(2)}(G_g) = 1 - \alpha_{2max}$$
$$\Delta^{(2)}(G_h) = 1 + \alpha_{3max}$$

We conclude from this that if $1 - \alpha_{2max} - \alpha_{2max}\,\alpha_{3max} \geqq 0$, then the circuit in Fig. 8(a) has a unique solution.

If in Fig. 8(a) $\alpha_{1max} = \alpha_{2max} = \alpha_{3max} = \alpha_{max}$, then the circuit has a unique solution for $\alpha_{max} \leqq (\sqrt{5}-1)/2$.

As an application of Theorem 2, we show that when we connect two circuits in some classes of circuits having a unique solution, the resulting circuit also has a unique solution.

In the following a CCCS or a VCVS one-port N_1 is said to have a unique solution if in both cases where the input port is open-circuited and short-circuited N_1 has a unique solution for all values of circuit parameters. Similarly, a CCCS or a VCVS two-port N_2 is said to have a unique solution if for any combination where the input port <u>and</u> the output port are open-circuited and/or short-circuited N_2 has a unique solution for all values of circuit parameters.

<u>Theorem 3</u>. If a CCCS (resp., VCVS) two-port having a unique solution is terminated in a CCCS (resp., VCVS) one-port having a unique solution, then the resulting one-port has a unique solution.

<u>Proof</u>. See Appendix 5.

<u>Theorem 4</u>. If two CCCS (resp., VCVS) two-port having a unique solution are connected in cascade, then the resulting two-port has a unique solution.

<u>Proof</u>. We can prove Theorem 4 in a similar way as that of Theorem 3 in Appendix 5.

These theorems can be used to synthesize many CCCS or VCVS circuit having a unique solution.

IV. Special Case Where $\alpha_{max} = 1$

Let us focus our attention on the special case where the maximum current gain α_{max} of CCCS's equals 1 in Theorem 1 since this is the most important situation in practice. We assume in this section that the current gain α_μ of each CCCS satisfies $0 < \alpha_\mu < 1$. Note that $\Delta^{(1)}(G_0) < 0$ means in this case that

$$|I + Q| < 0 \tag{10}$$

where $[I \vdots Q]$ is the fundamental cutset matrix of G_0.

Consider the 3 CCCS circuits in Figs. 9(a)-(c). Here, each diamond-shape symbol denotes a CCCS, each box labelled L, L_i (i=0,1,...,n) denotes an arbitrary connected circuit composed of dc sources and linear/nonlinear resistors and each box labelled C_i (i=1,2,...) denotes either the circuit in Fig. 10(a) or the circuit in Fig. 10(b).

Theorem 5. The CCCS circuits in Figs. 9(a)-(c) have a unique solution for all values of circuit parameters if they satisfy conditions 1) and 2) of Theorem 1.

Proof. See Appendix 6

Remark 4. The circuits in Figs. 9(a) and (b) can be regarded as a generalization of a grounded-base transistor circuit. The circuits in Fig. 10 include the Ebers-Moll model as a special case.

Remark 5. In Fig. 9(a) the directions of both controlling and controlled current-sources can be taken arbitrarily, but in Fig. 9(b) they must be assigned as shown in the figure.

Remark 6. There exists a VCVS version of Theorem 5, but it is omitted because the configurations are not very interesting.

Theorem 6. A circuit made of transistors (modelled by Ebers-Moll model), dc sources, linear and/or nonlinear resistors has a feedback structure in the sense defined in [1] if and only if by applying operations (I) and (II) in Section II to the associated graph we obtain a controlled source graph G_0 such that

$$\Delta^{(1)}(1,G_0) < 0$$

The detailed proof of Theorem 6 as well as its application to derive the Nielsen-Willson theorem will be given in a future paper.

V. Outline of the Proof of Theorem 2

Since the dual discussion holds for a CCCS circuit and a VCVS circuit, we will prove Theorem 2 only for a CCCS circuit.

5.1. Analytical condition for a solution to be unique

Consider a CCCS circuit N which contains k CCCS's and m nonlinear resistors. Then N can be represented as in Fig. 11 where the (2k+m)-port N_0 contains only linear resistors and dc sources. Let

$$V_a = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix} \qquad I_a = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_k \end{bmatrix}$$

$$V_b = \begin{bmatrix} v_{k+1} \\ v_{k+2} \\ \vdots \\ v_{2k} \end{bmatrix} \qquad I_b = \begin{bmatrix} i_{k+1} \\ i_{k+2} \\ \vdots \\ i_{2k} \end{bmatrix}$$

$$V_c = \begin{bmatrix} v_{2k+1} \\ \vdots \\ v_{2k+m} \end{bmatrix} \qquad I_c = \begin{bmatrix} i_{2k+1} \\ \vdots \\ i_{2k+m} \end{bmatrix}$$

The characteristics of the CCCS's and the nonlinear resistors are represented by

$$V_a = 0 \tag{11a}$$

$$I_b = AI_a \tag{11b}$$

$$-V_c = F(I_c) \tag{11c}$$

where

$$A = \text{diag}[\alpha_1, \alpha_2, \ldots, \alpha_k] \tag{12a}$$

satisfies

$$0 < \alpha_\mu < \alpha_{\mu\max}, \ \mu = 1, 2, \ldots, k \tag{12b}$$

and

$$F(I_c) = \begin{bmatrix} f_1(i_{2k+1}) \\ f_2(i_{2k+2}) \\ \vdots \\ f_m(i_{2k+m}) \end{bmatrix} \tag{13}$$

satisfies:

Assumption 2. f_μ ($\mu=1,\ldots,m$) are strictly monotone-increasing functions mapping R onto R.

Suppose for the moment the following assumption is satisfied.

Assumption 3. N_0 has the following impedance representation:

$$\begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = \begin{bmatrix} Z_{aa} & Z_{ab} & Z_{ac} \\ Z_{ba} & Z_{bb} & Z_{bc} \\ Z_{ca} & Z_{cb} & Z_{cc} \end{bmatrix} \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} + \begin{bmatrix} E_a \\ E_b \\ E_c \end{bmatrix}. \tag{14}$$

Equations (11)-(14) are the basic equations for our present analysis. Set.

$$\Gamma = \begin{vmatrix} Z_{aa}+Z_{ab}A & Z_{ac} \\ Z_{ca}+Z_{cb}A & Z_{cc}+D \end{vmatrix} \tag{15}$$

where D is a positive definite diagonal matrix, henceforth denoted by D (> 0).

Lemma 1. For any given values of linear resistors, the circuit in Fig. 11 has a unique solution for all A and all f_μ satisfying (12b) and Assumption 2 if and only if

(i) $\Gamma > 0$ for all A and all D (> 0), and (16a)

(ii) $|Z_{aa}| \neq 0$. (16b)

Proof. See Appendix 7.

Let K be a set of numbers $\{1,2,\ldots,k\}$ and let K_1 and K_2 denote a partition of K, i.e., $K_1 \cup K_2 = K$ and $K_1 \cap K_2 = \phi$. Let

$$\begin{bmatrix} Z_{aa} & Z_{ab} & Z_{ac} \\ \underbrace{Z_{ca}}_{k} & \underbrace{Z_{cb}}_{k} & \underbrace{Z_{cc+D}}_{m} \end{bmatrix} = [p_1,p_2,\ldots,p_k \vdots q_1,q_2,\ldots,q_k \vdots r_1,r_2,\ldots,r_m] \tag{17}$$

be denoted by 2k+m column vectors, p_1, p_2, ..., r_m, and let

$$\Gamma_\infty = |t_1,t_2,\ldots,t_k \vdots r_1,\ldots,r_m| \tag{18}$$

where

$$t_\mu = \begin{cases} p_\mu + q_\mu \alpha_{\mu\max} & \text{for } \mu \in K_1 \\ p_\mu & \text{for } \mu \in K_2 \end{cases} \tag{19}$$

Then we have:

Lemma 2. The condition (16a) is equivalent to the following conditions:

$$\Gamma_\infty \geqq 0 \text{ for all } D\ (> 0) \text{ and for any partition of } K \tag{20a}$$

and

$$\Gamma_\infty > 0 \text{ for some } D\ (> 0) \text{ and for at least one partition of } K \tag{20b}$$

This lemma can easily be proved by using Lemma A.2 in [3].

Since $|Z_{aa}| > 0$ (see (A7.3)), we have

$$\Gamma_\infty > 0 \text{ for } K_1 = \phi, \quad K_2 = K \text{ and } D \to \infty.^\dagger \tag{21}$$

Therefore the condition (20b) is always satisfied. So it remains to investigate only the condition (20a).

5.2. Topological Condition for Uniqueness

Let us analyze the condition for (20a) to be satisfied for all values of linear resistors and all D (> 0).

Let

$$\tilde{Z} = \begin{bmatrix} Z_{aa} & Z_{ab} & Z_{ac} \\ Z_{ba} & Z_{bb} & Z_{bc} \\ Z_{ca} & Z_{cb} & Z_{cc+D} \end{bmatrix} \tag{22}$$

Then $\tilde{Z}$ is the impedance matrix of the (2k+m)-port $\tilde{N}$ in Fig. 12, which is obtained from N_0 in Fig. 11 by (i) short-circuiting all voltage sources, (ii) open-circuiting all current sources, and (iii) connecting a resistor γ_μ (μ=1, 2,...,m) in series with the (2k+μ)-th port. It is clear that condition (20a) depends on $\tilde{Z}$ only.

The associated graph $\tilde{G}$ of the (2k+m)-port $\tilde{N}$ is defined as a graph obtained from N by replacing each resistor (including γ_μ), each port μ(μ=1,2,...,k), each port μ+k (μ=1,2,...,k), and each port 2k+μ (μ=1,2,...,m), respectively, by directed branches R_μ, a_μ, b_μ and c_μ.†† The direction of R_μ may be arbitrarily chosen. However, the directions of a_μ, b_μ, and c_μ must be chosen to be the same as those of the port currents. The graph $\tilde{G}$ is connected by Assumption 1.

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† $D \to \infty$ means that each diagonal element is positive and sufficiently large.

†† In this section and Appendix 8 we call the branches a_μ, b_μ, and c_μ, a-, b-, and c-branches, respectively. An a-branch and a b-branch are branches which are called an input and an output branch of a CCCS in the previous sections.

For the moment let us assume that

Assumption 4. There exists no loop consisting exclusively of a-, b-, and c-branches.

The case where Assumption 4 does not hold will be treated in Section 5.3. Let

$$m_0 = \text{rank of } \tilde{G} \text{-total number of a-, b-, and c-branches} \tag{23}$$

From Assumption 4 it follows that $m_0 \geqq 0$. We can modify $\tilde{G}$ by adding m_0 g-branches, g_μ ($\mu=1,2,\ldots,m_0$) so that all the a-, b-, c-, and g-branches form a tree, say T, of $\tilde{G}$. For simplicity we denote hereafter the modified graph by the same symbol $\tilde{G}$ as before.

Let the fundamental cutset matrix of G with respect to T be

$$C_f = [\overbrace{I}^{T} \,\vdots\, \overbrace{C_\ell}^{\bar{T}}]$$

and let the rows of C_f be arranged in the order of a-, b-, c-, and g-branches. Without loss of generality we will investigate the condition

$$\Gamma_\infty \geqq 0 \tag{24}$$

for

$$\begin{aligned} K_1 &= \{1,2,\ldots,k_1\} \quad (0 \leqq k_1 \leqq k) \\ K_2 &= \{k_1+1, k_1+2,\ldots,k\}. \end{aligned} \tag{25}$$

Set

$$K_2 = k - k_1. \tag{26}$$

Then C_ℓ can be written as in Fig. 13 where $M = \{1,2,\ldots,m\}$ and $M_0 = \{1,2,\ldots,m_0\}$ and a_{K1} means the set of branches a_μ ($\mu \in K_1$) and so on. Let

$$H = C_\ell R^{-1} C_L' \tag{27}$$

where the prime means the transpose of a matrix and R is a diagonal matrix whose diagonal elements are the values of linear resistors including γ_μ in Fig. 12. Note that $\tilde{Z}$ in (22) is the upper left-most $(2k+m) \times (2k+m)$ principal sub-matrix of H^{-1} (see Appendix 8).

Let B denote a matrix and let $\hat{B}$ denote the matrix obtained from B by adding the product of the $(s+\mu)$-th row ($\mu=1,2,\ldots,\hat{k}$) and λ_μ to the $(t+\mu)$-th row. We represent $\hat{B}$ by Fig. 14(a) where $\Lambda = \text{diag}\,[\lambda_1,\lambda_2,\ldots,\lambda_{\hat{k}}]$. Similarly Fig. 14(b) denotes the matrix obtained from B by adding the product of the $(s+\mu)$-th column (μ=1,2,...,$\hat{k}$) and λ_μ to the ($t+\mu$)-th column.

<u>Lemma 3</u>.

$$\Gamma_\infty = |H|^{-1}\delta_0 \tag{28}$$

where δ_0 is the determinant of the submatrix shaded by oblique lines in Fig. 15 where $A_{K1} = \mathrm{diag}[\alpha_{1max},\alpha_{2max},\ldots,\alpha_{k_1max}]$.

<u>Proof</u>. See Appendix 8.

Let $C_L^{(1)}$ denote the matrix composed of the rows a_{K1}, b_{K1}, b_{K2} and g_{M0} of C_L (See Fig. 16). Since $|H| > 0$, it is sufficient to consider the sign of δ_0. By using (27), we can write δ_0 as

$$\delta_0 = |C_{L1}R^{-1}C'_{L2}| \tag{29}$$

where C_{L1} (resp., C_{L2}) is the submatrix in Fig. 17(a) (resp., 17(b)) shaded by oblique (resp., vertical) lines. Let R_0 denote an arbitrary set of $k+m_0$ resistor branches. Let $C_L^{(2)}$ denote the matrix, comprised of all rows of $C_L^{(1)}$ and columns corresponding to R_0. Let δ_1 (resp., δ_2) denote the determinant of the submatrix shaded by oblique (resp., vertical) lines in Fig. 18(a) (resp., 18(b)) and let

$$\delta = \delta_1\delta_2 \tag{30}$$

<u>Lemma 4</u>. We can choose the values of resistors so that

$$\Gamma_\infty < 0 \tag{31}$$

if and only if there exists an R_0 such that

$$\delta < 0. \tag{32}$$

Since the proof of this lemma is similar to that of Lemma 4 in [3], we will omit it.

By carrying out the following elementary operations (i)-(iii) appropriately, we can transform $C_L^{(2)}$ into $C_L^{(3)}$ in Fig. 19, where D_0 is a nonsingular diagonal matrix whose diagonal elements are ± 1 and

$$P = -I. \tag{33}$$

(i) Multiply some columns by ± 1
(ii) Add the above columns to other columns
(iii) Interchange the columns

Since

$$\begin{aligned}\delta_1 &= \varepsilon|D_0|\ |-A_{K1}Q+P| \\ \delta_2 &= \varepsilon|D_0|\ |P|\end{aligned} \qquad (\varepsilon=\pm 1) \tag{34}$$

we have

$$\delta = |I+A_{K1}Q|.$$

Lemma 5. Let $G^{(3)}$ be a graph obtained from $\tilde{G}$ in Section 5.2 by the following operations:

(i) Apply $\mathcal{S}(\cdot)$ to each c-branch and $\mathcal{O}(\cdot)$ to each g-branch.

(ii) Apply $\mathcal{S}(\cdot)$ to resistor branches belonging to R_0 and $\mathcal{O}(\cdot)$ to all of the remaining resistor branches.

(iii) Apply $\mathfrak{Z}(\cdot)$ to CCCS branches of CCCS k_1+1, $k_1+2,\ldots,k$.

Then $G^{(3)}$ is a connected graph with a tree a_{K1} and has the fundamental cutset matrix

$$C_f^{(3)} = \begin{array}{cc} & \begin{array}{cc} a_{K1} & b_{K1} \end{array} \\ a_{K1} & \left[\begin{array}{c:c} I & Q \end{array}\right] \end{array}$$

if and only if $\delta \neq 0$.

We omit the proof of this lemma, since it is similar to that of Lemma 5 in [3].

From Lemmas 5 and 4 we obtain Theorem 2.

5.3. On Assumptions 3 and 4

Assumption 3 means that there exists no cutset consisting exclusively of a-, b-, and c- branches. We need not consider the case where the cutset includes an a-branch or a c-branch. The reason is the same as that of Appendix 8 in [3]. If there exists a cutset of b-branches only, then the voltages of controlled sources corresponding to the b-branches included in this cutset cannot be determined uniquely. Thus Condition 2) of Theorems 1 and 2 is necessary.

The dual discussion holds for Assumption 4.

Appendix 1. On the Remarks 1 and 2

We consider only Remark 2 because the same discussion holds for Remark 1. Let us consider the case of CCCS circuits and let G_0 be a controlled source graph in which input branches 1, 2, ..., n form a tree of G_0. Suppose that the branch $\hat{1}$ is a self-loop. Then the fundamental cutset matrix of G_0 is written as

$$C_f = [I \vdots Q] = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ n \end{array} \begin{array}{cccc|c|c} 1 & 2 & \cdots & n & \hat{1} & \hat{2} \cdots \hat{n} \\ \hline 1 & & & & 0 & \\ \hline & 1 & & & & \\ & & \ddots & & 0 & Q_1 \\ & & & 1 & & \end{array} \qquad \text{(A1.1)}$$

Therefore if $|I+AQ| < 0$ then $|I+A_1Q_1| < 0$ where A_1 is the matrix A with the first row and column deleted. Now $[I \vdots Q_1]$ is the fundamental cutset matrix of the graph G_1 obtained from G_0 by applying operation $\mathfrak{z}(1)$. Thus if $\Delta(G_0) < 0$ then there exists a graph G_1 such that G_1 has fewer vertices than G_0 and $\Delta(G_1) < 0$.

Suppose next that the branch 1 is a bridge. Then the fundamental cutset matrix of G_0 is given by

$$C_f = [I \vdots Q] = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ n \end{array} \begin{array}{cccc|c|c} 1 & 2 & \cdots & n & \hat{1} & \hat{2} \cdots \hat{n} \\ \hline 1 & & & & 0 & 0 \\ \hline & 1 & & & & \\ & & \ddots & & & Q_1 \\ & & & 1 & & \end{array} . \qquad \text{(A1.2)}$$

Therefore if $|I+AQ| < 0$ then $|I+A_1Q_1| < 0$ where A_1 is the same as in the above. Thus we have the same conclusion as before.

Appendix 2. Proof of Corollary 1.1

Let

$$f(\lambda) = |\lambda I + Q| \tag{A2.1}$$

where $Q = [q_{ij}]$ is a totally unimodular matrix of order n. We have, of course,

$$f(\lambda) > 0 \text{ for } \lambda \text{ sufficiently large} \tag{A2.2}$$

$$\begin{aligned}\Delta^{(1)}(\alpha_{max}) &= |I + \alpha_{max} Q| \\ &= \alpha_{max}^n \, |\alpha_{max}^{-1} I + Q| \\ &= \alpha_{max}^n \, f(\alpha_{max}^{-1})\end{aligned} \tag{A2.3}$$

Suppose that $f(\lambda) = 0$ does not have a positive root. Then by (A2.2) we have

$$\Delta^{(1)}(\alpha_{max}) > 0 \tag{A2.4}$$

So we consider the case where $f(\lambda) = 0$ has a positive root. Let λ_0 (> 0) be the maximum positive root of $f(\lambda) = 0$. Then we have

$$f(\lambda) \geq 0 \quad \text{for} \quad \lambda_0 \leq \lambda < \infty.$$

Note that λ_0 is an eigenvalue of the matrix $-Q$.

The following lemma is well-known [8].

Lemma A.1. Let $B = [b_{ij}]$ be an nxn complex matrix and let λ_μ ($\mu=1,\ldots,n$) be eigenvalues of the matrix B. Then

$$|\lambda_\mu| \leq \min \left[\max_{1 \leq i \leq n} \left[\sum_{j=1}^{n} |b_{ij}| \right], \max_{1 \leq j \leq n} \left[\sum_{i=1}^{n} b_{ij} \right] \right] \tag{A2.6}$$

Since Q is totally unimodular, we have by definition

$$q_{ij} = 0, \pm 1 \tag{A2.7}$$

Therefore it follows from Lemma A.1, (A2.6) and (A2.7) that

$$0 < \lambda_0 \leq n. \tag{A2.8}$$

From (A2.3), (A2.5) and (A2.8) we conclude that

$$\Delta^{(1)}(\alpha_{max}) \geq 0 \quad \text{for} \quad \alpha_{max} \leq 1/n \tag{A2.9}$$

Now consider the circuit described in Corollary 1.1. From the associated graph of the circuit we can obtain in general many controlled source graphs.

Some may contain k pairs of controlled source branches and others contain less than k pairs of controlled source branches. However if $\alpha_{max} < 1/k$ then

$$|I + \alpha_{max} Q| \geq 0$$

for the matrix Q of order n ($\leq$ k) since 1/n > 1/k (> α_{max}).

This completes the proof of Corollary 1.1.

Appendix 3. A Comment on Corollary 1.1

For example, consider a circuit N from which we can derive a controlled source graph G_0 such that the main part of the fundamental cutset or loop matrix of G_0 is given by

$$Q = \begin{bmatrix} -1 & -1 & \dots & -1 \\ -1 & -1 & \dots & -1 \\ \vdots & & & \\ -1 & \dots & \dots & -1 \end{bmatrix} \tag{A3.1}$$

It is apparent that such a graph G_0 and a circuit N exists. For the matrix Q of (A3.1), we have

$$|I + \alpha_{max} Q| < 0 \quad \text{for} \quad \alpha_{max} > 1/k. \tag{A3.2}$$

Appendix 4. Proof of Corollary 2.1

Let

$$f(\lambda) = |\lambda I + AQ| \tag{A4.1}$$

where A and Q are given by (1) and (4). For λ sufficiently large, we have

$$f(\lambda) > 0 \tag{A4.2}$$

Suppose $f(\lambda) = 0$ has a positive root (otherwise we have $f(\lambda) > 0$ for λ real) and let λ_0 be the maximum positive root. Of course λ_0 is an eigenvalue of the matrix $-AQ$. Then we have

$$f(\lambda) \geqq 0 \quad \text{for} \quad \lambda_0 \leqq \lambda < \infty \tag{A4.3}$$

Let $B = [b_{ij}] = AQ$. Then by (9) we have

$$\sum_{i=1}^{m} |b_{ij}| \leqq \sum_{i=1}^{n} \alpha_{\mu_i \max} \leqq \sum_{i=1}^{k} \alpha_{i\max} \leqq 1 \tag{A4.4}$$

Therefore it follows from Lemma A.1 in Appendix 2 and (A4.4) that

$$\lambda_0 \leqq 1 \tag{A4.5}$$

In particular by setting $\lambda = 1$ in (A4.3), we have

$$|I + AQ| \geqq 0, \tag{A4.6}$$

completing the proof of Corollary 2.1.

Appendix 5. Proof of Theorem 3

Let us consider the CCCS one-port N in Fig. A.1 where $N^{(1)}$ (resp., $N^{(2)}$) is a one-port (resp., two-port) with a unique solution. Let $N^{(1)}$ (open) (resp., $N^{(1)}$ (short)) denote the one-port $N^{(1)}$ with the input port open-circuited (resp., short-circuited) and $N^{(2)}$ (open; short) denote the two-port $N^{(2)}$ with the input port open-circuited and the output port short-circuited. Similarly $N^{(2)}$ (short; open) etc., are defined.

First consider N(open). By applying operations (I) and (II) to the associated graph of N (open) we get some graphs G_0 in which input branches of CCCS's form a tree of G_0. The graph G_0 can be partitioned into two parts; one part belongs to $N^{(1)}$ and is called $G_0^{(1)}$ and another to $N^{(2)}$ and is called $G_0^{(2)}$ (see Fig. A.2). There exist two cases to be considered.

(i) The input branches of CCCS's form a tree of $G_0^{(1)}$.

In this case the main part Q of the fundamental cutset matrix of G_0 is given by

$$Q = \left[\begin{array}{c|c} Q_1 & /// \\ \hline 0 & Q_2 \end{array}\right] \tag{A5.1}$$

where the rows of Q_1 (resp., Q_2) correspond to the input branches in $G_0^{(1)}$ (resp., $G_0^{(2)}$). By (A5.1) we have

$$|I+AQ| = |I+A_1Q_1|\ |I+A_2Q_2|$$

Since Q_1 (resp., Q_2) is the main part of the fundamental cutset matrix of the graph $G_0^{(1)}$ (resp., $G_0^{(2)}$ with 2-2' short-circuited) which is obtained from $N^{(1)}$ (open) (resp., $N^{(2)}$ (open; short)), we have

$$|I+A_1Q_1| \geqq 0 \tag{A5.2}$$

and

$$|I+A_2Q_2| \geqq 0 \tag{A5.3}$$

by definition. Therefore we conclude

$$|I+AQ| \geqq 0 \tag{A5.4}$$

(ii) The input branches of CCCS's form a tree of $G_0^{(2)}$.

In this case we have

$$Q = \left[\begin{array}{c|c} Q_1 & 0 \\ \hline /// & Q_2 \end{array}\right]$$

where Q_1 (resp., Q_2) is the main part of the fundamental cutset matrix of the graph $G^{(1)}$ with 2-2' short-circuited (resp., $G_0^{(2)}$) which is obtained from $N^{(1)}$ (short) (resp., $N^{(2)}$ (open; open)). Therefore it follows that

$$|I+A_1Q_1| \geqq 0$$

$$|I+A_2Q_2| \geqq 0$$

Thus we have

$$|I+AQ| \geqq 0 \tag{A5.5}$$

From (A5.4) and (A5.5) we conclude that N (open) has a unique solution.

Next consider N (short). Similar discussion holds in this case and we conclude that N (short) has a unique solution. Thus we have Theorem 3.

Appendix 6. Proof of Theorem 5

Let G_0 be any controlled source graph obtained from the associated graph of the circuits in Fig. 9 by applying operations (I) and (II) and assume that the input branches of the CCCS's form a tree of G_0.[†] Then, to prove Theorem 5, it is sufficient to show that $\Delta^{(1)}(G_0) = |I+Q| \geqq 0$. Here $[I \vdots Q]$ is the fundamental cutset matrix of G_0. Referring to Remark 1 in Section III we can assume without loss of generality that G_0 satisfies:

Assumption A.1. No input branch is a bridge.

Assumption A.2. No output branch is a self-loop.

We will prove the theorem for each circuit in Fig. 9.

Case 1: Circuit in Fig. 9(a).

Without loss of generality we assume G_0 consists of n pairs of CCCS branches, $(1,\hat{1})$, $(2,\hat{2})$, ..., $(n,\hat{n})$. By referring to Assumptions A.1 and A.2 and the configuration in Fig. 9(a) and by noting the input branches form a tree of G_0, we see that in G_0 each input branch is in parallel with only one output branch. Thus G_0 can be drawn generally as in Fig. A.3 where $(\hat{\mu}_1,\hat{\mu}_2,\ldots,\hat{\mu}_n)$ is a permutation of $(\hat{1},\hat{2},\ldots,\hat{n})$. The directions of i and $\hat{\mu}_i$ in Fig. A.3 may or may not be identical. The main part (that is, Q) of the fundamental cutset matrix of the graph in Fig. A.3 can be written (by remembering the CCCS branches appropriately) as the direct sum of the following types of matrices.

$$\left.\begin{aligned} S_1 &= [\varepsilon] \\ S_2 &= \begin{bmatrix} 0 & \varepsilon_1 & & & \\ & 0 & \varepsilon_2 & & \bigcirc \\ & & 0 & \ddots & \\ & \bigcirc & & \ddots & \varepsilon_{p-1} \\ \varepsilon_p & & & & 0 \end{bmatrix} \quad (p \geqq 2) \end{aligned}\right\} \tag{A6.1}$$

$\varepsilon, \varepsilon_i = \pm 1.$

Since

$$\begin{aligned} &1 + \varepsilon = 0 \text{ or } 2 \\ &|I + S_2| = 1 + (-1)^{p+1}\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p = 0 \text{ or } 2 \end{aligned} \tag{A6.2}$$

[†]The phrase "of the CCCS's" will be omitted hereafter.

we have

$$|I+Q| \geqq 0 \tag{A6.3}$$

for G_0. We therefore conclude that we cannot derive a controlled source graph G_0 such that $\Delta^{(1)}(G_0) < 0$ for the circuit in Fig. 9(a).

Case 2: Circuit in Fig. 9(b).

Without loss of generality we can replace each L_i $(i=0,1,\ldots,m)$ in Fig. 9(b) by a single resistor branch, r_i. When we apply operations (I) and (II) to the associated graph there occur three cases:

(i) All r_i $(i=0,1,\ldots,m)$ are short-circuited.

This case corresponds to a special case of Fig. 9(a). Therefore we cannot derive any controlled source graph G_0 such that $\Delta^{(1)}(G_0) < 0$.

(ii) Some of r_i $(i=1,2,\ldots,m)$ are open-circuited.

In this case we cannot obtain a graph G_0 in which the input branches form a tree of G_0. For, if r_i is open-circuited, then an input branch cannot reach one of the terminal points of the branch r_i, unless all output branches connected to the branch r_i are open-circuited. Even in the latter case an isolated point remains (and therefore G_0 cannot be connected). So we don't need to consider this case.

(iii) r_0 is open-circuited and r_i $(i=1,2,\ldots,m)$ are short-circuited.

As before we assume without loss of generality that G_0 consists of n pairs of CCCS branches, $(1,\hat{1})$, $(2,\hat{2})$, ..., $(n,\hat{n})$. Let the common initial vertex of all input branches be v_0 and a terminal vertex of the branch i be v_i. Since the input branches form a tree, all v_i $(i=0,1,\ldots,n)$ are distinct. All output branches have a common vertex (grounded terminal), say v. The vertex v may be v_0. The case $v = v_0$ reduces to that in Fig. 9(a). Therefore we assume $v \neq v_0$. Without loss of generality we assume $v = v_n$. The terminal vertex of each output branch is one of v_i $(i=0,1,\ldots,n-1)$. By referring to Assumptions A.1 and A.2, it suffices to consider only the following two cases:

(a) Each v_i $(i=0,1,\ldots,n-1)$ is a terminal vertex of some output branch (see Fig. A.4).

In this case the main part, Q, of the fundamental cutset matrix of G_0 can be written as

$$
Q = \left[\begin{array}{c|c|c|c}
Q_1 & \bigcirc & \bigcirc & \bigcirc \\
\hline
\bigcirc & Q_2 & \bigcirc & \bigcirc \\
\hline
\bigcirc & \bigcirc & \ddots & \bigcirc \\
\hline
\bigcirc & \bigcirc & \bigcirc & \\
\cline{1-3}
-1-1\ldots & -1-1\ldots & -1-1\ldots-1 & Q_0
\end{array}\right] \tag{A6.4}
$$

after we renumber the CCCS branches appropriately. Here, Q_1, Q_2, ... are matrices of the type S_1 or S_2 (in which ε and ε_i are all 1) in Case 1 and Q_0 is

$$
Q_0 = \begin{bmatrix}
0 & 1 & & & & \\
& 0 & 1 & & \bigcirc & \\
& & 0 & \ddots & & \\
& \bigcirc & & \ddots & \ddots & \\
& & & & 0 & 1 \\
-1 & -1 & -1 & \ldots & & -1
\end{bmatrix} . \tag{A6.5}
$$

Since

$$
\begin{aligned}
|I+Q_i| &= 0 \text{ or } 2 \quad (i=1,2,\ldots) \\
|I+Q_0| &= 0 \text{ or } 1
\end{aligned} \tag{A6.6}
$$

we have

$$
|I + Q| \geqq 0.
$$

(b) The vertex v_0 is not a terminal vertex of any output branch.

In this case G_0 can be drawn as in Fig. A.5, where $(\mu_1,\mu_2,\ldots,\mu_n)$ is a permutation of $(\hat{1},\hat{2},\ldots,\hat{n})$.

Assume that the main part Q of the fundamental cutset matrix of G_0 can be written as

$$
Q = \left[\begin{array}{c|c}
Q_1 & 0 \\
\hline
/\!/\!/\!/ & Q_2
\end{array}\right] \tag{A6.7}
$$

where Q_1 and Q_2 are square matrices. Then

$$|I+Q| = |I+Q_1|\ |I+Q_2|$$

holds. If $|I+Q| < 0$, then either $|I+Q_1| < 0$ or $|I+Q_2| < 0$ holds. Since both $[I \vdots Q_1]$ and $[I \vdots Q_2]$ are the fundamental cutset matrices of the graphs obtained by applying operation $\mathcal{Z}(\cdot)$ to G_0, it suffices to consider Q_1 and Q_2 instead of Q itself. So we can assume without loss of generality that:

<u>Assumption A.3</u>. Q cannot be written as in (A6.7) even by renumbering the CCCS branches appropriately. Then in Fig. A.5 we have to consider four cases depending on the value of μ_1 and μ_2.

(b.1) $\mu_1 = 1$ and $\mu_2 = n$

In this case by Assumption A.3 we have

$$n = 2$$

and Fig. A.5 becomes Fig. A.6. Since in this case

$$Q = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \tag{A6.8}$$

we have

$$|I+Q| = 1.$$

(b.2) $\mu_1 = 1$ and $\mu_2 \neq n$

In this case we can draw Fig. A.5 as Fig. A.7 without loss of generality. Then we have

$$Q = \begin{bmatrix} 1 & 1 & & & & \\ & 0 & 1 & & \bigcirc & \\ & & 0 & 1 & & \\ & \bigcirc & & \ddots & \ddots & \\ & & & & 0 & 1 \\ -1 & -1 & \cdots & & -1 & -1 \end{bmatrix} \tag{A6.9}$$

from which we have

$$|I+Q| = \begin{bmatrix} 2 & 1 & & & & \\ & 1 & 1 & & \bigcirc & \\ & & 0 & 1 & & \\ & \bigcirc & & \ddots & \ddots & \\ & & & & 1 & 1 \\ -1 & -1 & \cdots & & -1 & 0 \end{bmatrix} = 1 > 0 \tag{A6.10}$$

(b.3) $\mu_1 \neq 1$ and $\mu_2 = n$

Without loss of generality Fig. A.5 can be redrawn as in Fig. A.8. Then the main part Q of the fundamental cutset matrix of G_0 is given by

$$Q = \begin{bmatrix} 0 & 1 & & & & & 1 \\ & 0 & 1 & & & & \\ & & 0 & 1 & & \bigcirc & \\ & & & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \\ & & & & & & 1 \\ 1 & & & & & 0 & 0 \\ -1 & -1 & \cdot & \cdot & \cdot & -1 & -1 \end{bmatrix} \tag{A6.11}$$

Therefore we have

$$I+Q = \left.\begin{bmatrix} 1 & 1 & & & & & \\ & 1 & 1 & & & & \\ & & 1 & 1 & & \bigcirc & \\ & & & \ddots & \ddots & & \\ & \bigcirc & & & \ddots & 1 & 0 \\ & & & & & 1 & 0 \\ -1 & -1 & \cdot & \cdot & \cdot & -1 & 0 \end{bmatrix}\right\} n$$

$$= \begin{cases} 1 & \text{when } n \text{ is even} \\ 0 & \text{otherwise} \end{cases} \tag{A6.12}$$

(b.4) $\mu_1, \mu_2 \neq 1, n$

This case is more complicated. Even in this case, however, we can verify by the same consideration mentioned above that $|I+Q| \geq 0$ holds.

In any case case we have $|I+Q| \geq 0$.

Case 3: Circuit in Fig. 9(c).

As in Case 2 we can replace each L_i $(i=0,1,\ldots,4)$ in Figs. 10(a) and (b) by a resistor branch, r_i. First we consider a simple case where m = 1 in Fig. 9(c) and where C_1 is a circuit in Fig. 10(b). Suppose that we apply operation $O(r_0)$ to the associated graph G. In addition we apply operations (I) and (II) such that the resulting graph G_0 has a tree composed of the input branches only. Then G_0 is one of three graphs in Fig. A.9. In Fig. A.9(a) we have

$$Q = \left[\begin{array}{cc|c} 1 & -1 & \\ -1 & 1 & \\ \hline \multicolumn{2}{c|}{\bigcirc} & \end{array}\right] \tag{A6.13}$$

and in Fig A.9(b) and (c) we have

$$Q = \left[\begin{array}{c|c} -1 & \\ \hline \bigcirc & \end{array}\right] \quad . \tag{A6.14}$$

In any case we have

$$|I+Q| = 0$$

Thus if we apply $O(r_0)$ to the associated graph, then we get $|I+Q| = 0$. Even when we replace the circuit in Fig. 10(b) by that in Fig. 10(a), the same conclusion holds if we apply $O(r_0)$.

We will next consider the general case where $m \geqq 1$. Even in this case we get the same conclusion so long as we apply $O(\cdot)$ to <u>some</u> r_0. Therefore it suffices to consider only the case where we apply $S(\cdot)$ to every r_0 in Figs. 10(a) and (b). Since this case is a special case of Fig. 9(a), we conclude $\Delta^{(1)}(G_0) \geqq 0$.

This completes the proof of Theorem 5.

Appendix 7. Proof of Lemma 1

Necessity: Substituting (11) into (14) we have

$$\begin{bmatrix} Z_{aa}+Z_{ab}A & Z_{ac} \\ Z_{ca}+Z_{cb}A & Z_{cc} \end{bmatrix} \begin{bmatrix} I_a \\ I_c \end{bmatrix} + \begin{bmatrix} 0 \\ F(I_c) \end{bmatrix} = - \begin{bmatrix} E_a \\ E_c \end{bmatrix} \tag{A7.1}$$

If $|Z_{aa}| = 0$, then the rank of the matrix $[Z_{aa}\ Z_{ab}\ Z_{ac}]$ is less than k because Z is a positive semi-definite matrix. This means that if $|Z_{aa}| = 0$, then the first k equations of (A7.1) are not satisfied for some E_a. Therefore we see that

$$|Z_{aa}| \neq 0 \tag{A7.2}$$

is necessary for the circuit to have a unique solution. Since Z as well as Z_{aa} is a semi-positive definite matrix, (A7.2) means

$$|Z_{aa}| > 0. \tag{A7.3}$$

Consider the linear case where

$$F(I_c) = DI_c$$
$$D = \mathrm{diag}[d_1, d_2, \ldots, d_m] \qquad 0 < d_i < \infty \tag{A7.4}$$

Then (A7.1) becomes

$$\begin{bmatrix} Z_{aa}+Z_{ab}A & Z_{ac} \\ Z_{ca}+Z_{cb}A & Z_{cc}+D \end{bmatrix} \begin{bmatrix} I_a \\ I_c \end{bmatrix} = - \begin{bmatrix} E_a \\ E_c \end{bmatrix} \tag{A7.5}$$

From (A7.5) we see the condition

$$\Gamma \neq 0 \text{ for all } A \text{ and all } D\ (> 0) \tag{A7.6}$$

is necessary for (A7.5) to have a unique solution. Under the condition (A7.3), (A7.6) implies (16a). Thus (16) is a necessary condition.

Sufficiency: Suppose that (16) is satisfied. Then (A7.3) holds. If $|Z_{aa}+Z_{ab}A| < 0$ for some A, then for sufficiently large D, $\Gamma < 0$ follows, a contradiction. Therefore we see $|Z_{aa}+Z_{ab}A| \geq 0$. If $|Z_{aa}+Z_{ab}A| = 0$ for some A, we can choose A such that $|Z_{aa}+Z_{ab}A| < 0$ because A belongs to an open set and because (A7.3) holds. Therefore we conclude that (16a) implies $|Z_{aa}+Z_{ab}A| > 0$.

From (A7.1) we have

$$F(I_c) + BI_c = b \tag{A7.7}$$

where

$$B = Z_{cc} - (Z_{ca}+Z_{cb}A)(Z_{aa}+Z_{ab}A)^{-1}Z_{ac}$$
$$b = - E_c + (Z_{ca}+Z_{cb}A)(Z_{aa}+Z_{ab}A)^{-1}E_a \tag{A7.8}$$

Under the condition $|Z_{aa}+Z_{ab}A| > 0$, Eq. (16a) means

$$|B+D| > 0 \text{ for all } D > 0. \tag{A7.9}$$

Applying Lemma A.1 in [3], we conclude that (A7.7) has a unique solution for all f_μ.

This completes the proof.

Appendix 8. Proof of Lemma 3

First we will describe the relation between H in (27) and $\tilde{Z}$. In order to calculate $\tilde{Z}$, we connect a current source j to each of the a-, b-, c-, and g-branches. Here $j_\mu = 0$ ($\mu=2k+m+1, 2k+m+2, \ldots, 2k+m+m_0$). Let the voltage of each current source be u_i. Then we have a standard cutset equation

$$-HU = J \tag{A8.1}$$

where

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \\ \\ u_{2k+m+m_0} \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} j_1 \\ j_2 \\ \vdots \\ j_{2k+m} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{A8.2}$$

From (A8.1) we see that $\tilde{Z}$ is the upper left-most $(2k+m) \times (2k+m)$ principal submatrix of H^{-1}. Now Γ_∞ is the determinant of the submatrix shaded by oblique lines in Fig. A.10.

There exists a well-known relation between the minors of a matrix and of its inverse matrix (see Lemma A.3 in [3]). By using Lemma A.3 in [3], we will describe Γ_∞ in terms of H.

We will write Γ_∞ symbolically as follows:

$$\Gamma_\infty = \tilde{Z}\begin{pmatrix} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & k_1 \\ 1 \oplus \alpha_{1max}(k+1), & 2 \oplus \alpha_{2max}(k+2) & \cdots & k_1 \oplus \alpha_{k_1 max}(k+k_1), \end{array} \\ k_1+1, \ldots, k, 2k+1, \ldots, 2k+m \\ k_1+1, \ldots, k, 2k+1, \ldots, 2k+m \end{array} \end{pmatrix} \tag{A8.3}$$

The upper line enclosed within the parenthesis denotes suffixes of rows included in Γ_∞ and the lower line denotes suffixes of columns included in Γ_∞. Here $\mu \oplus \alpha_{\mu max}(k+\mu)$ denotes that the μ-th column of Γ_∞ is the sum of the μ-th column of $\tilde{Z}$ and the product of the $(k+\mu)$-th column and $\alpha_{\mu max}$ (see Fig. A.10). Therefore Γ_∞ in (A8.3) can be expanded as follows:

$$\Gamma_\infty = \tilde{Z}\begin{pmatrix} 1,2,\ldots,k,\ 2k+1,\ \ldots,\ 2k+m \\ 1,2,\ldots,k,\ 2k+1,\ldots,2k+m \end{pmatrix}$$

$$+ \alpha_{1max}\tilde{Z}\begin{pmatrix} 1,2,\ldots,k,\ 2k+1,\ldots,2k+m \\ k+1,2,\ldots,k,\ 2k+1,\ldots,2k+m \end{pmatrix}$$

$$+ \alpha_{2max}\tilde{Z}\begin{pmatrix} 1,\ 2,\ 3,\ldots,k,\ 2k+1,\ldots,2k+m \\ 1,k+2,3,\ldots,k,\ 2k+1,\ldots,2k+m \end{pmatrix} \qquad \text{(A8.4)}$$

$$\vdots$$

$$+ \alpha_{k_1max}\tilde{Z}\begin{pmatrix} 1,\ldots,k_1-1,k_1,k_1+1,\ldots,k,2k+1,\ldots,2k+m \\ 1,\ldots,k_1,k+k_1,k_1+1,\ldots k,2k+1,\ldots 2k+m \end{pmatrix}$$

$$+ \alpha_{1max}\alpha_{2max}\tilde{Z}\begin{pmatrix} 1,\ 2,\ 3,\ldots,k,\ 2k+1,\ldots,2k+m \\ k+1,k+2,3,\ldots,k,\ 2k+1,\ldots,2k+m \end{pmatrix}$$

$$+ \ldots$$

$$\vdots$$

$$+ \alpha_{1max}\alpha_{2max},\ldots,\alpha_{k_1max}\tilde{Z}\begin{pmatrix} 1,\ 2,\ \ldots,k_1,\ k_1+1,\ldots,k,2k+1,\ldots,2k+m \\ k+1,k+2,\ldots,k+k_1,k_1+1,\ldots,k,2k+1,\ldots 2k+m \end{pmatrix}$$

By applying Lemma A.3 in [3] to each term in (A8.4), we get

$$\Gamma_\infty = |H|^{-1}\left[H\begin{pmatrix} k+1,\ldots,2k,2k+m+1,\ldots,2k+m+m_0 \\ k+1,\ldots,2k,2k+m+1,\ldots,2k+m+m_0 \end{pmatrix}\right.$$

$$- \alpha_{1max}H\begin{pmatrix} k+1,k+2,\ldots,2k,\ldots \\ 1,k+2,\ldots,2k,\ldots \end{pmatrix}$$

$$- \alpha_{2max}H\begin{pmatrix} k+1,k+2,k+3,\ldots,2k,\ldots \\ k+1,\ 2,\ k+3,\ldots,2k,\ldots \end{pmatrix} \qquad \text{(A8.5)}$$

$$- \ldots$$

$$\vdots$$

$$\left.\pm\ \alpha_{1max}\alpha_{2max},\ldots,\alpha_{k_1max}H\begin{pmatrix} k+1,\ldots,k+k_1,k+k_1+1,\ldots,2k,\ldots \\ 1,\quad k_1,\ k+k_1+1,\quad 2k,\ldots \end{pmatrix}\right]$$

Now (A8.5) can be expressed compactly as follows:

$$\Gamma_\infty = |H|^{-1}H\begin{pmatrix} k+1, & \ldots,k+k_1, & k+k_1+1, \\ -\alpha_{1max}(1)\oplus(k+1), & -\alpha_{k_1max}(k_1)\oplus(k+k_1), & k+k_1+1 \end{pmatrix}$$

$$\left.\begin{matrix} \ldots,2k,2k+m+1,\ldots,2k+m+m_0 \\ \ldots,2k,2k+m+1,\ldots,2k+m+m_0 \end{matrix}\right) \tag{A8.6}$$

This gives the determinant of the shaded part in Fig. 15.

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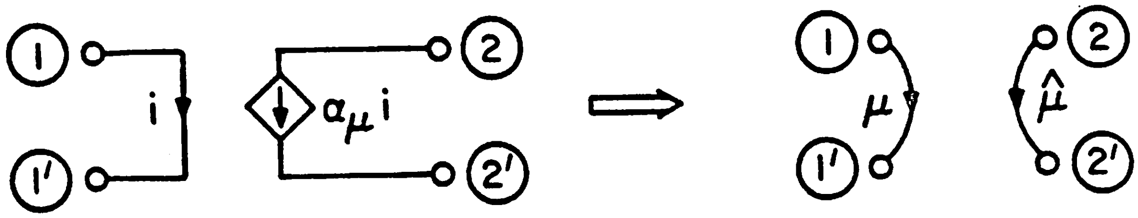
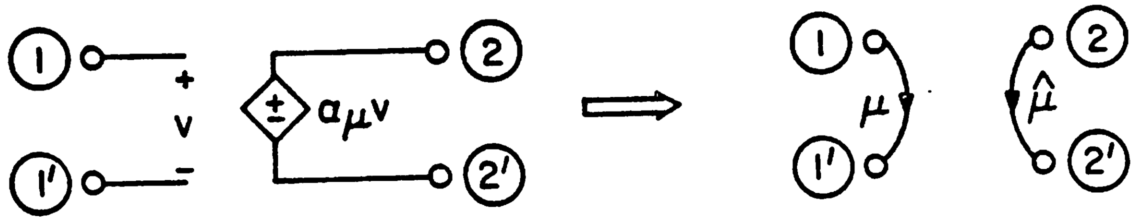
Figure Captions

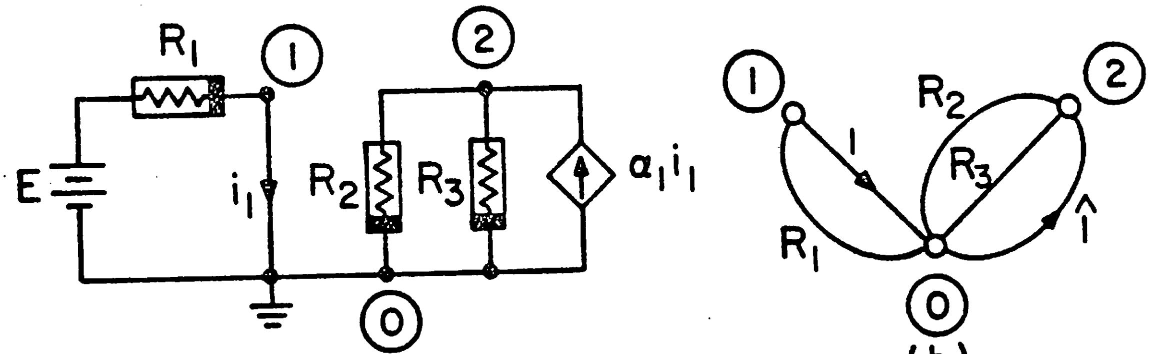


(a) $\alpha_\mu > 0$



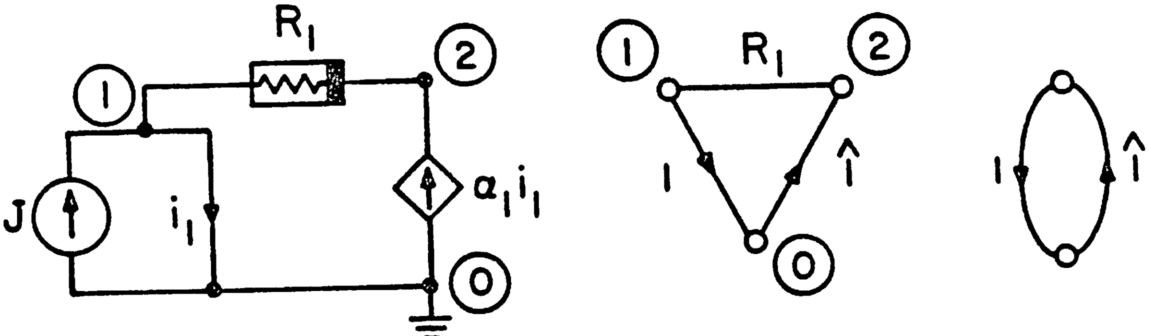
(b) $\alpha_\mu > 0$

Fig. 1



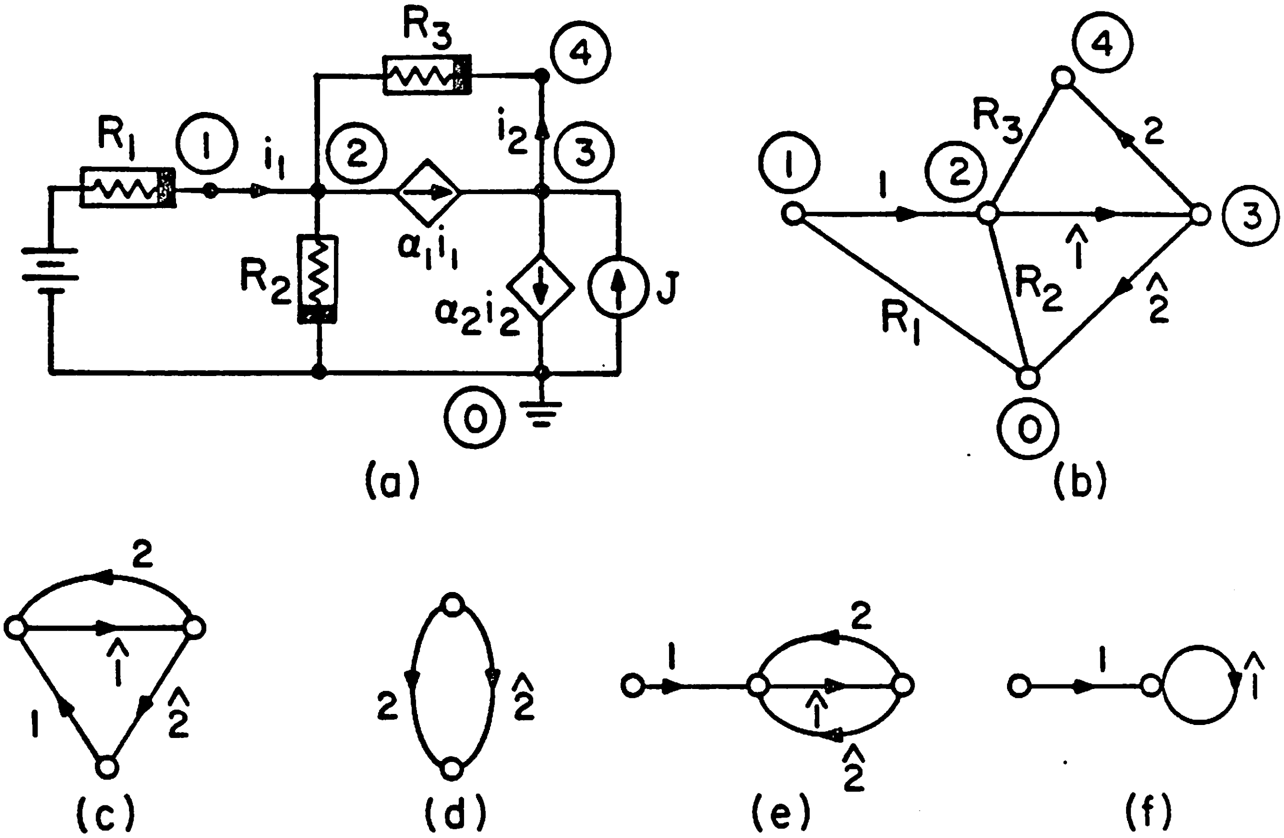
(a) (b)

Fig. 2



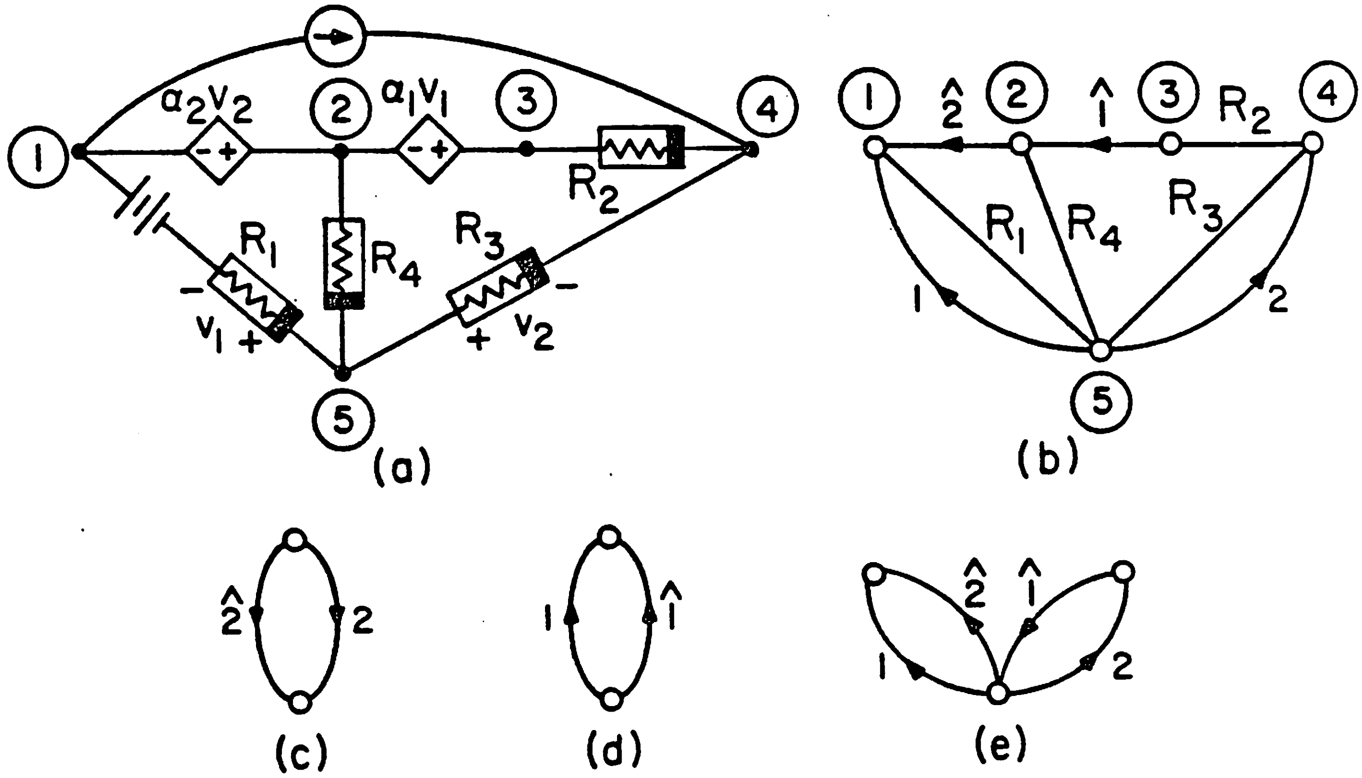
(a) (b) (c)

Fig. 3



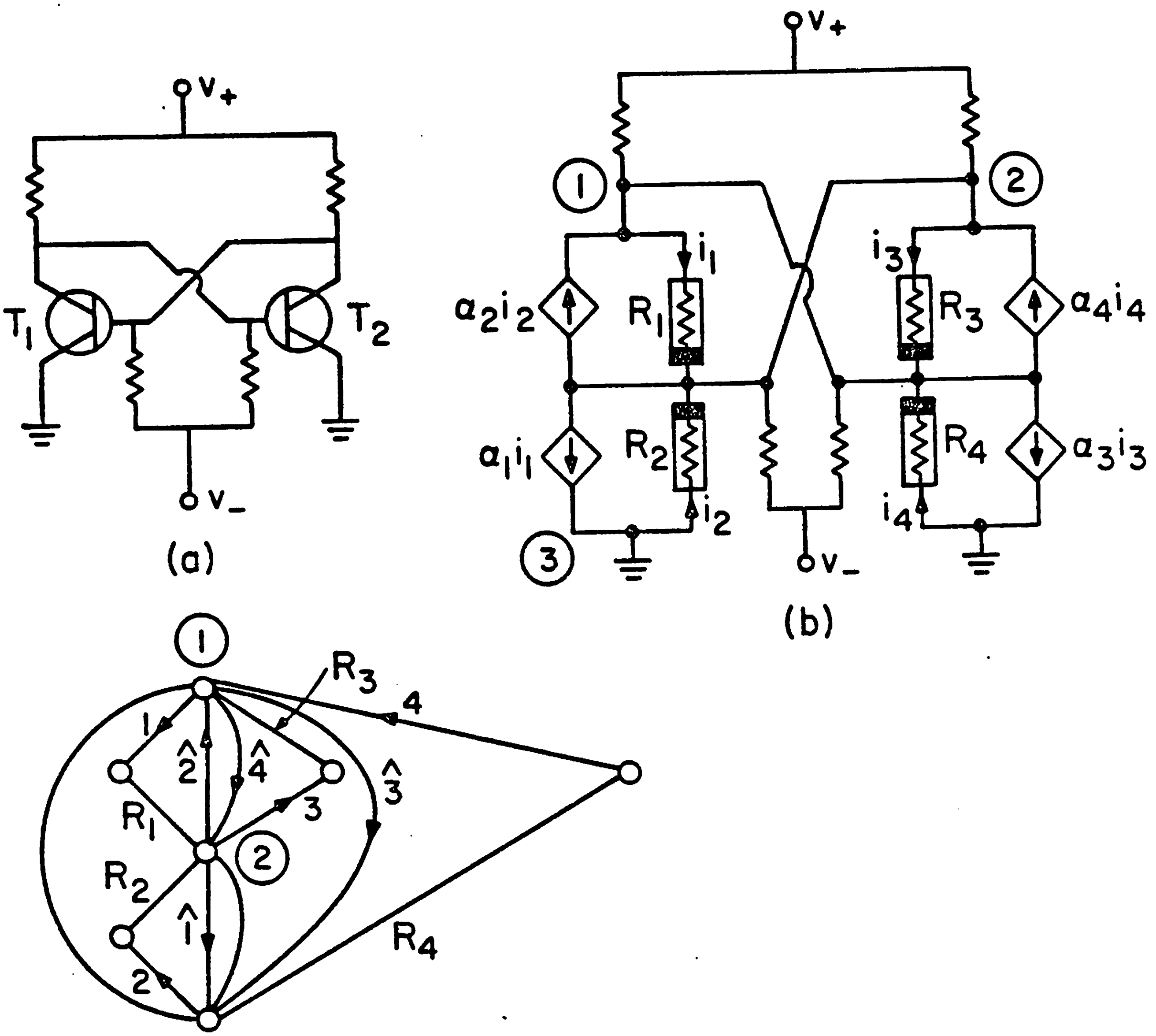
R3
4
R1
1
i1
2
i2
3
α1i1
R2
α2i2
J
0
(a)
4
R3
2
1
1
2
3
R2
R1
0
(b)
2
1
2
(c)
2
2
(d)
1
2
1
2
(e)
1
1
(f)

Fig. 4



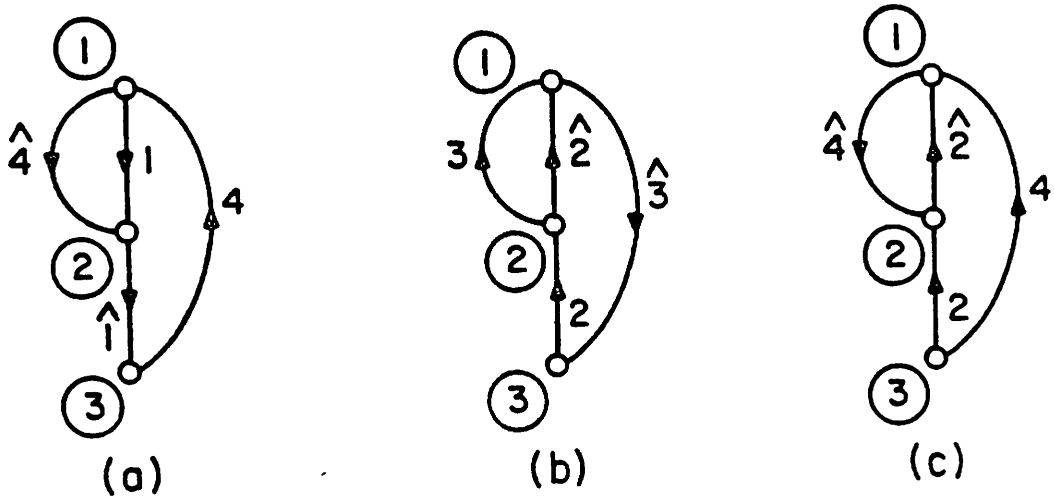
α2v2
2
α1v1
3
1
4
R2
R1
R4
R3
v1
v2
5
(a)
1
2
2
1
3
R2
4
R1
R4
R3
1
2
5
(b)
2
2
(c)
1
1
(d)
2
1
1
2
(e)

Fig. 5



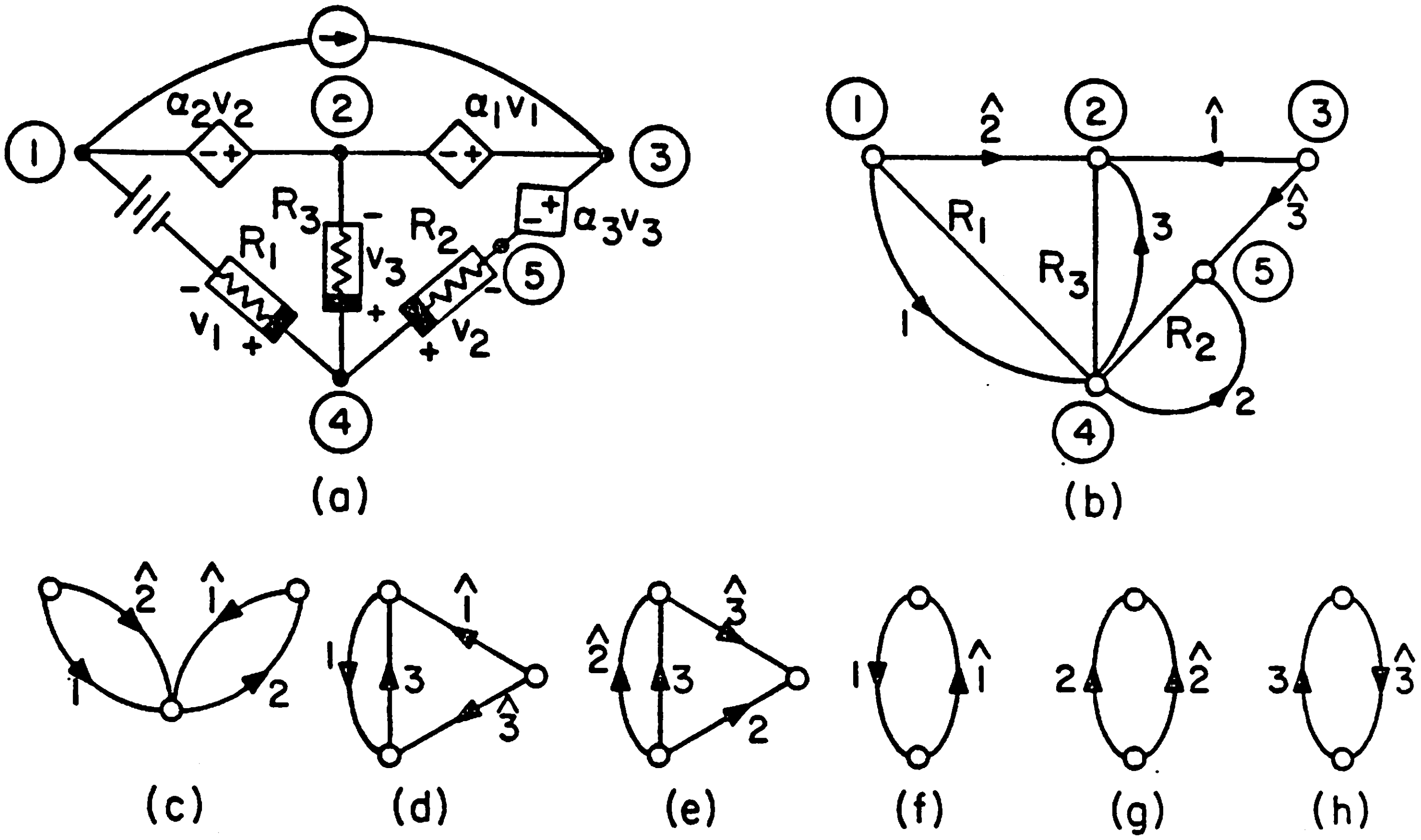
v+
T1
T2
v−
(a)
v+
1
2
i1
i3
α2i2
R1
R3
α4i4
α1i1
R2
R4
α3i3
i2
i4
3
v−
(b)
1
R3
4
1
2̂
4̂
3̂
3
R1
2
R2
1̂
R4
2
(c)

Fig. 6



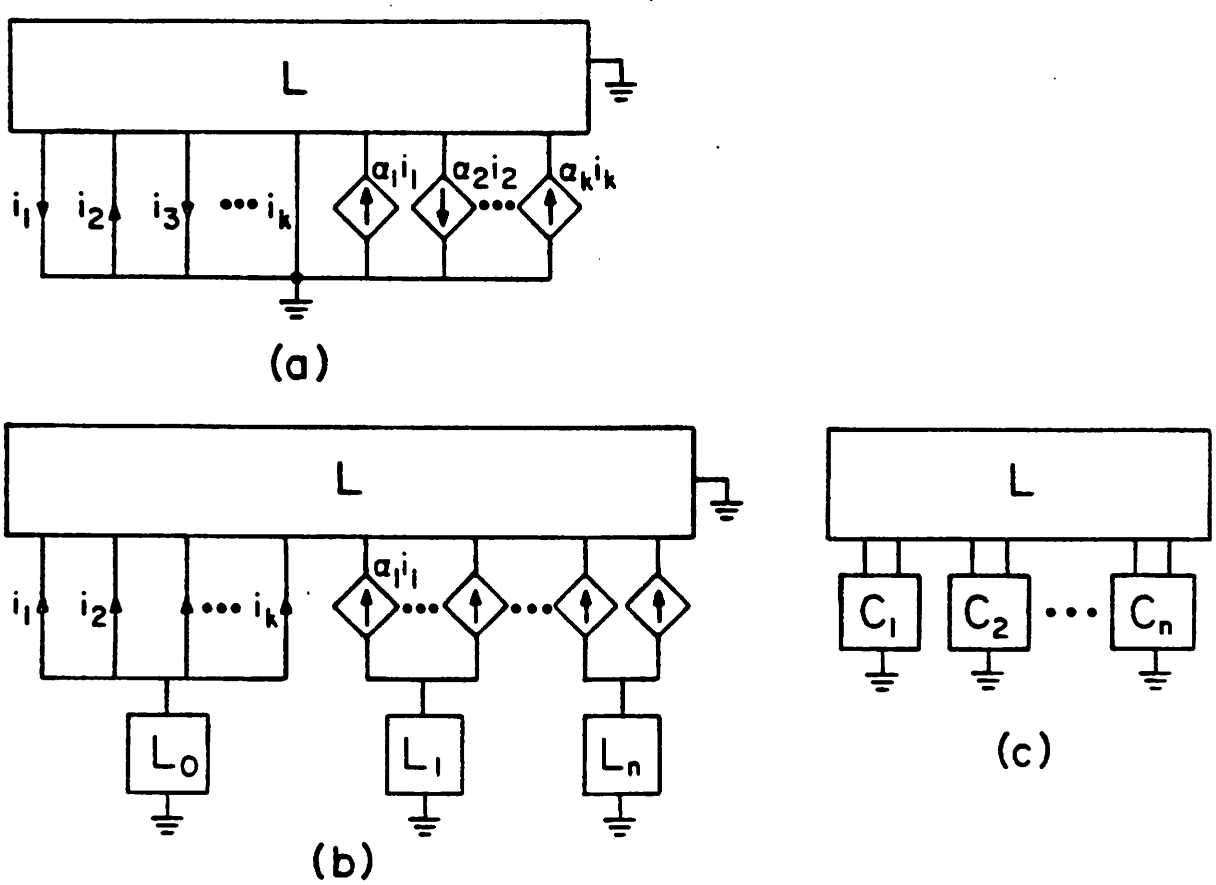
1
4̂
1
4
2
1̂
3
(a)
1
3
2̂
3̂
2
2
3
(b)
1
4̂
2̂
4
2
2
3
(c)

Fig. 7



1
2
3
4
5
$\alpha_2 v_2$
$\alpha_1 v_1$
$\alpha_3 v_3$
R_1
R_2
R_3
v_1
v_2
v_3
(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)

Fig. 8



L
i_1
i_2
i_3
i_k
$\alpha_1 i_1$
$\alpha_2 i_2$
$\alpha_k i_k$
(a)
L_0
L_1
L_n
C_1
C_2
C_n
(b)
(c)

Fig. 9

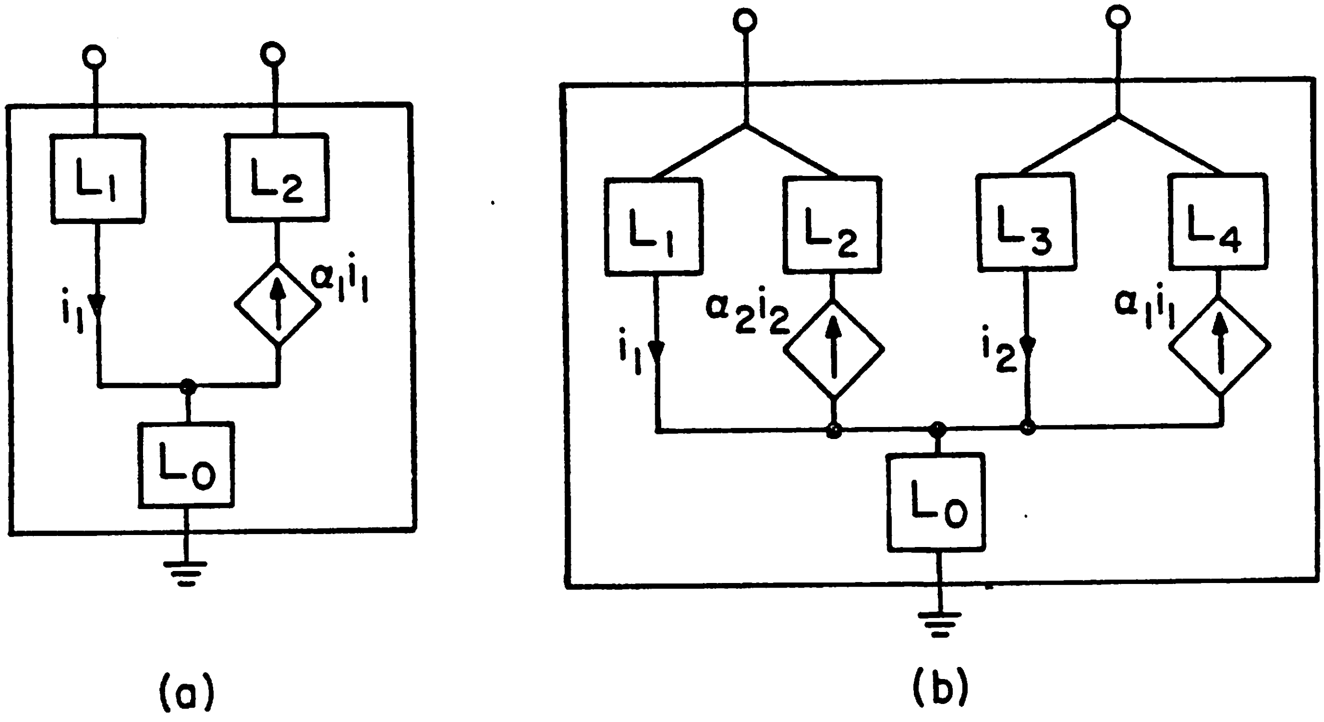


Fig. 10

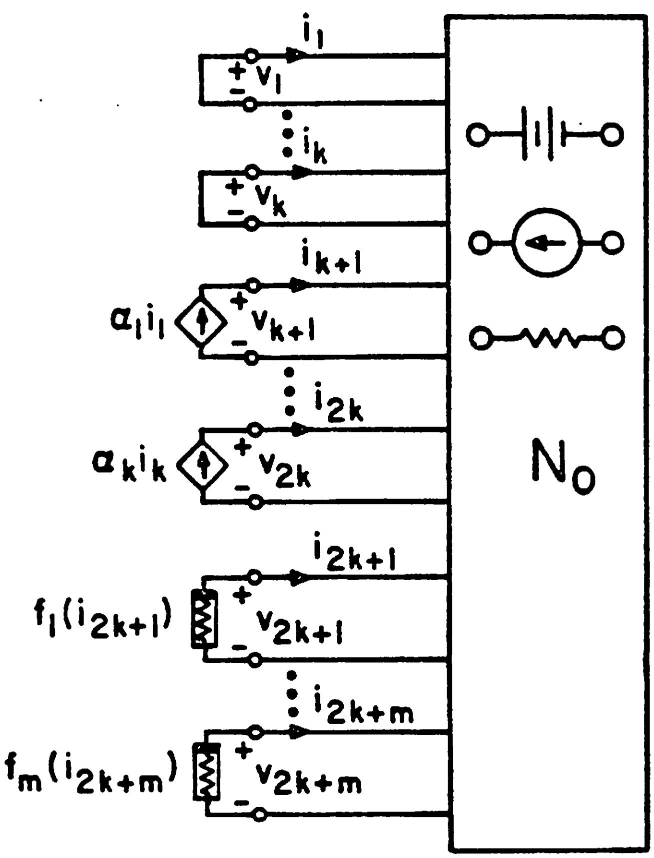


Fig. 11

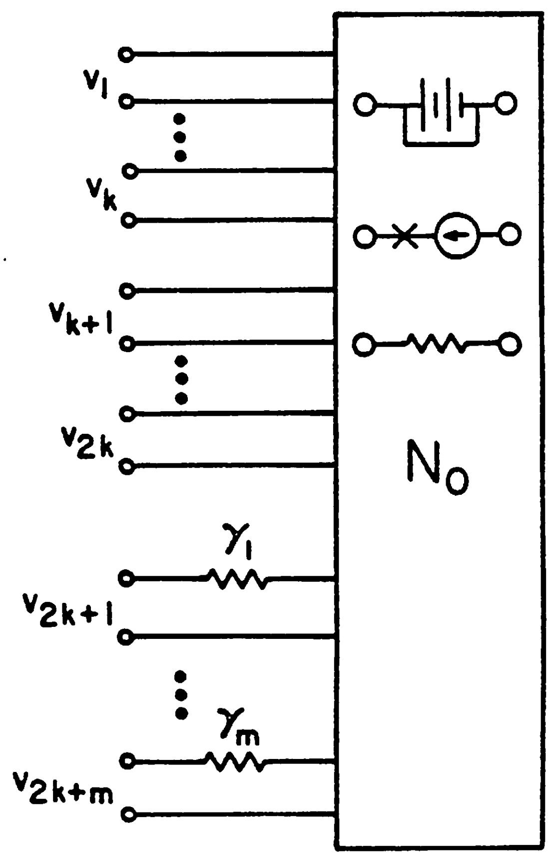


Fig. 12

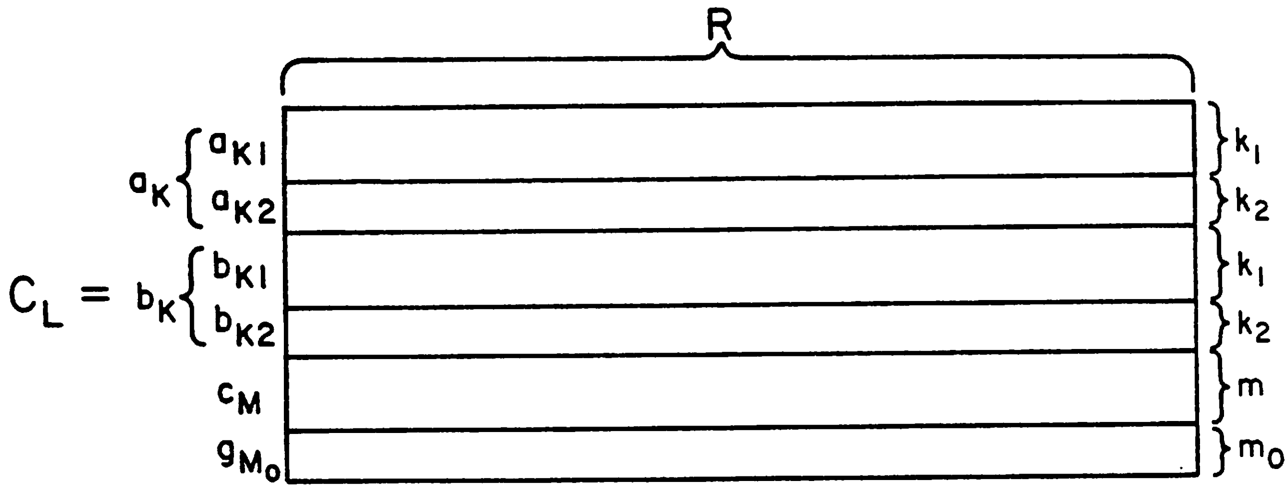


Fig. 13

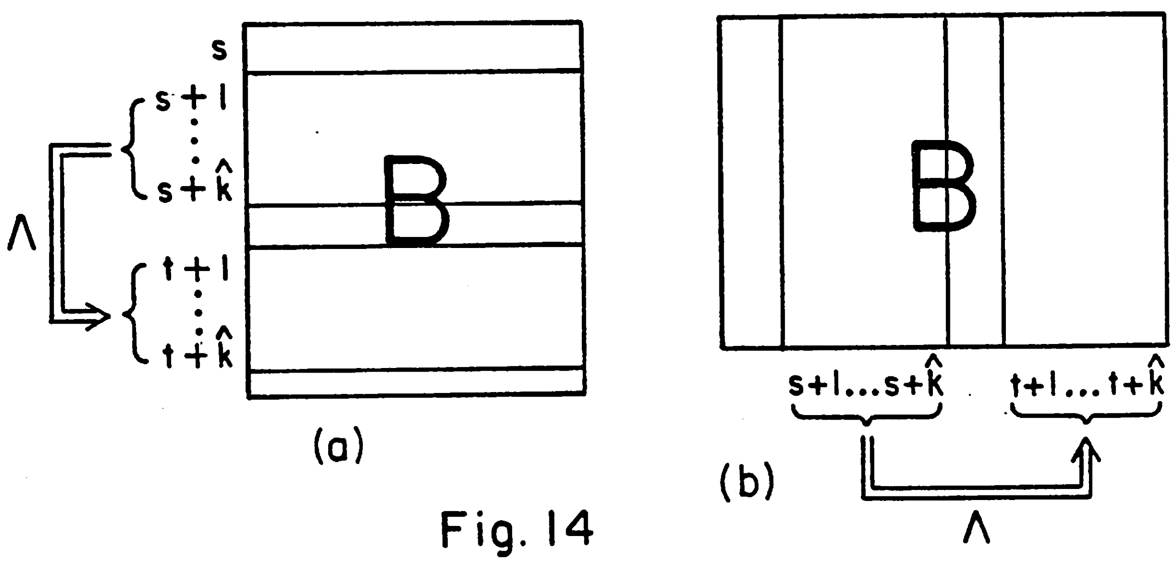


Fig. 14

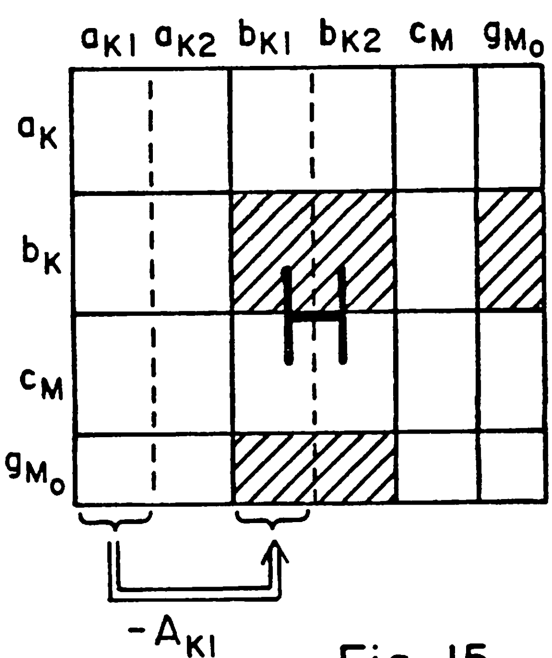


Fig. 15

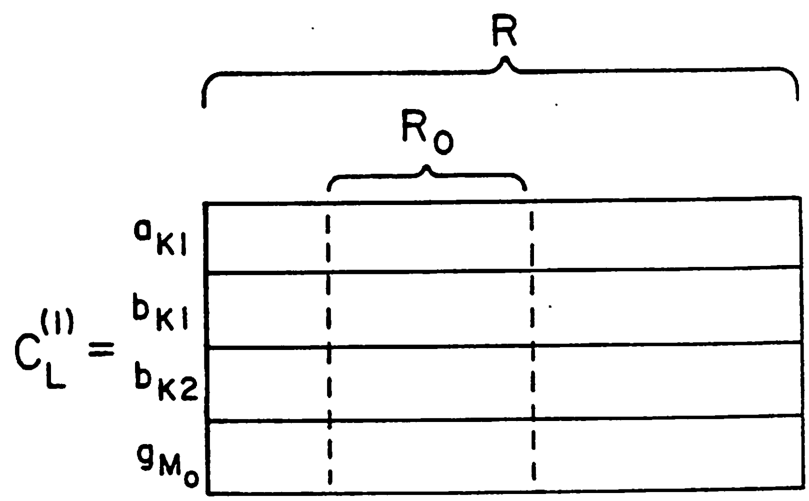


Fig. 16

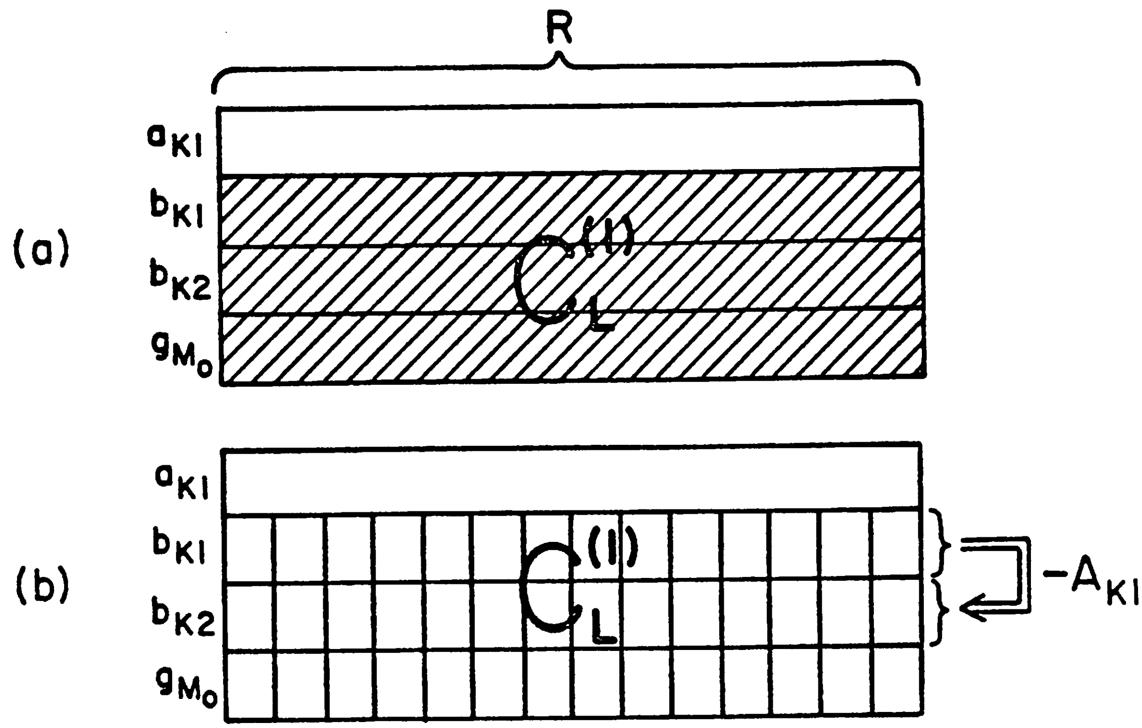


Fig. 17

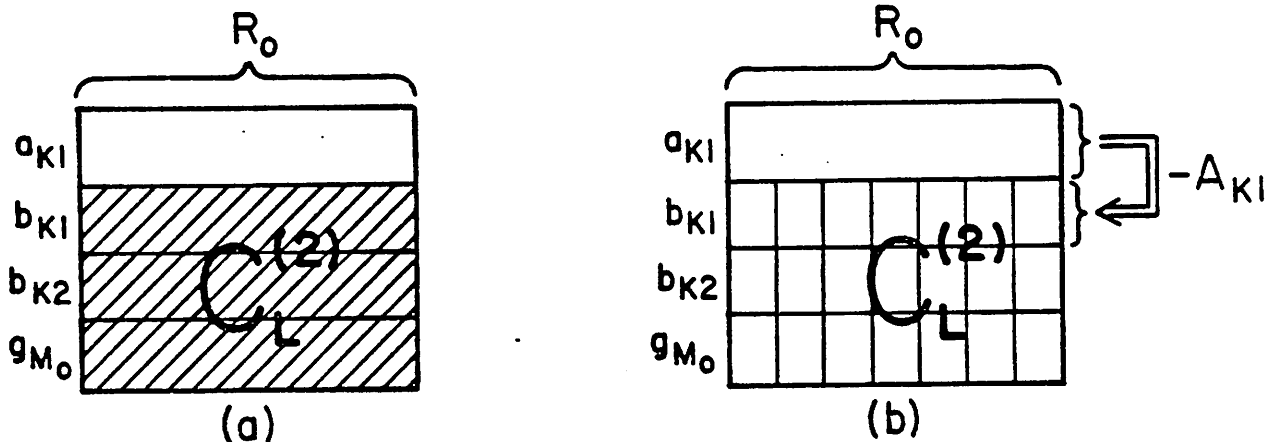


Fig. 18

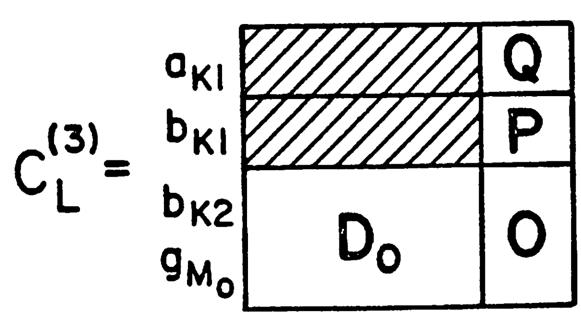


Fig. 19

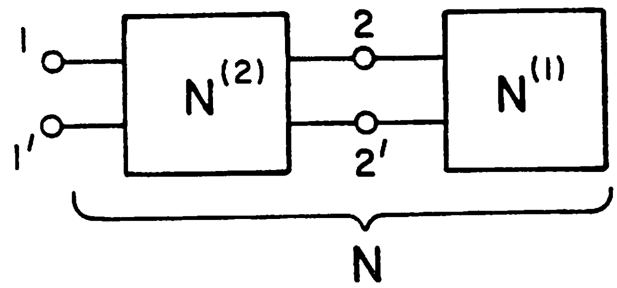


Fig. A.1

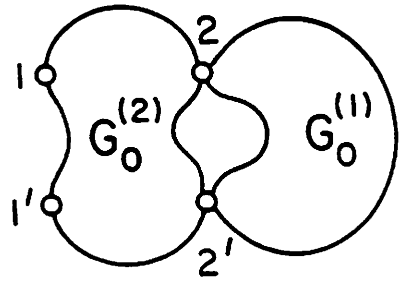


Fig. A.2

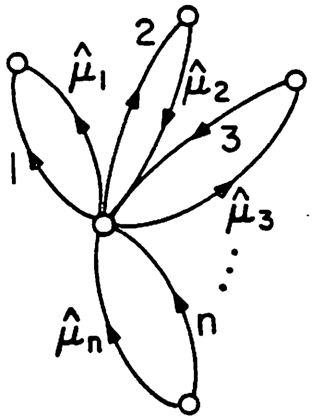


Fig. A.3

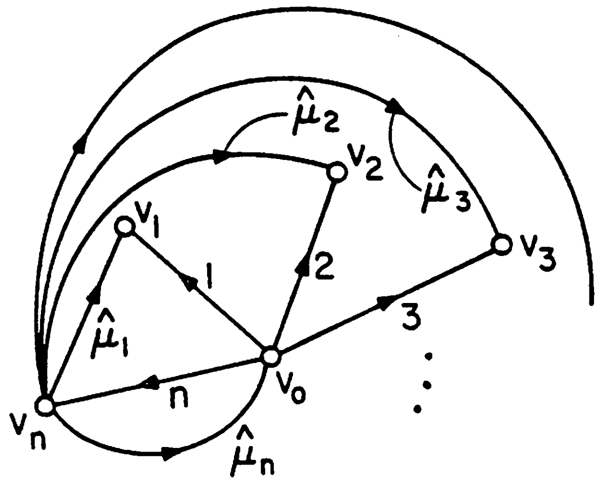


Fig. A.4

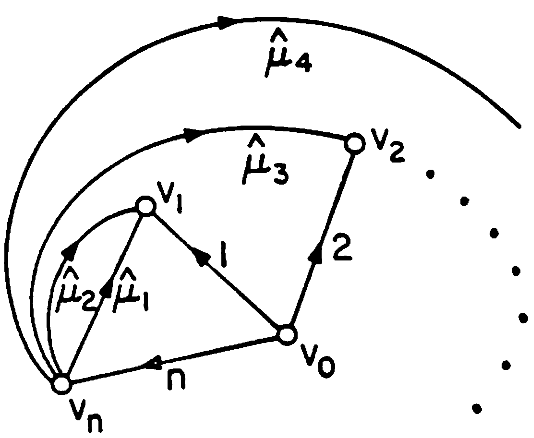


Fig. A.5

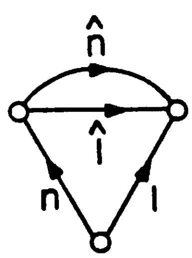


Fig. A.6

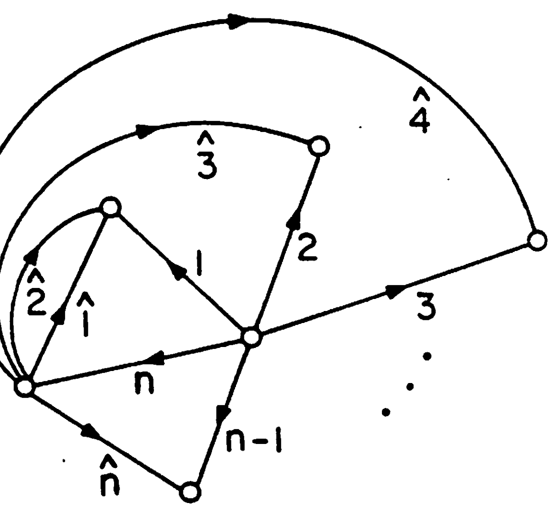


Fig. A.7

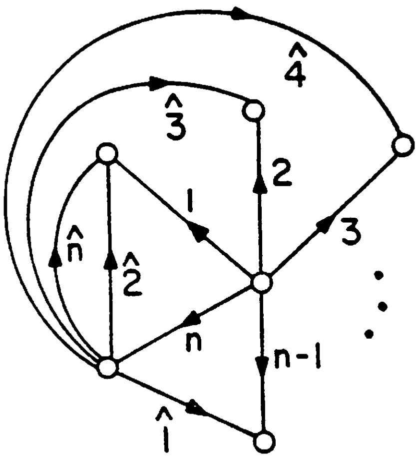


Fig. A.8

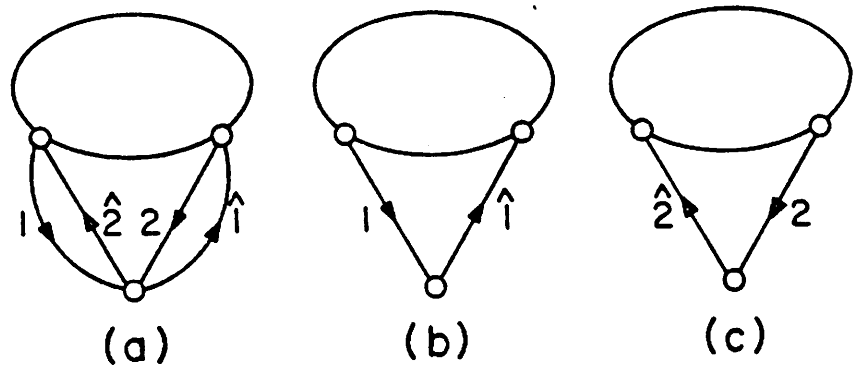


Fig. A.9

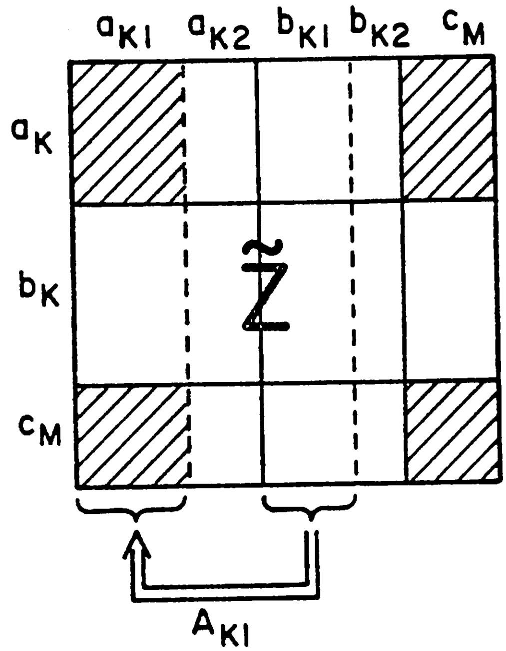


Fig. A.10