CONDITIONED INVARIANT SUBSPACES, DISTURBANCE DECOUPLING AND SOLUTIONS OF RATIONAL MATRIX EQUATIONS

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Abstract:

Conditioned invariant subspaces are introduced both in terms of output injection and in terms of state estimation. Various properties of these subspaces are explored and the problem of disturbance decoupling by output injection (OIP) is defined. It is then shown that OIP is equivalent to the problem of disturbance decoupled estimation as introduced in [1] & [2]. Both solvability conditions and a description of solutions for a class of rational matrix equations of the form X(s)M(s) = Q(s) on several ways are given in state space form. Finally, the problem of output stabilization with respect to a disturbance is briefly addressed.

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1. Introduction:

Undoubtedly, the most important concepts used in geometric control theory are *conditioned invariant { or (A,B)-invariant }* subspaces and *controllability* subspaces introduced by G. Basile & G. Marro in [11] and W. M. Wonham & A. S. Morse in [12]. A further generalization of the controllability subspace is the *stabilizability* subspace introduced by M. L. J. Hautus [5]. Several examples where these subspaces arise can be found in *disturbance decoupling by state feedback (DDP)* [4], *tracking & regulation,* and *non-interacting control, etc.* Chapter 4 & 5 of W. M. Wonham [4] has a complete treatment on this topic. In particular, using these subspaces and the DDP schemes, one can solve a class of rational matrix equations M(s) X(s) = Q(s) in state space form, where M(s) $\in \mathbf{R}_{p,o}(s)^{n_1 \times n}$ and $Q(s) \in \mathbf{R}_{p,o}(s)^{n_1 \times n_2}$. Here $\mathbf{R}_p(s)$ and $\mathbf{R}_{p,o}(s)$ denote respectively the ring of proper and strictly proper rational function of s, with real coefficients. Depending on the structures of M(s) and Q(s), one might obtain solutions X(s) belonging to one of the following rings: (1) $\mathbf{R}_p(s)$ or $R_{p,o}(s)$, (2) $\mathbf{R}(o)$ or $\mathbf{R}_o(o)$, the subring of elements of $\mathbf{R}_p(s)$, or of $\mathbf{R}_{p,o}(s)$ that are analytic in $\mathbb{C}_+$, and (3) $\mathbf{R}(u)$ or $\mathbf{R}_o(u)$, the subring of elements of $\mathbf{R}(o)$; or $\mathbf{R}_o(o)$ that are analytic in a region u of $\mathbb{C}$. In the algebraic approach to the design of multivariable control systems, one frequently encounters solutions of rational matrix equations of this kind and its dual form X(s) M(s) = Q(s) with unknown X(s). See for example [5] and [13].

The dual notion of controlled invariance, namely conditioned invariant subspaces had suffered considerable neglect until the work of Schumacher [15], even though it was introduced at the same time as controlled invariance and has importance in its own right. Recently, in [1] and [2], J. C. Willems and C. Commault introduced complementary detectability and complementary observability subspaces based on observer design and disturbance decoupled estimation. Also, in the literature, DDEP (*the disturbance decoupled estimation problem)* was commonly suggested to be *the dual* of DDP.

But in this paper, we will propose a new scheme - *disturbance decoupling by output injection (OIP)* - as *the dual of DDP.* Based on output injection, we will introduce the concepts of *conditioned invariant, complementary detectability* and *complementary observability* subspaces. Although the new scheme is completely equivalent to DDEP, in some sense, the

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subspaces we define have a better physical interpretation under the output injection scheme.

In section 1, we will give complete characterizations of these subspaces and the results of Lemma 4 and Proposition 8, 9, 11, & 12 are new. In section 5, we will propose a *direct method* of solving rational matrix equation X(s) M(s) = Q(s) in state space form using OIP and DDEP. Finally in section 6, we will generalize OIP to the problem of output stabilization with respect to disturbance and a direct dualization of this section leads to section 4 of [5].

The following notations will be standard throughout the paper: given a triple of maps (A, B, C), where A : $X \to X$, B : $U \to X$ and C : $X \to Y$, with $X = \mathbf{R}^n$, $U = \mathbf{R}^{n_i}$ and $Y = \mathbf{R}^{n_o}$, $\langle A \mid \mathrm{Im}B \rangle = \mathrm{Im}B + A\,\mathrm{Im}B + A^2\,\mathrm{Im}B + \dots + A^{n-1}\,\mathrm{Im}B$, denote the controllable subspace of the pair (A, B) and $\langle \mathrm{Ker}C \mid A \rangle = \mathrm{Ker}C \cap A^{-1}\,\mathrm{Ker}C \cap A^{-2}\,\mathrm{Ker}C \cap \dots \cap A^{-n+1}\,\mathrm{Ker}C$, denote the unobservable subspace of the pair (C,A).

Let $V \subset X$, call vectors $x,\ y \in X$ *equivalent mod* V if $x - y \in V$. We define the quotient space X/V (or X mod V) as the set of equivalent classes:

$$\bar{x} = (y : y \in X, y - x \in V) \qquad x \in V \tag{1-1}$$

Now let the map C be defined as before and suppose $V \subset \mathrm{Ker}C$. If P : $X \to X/V$ is the canonical projection, we claim there is a unique map $\bar{C} : X/V \to Y$ such that

$$C = \bar{C}P \tag{1-2}$$

thus, C " factors through " X/V. Also, assume $V \subset X$ and V is said to be invariant under the map A if

$$AV \subset V \tag{1-3}$$

Write $\bar{X} = X/V$, and $P : X \to \bar{X}$ be the canonical projection. There exists an unique map $\bar{A}$: $\bar{X} \to \bar{X}$ such that $\bar{A}P = PA$. Here $\bar{A}$ (or A mod V) is called the map induced by A in X/V. Please see [4] for further geometrical preliminaries.

2. Conditioned Invariant Subspaces

Conditioned invariant subspaces have been introduced before [1], [2] & [3]. They play a dual role to controlled invariant subspaces and can be introduced from this point of view. However, it is more natural to view them either in the context of output injection or in the context of observer design. In this section, we will use both output injection and observer design schemes to develop the theory of conditioned invariant subspaces and consequently show the relation between these two approaches. We consider a linear, finite dimensional, deterministic system given by

$$\dot{x}(t) = Ax(t)$$

and encouragement in this research.

$$y(t) = Cx(t) \tag{2-1}$$

Where x(t) $\in X = \mathbf{R}^n$ is the state of the system, y(t) $\in Y = \mathbf{R}^{n_o}$ is measured outputs and C, A are maps of appropriate dimensions.

Definition 1 : A subspace $S \subset X$ is said to be *conditioned invariant*, if for all $x_0 \in S$, there exists a constant output injection L : $Y \rightarrow X$ in (2-1) that renders x(t) in S for all t $\geq$ 0, i.e., after incorporation of output injection map L, $\dot{x}(t) = Ax + Ly$, and $x(t) = e^{(A+LC)t}x_0 \in S$ for all t.

Remark: Let $\underline{S}$ denote the class of conditioned invariant subspaces, then definition 1 implies that $S \in \underline{S}$ if there exists a map L : $Y \rightarrow X$ such that (A + LC) $S \subset S$.

The following characterizations of conditioned invariant subspaces are equivalent.

First, we consider a state observer for the system (2-1) of the form:

$$\dot{w}(t) = Kw(t) + Fy(t) \tag{2-2}$$

Where $w(t) \in W = \mathbf{R}^{\bar{n}}$ is the state of the observer; K $\in \mathbf{R}^{\bar{n}\times\bar{n}}$ and F $\in \mathbf{R}^{\bar{n}\times n_o}$. Then we have

Lemma 2: A subspace $S \subset X$ is conditioned invariant if and only if there exist maps K & F in (2-2) such that w(0) = x(0)mod S implies that w(t) = x(t)mod S for all t $\geq$ 0.

Remark: Let P: $X \rightarrow X/S$ be the canonical projection. By Lemma 2, $S \in \underline{S}$ if and only if there exist maps K & F such that Px(t) satisfies (2-2).

Second, we have

Lemma 3: A subspace $S \subset X$ is conditioned invariant if and only if $A(S \cap \mathrm{KerC}) \subset S$.

Proof: We show that Definition 1 implies Lemma 2; Equation (2-2) implies Lemma 3 and that A($S \cap$ KerC) $\subset S$ implies Definition 1. This completes the proof of Lemma 2 and Lemma 3 together. (i). Suppose that S is conditioned invariant, then by definition there exists a map L : $Y \rightarrow X$ such that (A + LC) $S \subset S$. Let K = (A + LC) mod S, F = - L mod S and in (2-1) we have:

$$P\dot{x}(t) = PAx(t) = P(A+LC-LC)x(t) = (A+LC)\mathrm{mod}S\ x(t) - PLy(t) = KPx(t) + Fy(t)$$

which shows necessity part of Lemma 2.

(ii). From the above arguments we know that if S is conditioned invariant, then there exists an observer of the form (2-2) such that $w(t) - Px(t) = 0$ for all t. Thus, for all $x \in S \cap$ KerC, it follows that $P\dot{x}(t) = PAx(t) = KPx(t) + FCx(t) = 0$ and A $x(t) \in$ KerP = S for all t. This establishes the necessity part of Lemma 3.

(iii). Lemma 3 implies Definition 1. Suppose that $S \subset X$ satisfies A($S \cap$ KerC) $\subset S$. Then, let $\{v_1, \cdots v_r\}$ be a basis for $S \cap$ KerC and complete this basis for S by $\{v_1, \cdots v_r, v_{r+1}, \ldots v_q\}$. Define $L_0 : CS \rightarrow X$ by

$$Av_i = -L_0Cv_i, \qquad i = r+1, \cdots q. \tag{2-3}$$

and let L be any extension of L_0 to Y. Then, $(A + LC)v_i \subset S$, i = 1, ... q, and (A + LC) $S \subset S$. This proves sufficiency part of Lemma 3. □

Thus, we have three alternative ways to characterize conditioned invariant subspaces, namely a subspace $S \subset X$ in (2-1) is conditioned invariant if and only if one of the following conditions holds: (1) there exists L $: Y \rightarrow X$ such that $(A + LC)S \subset S$, (2) $A(S \cap \text{Ker}C) \subset S$ and (3) there exists an observer of the form (2-2) such that $Px(t)$ satisfies the equation with P $: X \rightarrow X/S$.

Characterization (2) is commonly referred to as the *geometric characterization;* characterization (1) the *injection characterization,* and (3) the *observer characterization*. By (2), conditioned invariant subspace is also referred to as (C,A)-invariant subspaces and both terminologies will be used interchangeably in the sequel. The set of maps that satisfy (1) are called the " *friends of* S " denoted as $\underline{L}(S)$.

Remark: A simpler version of Lemma 2 without using projections is:

<u>*Lemma*</u> *2':* Given a map H : $X \rightarrow Z$ with $Z = \mathbf{R}^{n_c}$, if there exists an observer of the form (2-2) and $w(t) - Hx(t) = 0$ for all t, then S = KerH is conditioned invariant.

A further generalization of Lemma 2' leads to:

<u>*Lemma*</u> 4: Suppose H is the map defined as in Lemma 2', and S = KerH, which may or may not be conditioned invariant. If there exist maps K, F & N such that:

$$\dot{w}(t) = Kw(t) + Fy(t) \tag{2-4}$$

and $Nw(t) - H\,x(t) = 0$, then S is conditioned invariant.

Proof: Without loss of generality, we may assume that the pair (N, K) is observable for otherwise we could factor out the unobservable kernel of (N, K). Now for all $x \in S \cap \text{Ker}C$, $Nw(t) - Hx(t) = 0$ implies that $Nw(t) = 0$ for all t. Thus, $w(t)$ and therefore $\dot{w}(t) \in \text{Ker}N$ for all t. It then follows that $H\dot{x}(t) = N\dot{w}(t) = 0$ and by (2-1) we have $H\dot{x}(t) = HAx(t) = 0$. This shows that $A\,x(t) \in \text{Ker}H = S$ and by Lemma 2, S is conditioned invariant. Actually from (2-4), if $x \in S \cap \text{Ker}C$, then $\dot{w}(t) \in \text{Ker}N$ implies that $w(t) = 0$ for all t since (N, K) is observable and $y(t) = 0$. □

We now wish to define when the output injection in (2-1) will be called stable. Towards this goal, let us first fix some (symmetric) part of the complex plane that we wish to associate the word " *stable* " with; write $\mathbb{C}_g$ for this part of $\mathbb{C}$, and $\mathbb{C}_b$ for its complement in $\mathbb{C}$. For an operator A acting on a linear space X, we write $X_g(A)$ for the subspace of X spanned by the (generalized) eigenvectors of A associated with eigenvalues in $\mathbb{C}_g$, and define X_b (A) analogously. We recall that, if S is an A-invariant subspace of X, the operator A mod S (or $\bar{A}$), induced by A on $X/\,S$ is stable if and only if X_b (A) $\subset S$.

Definition 5: A conditioned invariant subspace S_g is said to be a *complementary detectability* subspace (relative to $\mathbb{C}_g$) if i) it is conditioned invariant, and ii) there exists L $\in \underline{L}$ (S_g) such that σ (A + LC) mod $S_g \subset \mathbb{C}_g$.

Similarly, we have

Definition 6: A conditioned invariant subspace N is said to be a *complementary observability* subspace if i) it is conditioned invariant and ii) for any given set Λ of ρ symmetric complex numbers, with $\dim(N) = n - \rho$, there exists a map L $\in \underline{L}(N)$ such that σ (A + LC) mod N $= \Lambda$.

Definitions 5 & 6 will become clearer as we get into the problem of disturbance decoupling by output injection. Denote the family of complementary detectability (complementary observability) subspaces by $\underline{S}_g$ ($\underline{N}$). We have, as a consequence of Lemma 2:

Proposition 7: $S_g \in \underline{S}_g$ (or $N \in \underline{N}$) if and only if there exist maps K & F with σ (K) $\subset \mathbb{C}_g$ (or σ (K) can be arbitrarily assigned in $\mathbb{C}$) such that $Px(t)$ satisfies $\dot{w}(t) = Kw(t) + Fy(t)$.

Proof: Similar to the proof of Lemma 2, except with the extra requirement that σ (A + LC) mod S_g and σ (K) belong to $\mathbb{C}_g$ (or assignable in $\mathbb{C}$).

Let S_g be any conditioned invariant subspace, and let maps L : $Y \rightarrow X$, G : $Y \rightarrow \hat{Z}$ with $\hat{Z} = \mathbf{R}^{\hat{n}}$ for some $\hat{n}$, satisfy L $\in \underline{L}(S_g)$ and KerGC $= S_g +$ KerC. Such choice of G is possible because KerC $\subset$ KerGC for any G : $Y \rightarrow \hat{Z}$ and KerC $\subset$ KerC + S_g. Denote the nonempty set of such pairs of maps L & G by Φ (S_g).

Proposition 8: Assume $S_g \subset X$ (or $N \subset X$) is a conditioned invariant subspace and assume (L,G) $\in \Phi$ (S_g) { or Φ (N)}, then S_g (or N) is a complementary detectability (or a complementary observability) subspace if and only if (A + LC, GC) mod S_g (or mod N) is detectable (or observable).

Proof: The if part is obvious because by detectability of (A + LC, GC) mod S_g there exists a map $\hat{L}$ such that σ (A + LC + $\hat{L}$ GC) mod $S_g = \sigma$ (A + (L + $\hat{L}$G)C)mod S_g belongs to $\mathbb{C}_g$. In order to prove the necessity part, we notice that if (A + $\hat{L}$ C) mod $S_g \subset \mathbb{C}_g$ for some $\hat{L}$, we must have

$$(\hat{L} - L)CS_g \subset S_g \subset S_g + \mathrm{KerC} \tag{2-5}$$

Hence, $\hat{L}C$ = LC + L_0GC, for some L_0. Define L_1 = (L + L_0G), it follows σ (A + L_1 C) mod $S_g = \sigma$ (A + $\hat{L}$C) mod $S_g \subset \mathbb{C}_g$ and A + L_1C = A + LC + L_0GC. □

Remark: Proposition 8 is a direct dualization of proposition 2.16 of Hautus [5].

Consequently, from Lemma 4 we have:

Proposition 9: Let S_g be any conditioned invariant subspace of X in system (2-1). S_g is a complementary detectability subspace if and only if the observer (2-2) in Lemma 4 is stable, i.e., σ (K) $\subset \mathbb{C}_g$ and P: $X \rightarrow X / S_g$.

Proof: Analogous to the proof of Lemma 3.

If poles of K can be placed arbitrarily, then S_g is a complementary observability subspace.

Lemma 10: The class of (C, A)-invariant subspaces is closed under the operation of subspace intersection, i.e., $S_1, S_2 \in \underline{S} \rightarrow S_1 \cap S_2 \in \underline{S}$.

Proof: $S_i \in \underline{S}$ implies A($S_i \cap$ KerC) $\subset S_i$, for i = 1,2. But, $S_1 \cap S_2 \subset S_1$ and $S_1 \cap S_2 \subset S_2$. Thus,

$$A(S_1 \cap S_2 \cap \mathrm{KerC}) \subset A(S_1 \cap \mathrm{KerC}) \subset S_1 \tag{2-6}$$

$$\text{and } A(S_1 \cap S_2 \cap \mathrm{KerC}) \subset A(S_2 \cap \mathrm{KerC}) \subset S_2 \tag{2-7}$$

Combining (2-6) with (2-7) we have: A($S_1 \cap S_2 \cap$ KerC) $\subset S_1 \cap S_2$. □

Let Ψ be a given subspace of X, which may or may not be conditioned invariant. By Lemma 10, there exists an infimal (C,A)-invariant subspace that contains Ψ, denoted by S^*_Ψ, such that:

(i). $S^*_\Psi \in \underline{S}$,

(ii). $\Psi \subset S^*_\Psi$

(iii). $S \in \underline{S}$ and $\Psi \subset S$ implies that $S^*_\Psi \subset S$.

S^*_Ψ is just the intersection of all conditioned invariant subspaces that contain Ψ. The following proposition gives an explicit expression for S^*_Ψ

Proposition 11: $S^*_\Psi \subset X$ is the infimal (C,A)- invariant subspace containing Ψ if and only if

$$S^*_\Psi = \Psi + \langle A \mid \Psi \cap \mathrm{KerC} \rangle \tag{2-8}$$

Proof: We have to show that S^*_Ψ satisfies the three requirement in its definition.

(i). Construct a basis $\{\nu_1, \ldots, \nu_r\}$ for $\Psi \cap$KerC and extends this basis for Ψ to $\{\nu_1, \ldots \nu_r, \nu_{r+1}, \cdots \nu_k\}$. Complete the basis for S^*_Ψ to $\{\nu_1, \ldots, \nu_r, \nu_{r+1}, \ldots, \nu_k, \nu_{k+1}, \ldots, \nu_p\}$. We define a map L_o such that

$$A\nu_i = -L_o C \nu_i, \qquad \text{for } i = r+1, \ldots k$$

$$\text{and } L_o C \nu_i = 0, \qquad \text{for } i = k+1, \ldots, p \tag{2-9}$$

Since $\langle A \mid \Psi \cap \mathrm{KerC} \rangle$ is A-invariant, it follows from (2-9) and the construction of the basis set that

$$(A + L_o C)\nu_i = A\nu_i \in \langle A \mid \Psi \cap \mathrm{KerC} \rangle \subset S^*_\Psi, \text{ for } i = 1, \ldots, r, k+1, \ldots, p \tag{2-10}$$

and therefore

$$(A + L_o C)\nu_i \in S^*_\Psi, \qquad \text{for } i = 1, \ldots, p \tag{2-11}$$

Let L be the completion of L_o we have

$$(A+LC)S_\Psi^* \subset S_\Psi^* \tag{2-12}$$

This shows that $S_\Psi^* \in \underline{S}$.

(ii). $\Psi \subset S_\Psi^*$ is obvious from (2-8).

(iii). Suppose that $S \in \underline{S}$ is any conditioned invariant subspace that contains Ψ, then

$$\begin{aligned} S_\Psi^* &= \Psi + \langle A \mid \Psi \cap \mathrm{KerC} \rangle \\ &\subset \Psi + \langle A \mid S \cap \mathrm{KerC} \rangle \quad (\text{since } \Psi \subset S) \\ &\subset \Psi + S \quad (S \in \underline{S}) \\ &\subset S \end{aligned} \tag{2-13}$$

This completes our proof. □

In an analogous fashion, we have the following proposition for N_Ψ^*, the infimal complementary observability subspace that contains Ψ .

Proposition 12: N_Ψ^* is the infimal complementary observability subspace containing Ψ if and only if

$$N_\Psi^* = \langle S_\Psi^* + \mathrm{KerC} \mid A + LC \rangle \tag{2-14}$$

for $L \in \underline{L}(S_\Psi^*)$

Remark: Proposition 12 could also be obtained through a direct dualization of Theorem 5-5, Wonham [4].

Proof: (i). The subspace $\langle \mathrm{KerC} \mid A \rangle = \langle \mathrm{KerC} \mid A + LC \rangle$ for $L \in \underline{L}(S_\Psi^*)$ clearly belongs to N_Ψ^*. This implies that N_Ψ^* is complementary observable and therefore $N_\Psi^* \in \underline{N}$.

(ii). Since in (2-14) $L \in \underline{L}(S_\Psi^*)$, then $(A + LC)^{-i}(S_\Psi^* + \mathrm{KerC}) \supset (A + LC)^{-i} S_\Psi^* = S_\Psi^*$, for i = 1, ... n-1 and it follows that $\Psi \subset N_\Psi^*$.

(iii). Let N be any complementary observability subspace that contains Ψ. Then, $S_\Psi^* \subset N$ and $\langle \mathrm{KerC} \mid A \rangle \subset N$, i.e., N contains the largest A-invariant factors of KerC. Let V_1 be the largest A-invariant factor in KerC and V_2 be its direct summand, i.e., $V_1 + V_2 = \mathrm{KerC}$. We claim that $(S_\Psi^* + \mathrm{KerC}) \cap (A + LC)^{-1}(S_\Psi^* + \mathrm{KerC}) \subset N_\Psi^*$. To see this, if $v_2 \, (\in V_2) \in (A + LC)^{-1}(S_\Psi^* + \mathrm{KerC})$, then $(A + LC)v_2 = Av_2 \in S_\Psi^* + \mathrm{KerC}$. Since V_1 is the largest A-invariant factor in KerC, $Av_2 \in S_\Psi^*$. Thus $v_2 \in S_\Psi^* \subset N$. This completes our proof of $N_\Psi^* \subset N$. □

Lemma 13: S_g^* is invariant in the sense that $(A + LC)S_g^* \subset S_g^*$ for every L: $Y \to X$.

Proof: Since S_g^* is the infimal complementary detectability subspace, it follows that $S_g^* \subset \langle \mathrm{KerC} \mid A \rangle$ and thus $S_g^* \subset \mathrm{Ker\ C}$. But $A(S_g^* \cap \mathrm{KerC}) \subset S_g^*$ implies that $AS_g^* \subset S_g^*$ and therefore, $(A + LC)\, S_g^* = A\, S_g^* \subset S_g^*$ for every map L: $Y \to X$. □

The following algorithms are conceptual algorithms for computing S_{Ψ}^{*} and N_{Ψ}^{*}. Please refer to Moore & Laub [9] for numerically stable algorithms.

Define two sequences by:

$$S^0 = 0, \qquad\qquad N^0 = X$$

$$S^{\mu+1} = \Psi + A(S^{\mu} \cap \mathrm{KerC}) \qquad N^{\mu+1} = \Psi + (A^{-1}N^{\mu}) \cap \mathrm{KerC}$$

Then, $S_{\Psi}^{*} = S^{\infty} = S^{n-\dim\Psi+1}$, $N^{\infty} = N^{n-\dim\Psi}$ and $N^{*} = S_{\Psi}^{*} + N^{\infty}$.

We first observe that the sequence S^{μ} is nondecreasing, and $S^{\mathrm{codim}\Psi+1} = S^{\infty} = \lim_{\mu\to\infty} S^{\mu}$. If $\{S^{\mu+1} = S^{\mu}\}$, then $\{S^{\mu} = S^{\infty}\}$. Similarily, the N^{μ} sequence is nonincreasing, $N^{\mathrm{codim}\Psi} = N^{\infty} = \lim_{\mu\to\infty} N^{\mu}$ and $\{N^{\mu+1} = N^{\mu}\}$ implies that $\{N^{\mu} = N^{\infty}\}$. A careful inspection reveals that the above sequences are just decompositions of (2-9) and (2-14) in Proposition 11 and 12 respectively.

3. Disturbance Decoupling by Output Injection

We consider the following dynamical system:

$$\dot{x}(t) = Ax(t) + Eq(t)$$

$$y(t) = Cx(t) \tag{3-1}$$

$$z(t) = Hx(t)$$

Where $q(t) \in \mathbf{R}^{n_i}$ is the disturbance, $z(t) \in \mathbf{R}^{n_c}$ is the outputs to be controlled.

The problem of disturbance decoupling by output injection (OIP) is: find a constant injection map $L : Y \to X$ in (3-1) so that the transfer function from the disturbance term q(t) to the controlled outputs z(t) is identically zero.

After incorporating output injection in (3-1) we have:

$$\dot{x}(t) = (A+LC)x(t) + Eq(t) \tag{3-2}$$

and OIP requires that:

$$z(t) = H\int_0^t e^{(A+LC)(t-\tau)}Eq(\tau)d\tau = 0 \tag{3-3}$$

(3-3) is satisfied if there exists a conditioned invariant subspace S that containing ImE is contained in KerH. Namely, that there exists a map $L : Y \to X$ such that

$$\langle A+LC \mid \mathrm{ImE}\rangle \subset \mathrm{KerH} \tag{3-4}$$

Theorem 14: *OIP is solvable if and only if $S^*_{ImE} \subset KerH$.*

Proof: ($\rightarrow$) If L solves OIP, then the subspace

$$S = \langle A + LC \mid \mathrm{ImE} \rangle \tag{3-5}$$

belongs to KerH. It then follows that $S^*_{\mathrm{ImE}} \subset S \subset$ KerH.

($\leftarrow$) Choose L $\in \underline{L}(S^*_{\mathrm{ImE}})$, we have from ImE $\subset S^*_{\mathrm{ImE}}$

$$\langle A+LC \mid \mathrm{ImE} \rangle \subset \langle A+LC \mid S^*_{\mathrm{ImE}} \rangle = S^*_{\mathrm{ImE}} \subset \mathrm{KerH}$$

□

Let us now formulate the stability requirement. Since the exponents appearing in the dynamics of the controlled outputs are precisely the points in the spectrum of the operator (A + LC) mod S induced by (A + LC) on the quotient space X/S, we get as our second stability requirement:

$$\sigma(A+LC)\mathrm{mod}\ S \subset \mathbb{C}_g \tag{3-6}$$

Consequently, we have the problem of *disturbance decoupling by output injection with stability* (OIPS) to be solvable if both requirements (3-4) & (3-6) are met. For *OIP with pole placement* (OIPPP) one requires further, in addition to (3-4), that the poles of $(A+LC)\mathrm{mod}S$ be arbitrarily assignable in $\mathbb{C}$.

Theorem 15: *OIPS is solvable if and only if $S^*_{g,ImE} \subset KerH$;*
and OIPPP is solvable if and only if $N^*_{ImE} \subset KerH$.

Proof: Analogous to the proof of Theorem 14.

We now consider a slightly modified version of the output injection problem (OIP') with the controlled outputs of the form:

$$z(t) = Hx(t) - Ny(t) \tag{3-7}$$

(3-7) implies that we can subtract some linear combinations of the measured outputs from the controlled outputs and then decouple the disturbance from the new outputs.

Lemma 16: Let Ψ be a subspace of X, then $\Psi \cap$ KerC $\subset$ KerH if and only if there exists a map N such that $\Psi \subset$ Ker(H - NC).

Proof: Follows along the lines of the proof of Lemma 3.

It follows from Theorem 14, Theorem 15 and Lemma 16 that:

Theorem 17: *(a) OIP' is solvable if and only if $S^*_{ImE} \cap KerC \subset KerH$.*
(b) OIPS' is solvable if and only if $S^*_{g,ImE} \cap KerC \subset KerH$.

(c) OIPPP' is solvable if and only if $N^*_{I_mE} \cap KerC \subset KerH$.

Remark: Using Proposition 8, we can achieve two-step design in the solutions of OIPS and OIPPP. (i) Choose any map L $\in \underline{L}(S)$ so as to hold ImE in S and subsequently filter the measured outputs by G. Thus (A + LC , GC) mod S is detectable (or observable). (ii) Apply output injection L_1 with the filtered outputs to stabilize the whole process, i.e., σ (A + LC + L_1 GC) mod S belongs to $\mathbb{C}_g$ (or $\mathbb{C}_u$).

4. Disturbance Decoupled Estimation Problem

The problem of disturbance decoupled estimation (DDEP) has been treated in the literature, see for example Willems [1] and Schumacher [3]. Here in this section, we will use the theory developed in section 2 for conditioned invariant subspaces to give a simpler proof of the DDEP results and establish a relation between OIP and DDEP.

Consider again the dynamical system

$$\dot{x}(t) = Ax(t) + Eq(t)$$

$$y(t) = Cx(t), \qquad z(t) = Hx(t) \tag{4-1}$$

Where z(t) are the outputs to be estimated (*instead of controlled outputs in OIP). The problem of disturbance decoupled estimation* (DDEP) is the problem of constructing an observer of the form:

$$\dot{w}(t) = Kw(t) + Fy(t) \tag{4-2}$$

$$\hat{z}(t) = Nw(t) \tag{4-3}$$

such that the resulting estimation error

$$e(t) = \hat{z}(t) - z(t) \tag{4-4}$$

depends only on the initial conditions $w(0)$,$x(0)$ and not on the disturbance. The *problem of disturbance decoupled estimation with stability* (DDEPS) further requires that σ (K) $\subset$ $\mathbb{C}_g$ and *DDEP with pole placement* (DDEPPP) requires spectrum of the error dynamics be arbitrarily assignable in $\mathbb{C}$.

In comparison to OIP, where we desire to hold ImE in KerH through output injection, we need here to estimate the states lying in the space X/KerH with the estimation error independent of the disturbance.

<u>*Theorem*</u> 18: *(a). DDEP is solvable if and only if* $S^*_{ImE} \subset KerH$.

(b). DDEPS is solvable if and only if $S^*_{g,ImE} \subset KerH$.

(c). DDEPPP is solvable if and only if $N^*_{imE} \subset kerH$.

Remark: The solvability conditions for DDEP, DDEPS and DDEPPP are then precisely the same as that of OIP, OIPS and OIPPP respectively.

Before proving Theorem 18, we need the following important result:

<u>*Lemma*</u> 19: Let S be a subspace of X in (4-1) that <u>contains</u> ImE. Then, there exists an injection L $:Y \rightarrow X$ such that $<A+LC \mid \mathrm{ImE}> \subset S$ if and only if there exist maps K, F & N such that

$$\dot{w}(t) = Kw(t) + Fy(t) \tag{4-5}$$

$$\text{and } \; Nw(t) - Px(t) = 0 \tag{4-6}$$

where P: $X \rightarrow X/S$ is the canonical projection.

Proof: ($\rightarrow$) Let $S_1 = <\mathrm{A} + \mathrm{LC} \mid \mathrm{ImE}> \subset S$, then $S_1 \in \underline{S}$, and ImE $\subset S_1$. By Lemma 2, there exist maps K & F such that

$$\dot{w}(t) = Kw(t) + Fy(t) \tag{4-7}$$

$$\text{and } \; w(t) - P_1 x(t) = 0 \tag{4-8}$$

where $P_1 : X \rightarrow X/S_1$. Let P: $X \rightarrow X/S$ and $S_1 \subset S =$ KerP implies that there exists a unique $\overline{N} : X/S_1 \rightarrow X/S$ such that P $= \overline{N}P_1$. From (4-8), we have

$$\overline{N}w(t) - \overline{N}P_1 x(t) = \overline{N}w(t) - Px(t) = 0$$

($\leftarrow$) Using Lemma 4, S is then conditioned invariant and L $\in \underline{L}(S)$ will do the job. □

Stability and pole placement versions of Lemma 19 follow trivially from the fact that $\sigma(K) \subset \mathbb{C}_g$ and σ (A + LC) mod $S \subset \mathbb{C}_g$ or that both σ (K) and σ (A + LC) mod S must be arbitrarily placed in $\mathbb{C}$. For the sake of completeness, we state this as a lemma and leave the verifications to the reader.

<u>*Lemma*</u> 20: Let S_g (or N) be a given subspace of X that contains ImE. Then there exists an injection map L : $Y \rightarrow X$ as defined in Lemma 19 and σ (A + LC) mod $S_g \subset \mathbb{C}_g$ (or σ (A + LC) mod N is assignable in $\mathbb{C}$) if and only if there exist maps K, F & N as defined in (4-5), (4-6) , of course with P $:X \rightarrow X/S_g$ (or P: $X \rightarrow X/N$) , and that σ (K) $\subset \mathbb{C}_g$ (or assignable in $\mathbb{C}$).

Proof (of Theorem 18-a): Suppose that $S^*_{\mathrm{ImE}} \subset$ KerH. Write $S =$ KerH, and let L $\in$ $\underline{L}(S^*_{ImE})$. Then, <A + LC | ImE > $\subset$ <A + LC | S^*_{ImE} > $\subset$ KerH $= S$. Solvability of DDEP then follows from Lemma 19. Conversely, if there exist maps K, F & N that solve DDEP, then it is necessary that ImE $\subset$ KerH = S. Using Lemma 19 again our conclusion follows. □

The proofs of (Theorem 18-b) and of (Theorem 18-c) proceed analogously to the above argument.

Remark : By Theorem 18, holding disturbance for the system (4-1) in KerH is as good as estimating x mod KerH with estimation error independent of the disturbance.

Now, let us consider the modified version of DDEP (DDEP '), namely with the estimator of the form

$$\hat{z}(t) = Nw(t) + My(t) \tag{4-9}$$

Then, it may be be easily verified that

<u>*Corollary*</u> 21: (a) DDEP' is solvable if and only if $S^*_{\mathrm{ImE}} \cap$ KerC $\subset$ KerH.
(b) DDEPS' is solvable if and only if $S^*_{g,\mathrm{ImE}} \cap$ KerC $\subset$ KerH.
(c) DDEPPP' is solvable if and only if $N^*_{\mathrm{ImE}} \cap$ KerC $\subset$ KerH.

5. Solutions of X(s)M(s) = Q(s)

In the design of multivariable control systems, one frequently encounters the problem of solving rational matrix equations on various rings. In particular, solutions of $M_1(s)$ X(s) = $M_3(s)$, X(s) M_2 (s) = $M_3(s)$ and M_1(s) X(s) M_2 (s) = M_3 (s) are extremely important in the algebraic design methodology (see [10]). Using developments in geometric control theory, one is able to solve $M_1(s)X(s) = M_3(s)$ in state space form using the concept of (A, B) - invariance along with the scheme of disturbance decoupling by state feedback. In [13] a complete description to the solutions of $M_1(s)X(s) = M_3(s)$ in state space form was given. In parallel to the work of [13], we will give in this section complete solutions to the matrix equation $X(s)M_2(s) = M_3(s)$. In a later paper, we will treat the third problem, namely solutions of $M_1(s)X(s)M_2(s) = M_3(s)$ in state space form. In this section and the section that follows, we will fix $\mathbb{C}_g$ to be the open left half plane ($\mathbb{C}_-$). So when we say $\mathbb{C}_g$, we really means $\mathbb{C}_-$.

Given the equation:

$$X(s)M(s) = Q(s) \tag{5-1}$$

where M(s) $\in \mathbf{R}^{n_o \times n_i}_{p,o}(s), Q(s) \in \mathbf{R}^{n_c \times n_i}_{p,o}(s)$ and X(s) unknown, we want to solve for X(s) over one of the following rings: (1) $\mathbf{R}_{p,o}(s)$, (ii) $\mathbf{R}_{p,o}(o)$, (iii) $\mathbf{R}_{p,o}(u)$ and (iv) any of the above rings with no restriction that the solution be strictly proper, only that it be proper.

The solution of (5-1) proceeds in two steps: First, we check certain solvability conditions to be derived in this section; Second, if they are satisfied we give descriptions of the solution in state space form.

To verify solvability of (5-1), we concatenate M(s) with Q(s) to form a new transfer function G(s) and realize G(s) as in (5-2). Partition the $\hat{H}$ matrix as shown in (5-2).

$$G(s) = \begin{bmatrix} M(s) \\ Q(s) \end{bmatrix} = \hat{H}(sI-A)^{-1}E = \begin{bmatrix} C \\ H \end{bmatrix}(sI-A)^{-1}E \tag{5-2}$$

Thus, M(s) = $C(sI-A)^{-1}E$, Q(s) = $H(sI-A)^{-1}E$ and (5-1) is equivalent to

$$X(s)C(sI-A)^{-1}E = H(sI-A)^{-1}E \tag{5-3}$$

Recall now that the OIP is solvable if and only if the transfer function from $q(s)$ to $\hat{z}(s)$ is zero, i.e.

$$\hat{z}(s) = H(sI-A-LC)^{-1}Eq(s) = 0 \tag{5-4}$$

Using the identity $(I + PQ)^{-1} = I - P(I + QP)^{-1}Q$ in (5-4) we have:

$$\hat{z}(s) = H(sI-A)^{-1}[I-LC(sI-A)^{-1}]^{-1}Eq(s)$$

$$= H(sI-A)^{-1}[I + LC(I-(sI-A)^{-1}LC)^{-1}(sI-A)^{-1}]Eq(s)$$

$$= [H(sI-A)^{-1}E + H(sI-A-LC)^{-1}LC(sI-A)^{-1}E]q(s)$$

$$= 0 \tag{5-5}$$

(5-5) holds for all $q \in \mathbf{R}^{n_i}$ if and only if

$$H(sI-A)^{-1}E + H(sI-A-LC)^{-1}LC(sI-A)^{-1}E \equiv 0 \tag{5-6}$$

With X(s) = - H $(sI-A-LC)^{-1}L$ in (5-6), we see that there exists a correspondence between the solution of (5-1) and the solution of OIP (or DDEP). The following theorems in fact establish their exact equivalences.

<u>*Theorem*</u> 22: *(a). (5-1) is solvable over* $\mathbf{R}_{p,o}^{n_c \times n_o}(s)$ *if and only if* $S_{ImE}^{*} \subset KerH$.
(b). (5-1) is solvable over $\mathbf{R}_{p}^{n_c \times n_o}(s)$ *if and only if* $S_{ImE}^{*} \cap KerC \subset KerH$.

Proof (of Theorem 22-a): ($\rightarrow$) if there exists X(s) $\in \mathbf{R}_{p,o}^{n_c \times n_o}(s)$ that solves (5-1), then by (5-3)

$$[X(s)C(sI-A)^{-1}E - H(sI-A)^{-1}E]\hat{q}(s) = 0 \quad \text{for all } \hat{q}(s) \tag{5-7}$$

In the DDEP, y(s) = $C(sI-A)^{-1}E\ \hat{q}(s)$, z(s) = $H(sI-A)^{-1}E\ \hat{q}(s)$ and (5-7) implies that the DDEP is solvable with the observer defined by

$$\dot{w}(t) = Kw(t) + Fy(t) \tag{5-8}$$

$$\hat{z}(t) = Nw(t)$$

where $N(sI-K)^{-1}F = X(s)$. It then follows from Theorem 18 that $S^{*}_{\mathrm{ImE}} \subset$ KerH.

($\leftarrow$) If $S^{*}_{\mathrm{ImE}} \subset$ KerH, then both the OIP and the DDEP are solvable. X(s) = - H $(sI-A-LC)^{-1}$ L with L $\in \underline{L}(S^{*}_{\mathrm{ImE}})$, or X(s) = N $(sI-K)^{-1}$F, the observer transfer function, is a solution of (5-1). $\square$

The proof of (Theorem 22-b) follows from OIP' and DDEP'.

<u>*Theorem*</u> 23. *If (A,E) is stabilizable, then (5-1) is solvable over* $\mathbf{R}_{p,o}^{n_c \times n_o}(o)$ *if and only if* $S^{*}_{g,ImE} \subset KerH$.

Theorem 23 is the stable version of Theorem 22. The importance of this theorem lies in the fact that usually only stable solutions to (5-1) are needed in real control system design. By the theorem, existence of a strictly proper and stable X(s) that solves (5-1) is completely equivalent to the solvability of DDEPS or of OIPS. The proof of Theorem 23 proceeds in two steps. First we prove the following lemma:

<u>*Lemma*</u> 24: If the proper rational function X(s) is stable and the pair (A, E) is stabilizable, then

$$X(s)C(sI-A)^{-1}E - H(sI-A)^{-1}E = 0 \tag{5-9}$$

implies that

$$v(s) = X(s)C(sI-A)^{-1}x(0) - H(sI-A)^{-1}x(0) \tag{5-10}$$

is stable for all initial conditions $x(0)$.

The proof of Lemma 24 is somewhat lengthy but is straight forward.

Proof: Since X(s) is stable, we can only consider the case where $x(0) \in X_b(A)$. Again, X_b(A) denotes the subspace of X spanned by the (generalized) eigenvectors of A corresponding to eigenvalues in $\mathbb{C}_b$. For if $x(0) \in X_g$(A) , then stability of (5-10) is automatically guaranteed.

First, let's assume that $\nu \in X_b(A)$ is an generalized eigenvector of A of degree k corresponding to eigenvalue $\lambda \in \mathbb{C}_b$. Then, $(A - \lambda I)^i \nu \neq 0$ for $1 \leq i \leq k-1$ and $(A - \lambda I)^k \nu = 0$. Denote this generalized eigenvector chain by $\{\nu_i\}_{i=1}^{i=k}$, where $\nu_k = \nu$ and $\nu_i = (A-\lambda I)^{k-i} \nu$ for $i=1,\ldots,k-1$. Stabilizability of (A, E) implies that for each ν_i there exists an input $q_i \in \mathbb{R}^{n_i}$ such that

$$\nu_i = \int_0^t e^{A(t-\tau)} E q_i(\tau) d\tau, \qquad \text{for } 0<t<\infty \tag{5-11}$$

Taking the Laplace transform of the above expression we have

$$\frac{\nu_i}{s} = (sI-A)^{-1} E \, \hat{q}_i(s) \tag{5-12}$$

Thus, $\nu_i = (sI-A)^{-1} \gamma_i(s)$, where $\gamma_i(s) = s\hat{q}_i(s)$. Multiply equation (5-9) by $\gamma_i(s)$ to give

$$X(s)C(sI-A)^{-1}E\,\gamma_i(s) - H(sI-A)^{-1}E\,\gamma_i(s) = \{X(s)C - H\}\,\nu_i = 0 \tag{5-13}$$

Now, consider the expression $(sI - A)^{-1}x(0)$. Without loss of generality, we may assume that $x(0)$ is a linear combination of the generalized eigenvector chain $\{\nu_i\}_{i=1}^{i=k}$ alone, i.e., $x(0) = \sum_{i=1}^{i=k} \alpha_i \nu_i$ for some constant α_i 's. Thus

$$(sI-A)^{-1}x(0) = (\frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \cdots)x(0)$$

$$=(\frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \cdots)\sum_{i=1}^{i=k} \alpha_i \nu_i \tag{5-14}$$

In turn using the definition of generalized eigenvectors, we have

$$(sI-A)^{-1}x(0) = \sum_{i=1}^{i=k} \hat{\alpha}_i(s, \lambda)\nu_i \tag{5-15}$$

for some $\hat{\alpha}_i(s, \lambda)$, which is generally a function of both s and λ.

Using (5-15) we can rewrite (5-10) as

$$v(s) = \{X(s)C - H\} \sum_{i=1}^{i=k} \hat{\alpha}_i(s, \lambda)\nu_i \tag{5-16}$$

On the other hand, (5-13) implies that

$$\{X(s)C - H\} \sum_{i=1}^{i=k} \hat{\alpha}_i(s,\lambda)\nu_i = 0 \tag{5-17}$$

Which shows that (5-10) is identically zero for all initial conditions $x(0) \in X_b(A)$ and therefore completes our proof. □

We are now in the position to prove Theorem 23.

Proof (of Theorem 23):($\rightarrow$) By the OIP, $X(s) = -H(sI-A-LC)^{-1}L$ with $L \in \underline{L}(S^*_{g,\mathrm{ImE}})$ is a stable solution of (5-1).

($\leftarrow$) Using a minimal realization of X(s) of the form $N(sI-K)^{-1}F$, we may construct an observer using (N, K & F). By minimality of the realization, σ (K) = P{X(s)} $\subset$ $\mathbb{C}_g$; with P{X(s)} denoting the Smith-McMillan poles of X(s). Now, consider the Laplace transform of the error term e(t) = $\hat{z}(t)$ - z(t) in the solution of DDEPS:

$$e(s) = N(sI-K)^{-1}w(0) + N(sI-K)^{-1}FC(sI-A)^{-1}x(0) - H(sI-A)^{-1}x(0)$$

$$+ N(sI-K)^{-1}FC(sI-A)^{-1}Eq(s) - H(sI-A)^{-1}Eq(s) \tag{5-18}$$

By (5-3), the last two terms in the right hand side of (5-18) drop out. Also, σ (K) $\subset$ $\mathbb{C}_g$ implies that the first term in (5-18) is stable. It follows from Lemma 24 that the remaining two terms and therefore the whole expression in (5-18) is stable for all initial conditions $x(0)$. This shows that DDEPS is solvable and by Theorem (18-b) $S^*_{g,\mathrm{ImE}} \subset$ KerH. □

Theorem 23 gives necessary and sufficient condition for existence of a strictly proper stable X(s) that solves (5-1). For an *proper* and stable X(s) that also solves (5-1), we have the following result:

Corollary 25: If (A,E) is stabilizable, then there exists X(s) $\in \mathbb{R}_p^{n_c \times n_o}(o)$ that solves (5-1) if and only if $S^*_{g,\mathrm{ImE}} \cap$ KerC $\subset$ KerH.

For the solution of X(s) in (5-1) with pole placement we have:

Theorem 26: a). If (A,E) is reachable, then there exists X(s) $\in \mathbb{R}_{p,o}^{n_c \times n_o}(u)$ that solves (5-1) if and only if $N^*_{ImE} \subset KerH$.

b). With the reachability assumption of (A, E) hold, there exists X(s) $\in \mathbb{R}_p^{n_c \times n_o}(u)$ that solves (5-1) if and only if $N^*_{ImE} \cap KerC \subset KerH$.

Remark 1: To prove Theorem 26 is really no more than to duplicate the proof of Theorem 23.

Remark 2 : The result of Theorem 23 is no longer valid if the stabilizability condition is omitted. Consider the example $n_i = n_o = 1$, n = 2, $\mathbb{C}_g$ = { s | Res < 0}, H = [0, 1], C = [1, 0]

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Then, S = ImE = { $(x_1, 0)'$ | $x_1 \subset \mathbb{R}$ } is (C,A)-invariant (even A-invariant). We have (A + LC) $S \subset S$ for L' = [l_1, l_2] if and only if $l_2 = 0$. But then

$$(A+LC)=\begin{bmatrix} l_1 & 0 \\ 0 & 1 \end{bmatrix}$$

and σ (A + LC) mod S = {1} $\not\subset$ $\mathbb{C}_g$. Thus, $S^*_{g,\mathrm{ImE}} = \mathbb{R}^2$ which obviously does not belong to KerH. But, H$(sI-A)^{-1}E$ = 0, and $C(sI-A)^{-1}E$ = 1/s, (5-1) has a strictly proper stable solution X(s) = 0.

6. Output Stabilization with Respect to Disturbances

Disturbance decoupling by output injection as described in section 3 cannot proceed without the satisfaction of the necessary condition HE = 0. In this section, we shall be content with a more modest objective, namely, attempt to find an injection L: $Y \rightarrow X$, such that the noise to output map of the closed loop system is stable.

OSDP: Given the system (3-1), determine L: $Y \rightarrow X$ such that $H(sI-A-LC)^{-1}E$ is stable.

Theorem 27: *OSDP is solvable* if and *only* if $S^*_{ImE} \cap S^*_g \subset KerH$, *where* S^*_g *denote the infimal complementary detectability subspace.*

Proof: We choose G_0 , as in Proposition 8, such that $\mathrm{Ker}G_0C = S^*_g + \mathrm{Ker}C = \mathrm{Ker}C$ and $L_0 \in \underline{L}\ (S^*_{\mathrm{ImE}})$, then S^*_g, and therefore $S^*_g \cap S^*_{\mathrm{ImE}}$ is $(A+L_0C)$ -invariant (see Lemma 13). We choose a basis $(q_1, q_2, \cdots q_n)$ of X such that $(q_1, \ldots q_k)$ is a basis of $S^*_{\mathrm{ImE}} \cap S^*_g$, $(q_1, \ldots q_l)$ is a basis of S^*_{ImE} (where $l \geq k$), $(q_1, \ldots q_k, q_{l+1}, \cdots q_h)$ a basis of S^*_g. Accordingly, we can split matrices and vectors as:

$$A+L_0C = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}, \quad E = \begin{bmatrix} E_1 \\ E_2 \\ 0 \\ 0 \end{bmatrix}$$

$$G_0C = [0\ C_2\ 0\ C_4] \qquad H = [0\ H_2\ H_3\ H_4]$$

Since S^*_g is a complementary detectability subspace and $(L_0, G_0) \in \Phi\ (S^*_g)$ it follows that

$$\begin{bmatrix} A_{22} & A_{24} \\ 0 & A_{44} \end{bmatrix} \quad [C_2, C_4]$$

is detectable. Hence, (A_{22}, C_2) is detectable. Therefore, there exists L_2 such that $\sigma(A_{22}+L_2C_2) \subset \mathbb{C}_g$. If we define $\hat{L}' = [0, G_0'L_2', 0, 0]$, we see that H $[sI-A-(L_0+\hat{L})C]^{-1}$ E $= H_2(sI-A_{22}-L_2C_2)^{-1}E_2$ is stable if $S^*_g \cap S^*_{\mathrm{ImE}} \subset$ KerH. □

Corollary 28: a).There exists $X(s) \in \mathbf{R}_{p,o}^{n_c \times n_o}(s)$ such that X(s)M(s) - Q(s) is stable if and only if $S^*_g \cap S^*_{\mathrm{ImE}} \subset$ KerH.

b). there exists an $X(s) \in \mathbf{R}_{p,o}^{n_c \times n_o}(o)$ such that X(s)M(s) - Q(s) is stable if and only if $S^*_g \cap S^*_{g,\mathrm{ImE}} \subset$ KerH.

Remark: Let $e(s) = H(sI - A - LC)^{-1}E$ in the solution of OSDP and by (5-5), $e(s) = H(sI - A)^{-1}E + H(sI - A - LC)^{-1}LC(sI - A)^{-1}E$. Thus, the error term, i.e., X(s) M(s) - Q(s) is given by $H_2(sI - A_{22} - L_2C_2)^{-1}E_2$, with A_{22}, C_2, H_2, and L_2 defined in the proof of

Theorem 27.

7. Conclusions

We have introduced conditioned invariant, complementary detectability and complementary observability subspaces both in terms of output injection and in terms of observer design schemes. We have given detailed characterizations of the infimal (C,A)-invariant subspace that contains a subspace Ψ. Section 2 is a complete dualization of Chapters 4 and 5 of Wonham [4]. In sections 3 & 4, we have proposed the disturbance decoupling by output injection scheme - a natural dualization of the disturbance decoupling by state feedback (DDP) scheme and have given solutions to stability and pole placement versions of OIP. We have established a direct relation between OIP and DDEP.

We have solved X(s) M(s) = Q(s) in section 5 using OIP and DDEP: necessary and sufficient conditions for solvability of the equation with or without stability or pole placement are given and proven; and we have shown when strictly proper and when proper solutions exist to the above equation. We mention here that the techniques of [13] may be used to solve the transposed equation (5-1). Our techniques are however of independent interest. Finally, we have briefly addressed the problem of output stabilization with respect to the disturbance in the OIP scheme and have subsequently given necessary and sufficient conditions for existence of an X(s) such that the error term X(s) M(s) - Q(s) is stable.

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