SOLVING RATIONAL MATRIX EQUATIONS IN THE STATE SPACE WITH APPLICATIONS TO COMPUTER AIDED CONTROL SYSTEM DESIGN

by

A. K. Packard and S. S. Sastry

Memorandum No. UCB/ERL M85/13

26 February 1985

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

SOLVING RATIONAL MATRIX EQUATIONS IN THE STATE SPACE WITH APPLICATIONS TO COMPUTER AIDED CONTROL SYSTEM DESIGN

A.K. Packard § *and S.S. Sastry* § §

Electronics Research Laboratory
University of California, Berkeley, CA 94720

ABSTRACT

A method of solving a class of linear matrix equations over various rings is proposed, using results from linear geometric control theory. An algorithm, successfully implemented, is presented, along with nontrivial numerical examples. Applications of the method to the algebraic control system design methodology are discussed.

1. INTRODUCTION

This paper discusses the solution of the equation

$$P(s)\,Q(s) = H(s) \tag{1.1}$$

where $P(s) \in R_o^{r\times m}(s)$ and $H(s) \in R_o^{r\times d}(s)$ are given, and $Q(s)$ is to be determined. Here $R_o(s)$ and $R_p(s)$ will respectively denote strictly proper and proper rational functions of s, with real coefficients. In particular, we will be concerned with solutions $Q(s)$ having elements in $R_p(s)$. Recently, necessary and sufficient conditions for the existence of solutions on various rings have been derived. Conditions for strictly proper solutions are obtained in [1], while [2] covers stable, strictly proper solutions. (By stable, we mean the poles of $Q(s)$ are in some 'good' region of the complex plane.) In [3], conditions for nonproper solutions are given in terms of almost invariant subspaces. We extend the ideas of [1] and [2], and derive conditions for proper solutions, both with and without the

---

§ Department of Mechanical Engineering. This material is based on work supported under a National Science Foundation Graduate Fellowship.

§ § Department of Electrical Engineering and Computer Science.

Research also supported by NASA under grant NGL2-243. We would like to thank Li Zexiang, Jan Willems, Jan van Schuppen, and Jan Schumacher for several useful discussions.

stability criteria. Along with each sufficient condition is an explicit state space description of a solution $Q(s)$.

Next, we present an algorithm which checks the solvability conditions, and if solvable, generates the state space description of $Q(s)$ mentioned above. This algorithm relies on state of the art numerical software [4], and *does not* involve polynomial or rational function manipulations. For the most part, it is based on the results found in [5] and [6], and is simply a direct application of the theorems presented here. Often times, when a solution exists, but is not stable, the algorithm provides information that can be exploited to modify $H(s)$ so that a stable solution will exist. This nice feature is very useful in the application, and is illustrated in an example.

The format is geometric (as in [7]), and consequently, the solvability conditions require that a certain subspace, say W (constructible from the data, i.e. $P(s)$ and $H(s)$) be contained in another subspace, say V, (also constructible from P and H). In light of the extreme sensitivity of the idea of exact subspace containment, we introduce a *measure of containment*, $m(W,V)$ which quantifies (in a useful way) just how far away W is from being contained in V. This notion is applied to the **Disturbance Decoupling Problem with Stability** (chapter 5 in [7]), and in turn provides a degree of approximate solvability of $P(s)\,Q(s) = H(s)$ by a stable $Q(s)$ when no exact stable solution exists.

Finally, we discuss an aspect of the algebraic multivariable design approach, [8] and [9], and indicate how our results may be incorporated in a computer aided design of linear control systems. For various feedback configurations, the algebraic methodology simultaneously yields a parametrization of all stabilizing compensators, and the corresponding achievable input/output transfer functions. Choosing a compensator to obtain a desired I/O map involves solving equations of the form (1.1) on the ring of proper, stable rational functions. Our techniques carry out this computation in the state space, and generate a realization of the resulting compensator. The proposed compensator would then be subjected to other design constraints, such as robustness properties, to determine if it is indeed an acceptable controller. A simple example is given to demonstrate this application.

2. CONTROLLED INVARIANCE

Consider the linear system described by

$$\dot{x} = Ax + Bu \tag{2.1}$$

with $x(t) \in R^n =: X$, the state, and $u(t) \in R^m =: U$, the control input. The controllable subspace, $\mathrm{Im}B + A(\mathrm{Im}B) + \cdots + A^{n-1}(\mathrm{Im}B)$, will be denoted $\langle A \mid \mathrm{Im}\, B \rangle$. In some instances we will append to this system an output $y(t) \in R^r =: Y$ given as

$$y = Cx \tag{2.2}$$

and let, $(\mathrm{Ker}\ C) \cap A^{-1}(\mathrm{Ker}\ C) \cap \cdots \cap (A^{n-1})^{-1}(\mathrm{Ker}\ C)$,the unobservable subspace, be denoted by $\langle \mathrm{Ker}\, C | A \rangle$.

Let $V \subset X$ be a subspace. Then the following statements are equivalent (see appendix for proofs); furthermore, any subspace V satisfying one of these conditions, and hence all of them, will be called a *controlled invariant subspace* (relative to the system described by (2.1)), and we will denote $\underline{V}$ as the set of all controlled invariant subspaces of (2.1).

i) for all $x_0 \in V$, there exists a continuous control input $u:[0,\infty) \to R^m$ such that with $x(0) = x_0$, this control renders $x(t) \in V$ for all $t \geq 0$.

ii) $AV \subset V + \mathrm{Im}B$

iii) there exists a linear map $F: X \to U$ such that $(A+BF)V \subset V$. For a given V, we denote all the F's that do this by $\mathbf{F}(V)$,and call any $F \in \mathbf{F}(V)$ a *friend* of V.

These definitions can be found in [1] and [7], along with more detailed analysis of additional properties. It is easily seen, however, that (i) or (ii) imply that if $V_1 \in \underline{V}$ and $V_2 \in \underline{V}$, then $V_3 := V_1 + V_2 \in \underline{V}$. This closure of $\underline{V}$ under subspace addition implies that for any subspace K, there exists a unique subspace V_K^* satisfying

a) $V_K^* \in \underline{V}$.

b) $V_K^* \subset K$

c) if $V \in \underline{V}$ and $V \subset K$, then $V \subset V_K^*$.

For this reason, V_K^* is called the supremal controlled invariant subspace contained in K. A numerically efficient algorithm for computing V_K^* can be found in [4]. This brings us to a useful lemma concerning V_K^*.

Lemma 1 Let K be any subspace, and define Ψ as the set of all $x_0 \in X$, such that there is a continuous control $u:[0,\infty) \to R^m$ so that, with $x(0) = x_0$, the resulting state trajectory remains in K for all $t \geq 0$. Then $\Psi = V_K^*$.

proof: By definition, $V_K^* \subset \Psi$. For the reverse inclusion, note first that Ψ is a subspace. Also, since the condition holds at $t=0$, we must have $\Psi \subset K$. Now let $x_0 \in \Psi$ and let $u(\cdot)$ be the control that keeps the trajectory in K. Let $T \geq 0$ and otherwise be arbitrary. Define $\mathbf{x}_0 := x(T)$ and $\mathbf{u}(\tau) := u(\tau + T)$ for $\tau \geq 0$. Then $\mathbf{u}:[0,\infty) \to R^m$ is continuous and with $\mathbf{x}(0)=\mathbf{x}_0$, will render $\mathbf{x}(\tau) = x(\tau+T) \in K$ for all $\tau \geq 0$. By definition then, $\mathbf{x}_0 \in \Psi$. With $\mathbf{x}_0 = x(T)$ and T arbitrary, we get $x(t)$ *actually* remains in Ψ, for all $t \geq 0$. Therefore Ψ *is* controlled invariant, so that by the supremal properties of V_K^*, $\Psi \subset V_K^*$.

This is the cleanest characterization of V_K^*, just all the initial conditions that can be held in K using a continuous control. Condition (iii) implies that the control need not be open loop – for any $F \in \mathbf{F}(V_K^*)$, state feedback Fx will also work.

Next we address the stability issue, using the concept of stabilizability subspaces, in much the same manner as [2]. First though, some notation: C_g will denote a symmetric subset of the complex plane, containing at least one point of the real axis, and C_b will denote its complement. When we use the terms *stabilizable*, *detectable*, etc, we will mean relative to some predetermined C_g. $R_{g,o}(s)$, and $R_{g,p}(s)$ will respectively denote stable, strictly proper rational functions, and stable, proper rational functions. Finally, a continuous function $z:[0,\infty) \to R^l$ will be called C_g *stable* if $L[z(t)] \in R_{g,o}^l(s)$.

3. STABILIZABILITY SUBSPACES

Let $V \subset X$ be a subspace, then the following are equivalent (see appendix); also, any subspace V satisfying these conditions will be called a *stabilizability subspace*, and we write $V \in \underline{V}_g$ where $\underline{V}_g$ is the set of all stabilizability subspaces.

i) for all $x_0 \in V$, there exists a C_g stable control $u:[0,\infty) \to R^m$, so that with $x(0) = x_0$, $x(t) \in V$ for all $t \geq 0$ and $x(\cdot)$ is C_g stable

ii) there exists a linear map $F:X \to U$ such that $(A+BF)V \subset V$, and $\sigma(A+BF|_V) \subset C_g$.

Note from (ii) that X is a stabilizability subspace if and only if the pair (A,B) is stabilizable. Therefore, if (A,B) is stabilizable, we can apply (i) to initial conditions of the form $\{e_1, e_2, \ldots, e_n\}$ where e_i is the i'th canonical basis vector of R^n. Collecting up each resulting C_g stable $x_i(\cdot)$, and $u_i(\cdot)$, we get $X(s) \in R_{g,o}^{n\times n}(s)$, and $U(s) \in R_{g,o}^{m\times n}(s)$ such that

$$X(s) = (sI-A)^{-1} + (sI-A)^{-1}BU(s).$$

The dual translates as; if (C,A) is detectable, then there exists $Z(s) \in R_{g,o}^{n\times n}(s)$ and $W(s) \in R_{g,o}^{n\times r}(s)$ satisfying

$$Z(s) = (sI - A)^{-1} + W(s)C(sI-A)^{-1}. \tag{3.1}$$

Again (i) implies that $\underline{V}_g$ is closed under subspace addition, and hence for any subspace K, there is a unique supremal stabilizability subspace contained in K, which we will denote by $V_{g,K}^{*}$. Reliable computation of $V_{g,K}^{*}$ is discussed in [4]. The analog of Lemma 1 is then;

<u>Lemma 2</u> Let K be a subspace, and define Ω as the set of all $x_o \in X$, such that there exists a C_g stable control input $u(\cdot)$, so that with $x(0) = x_o$, this control renders $x(t) \in K$ for all $t \geq 0$ and $x(\cdot)$ is C_g stable. Then $\Omega = V_{g,K}^{*}$.

proof: Exactly like lemma 1, noting that **u** and **x** will still be C_g stable signals. ▪

The next lemma uses (3.1) to relax the condition in Lemma 2 of having to verify that the state trajectory $x(\cdot)$ is C_g stable.

<u>Lemma 3</u> Let $V \subset X$ be a subspace and the pair (C,A) be detectable, and suppose that for all $x_o \in V$, there is a C_g stable input that results in $x(t) \in \operatorname{Ker} C$ for all $t \geq 0$. Then $V \subset V_{g,\operatorname{Ker} C}^{*}$.

proof: Let $x_o \in V$, and let $u(\cdot)$ be the C_g stable input that keeps $x(t)$ in $\operatorname{Ker} C$ for all t. Then taking Laplace transforms gives

$$\hat{x}(s) = (sI - A)^{-1}(x_o + B\hat{u}(s)) \tag{3.2}$$

while detectability gives

$$(sI - A)^{-1} = Z(s) - W(s)C(sI-A)^{-1} \tag{3.3}$$

with $Z(s)$ and $W(s)$ stable, strictly proper rational matrices. Substituting (3.3) into (3.2) gives

$$\hat{x}(s) = Z(s)(x_o + B\hat{u}(s)) - W(s)C\hat{x}(s) \tag{3.4}$$

where the last term is zero since $x(t) \in \operatorname{Ker} C$ for all t. With $Z(s)$ and $\hat{u}(s)$ stable and strictly proper, we see that $x(\cdot)$ *is* a C_g stable signal, so by definition of Ω above, $x_o \in \Omega = V_{g,\operatorname{Ker} C}^{*}$ giving that $V \subset V_{g,\operatorname{Ker} C}^{*}$, as desired. ▪

Consequently, under the detectability assumption, $V_{g,\operatorname{Ker} C}^{*}$ is all the initial conditions that can be held in Ker C, using a C_g stable control $u(\cdot)$.

Before proving our main results, we need one simple result from linear algebra.

Lemma 4 Let X, U and D be finite dimensional vector spaces, and suppose $V \subset X$ is a subspace, and $E{:}D \to X$ and $B{:}U \to X$ are linear maps. Then, there exists a linear map $L{:}D \to U$ such that $\mathrm{Im}(BL+E) \subset V$ *if and only if* $\mathrm{Im}\, E \subset V + \mathrm{Im}\, B$.

proof: $\to$ is obvious; for $\leftarrow$, let $\{d_1, d_2, \ldots, d_s\}$ be a basis for D. For each $i \in \underline{s}$, there exists a $v_i \in V$ and a $u_i \in U$ such that

$$Ed_i = v_i + Bu_i .$$

Define L as the linear map with action on D as $Ld_i = -u_i$, for all $i \in \underline{s}$. Then $(BL+E)d_i = v_i \in V$ for all $i \in \underline{s}$. ▪

4. SOLVABILITY CONDITIONS

Theorem 1 Consider the rational matrix equation

$$C(sI-A)^{-1}BQ(s) = C(sI-A)^{-1}E \qquad (4.1)$$

where $C \in R^{r\times n}$, $A \in R^{n\times n}$, $B \in R^{n\times m}$, $E \in R^{n\times d}$ are given, and $Q(s)$ (of dimension $m\times d$) is unknown. Then if $\mathbf{M}[J]$ denotes all matrices with elements in some given ring J, we have

i) there exists a $Q \in \mathbf{M}[R]$ solving (4.1) *if and only if* $\mathrm{Im}E \subset \langle \mathrm{Ker}\ C|A\rangle + \mathrm{Im}B$.

ii) there exists a $Q \in \mathbf{M}[R_o(s)]$ solving (4.1) *if and only if* $\mathrm{Im}E \subset V^*_{\mathrm{Ker}\ C}$.

iii) there exists a $Q \in \mathbf{M}[R_p(s)]$ solving (4.1) *if and only if* $\mathrm{Im}E \subset V^*_{\mathrm{Ker}\ C} + \mathrm{Im}B$.

Furthermore, if (C,A) is detectable then

iv) there exists a $Q \in \mathbf{M}[R_{g,o}(s)]$ solving (4.1) *if and only if* $\mathrm{Im}E \subset V^*_{g,\mathrm{Ker}\ C}$.

v) there exists a $Q \in \mathbf{M}[R_{g,p}(s)]$ solving (4.1) *if and only if* $\mathrm{Im}E \subset V^*_{g,\mathrm{Ker}\ C} + \mathrm{Im}B$.

proof:

i) $\to$ We must have $C(sI-A)^{-1}(BQ-E) \equiv 0$, which implies that there is a $Q{:}R^d \to R^m$ with $\mathrm{Im}(BQ-E) \subset \langle \mathrm{Ker}\ C|A\rangle$. By Lemma 4, then $\mathrm{Im}E \subset \langle \mathrm{Ker}\ C|A\rangle + \mathrm{Im}B$. ▪

$\leftarrow$ Construct, by Lemma 4, a $Q \in R^{m\times d}$ so that $\mathrm{Im}(BQ-E) \subset \langle \mathrm{Ker}\ C|A\rangle$. Then this Q works. ▪

ii) → We have a strictly proper, rational $Q(s)$ solving

$$C(sI-A)^{-1}B(-Q(s)) + C(sI-A)^{-1}E \equiv 0$$

so that for all $q \in R^d$ we get

$$C\left[(sI-A)^{-1}B(-Q(s)q) + (sI-A)^{-1}Eq\right] \equiv 0.$$

This implies that for any initial condition in ImE (here just Eq), there is a continuous control, namely the laplace inverse of $-Q(s)q$, that holds $x(t) \in$ Ker C for all $t \geq 0$. By Lemma 1, Im$E \subset V^*_{Ker\ C}$. ▪

← Choose any $F \in \mathbf{F}(V^*_{Ker\ C})$ and verify that

$$Q(s) := -F(sI-A-BF)^{-1}E$$

is a solution, since $C(sI-A-BF)^{-1}E \equiv 0$ for all $F \in \underline{F}(V^*_{Ker\ C})$. ▪

iii) → Since $Q(s)$ is a proper rational matrix, it can be written as $Q(s) = L + U(s)$, where $L \in \mathbf{M}[R]$ and $U(s) \in \mathbf{M}[R_o(s)]$. Then

$$C(sI-A)^{-1}B(-U(s)) + C(sI-A)^{-1}(-BL+E) \equiv 0.$$

Just as in (ii), this means that all initial conditions in Im$(-BL+E)$ can be held in Ker C using a continuous input. Hence, there exists an $L: R^d \to R^m$ such that Im$(-BL+E) \subset V^*_{Ker\ C}$, so that using Lemma 4 gives Im$E \subset V^*_{Ker\ C}$ + ImB.

← Construct $L \in R^{m \times d}$ so that Im$(BL+E) \subset V^*_{Ker\ C}$ and let $F \in \mathbf{F}(V^*_{Ker\ C})$. Then, verify that

$$Q(s) := -L - F(sI-A-BF)^{-1}(BL+E)$$

is a proper solution. ▪

iv) → Identically to (ii), using detectability and Lemma 3 instead of Lemma 1. ▪

← Choose $F \in \mathbf{F}(V^*_{g,Ker\ C})$ so that $\sigma(A+BF|_{V^*_{g,Ker\ C}}) \subset C_g$. Then $Q(s) := -F(sI-A-BF)^{-1}E$ is a solution, and since Im$E \subset V^*_{g,Ker\ C}$, it is also stable. Note that detectability is not used in this direction. ▪

v) $\rightarrow$ Similar to (iii), using Lemma 4 and (iv). ▪

$\leftarrow$ Choose $L \in R^{m \times d}$ so that $\mathrm{Im}(BL+E) \subset V^*_{g,Ker\ C}$, and let $F \in \mathbf{F}(V^*_{g,Ker\ C})$ with $\sigma(A+BF|_{V^*_{g,Ker\ C}}) \subset C_g$. Then $Q(s) := -L - F(sI-A-BF)^{-1}(BL+E)$ is a solution, and also is stable. Again, detectability is not used. ▪

5. MEASURE OF SUBSPACE CONTAINMENT

Let W and V be two subspaces of R^n, and let S be the orthogonal complement of V. Also, let U_1 be a real matrix (with n rows) whose orthonormal columns span W and let U_2 be a real matrix whose orthonormal columns span V. Then if $x = v + s$ with $v \in V$ and $s \in S$, we get that $(I - U_2 U_2^T)x = s$. Intuitively then, we define the *measure of containment of W by V* as

$$m(W,V) := \underset{|x| \le 1, x \in W}{\mathrm{supremum}} |(I - U_2 U_2^T)x|$$

which can be rewritten as

$$m(W,V) = \|(I - U_2 U_2^T) U_1\|.$$

Here $|\cdot|$ is the Euclidean norm in R^n, and $\|\cdot\|$ is the corresponding induced norm on linear maps. Note that $0 \le m(W,V) \le 1$ for all subspaces W and V, and that $m(W,V) = 0$ if and only if $W \subset V$. The above ideas are found in [10], where m is referred to as a *distance between subspaces* (the word distance carries an implication of symmetry, which this measure does not have, hence the name change). The usefulness of the definition though, is apparent in the following lemma.

Lemma 5 Let $E: R^d \rightarrow R^n$ be linear. (E will also stand for the matrix representation relative to the canonical basis in R^d and R^n), and let V be a subspace of R^n, with $m(Im\ E, V) = \delta$. Then there exists two linear maps, E_1 and E_2 satisfying

a) $E_1 + E_2 = E$

b) $\mathrm{Im}\, E_1 \subset V$

c) $\|E_2\| \le \delta \|E\|$.

In imprecise words, E can be broken into two parts, one which remains in V, and one whose size is in some way related to δ .

proof: Let $\mathbf{V}$ be a matrix whose orthonormal columns span V and define $E_1 := \mathbf{V}\mathbf{V}^T E$ and $E_2 := (I - \mathbf{V}\mathbf{V}^T)E$, thus taking care of (a) and (b). Now

$$\|E_2\| := \underset{|x|=1}{\mathrm{supremum}} |(I - \mathbf{V}\mathbf{V}^T)Ex|$$

$$\leq \underset{e \in Im E, |e| \leq |E|}{supremum} |(I - VV^T)e|$$

$$= \|E\| \, m(Im\, E, V)$$

$$= \delta \|E\| \quad \blacksquare$$

For an application of this measure, consider the linear system

$$\dot{x} = Ax + Bu + Eq$$

$$y = Cx$$

where $q:[0,\infty) \to R^d$ is a disturbance. It is well known, [7], that using state feedback, we can stabilize the system and decouple the disturbance from the output if and only if (A,B) is stabilizable and Im$E \subset V^*_{g,Ker\,C}$.

Suppose that (A,B) is stabilizable, but

$$0 < m(Im\, E, V^*_{g,Ker\,C}) =: \delta$$

so that the condition Im$E \subset V^*_{g,Ker\,C}$ is *not* met. By using a stabilizing friend of $V^*_{g,Ker\,C}$ (a $F \in \mathbf{F}(V^*_{g,Ker\,C})$ such that not only is the restriction of $A+BF$ to $V^*_{g,Ker\,C}$ stable, but all of $A+BF$ is stable), can we expect a reasonable degree of disturbance rejection? As one would hope, the answer is yes, provided δ is small enough. To see this, decompose E into E_1 and E_2 as in the lemma, and note that for F a stabilizing friend of $V^*_{g,Ker\,C}$, we get

$$\|Ce^{(A+BF)t}E\| = \|Ce^{(A+BF)t}E_1 + Ce^{(A+BF)t}E_2\|$$

$$= \|Ce^{(A+BF)t}E_2\| \quad \textit{since } \mathrm{Im}\,E_1 \subset V^*_{g,Ker\,C} \subset Ker\ C$$

$$\leq \|Ce^{(A+BF)t}\| \, \delta \, \|E\|$$

$$\leq Me^{-\lambda t} \delta \|E\|$$

for some $M, \lambda \geq 0$, since $Ce^{(A+BF)t}$ is exponentially stable. If δ is small (say 10^{-2} or 10^{-3}), this can indeed be a useful amount of disturbance rejection. Actually, a tighter bound than $\delta\|E\|$ will be $\|E_2\|$, which is easily calculated. Choosing F so as to minimize some norm of $Ce^{(A+BF)t}E$ is a harder problem; here we have simply shown that with a certain F implemented, slight variations in E, which may pop ImE out of $V^*_{g,Ker\,C}$, do not have drastic consequences.

6. ALGORITHM

The procedure for carrying out the results of Theorem 1 is straightforward, and a rough outline is listed here.

i) $P \in R_o^{r \times m}(s)$ and $H \in R_o^{r \times d}(s)$ are placed side by side as

$$\left[P(s) \,\vdots\, H(s) \right] =: T \in R_o^{r \times (m+d)}(s)$$

and T is realized as $C(sI-A)^{-1}G$ with $C \in R^{r \times n}, A \in R^{n \times n}$, and $G \in R^{n \times (m+d)}$. There is no restriction on the minimality of the realization, however if concern centers around the existence of a stable solution, then we must have the pair (C,A) detectable. Since there are straightforward methods for obtaining observable realizations, this poses no problems.

ii) $G \in R^{n \times (m+d)}$ is partitioned as

$$\left[B \,\vdots\, E \right] := G$$

so that $B \in R^{n \times m}$ and $E \in R^{n \times d}$. Now $P(s) = C(sI-A)^{-1}B$ and $H(s) = C(sI-A)^{-1}E$, hence Theorem 1 is applicable.

iii) Via [5] and [6], matrices $\mathbf{V}^*_{g,Ker\,C}$ and $\mathbf{V}^*_{Ker\,C}$, whose columns span $V^*_{g,Ker\,C}$ and $V^*_{Ker\,C}$, are obtained using the data A, B, and C from steps (i) and (ii). In this process, other important quantities are calculated, namely, a list of the transmission zeros of the triple (A,B,C), and matrix representations of $F|_{V_g^*}$ and $F|_{V^*}$,(respectively denoted $\mathbf{W}_g$ and $\mathbf{W}$), which are the restrictions to $V^*_{g,Ker\,C}$ and $V^*_{Ker\,C}$ that feedback friends must satisfy.

iv) Two new matrices are formed by appending the m columns of B to both $\mathbf{V}^*_{g,Ker\,C}$ and $\mathbf{V}^*_{Ker\,C}$. This looks like

$$\mathbf{Y}^*_{g,Ker\,C} := \left[\mathbf{V}^*_{g,Ker\,C} \,\vdots\, B \right]$$

$$\mathbf{Y}^*_{Ker\,C} := \left[\mathbf{V}^*_{Ker\,C} \,\vdots\, B\right]$$

Hence $span\ \mathbf{Y}^*_{g,Ker\,C} = V^*_{g,Ker\,C} + \mathrm{Im}B$, and likewise for $\mathbf{Y}^*_{Ker\,C}$.

v) Calculate the following containment measures, and verify if any are zero.

a1) $m(\mathrm{Im}E\,, span\,\mathbf{V}^*_{g,Ker\,C})$

a2) $m(\mathrm{Im}E\,, span\,\mathbf{Y}^*_{g,Ker\,C})$

b1) $m(\mathrm{Im}E\,, span\,\mathbf{V}^*_{Ker\,C})$

b2) $m(\mathrm{Im}E\,, span\,\mathbf{Y}^*_{Ker\,C})$

vi) Assuming at least one of the above is zero, generate the necessary matrices for the solution $Q(s)$.

- Friends of $V^*_{g,Ker\,C}$ and $V^*_{Ker\,C}$ are computed using the data from (iii), namely as a real $m \times n$ matrix, F must satisfy

$$F\,\mathbf{V}^*_{g,Ker\,C} = \mathbf{W}_g \quad or \quad F\,\mathbf{V}^*_{Ker\,C} = \mathbf{W}$$

depending on whether $V^*_{g,Ker\,C}$ (cases (a1) and (a2)) or $V^*_{Ker\,C}$ (cases (b1) and (b2)) is to be made closed loop invariant. The additional freedom in choosing F can be used to adjust the spectrum of $A+BF$ "above" V^*. See Prop. 4.1 on page 92 of [7] for details.

- In the case of (a2) or (b2), construct L so that $\mathrm{Im}(BL+E) \subset V^*_{g,Ker\,C}$ or $\mathrm{Im}(BL+E) \subset V^*_{Ker\,C}$.

With F, and L if necessary, so determined, a solution is given by the results in theorem 1.

vii) Immediate model order reduction on the solution $Q(s)$ is possible. For example, consider case (b2), as all the other cases are handled similarly.

Since $\mathrm{Im}(BL+E) \subset V^*_{Ker\,C}$, and $(A+BF)\,V^*_{Ker\,C} \subset V^*_{Ker\,C}$, a basis chosen for $V^*_{Ker\,C}$, and augmented with a basis for some S such that $V^*_{Ker\,C} \oplus S = X$ will result in the following matrices;

$$F = \left[F|_{V^*}\ F|_S\right] \quad A+BF = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \quad BL+E = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}$$

giving a solution $Q(s) = -L - F|_{V^*}(sI - \bar{A}_{11})^{-1}\bar{B}_1$, which has order equal to $\dim(V^*_{Ker\,C})$.

Furthermore, if the transmission zeros of (A,B,C) are distinct, and the basis vectors for $V^*_{Ker\,C}$ (and columns of $\mathbf{V}^*_{Ker\,C}$) are just the closed loop eigenvectors of $V^*_{Ker\,C}$, then $F|_{V^*}$ will just be $\mathbf{W}$, (hence F need not even be calculated) and $\bar{A}_{11}$ will be diagonal (or contain 2×2 blocks for complex roots) with the spectrum of $A+BF|_{V^*_{Ker\,C}}$ appearing on the diagonal.

Remark: Let H_i denote the i'th column of $H(s)$, and let J denote one of the following rings,

$$R_p(s)\,,\, R_o(s)\,,\, R_{g,p}(s)\,,\, R_{g,o}(s).$$

Then there exists a $Q(s) \in J^{m \times d}$ solving $P(s)\,Q(s) = H(s)$ if and only if for each $i \in \underline{d}$, there is a $Q_i(s) \in J^{m \times 1}$ solving $P(s)\,Q_i(s) = H_i(s)$. Hence, if a specific $\bar{H}(s)$ is desired, it is best to solve column by column, to more finely determine the problem entries of $\bar{H}(s)$. This amounts to checking the containment measures of part (iv) separately for each column of E.

7. APPROXIMATE SOLUTIONS

The results of section 5 can be used in conjunction with the algorithm in section 6 so that under special circumstances, we can obtain a $Q_a(s) \in \mathbf{M}[R_{g,o}(s)]$ (or $\mathbf{M}[R_{g,p}(s)]$) that makes

$$P(s)\,Q_a(s) - H(s)$$

small, when none of the containment measures are zero.

Suppose that $0 < m(\mathrm{Im}\,E, V^*_{g,Ker\,C}) =: \delta$, and (A,B) is stabilizable. In general it will be hard to conclude any structural properties of the pair (A,B), since B is merely a portion of the input matrix of a realization of $[\; P(s) \vdots H(s) \;]$. If however, $H(s)$ is stable, and the unstable poles of $P(s)$ are realized minimally, (A,B) will be stabilizable. This can be done easily if all of the unstable poles of $P(s)$ are simple poles, using a Gilbert realization. We note that if both P and H are stable, then $\sigma(A) \subset C_g$, and (A,B) is trivially a stabilizable pair.

With this in mind, consider any stabilizing friend F of $V^*_{g,Ker\,C}$. Then

$$Q_a(s) := -F(sI - A - BF)^{-1}E$$

is an approximate solution in the following sense.

Certainly

$$H(s)-P(s)Q_a(s)=C(sI-A-BF)^{-1}E.$$

Taking laplace inverses, noting that $A+BF$ is stable, gives, for some $M,\lambda>0$, (see section 5)

$$\|L^{-1}(H-PQ_a)\| \leq Me^{-\lambda t}\,\delta\,\|E\|$$

which leads to

$$\sup_{\omega\in R}\|(H-PQ_a)(j\omega)\| \leq \frac{\delta M\|E\|}{\lambda}.$$

Unfortunately, since Im E is not contained in $V^*_{g,Ker\,C}$, the model reduction as described in part (vii) of the algorithm does not apply. Consequently, $Q_a(s)$ has order equal to n, which can be quite high.

For the proper case, let E_1 be the orthogonal projection of E onto $V^*_{g,Ker\,C}+ImB$, and calculate L so that Im $(BL+E_1)\subset V^*_{g,Ker\,C}$ and let F be any stabilizing friend of $V^*_{g,Ker\,C}$. Similar reasoning then yields that

$$Q_a(s):=-L-F(sI-A-BF)^{-1}(BL+E)$$

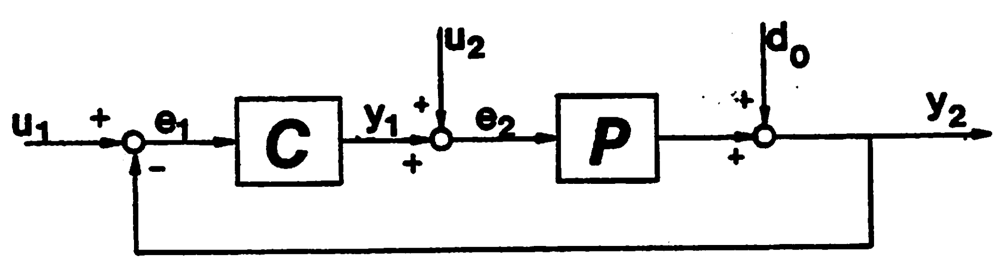
is an approximate solution.

1. ALGEBRAIC MULTIVARIABLE DESIGN METHODOLOGY FOR STABLE PLANTS

1.1. REVIEW OF THE METHOD

In this section, we give a short review of the results for design with stable plants. The details and proofs can be found in [8].

Consider the feedback system, called $\Sigma_1(P,C)$



with the assumption that $P(s) \in R_o^{r \times m}(s)$ and $C(s) \in R_p^{m \times r}(s)$. These two assumptions imply that $\lim_{s \to \infty} P(s)\,C(s) = 0$, and that $I + P(s)\,C(s) \in \mathbf{M}[R_p(s)]$. Consequently, dropping the s dependence for clarity, $(I+PC)^{-1}$ is a well defined element of $\mathbf{M}[R_p(s)]$, and we define $Q(s) := C(I+PC)^{-1}$. Note that if $C(s)$ is (strictly) proper, then $Q(s)$ is (strictly) proper also. The following are useful identities.

$$I - PQ = (I + PC)^{-1}$$

$$I - QP = (I + CP)^{-1}.$$

Since Q is proper, $(I - PQ)^{-1} \in \mathbf{M}[R_p(s)]$, and we can solve for $C(s)$ in terms of $Q(s)$, namely

$$C = Q(I - PQ)^{-1} = (I - QP)^{-1}Q.$$

From here, we see that $Q(s)$ proper implies that $C(s)$ is proper, and $Q(s)$ strictly proper implies that $C(s)$ is strictly proper.

Writing loop equations yields

$$P[C(u_1 - y_2) + u_2] + d_o = y_2$$

$$C[-P(y_1 + u_2) - d_o] + u_1 = y_1$$

$$-P(Ce_1 + u_2) - d_o + u_1 = e_1$$

$$C[-(Pe_2 + d_o) + u_1] + u_2 = e_2$$

which after some algebra, are rewritten as

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} Q & -QP & -Q \\ PQ & P(I-QP) & -(I-PQ) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ d_o \end{bmatrix} \quad \textit{or just} \quad \mathbf{y} = \mathbf{H}_{yu}\,\mathbf{u}$$

and

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} I-PQ & -P(I-QP) & -(I-PQ) \\ Q & I-QP & -Q \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ d_o \end{bmatrix}, \quad \mathbf{e} = \mathbf{H}_{eu}\,\mathbf{u}.$$

We will consider only closed loop systems with proper transfer functions between exogenous inputs and the outputs and error signals. Using the earlier fact relating the properness of $C(s)$ and $Q(s)$, it is apparent that all closed loop transfer functions are proper if and only if $C(s)$ is proper.

Now to discount *internal* instabilities in $C(s)$ and $P(s)$, suppose that both C and P have underlying state space descriptions that are both stabilizable and detectable. Then we define $\Sigma_1(P,C)$ to be C_g stable if the rational matrix $\mathbf{H}_{yu} \in \mathbb{M}[R_{g,p}(s)]$.

The main design parametrization theorem for stable plants P is

Theorem 2 Suppose $P(s) \in \mathbb{M}[R_{g,o}(s)]$ and is connected with a $C(s) \in \mathbb{M}[R_p(s)]$ in the $\Sigma_1(P,C)$ configuation. Then

i) $Q(s) \in \mathbb{M}[R_{g,p}(s)]$ *if and only if* $C(s) \in \mathbb{M}[R_p(s)]$ and $\Sigma_1(P,C)$ is C_g stable.

ii) $Q(s) \in \mathbb{M}[R_{g,o}(s)]$ *if and only if* $C(s) \in \mathbb{M}[R_o(s)]$ and $\Sigma_1(P,C)$ is C_g stable.

Our application is based on the next lemma.

Lemma 6 Given $P(s) \in \mathbb{M}[R_{g,o}(s)]$, let $\underline{H}_{y_2u_1}$ denote the set of all achievable input/output maps, $H_{y_2u_1}(s)$ (between u_1 and y_2), with the restriction that the resulting $\Sigma_1(P,C)$ system be C_g stable. Then

i) using strictly proper compensators $C(s)$,

$$\underline{H}_{y_2u_1} = \{ PQ : Q(s) \in \mathbb{M}[R_{g,o}(s)] \}.$$

ii) using proper compensators $C(s)$,

$$\underline{H}_{y_2u_1} = \{ PQ : Q(s) \in \mathbb{M}[R_{g,p}(s)] \}.$$

8.2. APPLICATION AND DESIGN OF COMPENSATOR

The boxed statement summarizes an obvious application.

It is possible to achieve a desired input/output response, $\overline{H}(s) = H_{y_2u_1}(s)$, with a overall stable closed loop system, *if and only if* there is a proper, stable $Q(s)$ satisfying $PQ = \overline{H}$. The theory in sections 2 through 6 specifies exactly when this happens, and gives a constructive method of obtaining the solution $Q(s)$. Given a stable solution, it is only a matter of building $C(s) = (I-QP)^{-1}Q$ to render an I/O response $H_{y_2u_1}(s)$ equal to $\overline{H}(s)$.

Next, we study a very easily obtained realization of the required compensator $C(s)$, given realizations for the stable plant $P(s)$, and the stable solution $Q(s)$, which satisfies $PQ = \overline{H}$. We begin with a simple lemma concerning the stabilizability and detectability of a specific interconnection of two linear systems.

Lemma 7 Let $\mathbf{R}_T := \{A_T, B_T, C_T\}$ be a realization of $T(s) \in R_{g,o}^{r\times m}(s)$ with $\sigma(A_T) \subset C_g$, and let $\mathbf{R}_G := \{A_G, B_G, C_G, D_G\}$ be a stabilizable and detectable realization of $G(s) \in R_p^{m\times r}(s)$. Then $\mathbf{R}_C := \{A_C, B_C, C_C, D_C\}$, defined as,

$$A_C := \begin{bmatrix} A_G & B_G C_T \\ B_T C_G & A_T + B_T D_G C_T \end{bmatrix}$$

$$B_C := \begin{bmatrix} B_G \\ B_T D_G \end{bmatrix}$$

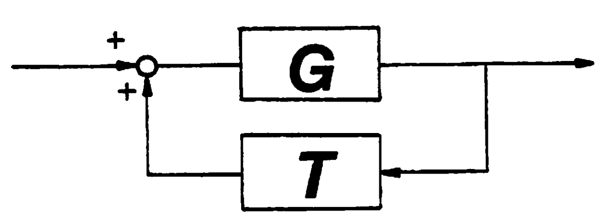
$$C_C := \begin{bmatrix} C_G & D_G C_T \end{bmatrix}$$

$$D_C := \begin{bmatrix} D_G \end{bmatrix}$$

is a stabilizable and detectable realization of

$$C(s) := G(I-TG)^{-1} \equiv (I-GT)^{-1}G.$$

proof: It is straightforward to show this *is* a realization of $C(s)$ simply by considering the feedback configuration



and writing down the obvious state equations.

For stabilizability, let F be such that $\sigma(A_G+B_GF) \subset C_g$, (by hypothesis, such an F exists), and then define

$$F_C := \left[F \,\vdots\, -C_T \right]$$

Then $\sigma(A_C+B_CF_C) = \sigma(A_G+B_GF) \cup \sigma(A_T) \subset C_g$, so by definition, (A_C,B_C) is a stabilizable pair.

For detectability, let L be such that $\sigma(A_G+LC_G) \subset C_g$, and choose

$$L_C := \begin{bmatrix} L \\ -B_T \end{bmatrix}$$

Then $\sigma(A_C+L_CC_C) \subset C_g$, so (C_C,A_C) is a detectable pair. ▪

Now back to our situation: Let $\mathbf{R}_Q := (A_Q,B_Q,C_Q,D_Q)$ be a realization of $Q(s)$, and $\mathbf{R}_P := (A_P,B_P,C_P)$ be a realization of $P(s)$. In this application, both $P(s)$ and $Q(s)$ are stable, so that we might as well take $\sigma(A_Q) \subset C_g$ and $\sigma(A_P) \subset C_g$. Trivially then, $\mathbf{R}_G$ is both stabilizable and detectable, hence applying lemma 7 gives that $\mathbf{R}_C$, defined by

$$A_C := \begin{bmatrix} A_Q & B_QC_P \\ B_PC_Q & A_P+B_PD_QC_P \end{bmatrix}$$

$$B_C := \begin{bmatrix} B_Q \\ B_PD_Q \end{bmatrix}$$

$$C_C := \begin{bmatrix} C_Q & D_QC_P \end{bmatrix}$$

$$D_C := \begin{bmatrix} D_Q \end{bmatrix}$$

is a stabilizable and detectable realization of $C(s)$, which is exactly what we need. Recall $C(s)$ need not be stable, however, in addition to the properness requirement, the other assumption on C is that it have no unstable hidden modes. This particular realization has none, and hence can be used to build

$C(s)$, successfully completing the design.

8.3. LIMITATIONS

An important limitation on $H_{y_2u_1}$ is imposed by the C_b Smith McMillan zeros of the plant $P(s)$.

Lemma 8 Suppose $P(s)$ is stable, and $C(s)$ is chosen so that $\Sigma_1(P,C)$ is stable. Let $\overline{H}(s)$ denote the input/output transfer function relating y_2 to u_1, and assume $\overline{H}(s)$ is nonsingular. Note that by lemma 6, we must have $P(s)\,Q(s)=\overline{H}(s)$ for some stable $Q(s)$. Then

$$Z[P(s)]\cap C_b \subset Z[\overline{H}(s)]$$

where $Z[\cdot]$ stands for the Smith McMillan zeros of the argument.

proof: Since $\overline{H}(s)$ is square and nonsingular, we know that $z\in Z[\overline{H}(s)]$ if and only if z is a pole of $\overline{H}^{-1}(s)$. Now let (D_{pl},N_{pl}) be a left coprime factorization of $P(s)$ (over the ring of polynomials, $R[s]$), and let (D_{qr},N_{qr}) be a right coprime factorization of $Q(s)$. Then

$$\overline{H}(s)=D_{pl}^{-1}N_{pl}N_{qr}D_{qr}^{-1}$$

and since $\overline{H}(s)$ is nonsingular, we get

$$(N_{pl}N_{qr})^{-1}=D_{qr}^{-1}\overline{H}^{-1}D_{pl}^{-1}.$$

Suppose λ is a 'bad' ($\lambda\in C_b$) Smith McMillan zero of $P(s)$. Then with (D_{pl},N_{pl}) left coprime, $N_{pl}[s]$ must lose rank at $s=\lambda$, so that $(N_{pl}N_{qr})^{-1}$ has a pole at $s=\lambda$. Now D_{qr}^{-1} and D_{pl}^{-1} have no poles at $s=\lambda$ since Q and P are stable. Therefore $\overline{H}^{-1}$ must have a pole at λ, otherwise equality in (7.1) will not hold. ▪

8.4. MODIFYING $\overline{H}(s)$

Suppose $\overline{H}(s)$ is square, nonsingular, and ***diagonal***, reflecting a desire to decouple the I/O response so that the i'th entry of u_1 affects only the i'th entry of y_2. In this case, the Smith-McMillan zeros of $\overline{H}$ are easily known; they are just the zeros of each scalar entry $\bar{h}_{ii}$ of $\bar{H}(s)$.

Hence, using the column by column procedure mentioned in section 6, one can systematically add combinations of the unstable transmission zeros to the numerator of $\bar{h}_{ii}$ until a stable solution is reached. This procedure will be displayed in the design example calculation found in section 11.3.

9. CONCLUDING REMARKS

In this paper we have discussed the application of simple ideas from linear geometric control theory to solve equations of the form

$$P(s)\,Q(s) = H(s) \tag{9.1}$$

for $Q(s)$ belonging to various rings, most notably, the ring of proper, stable rational functions. The complete dual to this problem is worked out in [11], using (C,A) (or conditioned) invariant subspaces. Also of interest is the solution $Q(s)$ of the equation

$$P(s)\,Q(s)\,T(s) = H(s) \tag{9.2}$$

given strictly proper P, T, and H. Various aspects of this problem are discussed in [3], [12] and [13]. Presently, we are working on conditions for stable solutions $Q(s)$, and the appropriate software for the computation.

We have applied our techniques to the algebraic approach to control system design for stable plants. Extensions to unstable plants using the solution to (9.2) and the approach of [9] are being worked out.

We also addressed the numerical sensitivity of our solution technique for solving (9.1), and when an exact solution was not possible, we gave an "approximate" solution and the degree of the approximation. The question of approximation is an interesting one, and has connections with matrix interpolation problems [14].

10. REFERENCES

[1] J.C. Willems, *Almost Invariant Subspaces: An Approach to High Gain Feedback Design- Part I: Almost Controlled Invariant Subspaces*, IEEE Trans. Automatic Control, AC-26, 1981, pp. 235-252.

[2] M.L.J. Hautus, *(A,B) Invariant and Stabilizability Subspaces, a Frequency Domain Description*, Automatica, vol. 16, 1980, pp. 703-707.

[3] J.C. Willems, *Almost Invariant Subspaces: An Approach to High Gain Feedback Design- Part II: Almost Conditionally Invariant Subspaces*, IEEE Trans. Automatic Control, AC-27, 1982, pp. 1071-1085.

[4] B.S. Garbow *et al.*, *Matrix Eigensystems Routines- EISPACK Guide Extension*, Lecture notes in Computer Science, vol. 51, Berlin, Springer Verlag, 1977.

[5] B.C. Moore and A.J. Laub, *Computation of Supremal (A,B) Invariant and Controllability Subspaces*, IEEE Trans. Automatic Control, AC-23, 1978, pp. 783-792.

[6] A.J. Laub and B.C. Moore, *Calculation of Transmission Zeros using QZ techniques*, Automatica, vol. 14, 1978, pp. 557-566.

[7] W.M. Wonham, *Linear Multivariable Control, a Geometric Approach*, 2nd ed., Springer Verlag, New York, 1979.

[8] F.M. Callier and C.A. Desoer, *Multivariable Feedback Systems*, Springer Verlag, New York, 1982.

[9] C.A. Desoer and C.L. Gustafson, *Algebraic Theory of Linear Multivariable Feedback Systems*, IEEE Trans. Automatic Control, AC-29, 1984, pp. 909-917.

[10] S.S. Sastry and P. Varaiya, *Coherency for Interconnected Power Systems*, IEEE Trans. Automatic Control, AC-26, 1981, pp. 218-226.

[11] Li Zexiang and S.S. Sastry, *(C,A) Invariant Subpaces and the Solution of Rational Matrix Equations in the State Space*, in preparation.

[12] J.M. Schumacher, *Compensator Synthesis using (C,A,B) Pairs, IEEE Trans. Automatic Control, AC-25, 1980, pp. 1133-1138.*

[13] D.Y. Ohm, S.P. Bhattacharyya, and J.W. Howze, *Transfer Function Conditions for (C,A,B) Pairs*, IEEE Trans. Automatic Control, AC-29, 1984, pp. 172-174.

[14] M. Vidysagar, *Control System Synthesis: A Factorization Approach*, M.I.T. Press, 1985, esp. chap. 6.

11. APPENDIX

11.1 PROOFS

The appendix is devoted to proving the equivalences of (i), (ii), and (iii) in section 2, and (i) and (ii) in section 3. For the most part, these proofs follow along the lines of [2], and [7].

Before proceeding with the equivalences for controlled invariance, we need this next lemma.

Lemma 9 Let X be a normed, finite dimensional, real linear space, and let $V \subset X$ be a subspace. Suppose $x:[0,\infty) \to X$ is differentiable (one sided at 0), and $x(t) \in V$ for all $t \in [0,\infty)$. Then $\dot{x}(t) \in V$ for all $t \in [0,\infty)$.

proof: Let $\mathbf{P}$ be any linear map on $X \to R$ with $V = \operatorname{Ker} \mathbf{P}$. Then $f(t) := \mathbf{P}(x(t)) = 0$ for all t. By chain rule, noting that $\mathbf{P}$ is linear, we get $\mathbf{P}\dot{x}(t) = 0$ for all $t \in [0,\infty)$, so that $\dot{x}(t) \in \operatorname{Ker} \mathbf{P} = V$ for all t. ▪

Onto the equivalences:

(i) → (ii) Let $x_0 \in V$ and let $u(\cdot)$ be the control that holds x in V. With u continuous, x is certainly differentiable on $[0,\infty)$. At zero, then, $\dot{x} = Ax_0 + Bu(0)$, so that $Ax_0 \in V + \operatorname{Im} B$. With x_0 arbitrary in V, (ii) follows. ▪

(ii) → (iii) Let $V \oplus W = X$ and let $\{v_1, v_2, \ldots, v_l\}$ be a basis for V. Then for each $i \in \underline{l}$, there is a s_i in V and a u_i in U such that

$$Av_i = s_i + Bu_i.$$

Define F as the linear map with action on V given by

$$Fv_i = -u_i$$

and action on W can be specified arbitrarily. Then $(A+BF)v_i = s_i \in V$ for all $i \in \underline{l}$ as desired. ▪

(iii) → (i) Let $u := Fx$, for some $F \in \mathbf{F}(V)$, then

$$x(t) = e^{(A+BF)t} x_0 \qquad (11.1)$$

and if $x_0 \in V$, we get $x(t) \in V$ for all t, since $(A+BF)V \subset V$. ▪

The equivalences in section 3 are a bit harder to verify as one would expect. In fact, to make our job easier, we will slip in between (i) and (ii), a new equivalence (i'') which reads as,

(i'') a) $\mathbf{F}(V)$ is a nonempty set.

b) There exists a linear map $G: U \to U$ such that $\mathrm{Im}(BG) = \mathrm{Im}\, B \cap V$ and $Bv \in V$ will imply that $v \in \mathrm{Im}\, G$.

c) Furthermore, if $F_o \in \mathbf{F}(V)$, then $A+BF_o$ and BG, with G chosen as above, appear as (with a basis in V and some S, such that $V \oplus S = X$)

$$A+BF_o = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \qquad BG = \begin{bmatrix} (BG)_1 \\ 0 \end{bmatrix}$$

and the pair $(\bar{A}_{11}, (BG)_1)$ is stabilizable.

We will show the equivalences by proving (i) $\to$ (i'') $\to$ (ii) $\to$ (i), but we need a linear algebra result, namely Lemma 10.

<u>Lemma 10</u> Consider finite dimensional linear spaces X and U, a linear map $B: U \to X$, and a subspace $V \subset X$. Then there exists a linear map $G: U \to U$ such that $\mathrm{Im}\,(BG) = \mathrm{Im}\, B \cap V$ and if $u \in U$ such that $Bu \in V$, then $u \in \mathrm{Im}\, G$.

proof: Choose $T \subset X$, and $W \subset X$ so that $(\mathrm{Im}\, B \cap V) \oplus T = V$ and $(\mathrm{Im}\, B \cap V) \oplus W = \mathrm{Im}\, B$. Then it will be possible to choose U_1 and U_2 such that $U_1 \oplus U_2 \oplus \mathrm{Ker}\, B = U$, and appropriate base vectors in X and U, so that B appears as

$$B = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then check that

$$G := \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \end{bmatrix}$$

does the job. ▪

Now, back to the equivalences:

(i) → (i'')

a) This is obvious, since (i) certainly implies that V is controlled invariant, and hence has at least one friend.

b) Simply apply Lemma 10: also note that Lemma 10 really is independent of our problem.

c) For clarity, let $l := \dim(V)$, then identify $V = R^l$, and $S = R^{n-l}$, so $S \oplus V = X = R^n$. If $x \in X$, then there exist unique $v \in V \subset X$ and $s \in S \subset X$ such that $x = v + s$, and we write this equality as $x = \begin{bmatrix} v \\ s \end{bmatrix}$.

Let $v_0 \in R^l$ be arbitrary, and define

$$x_0 := \begin{bmatrix} v_0 \\ 0 \end{bmatrix}$$

Under the action of a control $u(\cdot)$, with $x(0) = x_0$, we can decompose the state trajectory $x(t)$ as

$$x(t) := \begin{bmatrix} v(t) \\ s(t) \end{bmatrix}$$

Now, for a contradiction, suppose the pair $(\bar{A}_{11}, (BG)_1)$ is not stabilizable. Then there exists a $\lambda \in C_b$, and $\xi \in R^l$, $\xi \neq 0$ such that

$$\xi^T \begin{bmatrix} \lambda I - \bar{A}_{11} & (BG)_1 \end{bmatrix} = 0$$

so that

$$\xi^T \lambda = \xi^T \bar{A}_{11} \quad , \quad \xi^T (BG)_1 = 0. \tag{11.2}$$

Since x_0 is in fact an element of V, by hypothesis of (i), we can find a continuous control $u(\cdot)$ that makes $x(\cdot)$ a C_g stable signal, and keeps $x(t) \in V$ for all $t \geq 0$. In terms of $v(t)$ and $s(t)$, we need $s \equiv 0$ and $v(\cdot)$ must be C_g stable. In X, the trajectory satisfies

$$\dot{x} = Ax + Bu$$

or for any F of appropriate dimension

$$\dot{x} = (A+BF)x + B(u - Fx). \tag{11.3}$$

Pick $F = F_o \in \mathbf{F}(V)$, then adapted to our basis, (11.3) looks like

$$\begin{bmatrix} \dot{v} \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} v \\ 0 \end{bmatrix} + B\left(u - F_o \begin{bmatrix} v \\ 0 \end{bmatrix}\right) \tag{11.4}$$

Rearranging (11.4) yields

$$B\left(u - F_o \begin{bmatrix} v \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \dot{v} - \bar{A}_{11}v \\ 0 \end{bmatrix} \tag{11.5}$$

Using (b), there is a $\bar{u}(\cdot)$ such that

$$B\left[u(t) - F_o \begin{bmatrix} v(t) \\ 0 \end{bmatrix}\right] = \begin{bmatrix} (BG)_1 \\ 0 \end{bmatrix} \bar{u}(t) \tag{11.6}$$

for all $t \geq 0$. Combining (11.6) and (11.4) means that $v(t)$ satisfies

$$\dot{v} = \bar{A}_{11}v + (BG)_1\bar{u} \quad , \quad v(0) = v_o. \tag{11.7}$$

Premultiply this by ξ^T, the constant vector from (11.2), to obtain

$$\frac{d}{dt}(\xi^T v) = \lambda(\xi^T v).$$

This then integrates to

$$\xi^T v(t) = e^{\lambda t}\, \xi^T v_o. \tag{11.8}$$

Since $\xi \neq 0$, we can choose our $v_o \in V$, so that the right hand side of (11.8) is not identically zero. However, this will result in a contradiction, because we know that $v(t)$ is made up of exponentials from C_g, and the right hand side is an exponential in C_b. Therefore, $(\bar{A}_{11},(BG)_1)$ is a stabilizable pair. ▪

(i'') → (ii) Since $(\bar{A}_{11},(BG)_1)$ is a stabilizable pair, there exists a $L: V \to U$ such that

$$\sigma(\bar{A}_{11}+(BG)_1 L) \subset C_g.$$

It is easy to verify that

$$F := F_o + G\left[L \,\vdots\, 0\right]$$

is a friend of V and that $\sigma(A+BF|_V) \subset C_g$ as desired. ▪

(ii) → (i) Let the control u be $u(t) := Fx(t)$, where F satisfies the hypothesis of (ii), then the trajectory will satisfy

$$x(t) = e^{(A+BF)t} x_o.$$

Like (11.1), for all $x_o \in V$, we get that $x(t) \in V$ for all t, and here, since $A+BF$ is C_g stable on V, $x(t)$ is C_g stable. ▪

Remark: The notation (i'') for the added equivalence is not without reason. Note that in going from (i) → (i''), $u(\cdot)$ being C_g stable is not used. Hence, we do not need it. Therefore we are led to formulate (i'), fitting between (i) and (i''), as

(i') For all x_o in V, there exists a continuous control $u(\cdot)$, such that with $x(0) = x_o$, $x(t) \in V$ for all $t \geq 0$ and $x(\cdot)$ is C_g stable.

Certainly then, (i) → (i') → (i'') → (ii) → (i), so that this is indeed equivalent.

11.2 COMPUTATIONAL CONSIDERATIONS

It is apparent that our ability to solve these rational matrix equations hinges precariously on our ability to compute the subspace $V^*_{Ker\,C}$. In [5], the procedures for calculating $V^*_{Ker\,C}$ are discussed, as are the numerical difficulties that plague the computation. These problems of course play a role in our computations too.

For instance, computing $V^{*}_{Ker\,C}$ basically involves solving a certain generalized eigenvalue problem

$$Lv = \lambda Mv$$

where in this case, M is singular. In this application, the set of generalized eigenvectors is used to calculate basis vectors for $V^{*}_{Ker\,C}$. The corresponding generalized eigenvalues are the subset of $\sigma(A+BF)$ that is fixed for all friends F of $V^{*}_{Ker\,C}$, and in fact are the transmission zeros of the triple (A,B,C).

Suppose $\bar{\lambda}$ is a repeated generalized eigenvalue, say multiplicity m, with only 1 generalized eigenvector (could generalize this to some $l < m$, but the point being made will be the same). In order to completely determine a basis for $V^{*}_{Ker\,C}$, we must calculate the generalized principle vectors, namely vectors $v_2, \ldots, v_m$ that satisfy the recursive condition

$$(L - \bar{\lambda}M)v_i = \bar{\lambda}Mv_{i-1} \qquad i = 2,\ldots,m \quad v_1 := v.$$

This computation is difficult and is a source of error. More importantly, this situation means that $(sI - (A+BF)|_{V^*})^{-1}$ will have a nonsimple pole at $s = \bar{\lambda}$ for every friend F of V^*. This interesting fact implies that many rational matrix equations will probably lead to this situation, namely those that necessitate a nonsimple pole in the solution $Q(s)$.

For example, consider

$$P(s) = \frac{s+1}{(s+2)(s+3)} \qquad H(s) = \frac{1}{(s+1)}$$

which is solvable by inspection. A typical realization will be

$$A = diag(-1, -2, -3)$$

$$B = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

$$C = [1 \;\; 1 \;\; 1]$$

$$E = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

It is trivial to show that -1 is a repeated transmission zero of the triple (A, B, C), but has only one generalized eigenvector. The generalized principal vector must be determined to obtain a basis for $V^*_{Ker\, C}$ and consequently complete the solution.

On the other hand, the computation in Lemma 4 is straightforward, and is easily done using the subroutine MINFIT found in [4].

DATA:

- $E \in R^{n \times d}$, $B \in R^{n \times m}$; matrix representations of the maps E and B, relative to basis sets chosen in X, U, and D.
- $V \in R^{n \times v}$; a matrix whose columns span V.

PROCEDURE:

- using MINFIT, solve for $W \in R^{(v+m) \times d}$ that satisfies

$$\begin{bmatrix} V & B \end{bmatrix} W = E.$$

- by result of the lemma, this will have an exact solution W. Partition this solution as

$$W = \begin{bmatrix} N \\ -L \end{bmatrix}$$

  where $L \in R^{m \times d}$, and this is the matrix representation of the desired map L.

The measure of containment, $m(W, V)$, as defined in section 5, is readily calculated using 3 singular value decompositions, one each to determine U_1 and U_2 and one more for the evaluation of the norm. Using single precision, with the number of rows in U_1 and U_2 varying between 4 and 30, exact containment is characterized by a measure of roughly 10^{-6}.

11.2 NUMERICAL EXAMPLES

The following examples illustrate nearly all the facets of the paper. First, define $P(s)$ as

$$\begin{bmatrix} \frac{4}{(s+2)(s+4)} & \frac{6}{s+5} & 0 \\ 0 & \frac{3(s-2)}{(s+1)(s+5)} & \frac{5}{s+6} \\ \frac{5(s-3)}{(s+3)(s+5)} & \frac{-1}{s+1} & \frac{10}{(s+2)(s+7)} \end{bmatrix}$$

and let

$$H(s) = diag\left[\frac{20}{(s+4)(s+5)}, \frac{18}{(s+3)(s+6)}, \frac{7}{(s+2)(s+3.5)}\right]$$

Suppose that a stable (here C_g will be the open left half plane), proper solution $Q(s)$ is needed, at the possible expense of modifying $H(s)$ as outlined in section 8.5. Proceeding with step (i) and (ii) of the algorithm, we use a simple Gilbert realization to obtain

$$A = diag\,(-2.0, -2.0, -4.0, -5.0, -5.0, -5.0, -1.0, -6.0, -3.0, -3.0, -7.0, -3.5)$$

$$B = \begin{bmatrix} 0 & 0 & 0.888 \\ -1.41 & 0 & 0 \\ 0.446 & 0 & 0 \\ 0 & 1.395 & 0 \\ -4.47 & 0 & 0 \\ 0 & -2.13 & 0 \\ 0 & 1.569 & 0 \\ 0 & 0 & -1.79 \\ -3.87 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1.414 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -1.41 & 0 & -4.48 & 4.56 & 0 & 0.171 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.350 & 0 & -2.23 & -1.43 & -2.79 & 0 & -2.45 & 0 & 0 \\ 0 & -1.41 & 0 & 0 & -4.47 & 0 & -0.64 & 0 & 3.87 & 0 & -1.41 & -2.16 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 0 & 2.07 \\ 0 & 0 & 0 \\ -4.46 & 0 & 0 \\ -4.36 & 0 & 0 \\ 0 & 0 & 0 \\ -0.68 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2.15 & 0 \\ 0 & 0 & 0 \\ 0 & -2.45 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2.16 \end{bmatrix}$$

so that $P(s) = C(sI-A)^{-1}B$ and $H(s) = C(sI-A)^{-1}E$.

In the process of calculating the basis vectors for $V^*_{g,Ker\,C}$ and $V^*_{Ker\,C}$, the transmission zeros of the triple (A,B,C) are found to be

$$\begin{matrix} 2.57 \\ -1.07+j0.839 \\ -1.07-j0.839 \\ -2.53 \\ -3.00 \\ -3.50 \\ -4.04 \\ -5.00 \\ -6.99 \end{matrix}$$

The containment measures for a stable solution are

$$m(ImE, V^*_{g,Ker\,C}) = 0.445$$
$$m(ImE, V^*_{g,Ker\,C} + ImB) = 0.083$$

while dropping the stability restriction yields

$$m(ImE, V^*_{Ker\,C}) = 1.67 \times 10^{-6}$$
$$m(ImE, V^*_{Ker\,C} + ImB) = 2.00 \times 10^{-6}.$$

This leads to two conclusions: first, a stable solution does not exist, and second, a strictly proper one does. Adding the unstable transmission zero to some or all of the diagonal entries of $H(s)$ should take care of the stability requirement, however if no poles are added to $H(s)$, then the relative degree of each nonzero entry of $H(s)$ will decrease by one. Intuitively, strictly proper solutions will no longer exist, but proper solutions will.

Indeed, the column by column procedure reveals that the unstable zero at $s=2.57$ must be added to each, hence we modify $H(s)$ to be

$$H(s) = diag\left[\frac{8(s-2.57)}{(s+4)(s+5)}, \frac{7(s-2.57)}{(s+3)(s+6)}, \frac{3(s-2.57)}{(s+2)(s+3.5)} \right]$$

Consequently, the matrices B, C, and E change and the new containment measures are

$$m(ImE, V^*_{g,Ker\,c}) = 0.528$$
$$m(ImE, V^*_{g,Ker\,c} + ImB) = 5.48 \times 10^{-8}.$$
$$m(ImE, V^*_{Ker\,c}) = 0.192$$

The intuition and subsequent modifications were correct, and while now there are no strictly proper solutions, a stable, proper solution is available. The solution $Q(s)$ is 8th order, and a realization is

$$A_Q = diag\left(\begin{bmatrix} -1.073 & 0.839 \\ -0.839 & -1.073 \end{bmatrix}, -6.987, -2.529, -4.039, -5.000, -3.000, -3.500\right)$$

$$B_Q = \begin{bmatrix} 0.854 & -0.0319 & 0.163 \\ 0.497 & -0.572 & 0.579 \\ -0.556 & 0.975 & -0.003050 \\ -0.598 & 0.739 & -0.344 \\ 4.195 & -0.0628 & 0.0766 \\ -4.815 & 0 & 0 \\ 0 & 1.620 & 0 \\ 0 & 0 & 0.724 \end{bmatrix}$$

$$C_Q = \begin{bmatrix} -0.755 & 2.304 & -1.086 & -0.586 & 0.245 & 0 & 0 & 0.339 \\ -0.798 & -1.506 & -0.0966 & -1.240 & -1.979 & 0 & 0 & 0.452 \\ -3.058 & 3.719 & 0.0432 & 3.097 & 4.815 & 2.201 & -4.815 & -0.995 \end{bmatrix}$$

$$D_Q = \begin{bmatrix} 0.2667 & 0 & 0.600 \\ 1.333 & 0 & 0 \\ -0.800 & 1.400 & 0 \end{bmatrix}.$$

With state space realizations available for $P(s)$, $H(s)$, and $Q(s)$, it is trivial to obtain a realization for $PQ - H$ and verify that the impulse response is identically zero as we expect.

The second example is similar, but we carry out the compensator design. Let the stable plant $P(s)$ be

$$\begin{bmatrix} \dfrac{2}{s+1} & \dfrac{2}{s+4} \\[2ex] \dfrac{1}{s+2} & \dfrac{4}{s+3} \end{bmatrix}.$$

and the desired I/O map

$$H(s) = diag\left(\frac{10}{(s+2)(s+5)}, \frac{12}{(s+3)(s+4)}\right)$$

Right away, we notice that every element of $H(s)$ has higher relative degree than every element of $P(s)$, so that a strictly proper solution seems feasible. Again, matrices A, B, C, and E are obtained, and the transmission zeros of (A, B, C) are

$$\begin{matrix} -2.00 \\ -2.13 \\ -4.00 \\ -4.54 \\ -5.00 \end{matrix}$$

There are no unstable transmission zeros, hence $V^*_{g.Ker\, C} = V^*_{Ker\, C}$. All of the containment measures are found to be zero, and the following is a stable, strictly proper, 5th order solution.

$$A_Q = diag\,(-4.535, -2.131, -4.0, -2.0, -5.0)$$

$$B_Q = \begin{bmatrix} -2.139 & -5.279 \\ 2.945 & 0.1909 \\ 0 & 4.546 \\ -2.526 & 0 \\ 3.789 & 0 \end{bmatrix}$$

$$C_Q = \begin{bmatrix} 5.279 & -0.6941 & 5.279 & 0 & 5.279 \\ -0.7993 & -1.146 & 0 & -1.320 & -0.8788 \end{bmatrix}$$

A simple 4th order realization of $P(s)$, coupled with the results of section 8.3 yields the following 9th order, strictly proper compensator.

$$A_C = \begin{bmatrix} -4.535 & 0 & 0 & 0 & 0 & 3.024 & 3.024 & 5.279 & 10.56 \\ 0 & -2.131 & 0 & 0 & 0 & -4.165 & -4.165 & -0.1909 & -0.3818 \\ 0 & 0 & -4.00 & 0 & 0 & 0 & 0 & -4.546 & -9.093 \\ 0 & 0 & 0 & -2.00 & 0 & 3.572 & 3.572 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5.00 & -5.358 & -5.358 & 0 & 0 \\ -7.466 & 0.9816 & -7.446 & 0 & -7.466 & -1.00 & 0 & 0 & 0 \\ 1.130 & 1.621 & 0 & 1.866 & 1.244 & 0 & -4.00 & 0 & 0 \\ -5.279 & 0.6941 & -5.279 & 0 & -5.279 & 0 & 0 & -2.00 & 0 \\ 1.598 & 2.292 & 0 & 2.639 & 1.760 & 0 & 0 & 0 & -3.00 \end{bmatrix}$$

$$B_C = \begin{bmatrix} -2.139 & -5.279 \\ 2.945 & 0.1909 \\ 0 & 4.546 \\ -2.526 & 0 \\ 3.789 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C_C = \begin{bmatrix} 5.279 & -0.6941 & 5.279 & 0 & 5.279 & 0 & 0 & 0 & 0 \\ -0.7993 & -1.146 & 0 & -1.320 & -0.8788 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Used in the $\Sigma_1(P,C)$ configuration, this compensator will render the overall closed loop system stable, and the I/O map between u_1 and y_2 will be $H(s)$.