NEW STABILITY THEOREMS FOR AVERAGING AND THEIR APPLICATIONS TO THE CONVERGENCE ANALYSIS OF ADAPTIVE IDENTIFICATION AND CONTROL SCHEMES

by

L-C. Fu, M. Bodson, and S. S. Sastry

Memorandum No. UCB/ERL M85/21

14 March 1985

COVER PAGE

NEW STABILITY THEOREMS FOR AVERAGING AND THEIR APPLICATIONS TO THE CONVERGENCE ANALYSIS OF ADAPTIVE IDENTIFICATION AND CONTROL SCHEMES

by

L-C. Fu, M. Bodson, and S. S. Sastry

Memorandum No. UCB/ERL M85/21

14 March 1985

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

TITLE PAGE

New Stability Theorems for Averaging and Their Application to the Convergence Analysis of Adaptive Identification and Control Schemes

*Li-Chen Fu, Marc Bodson and Shanker Sastry**

Electronics Research Laboratory
Department of Electrical Engineering & Computer Science
University of California, Berkeley CA 94720

ABSTRACT

We develop new stability theorems for the convergence analysis of a class of one and two-time scale time-varying nonlinear systems using averaging theory. These theorems are applied to a class of continuous time adaptive identifiers and model reference adaptive controllers to obtain estimates of the parameter rate of exponential convergence.

March 14, 1985

Keywords: *Averaging, Stability Analysis, Lyapunov Converse Theorems, Adaptive Identification and Control, Parameter Convergence.*

New Stability Theorems for Averaging and Their Application to the Convergence Analysis of Adaptive Identification and Control Schemes

*Li-Chen Fu, Marc Bodson and Shanker Sastry**

Electronics Research Laboratory
Department of Electrical Engineering & Computer Science
University of California, Berkeley CA 94720

1. Introduction

The method of averaging is concerned with differential equations of the form

$$\dot{x} = \epsilon f(t,x) \tag{1.1}$$

and relates the properties of solutions of system (1.1) with solutions of the autonomous "averaged" system

$$\dot{x}_{av} = \epsilon f_{av}(x_{av}) \tag{1.2}$$

$$f_{av}(x) = \lim_{T\to\infty} \frac{1}{T} \int_{s}^{s+T} f(t,x)\,dt \tag{1.3}$$

for sufficiently small values of the parameter ϵ. The method was proposed originally by Kryloff-Bogoliuboff- Mitropolskii [1], reformulated by Hale [2,3], developed subsequently by Sethna [4,5] and stated in a geometric form in Arnold [6], Guckenheimer and Holmes [7]. These results constitute a generalization of classical singular perturbation techniques such as those in Hoppensteadt [8] and are an extremely important tool for the state space analysis of systems with multiple time scales. These results have been used extensively in mathematical physics. From our viewpoint, as control theorists, we feel that the technique bears the promise of evolving into a "frequency domain" technique for the *state-space* trajectory analysis of some classes of nonlinear systems— to be distinguished from the Volterra approach for *input-output* functional expansions for non-linear systems. This theme will be developed in later work.

The current paper has two sets of contributions:

(A) We develop new theorems for averaging. With the exception of [4,5], all the aforementioned references make the assumption of almost periodicity for the right hand side of (1.2). We relax this assumption in Section 2 of this paper. Our theorems are rather different in hypothesis

*Research supported by NASA under grant NAG 2-243, the IBM Corporation under a Faculty Development Award and the Semiconductor Research Corporation. We would like to thank B. Riedle & P. Kokotovic, K. Astrom, S. Boyd, E. Bai and H. Ping for several valuable discussions.

and rather simpler than those of [4,5] in this regard. Another important contribution of Section 2 is to relate the stability of the averaged system (1.2) to the stability of the unaveraged system (1.1), using a converse theorem of Lyapunov. As such, these theorems are a considerable extension of the local stability theorems of Hale.

In Section 4, we extend all of these results to two-time scale state space systems and the results are generalizations in the sense mentioned above of those of Hale and Sethna.

(B) Our development of these theorems on averaging was heavily motivated by recent literature on the application of averaging techniques to adaptive control--notably the work of Krause, et al [9], Riedle and Kokotovic [10], Astrom [11]. Averaging methods have been more prevalent in the stochastic adaptive control literature, eg. Ljung [12] and the first attempts to apply averaging were made heuristically in [9], and increasingly rigorously in [11] and [10]. * The primary focus of the efforts in [9-11] is to use averaging to explain instability mechanisms in adaptive control arising from unmodelled dynamics, a phenomenon popularized by Rohrs et al [13]. In this paper, we content ourselves with applying our results on averaging theory along with techniques of generalized harmonic analysis introduced in Boyd-Sastry [14]. We study convergence rates of adaptive identification schemes and linearized adaptive control schemes without unmodelled dynamics and in the presence of persistent excitation. Estimates of convergence rates are of interest in the determination of optimal input signals for identification. In earlier work (Bodson-Sastry [15]), we also showed how persistent excitation guarantees a margin of robustness to unmodelled dynamics and established connections between the rate of convergence of the adaptive schemes and their robustness margins. A more detailed study of instability theorems for averaging and their application to understanding the mechanism of slow drift instability pointed out by Riedle-Kokotovic [16] will be presented in forthcoming work.

The results of Section 3 on the application of averaging theory to obtaining estimates of the convergence rates for adaptive identifiers are to our knowledge new , while those of Section 5 on convergence rates for adaptive control schemes in the relative degree 1 case are a small generalization of the results of [10] with a somewhat different focus.

*After this manuscript was written, new and related work of Kosut and Anderson [17] was communicated to us for system (1.1) with $f(t,x)$ linear in x, but with weaker conditions in the limit in (1.3).

2. Basic Averaging Theory

In this section, we consider differential equations of the form:

$$\dot{x} = \epsilon f(t,x,\epsilon) \qquad x(0)=x_0 \tag{2.1}$$

where $x \in R^n$, $t \geq 0$, $0 < \epsilon \leq \epsilon_0$, and f is piecewise continuous with respect to time. We will concentrate our attention on the behavior of the solutions in some closed ball B_h of radius h, centered at the origin.

For small ϵ, the variation of x with time is slow, as compared to the rate of time variation of f. Such systems can be conveniently studied using the *method of averaging* (see e.g. [1], [3], [6], [7]). The theory relies on the assumption of the existence of the mean value of $f(t,x,0)$ defined by the limit:

$$f_{av}(x) = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} f(\tau,x,0)d\tau \tag{2.2}$$

assuming that the limit exists uniformly in t and x. This is formulated more precisely in the following definition:

Definition 2.1 Mean Value of a Function, Convergence Function

The function $f(t,x,0)$ is said to have mean value $f_{av}(x)$ if there exists a continuous function $\gamma(T)$: $R_+ \to R_+$, strictly decreasing, such that $\gamma(T) \to 0$ as $T \to \infty$, and:

$$\left\| \frac{1}{T} \int_t^{t+T} f(\tau,x,0)d\tau - f_{av}(x) \right\| \leq \gamma(T) \tag{2.3}$$

for all $t, T \geq 0$, $x \in B_h$.

The function $\gamma(T)$ will be called the *convergence function*.

Note that the function $f(t,x,0)$ has mean value $f_{av}(x)$ if and only if the function:

$$d(t,x) = f(t,x,0) - f_{av}(x) \tag{2.4}$$

has zero mean value.

The following definition ([20], p 7) will also be useful:

Definition 2.2 Class K Function

A function $\alpha(\epsilon)$:$R_+ \to R_+$ belongs to class K ($\alpha(\epsilon) \in K$), if it is continuous, strictly increasing, and $\alpha(0)=0$.

It is common, in the literature on averaging, to assume that the function $f(t,x,\epsilon)$ is periodic in t, or almost periodic in t. Then, the existence of the mean value is guaranteed, without further assumption ([3], theorem 6, p 344). We do not make the assumption of (almost) periodicity, but consider instead the assumption of the existence of the mean value as the starting point of our analysis.

Note that if the function $d(t,x)$ is periodic in t, and is bounded, then the integral of the function $d(t,x)$ is also a bounded function of time. This is equivalent to saying that there exists a convergence function $\gamma(T)=a/T$ (i.e. of the order of $1/T$) such that (2.3) is satisfied. On the other hand, if the function $d(t,x)$ is bounded, but is only required to be almost periodic, then the integral of the function $d(t,x)$ need not be a bounded function of time, even if its mean value is zero ([3], p 346). The function $\gamma(T)$ is bounded (by the same bound as $d(t,x)$), and converges to zero as $T\rightarrow\infty$, but the convergence function need not be bounded by a/T as $T\rightarrow\infty$ (it may be of order $1/\sqrt{T}$ for example). In general, a zero mean function need not have a bounded integral, although the converse is true. In this paper, we do not make the distinction between the periodic, and the almost periodic case, but we do distinguish the bounded integral case from the general case, and indicate the importance of the function $\gamma(T)$ in the subsequent development.

Assuming the existence of the mean value for the original system (2.1), the *averaged system* is defined to be:

$$\dot{x}_{av}=\epsilon f_{av}(x_{av}) \qquad x_{av}(0)=x_0 \tag{2.5}$$

Note that the averaged system is autonomous and, for T fixed and ϵ varying, the solutions over intervals $[0,T/\epsilon]$ are identical, modulo a simple time scaling by ϵ.

We address the following two questions:

(i) the closeness of the response of the original and averaged systems,

(ii) the relationships between the stability properties of the two systems.

To compare the solutions of the original and of the averaged system, it is convenient to transform the original system in such a way that it becomes a *perturbed* version of the averaged system. An important lemma that leads to this result is attributed to Bogoliuboff and Mitropolskii ([1], p 450, and [3], lemma 4, p 346). We state a generalized version of this lemma.

Lemma 2.1 Approximate Integral of a Zero Mean Function

If: $d(t,x):R_+\times B_h\rightarrow R^n$ is a bounded function, piecewise continuous with respect to t, and has zero mean value with convergence function $\gamma(T)$,

Then: There exists $\xi(\epsilon)\in K$, and a function $w_\epsilon(t,x):R_+\times B_h\rightarrow R^n$ such that:

$$\|\epsilon w_\epsilon(t,x)\|\leq\xi(\epsilon) \tag{2.6}$$

$$\|\frac{\partial w_\epsilon(t,x)}{\partial t}-d(t,x)\|\leq\xi(\epsilon) \tag{2.7}$$

for all $t \geq 0$, $x \in B_h$. Moreover, $w_\epsilon(0,x)=0$, for all $x \in B_h$.

If, moreover: $\gamma(T)=a/T^r$ for some $a \geq 0$, $r \in (0,1]$,

Then: The function $\xi(\epsilon)$ can be chosen to be $2a\epsilon^r$.

The proof of *Lemma 2.1* is provided in the appendix. The construction of the function $w_\epsilon(t,x)$ is identical to that in [2.1], but the proof of (2.6), (2.7) is different, and leads to the relationship between the convergence function $\gamma(T)$ and the function $\xi(\epsilon)$.

The main point of *Lemma 2.1* is that, although the exact integral of $d(t,x)$ may be an unbounded function of time, there exists a bounded function $w_\epsilon(t,x)$, whose first partial derivative with respect to t is arbitrarily close to $d(t,x)$. Although the bound on $w_\epsilon(t,x)$ may increase as $\epsilon \to 0$, it increases slower than $1/\epsilon$, as indicated by (2.6).

It is necessary to obtain a function $w_\epsilon(t,x)$, as in *Lemma 2.1*, that has some additional smoothness properties. A useful lemma is given by Hale in [3] (lemma 5, p 349). For the price of additional assumptions on the function $d(t,x)$, the following lemma leads to stronger conclusions that are useful in the sequel.

Lemma 2.2 Smooth Approximate Integral of a Zero Mean Function

If: $d(t,x): R_+ \times B_h \to R^n$ is piecewise continuous with respect to t, has bounded and continuous first partial derivatives with respect to x, and $d(t,0)=0$ for all $t \geq 0$. Moreover, $d(t,x)$ has zero mean value, with convergence function $\gamma(T)||x||$, and $\dfrac{\partial d(t,x)}{\partial x}$ has zero mean value, with convergence function $\gamma(T)$,

Then: There exists $\xi(\epsilon) \in K$, and a function $w_\epsilon(t,x): R_+ \times B_h \to R^n$, such that:

$$||\epsilon w_\epsilon(t,x)|| \leq \xi(\epsilon)||x|| \tag{2.8}$$

$$||\frac{\partial w_\epsilon(t,x)}{\partial t} - d(t,x)|| \leq \xi(\epsilon)\,||x|| \tag{2.9}$$

$$||\epsilon \frac{\partial w_\epsilon(t,x)}{\partial x}|| \leq \xi(\epsilon) \tag{2.10}$$

for all $t \geq 0$, $x \in B_h$. Moreover, $w_\epsilon(0,x)=0$, for all $x \in B_h$.

If, moreover: $\gamma(T)=a/T^r$ for some $a \geq 0$, $r \in (0,1]$,

Then: the function $\xi(\epsilon)$ can be chosen to be $2a\epsilon^r$.

The proof of *Lemma 2.2* is provided in the appendix. The difference from *Lemma 2.1* is in the condition on the partial derivative of $w_\epsilon(t,x)$ with respect to x in (2.10), and the dependence on $||x||$ in (2.8), (2.9). These results will be necessary to derive the following theorems.

Note that if the original system is linear, i.e.:

$$\dot{x} = A(t)x \qquad x(0)=x_0 \tag{2.11}$$

for some $A(t): R_+ \to R^{n \times n}$, then the main assumption of *Lemma 2.2* is that there exists A_{av} such that $A(t) - A_{av}$ has zero mean value.

The following assumptions will hence forth be in effect.

(A1) $x=0$ is an equilibrium point of system (2.1), i.e. $f(t,0,0)=0$ for all $t \geq 0$. $f(t,x,\epsilon)$ is Lipschitz in x, i.e.:

$$\|f(t,x_1,\epsilon) - f(t,x_2,\epsilon)\| \leq l_1 \|x_1 - x_2\| \tag{2.12}$$

for all $t \geq 0$, $x_1, x_2 \in B_h$, $\epsilon \leq \epsilon_0$.

(A2) $f(t,x,\epsilon)$ is Lipschitz in ϵ, linearly in x, i.e.:

$$\|f(t,x,\epsilon_1) - f(t,x,\epsilon_2)\| \leq l_2 \|x\| \, |\epsilon_1 - \epsilon_2| \tag{2.13}$$

for all $t \geq 0$, $x \in B_h$, $\epsilon_1, \epsilon_2 \leq \epsilon_0$.

(A3) $f_{av}(0)=0$, and $f_{av}(x)$ is Lipschitz in x, i.e.:

$$\|f_{av}(x_1) - f_{av}(x_2)\| \leq l_{av} \|x_1 - x_2\| \tag{2.14}$$

for all x_1, $x_2 \in B_h$.

(A4) the function $d(t,x) = f(t,x,0) - f_{av}(x)$ satisfies the conditions of *Lemma 2.2*.

Lemma 2.3 Perturbation Formulation of Averaging

If: The systems (2.1), and (2.5) satisfy assumptions (A1)-(A4),

Then: There exist functions $w_\epsilon(t,x)$ and $\xi(\epsilon)$, as in *Lemma 2.2*, and a transformation of the form:

$$x = z + \epsilon w_\epsilon(t,z) \tag{2.15}$$

under which system (2.1) becomes:

$$\dot{z} = \epsilon f_{av}(z) + \epsilon p(t,z,\epsilon) \qquad z(0)=x_0 \tag{2.16}$$

where $p(t,z,\epsilon)$ satisfies:

$$\|p(t,z,\epsilon)\| \leq \psi(\epsilon) \|z\| \tag{2.17}$$

for some $\psi(\epsilon) \in K$, $\epsilon_1 > 0$, and for all $\epsilon \leq \epsilon_1$. Further, $\psi(\epsilon)$ is of the order of $\epsilon + \xi(\epsilon)$.

Comments

The proof of *Lemma 2.3* is provided in the appendix. A similar lemma can be found in [3] (lemma 3.2, p 192). Inequality (2.17) is a Lipschitz type of condition on $p(t,z,\epsilon)$, which is not found in [3], and results from the stronger conclusions of *Lemma 2.2*.

Lemma 2.3 is fundamental to the theory of averaging presented hereafter. It separates the error in the approximation of the original system by the averaged system $(x-x_{av})$ into two components: $x-z$ and $z-x_{av}$. The first component results from a pointwise (in time) transformation of variable. This component is guaranteed to be small by inequality (2.8). For ϵ sufficiently small ($\epsilon \leq \epsilon_1$), the transformation $z \rightarrow x$ is invertible, and as $\epsilon \rightarrow 0$, it tends to the identity transformation. The second component is due to the perturbation term $p(t,z,\epsilon)$. Inequality (2.17) guarantees that this perturbation is small as $\epsilon \rightarrow 0$.

At this point, we can relate the convergence of the function $\gamma(T)$ to the order of the two components of the error $x-x_{av}$ in the approximation of the original system by the averaged system. The relationship between the functions $\gamma(T)$ and $\xi(\epsilon)$ was indicated in *Lemma 2.1*. *Lemma 2.3* relates the function $\xi(\epsilon)$ to the error due to the averaging. If $d(t,x)$ has a bounded integral (i.e. $\gamma(T) \sim 1/T$), then both $x-z$ and $p(t,z,\epsilon)$ are of the order of ϵ with respect to the main term $f_{av}(z)$. In general, these terms go to zero as $\epsilon \rightarrow 0$, but possibly more slowly than linearly (as $\sqrt{\epsilon}$ for example). The proof of *Lemma 2.1* provides a direct relationship between the order of the convergence to the mean value, and the order of the error terms.

We now focus attention on the approximation of the original system by the averaged system. Consider first the following assumption:

(A5) x_0 is sufficiently small so that, for fixed T, and some $h' < h$, $x_{av}(t) \in B_{h'}$ for all $t \in [0, T/\epsilon]$ (this is possible, from (A3)).

Theorem 2.4 Basic Averaging Theorem

If: The original system (2.1), and the averaged system (2.5) satisfy assumptions (A1)-(A5),

Then: There exists $\psi(\epsilon)$ as in *Lemma 2.3* such that, given $T \geq 0$:

$$\|x(t) - x_{av}(t)\| \leq \psi(\epsilon)\, b_T \tag{2.18}$$

for some b_T, $\epsilon_T > 0$, and for all $t \in [0, T/\epsilon]$, $\epsilon \leq \epsilon_T$.

Proof: From *Lemma 2.2* and *Lemma 2.3*, we have that:

$$\|x - z\| \leq \xi(\epsilon)\|z\| \leq \psi(\epsilon)\|z\| \tag{2.19}$$

for $\epsilon \leq \epsilon_1$. On the other hand, we have that:

$$\frac{d}{dt}(z-x_{av})=\epsilon(f_{av}(z)-f_{av}(x_{av}))+\epsilon p(t,z,\epsilon) \qquad z(0)-x_{av}(0)=0 \tag{2.20}$$

for all $t\in[0,T/\epsilon]$, $x_{av}\in B_{h'}$, $h'<h$. We will now show that, on this time interval, and for as long as $x,z\in B_h$, the errors $(z-x_{av})$ and $(x-x_{av})$ can be made arbitrarily small by reducing ϵ.

Integrating (2.19):

$$\|z(t)-x_{av}(t)\| \leq \epsilon l_{av}\int_0^t \|z(\tau)-x_{av}(\tau)\|d\tau+\epsilon\psi(\epsilon)\int_0^t\|z(\tau)\|d\tau \tag{2.21}$$

Using the *Generalized Bellman-Gronwall Lemma* (see appendix):

$$\|z(t)-x_{av}(t)\| \leq \epsilon\psi(\epsilon)\int_0^t\|z(\tau)\|e^{\epsilon l_{av}(t-\tau)}d\tau \leq \psi(\epsilon)h\left(\frac{e^{\epsilon l_{av}T}-1}{l_{av}}\right)$$

$$:=\psi(\epsilon)\,a_T \tag{2.22}$$

Combining these results:

$$\begin{aligned}\|x(t)-x_{av}(t)\| &\leq \|x(t)-z(t)\|+\|z(t)-x_{av}(t)\| \\ &\leq \psi(\epsilon)\|x_{av}(t)\|+(1+\psi(\epsilon))\|z(t)-x_{av}(t)\| \\ &\leq \psi(\epsilon)(h+(1+\psi(\epsilon_1))\,a_T) \\ &:=\psi(\epsilon)b_T \end{aligned} \tag{2.23}$$

By assumption, $\|x_{av}(t)\|\leq h'<h$. Let ϵ_T (with $0<\epsilon_T\leq\epsilon_1$) such that $\psi(\epsilon_T)b_T<h-h'$. It follows, from a simple contradiction argument, that the estimate in (2.23) is valid for all $t\in[0,T/\epsilon]$, whenever $\epsilon\leq\epsilon_T$.

Comments

Theorem 2.4 establishes that the trajectories of the original and the averaged system are arbitrarily close on intervals $[0,T/\epsilon]$, as ϵ is reduced. The error is of the order of $\psi(\epsilon)$, and the order is related to the order of convergence of $\gamma(T)$. If $d(t,x)$ has a bounded integral (i.e. $\gamma(T)\sim 1/T$), then the error is of the order of ϵ.

It is important to remember that, although the intervals $[0,T/\epsilon]$ are unbounded, *Theorem 2.4* does not state that:

$$\|x(t)-x_{av}(t)\|\leq\psi(\epsilon)\,b \tag{2.24}$$

for all $t\geq 0$, and some b. Consequently, *Theorem 2.4* does not allow us to relate the stability of the original and of the averaged system. This relationship is investigated in *Theorem 2.5,* after a preliminary definition.

Definition 2.3 Exponential Stability, Rate of Convergence

The equilibrium point $x=0$ of a differential equation is said to be *exponentially stable*, with *rate of convergence* α ($\alpha>0$), if:

$$\|x(t)\| \leq m\|x(t_0)\|\, e^{-\alpha(t-t_0)} \tag{2.25}$$

for all $t \geq t_0 \geq 0$, $x(t_0) \in B_{h_0}$, and some $m \geq 1$.

We assume that $h_0 \leq h/m$, so that all trajectories are guaranteed to remain in B_h.

Theorem 2.5 Exponential Stability Theorem

If: The original and averaged systems satisfy assumptions (A1)-(A5), the function $f_{av}(x)$ has continuous and bounded first order partial derivatives in x, and $x=0$ is an exponentially stable equilibrium point of the averaged system,

Then: There exists $\epsilon_2>0$ such that the equilibrium point $x=0$ of the original system is exponentially stable for all $\epsilon \leq \epsilon_2$.

Proof: The proof relies on a converse theorem of Lyapunov for exponentially stable systems (see for example [20], p 273). Under the hypotheses, there exists a function $v(x_{av}): R^n \rightarrow R_+$, and strictly positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, such that, for all $x_{av} \in B_{h_0}$:

$$\alpha_1 \|x_{av}\|^2 \leq v(x_{av}) \leq \alpha_2 \|x_{av}\|^2 \tag{2.26}$$

$$\dot{v}(x_{av})\big|_{(2.5)} \leq -\epsilon\, \alpha_3 \|x_{av}\|^2 \tag{2.27}$$

$$\left\|\frac{\partial v}{\partial x_{av}}\right\| \leq \alpha_4 \|x_{av}\| \tag{2.28}$$

The derivative in (2.27) is to be taken along the trajectories of the averaged system (2.5). The function v is now used to study the stability of the perturbed system (2.16). Considering $v(z)$, inequalities (2.26) and (2.28) are still verified, with z replacing x_{av}. The derivative of $v(z)$ along the trajectories of (2.16) is given by:

$$\dot{v}(z)\big|_{(2.16)} = \dot{v}(z)\big|_{(2.5)} + \left(\frac{\partial v}{\partial z}\right)(\epsilon\, p(t,z,\epsilon)) \tag{2.29}$$

and, using previous inequalities (including those from *Lemma 2.3*):

$$\begin{aligned}\dot{v}(z)\big|_{(2.16)} &\leq -\epsilon\alpha_3\|z\|^2 + \epsilon\alpha_4\psi(\epsilon)\|z\|^2 \\ &\leq -\epsilon\left[\frac{\alpha_3-\psi(\epsilon)\alpha_4}{\alpha_2}\right] v(z)\end{aligned} \tag{2.30}$$

for all $\epsilon \leq \epsilon_1$. Let ϵ'_2 be such that $\alpha_3-\psi(\epsilon'_2)\alpha_4>0$, and define $\epsilon_2=\min(\epsilon_1,\epsilon'_2)$. Denote:

$$\alpha(\epsilon) := \frac{\alpha_3-\psi(\epsilon)\alpha_4}{2\alpha_2} \tag{2.31}$$

Consequently, (2.30) implies that:

$$v(x) \leq v(x(t_0))\, e^{-2\epsilon\alpha(\epsilon)(t-t_0)} \tag{2.32}$$

and:

$$\|x(t)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}}\, \|x(t_0)\|\, e^{-\epsilon\alpha(\epsilon)(t-t_0)} \tag{2.33}$$

Since $\alpha(\epsilon) > 0$ for all $\epsilon \leq \epsilon_2$, system (2.15) is exponentially stable. Using (L16), it follows that:

$$\|x(t)\| \leq \frac{1+\xi(\epsilon)}{1-\xi(\epsilon)} \sqrt{\frac{\alpha_2}{\alpha_1}}\, \|x(t_0)\|\, e^{-\epsilon\alpha(\epsilon)(t-t_0)} \tag{2.34}$$

for all $t \geq t_0 \geq 0$, $\epsilon \leq \epsilon_2$, and $x(t_0)$ sufficiently small that all signals remain in B_h. In conclusion, the original system is exponentially stable, with rate of convergence (at least) $\epsilon\alpha(\epsilon)$.

Comments

1) *Theorem 2.5* is a *local* exponential stability result. The original system will be *globally* exponentially stable, if the averaged system is globally exponentially stable, and provided that *all* assumptions are valid globally.

2) The proof of *Theorem 2.5* gives a useful bound on the rate of convergence of the original system. As ϵ tends to zero, $\epsilon\alpha(\epsilon)$ tends to $\frac{\epsilon}{2}\frac{\alpha_3}{\alpha_2}$, which is the bound on the rate of convergence of the averaged system that one would obtain using (2.26)-(2.27). In other words, the proof provides a bound on the rate of convergence, and this bound gets arbitrarily close to the corresponding bound for the averaged system, provided that ϵ is sufficiently small. This is a useful conclusion because it is in general very difficult to obtain a guaranteed rate of convergence for the original, nonautonomous system. The proof assumes the existence of a Lyapunov function satisfying (2.26)-(2.28), but does not depend on the specific function chosen. Since the averaged system is autonomous, such a function is usually easier to find than for the original system, and any such function will provide a bound on the rate of convergence of the original system for ϵ sufficiently small.

3) The conclusion of *Theorem 2.5* is quite different from the conclusion of *Theorem 2.4*. Since both x and x_{av} go to zero exponentially with t, the error $x - x_{av}$ also goes to zero exponentially with t. Yet, *Theorem 2.5* does not relate the bound on the error to ϵ. It is possible, however, to combine *Theorem 2.4* and *Theorem 2.5* to obtain a uniform approximation result, with an estimate similar to (2.24).

3. Averaging Theory Applied to Adaptive Identifiers

Brief Review of a Simple Adaptive Identifier

We consider an unknown plant, described by a single-input single-output, exponentially stable transfer function:

$$\hat{p}(s) = \frac{\hat{n}_p(s)}{\hat{d}_p(s)} \tag{3.1}$$

where $\hat{d}_p(s)$ is a monic polynomial of degree n (n is assumed to be known), and $\hat{n}_p(s)$ is a polynomial of degree less than or equal to n. The coefficients of the polynomials are unknown, and are to be obtained from the identifier.

The identifier considered here is an adaptive observer/identifier (see e.g. [21], [22]), and its structure is shown in Fig. 3.1. The filter blocks F_1 and F_2 generate smoothed derivatives of the input r, and of the output y_p of the plant. Each of these blocks has a transfer function:

$$(sI-\Lambda)^{-1}b = \frac{1}{\det(sI-\Lambda)} \begin{bmatrix} 1 \\ s \\ \cdot \\ \cdot \\ s^{n-1} \end{bmatrix} \quad \in R^n(s) \tag{3.2}$$

where $\Lambda \in R^{n \times n}$, $b \in R^n$, and $\det(sI-\Lambda)$ is a Hurwitz polynomial. The outputs of the filters are respectively $v^{(1)}$, $v^{(2)} \in R^n$. The signal y_0 is obtained through the adaptive gains $c(t)$, $d(t) \in R^n$, and $c_{n+1}(t) \in R$:

$$y_0 = c^T v^{(1)} + d^T v^{(2)} + c_{n+1} r \tag{3.3}$$

and it may be verified that there exists a unique choice of the adaptive gains, denoted c^*, d^*, and c^*_{n+1} such that the transfer function from the input r to the output y_0 is identical to the plant transfer function $\hat{p}(s)$. We define the parameter vector $\theta \in R^{2n+1}$:

$$\theta^T = (c^T, d^T, c_{n+1}) \tag{3.4}$$

and the signal vector $w \in R^{2n+1}$:

$$w^T = (v^{(1)T}, v^{(2)T}, r) \tag{3.5}$$

so that:

$$y_0 = \theta^T w \tag{3.6}$$

In the sequel, we will neglect the effect of the initial conditions of the plant, and of the filter blocks F_1, F_2. The results can be modified to take them into account, without any fundamental differences in the conclusions. We simply assume that the dynamics of the observers (determined by the eigenvalues of Λ) are faster than those of the identifier. The output of the plant is then given by an equation similar to that of the identifier:

$$y_p = \theta^{*T} w \tag{3.7}$$

where θ^* is the vector of "true" parameters corresponding to $\hat{p}(s)$. Defining the parameter error:

$$\phi = \theta - \theta^* \tag{3.8}$$

the output error $e_1 = y_p - y_0$ is given by:

$$e_1 = \phi^T w \tag{3.9}$$

It can be shown ([21], [22]) that, with the following adaptation law:

$$\dot{\phi} = -\Gamma e_1 w \tag{3.10}$$

where $\Gamma \in R^{n \times n} > 0$, the following propositions are true:

(i) if r is bounded, then $\lim_{t \to \infty} e_1(t) = 0$

(ii) if, moreover, w is *persistently exciting,* that is, if there exist constants α_1, α_2, $\delta > 0$, such that:

$$\alpha_1 I \leq \int_s^{s+\delta} w\, w^T\, dt \leq \alpha_2 I \quad \text{for all } s > 0 \tag{3.11}$$

then the parameter error also tends to zero, i.e.:

$$\lim_{t \to \infty} \phi(t) = 0 \tag{3.12}$$

and the convergence is exponential.

Application of the Averaging Theory

To apply the averaging theory developed in section 2, we will study the case when $\Gamma = \epsilon I$, i.e. when the update law (3.10) is given by:

$$\dot{\phi} = -\epsilon e_1 w \tag{3.13}$$

or, using eq. (3.9):

$$\dot{\phi} = -\epsilon\, w\, w^T \phi \tag{3.14}$$

Eq. (3.14) leads us to the following definition:

Definition 3.1 (Stationarity, Autocovariance): A signal $u: R_+ \to R^n$ is said to be *stationary* if the following limit exists uniformly in s:

$$R_u(\tau) := \lim_{T \to 0} \frac{1}{T} \int_s^{s+T} u(t) u^T(t+\tau) d\tau \ \in R^{n \times n} \tag{3.15}$$

in which instance, the limit $R_u(\tau)$ is called the *autocovariance* of u.

It may be verified that the autocovariance matrix of a stationary signal w is a positive semidefinite function $R_w(\tau)$, and that w is persistently exciting if and only if the autocovariance at 0 is positive definite [14]. Also, $R_w(\tau)$ can be written as the inverse Fourier transform of a positive *spectral measure* $S_w(d\nu)$:

$$R_w(\tau)=\int_{-\infty}^{\infty} e^{i\nu\tau}S_w(d\nu) \tag{3.16}$$

Further, if the input r is also stationary, $S_w(d\nu)$ can be computed, using the fact that the transfer function from r to w is given by:

$$q(s):=\begin{bmatrix} (sI-\Lambda)^{-1}b \\ (sI-\Lambda)^{-1}b\, p(s) \\ 1 \end{bmatrix} \in R^{2n+1}(s) \tag{3.17}$$

so that:

$$S_w(d\nu)=q(j\nu)q^*(j\nu)s_r(d\nu) \tag{3.18}$$

Using eqns. (3.16) and (3.17), we can conclude that:

$$R_w(0)=\int_{-\infty}^{\infty} q(j\nu)q^*(j\nu)s_r(d\nu)>0 \tag{3.19}$$

This in turn is assured [14] if the support of $s_r(d\nu)$ is greater than or equal to $2n+1$ points (the dimension of w = the number of unknown parameters = $2n+1$).

With these definitions, the averaged system corresponding to (3.14) is simply:

$$\dot{\phi}_{av}=-\epsilon R_w(0)\phi_{av} \tag{3.20}$$

This system is particularly easy to study, since it is linear, and when w is persistently exciting, $R_w(0)$ is a positive definite matrix.

A natural Lyapunov function for (3.14) is:

$$V(\phi_{av})=\frac{1}{2}\|\phi_{av}\|^2 \tag{3.21}$$

and:

$$-\epsilon\lambda_{min}(R_w(0))\|\phi_{av}\|^2 \le -\dot{V}(\phi_{av}) \le -\epsilon\lambda_{max}(R_w(0))\|\phi_{av}\|^2 \tag{3.22}$$

where λ_{min} and λ_{max} are respectively the minimum and maximum eigenvalues of $R_w(0)$. Thus, the rate of exponential convergence of the averaged system is at least $\epsilon\lambda_{min}(R_w(0))$, and at most $\epsilon\lambda_{max}(R_w(0))$. By the comments after theorem 2.5, we can conclude that the rate of convergence of the unaveraged system for ϵ small enough is close to the interval $[\epsilon\lambda_{min}(R_w(0)), \epsilon\lambda_{max}(R_w(0))]$.

Eq. (3.19) gives an interpretation of $R_v(0)$ in the frequency domain, and also a mean of computing an estimate of the rate of convergence of the adaptive algorithm, given the spectral content of the reference input. If the input r is periodic or almost periodic, the integral in (3.19) may be replaced by a summation. Since the transfer function $q(s)$ depends on the unknown plant being identified, the use of the averaged equation to determine the rate of convergence is more conceptual than practical. If, however, bounds on parameter values are available, then some bounds on $R_v(0)$ and on the corresponding rates of convergence can be deduced. These in turn can be used to determine the spectral content of the reference input that will optimize the rate of convergence of the identifier, given the physical constraints on r. Such a procedure is very reminiscent of the procedure indicated in [23] (chapter 6) for the design of input signals in identification. The autocovariance matrix defined here is similar to the *average information matrix* defined in [23] (p 134). Our interpretation is, however, in terms of rates of parameter convergence of the averaged system rather than in terms of parameter covariance.

To illustrate the conclusions of this section, we consider the following example:

$$\hat{p}(s) = \frac{2s+2}{s+3} \tag{3.23}$$

The filter is chosen to be $\det(sI-\Lambda) = (s+5)$. The "true" values of the parameters c_1, d_1, c_2 are -1.6, 0.4, and 2. Denote the parameter error as

$$\phi_1 = c_1 - c_1^* \quad , \quad \phi_2 = d_1 - d_1^* \quad , \quad \phi_3 = c_2 - c_2^*$$

Since the number of unknown parameters is 3, parameter convergence will occur when the support of $s_r(d\nu)$ is greater than or equal to 3 points. For the simulations, we considered an input of the form $a_0 + a_1\sin(\omega t)$. By virtue of (3.18) and (3.19), (3.20) now becomes

$$\begin{pmatrix} \dot{\phi}_{av1} \\ \dot{\phi}_{av2} \\ \dot{\phi}_{av3} \end{pmatrix} = -\epsilon \begin{pmatrix} a_0^2 + \dfrac{25a_1^2}{2(25+\omega^2)} & \dfrac{2}{3}a_0^2 + \dfrac{25(3+\omega^2)a_1^2}{(9+\omega^2)(25+\omega^2)} & a_0^2 + \dfrac{25a_1^2}{2(25+\omega^2)} \\ \dfrac{2}{3}a_0^2 + \dfrac{25(3+\omega^2)a_1^2}{(9+\omega^2)(25+\omega^2)} & \dfrac{4}{9}a_0^2 + \dfrac{50(1+\omega^2)a_1^2}{(9+\omega^2)(25+\omega^2)} & \dfrac{2}{3}a_0^2 + \dfrac{5(15+7\omega^2)a_1^2}{(9+\omega^2)(25+\omega^2)} \\ a_0^2 + \dfrac{25a_1^2}{2(25+\omega^2)} & \dfrac{2}{3}a_0^2 + \dfrac{5(15+7\omega^2)a_1^2}{(9+\omega^2)(25+\omega^2)} & a_0^2 + \dfrac{a_1^2}{2} \end{pmatrix}$$

$$\cdot \begin{pmatrix} \phi_{av1} \\ \phi_{av2} \\ \phi_{av3} \end{pmatrix} \tag{3.24}$$

With $a_0 = 2$, $a_1 = 2$ and $\omega = 4$, the eigenvalues of the averaged system (3.24) are computed to be -0.28ϵ, -0.64ϵ and -15.39ϵ. Figs 3.2 and 3.3 show the plots of the parameter errors of c_1 and d_1 for both the original and averaged systems with three different adaptation gains $\epsilon = 0.1, 0.5, 1$. Fig 3.4 is a plot of the Lyapunov function of (3.21) for both systems using a log scale. It illustrates the closeness of the rate of convergence of the two systems.

4. Averaging of Two-Time Scale Systems

Systems of the form (2.1) studied in section 2 are to be thought of as one time scale systems in that the entire state variable x is varying slowly in comparison with the rate of time variation of the right hand side of the differential equation. In this section, we will study averaging for the case when only some of the state variables are slowly varying.

Consider, for example, the system:

$$\dot{x} = \epsilon f(t,x,y) \qquad x(0)=x_0 \tag{4.1}$$

$$\dot{y} = Ay + \epsilon g(t,x,y) \qquad y(0)=y_0 \tag{4.2}$$

where $x \in R^n$ is called the *slow* state, $y \in R^m$ is called the *fast* state, and f,g are piecewise continuous functions of time.

The goal of averaging will be to approximate the evolution of the slow state. The system (4.1), (4.2) is not the most general two-time scale system. In fact, it is easily seen to be decoupled and linear at $\epsilon = 0$. The study of this special form is motivated by several applications. We will also study another special form later in this section. It is easy to see, from the proofs of this section and those of section 2, that f and g may be allowed to depend smoothly on ϵ as in (A2).

The averaged system for the slow state is:

$$\dot{x}_{av} = \epsilon f_{av}(x_{av}) \qquad x_{av}(0) = x_0 \tag{4.3}$$

where f_{av} is defined by the limit:

$$f_{av}(x) = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} f(\tau,x,0)d\tau \tag{4.4}$$

assuming that the limit exists uniformly in t and x.

The following assumptions will be in effect for (4.1), (4.2):

(B1) $x=0$, $y=0$ is an equilibrium point of system (4.1), (4.2), i.e. $f(t,0,0)=0$ and $g(t,0,0)=0$ for all $t \geq 0$. Both f and g are Lipschitz in x and y, i.e.:

$$||f(t,x_1,y_1) - f(t,x_2,y_2)|| \leq l_1||x_1 - x_2|| + l_2||y_1 - y_2|| \tag{4.5}$$

$$||g(t,x_1,y_1) - g(t,x_2,y_2)|| \leq l_3||x_1 - x_2|| + l_4||y_1 - y_2|| \tag{4.6}$$

for all $t \geq 0$, $x_1, x_2 \in B_h$, $y_1, y_2 \in B_h$.

(B2) $f_{av}(0) = 0$, and f_{av} is Lipschitz in x, i.e.:

$$||f_{av}(x_1) - f_{av}(x_2)|| \leq l_{av}||x_1 - x_2|| \tag{4.7}$$

for all $x_1, x_2 \in B_h$.

(B3) the function $d(t,x) = f(t,x,0) - f_{av}(x)$ satisfies the conditions of *Lemma 2.2*

(B4) $A \in R^{m \times m}$ is Hurwitz.

(B5) x_0 is sufficiently small that, for T fixed, and some $h' < h$, $x_{av}(t) \in B_{h'}$ for all $t \in [0, T/\epsilon]$ (this is possible, from (B2)). We will also assume that $y_0 \in B_{h'}$, the corresponding closed ball in R^m.

Theorem 4.1 Basic Averaging Theorem for Two-Time Scale System

If: The original system (4.1), (4.2), and the averaged system (4.3), satisfy assumptions (B1)-(B5),

Then: There exists $\psi(\epsilon) \in K$ such that, given $T \geq 0$:

$$\|x(t) - x_{av}(t)\| \leq \psi(\epsilon) b_T \tag{4.8}$$

for some b_T, $\epsilon_T > 0$, and for all $t \in [0, T/\epsilon]$, $\epsilon \leq \epsilon_T$, and y_0 sufficiently small. Further, $\psi(\epsilon)$ is of the order of $\epsilon + \xi(\epsilon)$ (as defined in *Lemma 2.2)*.

Proof: We first apply *Lemma 2.2,* and obtain a result similar to *Lemma 2.3.* Consider the transformation of variable:

$$x = z + \epsilon w_\epsilon(t,z) \tag{4.9}$$

with $\epsilon \leq \epsilon_1$. This transformation leads to:

$$\begin{aligned}\dot{z} = (I + \epsilon\frac{\partial w_\epsilon}{\partial z})^{-1}\epsilon \Big\{ & f_{av}(z) + (f(t,z,0) - f_{av}(z) - \frac{\partial w_\epsilon}{\partial t}) \\ & + (f(t,z+\epsilon w_\epsilon,0) - f(t,z,0)) \\ & + (f(t,z+\epsilon w_\epsilon,y) - f(t,z+\epsilon w_\epsilon,0))\Big\}\end{aligned} \tag{4.10}$$

or:

$$\dot{z} = \epsilon f_{av}(z) + \epsilon p_1(t,z,\epsilon) + \epsilon p_2(t,z,y,\epsilon) \qquad z(0) = x_0 \tag{4.11}$$

where:

$$\|p_1(t,z,\epsilon)\| \leq \frac{1}{1-\xi(\epsilon_1)}(\xi(\epsilon)l_{av} + \xi(\epsilon) + \xi(\epsilon)l_1)\ \|z\| := \xi(\epsilon)\, k_1 \|z\| \tag{4.12}$$

and:

$$\|p_2(t,z,y,\epsilon)\| \leq \frac{1}{1-\xi(\epsilon_1)} l_2 \|y\| := k_2 \|y\| \tag{4.13}$$

We now estimate the error $x-x_{av}$, following a proof similar to the proof of *Theorem* 2.4. First, we have that:

$$\|x-z\| \le \xi(\epsilon)\|z\| \tag{4.14}$$

Then, the error $z-x_{av}$ can be estimated from:

$$\frac{d}{dt}(z-x_{av}) = \epsilon(f_{av}(z)-f_{av}(x_{av})) + \epsilon p_1(t,z,\epsilon) + \epsilon p_2(t,z,y,\epsilon) \qquad z(0)-x_{av}(0)=0 \tag{4.15}$$

for all $t\in[0,T/\epsilon]$, $x_{av}(t)\in B_{h'}$, $h'<h$. As in the proof of Theorem 2.4, we will show that, on this interval, and for as long as $x,z\in B_h$, the errors $z-x_{av}$, and $x-x_{av}$ can be made arbitrarily small by reducing ϵ.

Integrating (4.15):

$$\|z(t)-x_{av}(t)\| \le \epsilon l_{av}\int_0^t \|z(\tau)-x_{av}(\tau)\|d\tau + \epsilon\xi(\epsilon)k_1\int_0^t \|z(\tau)\|d\tau + \epsilon k_2\int_0^t \|y(\tau)\|d\tau \tag{4.16}$$

Further, $y(t)$ can be calculated from (4.2):

$$y(t) = e^{At}y_0 + \epsilon\int_0^t e^{A(t-\tau)}g(\tau,x,y)d\tau \tag{4.17}$$

Since A is Hurwitz, we have that:

$$\|e^{At}\| \le m\, e^{-\lambda t} \tag{4.18}$$

for some $m,\lambda>0$, and:

$$\|y(t)\| \le m\,\|y_0\|\,e^{-\lambda t} + \epsilon m\int_0^t e^{-\lambda(t-\tau)}(l_3\|x(\tau)\| + l_4\|y(\tau)\|)d\tau \tag{4.19}$$

or:

$$\|e^{\lambda t}y(t)\| \le m\,\|y_0\| + \epsilon m l_3\int_0^t e^{\lambda\tau}\|x(\tau)\|d\tau + \epsilon m l_4\int_0^t \|e^{\lambda\tau}y(\tau)\|d\tau \tag{4.20}$$

Applying the *Generalized Bellman-Gronwall Lemma:*

$$\|e^{\lambda t}y(t)\| \le m\,\|y_0\|\,e^{\epsilon m l_4 t} + \int_0^t \epsilon m l_3 e^{\lambda\tau}\|x(\tau)\|e^{\epsilon m l_4(t-\tau)}d\tau \tag{4.21}$$

Define $\lambda(\epsilon)=\lambda-\epsilon m l_4$, and ϵ_3 $(0<\epsilon_3\le\epsilon_1)$ so that $\lambda(\epsilon)>0$ for $\epsilon\le\epsilon_3$. It follows that:

$$\|y(t)\| \le m\,h\,e^{-\lambda(\epsilon)t} + \epsilon m l_3 h/\lambda(\epsilon) \tag{4.22}$$

Using this estimate in (4.16), and using the *Generalized Bellman-Gronwall Lemma* again:

$$\|z(t)-x_{av}(t)\| \leq \int_0^t \left[\xi(\epsilon)k_1h + mk_2he^{-\lambda(\epsilon)\tau} + \frac{\epsilon mk_2l_3h}{\lambda(\epsilon)} \right] \epsilon e^{\epsilon l_{av}(t-\tau)} d\tau$$

$$\leq (\epsilon + \xi(\epsilon)) \left[k_1h + \frac{mk_2h\, l_{av}}{\lambda(\epsilon)+\epsilon l_{av}} + \frac{mk_2l_3h}{\lambda(\epsilon)} \right] \left[\frac{e^{l_{av}T}}{l_{av}} \right]$$

$$:= \psi(\epsilon)\, a_T \tag{4.23}$$

As in *Theorem 2.4*, it follows that, for some b_T:

$$\|z(t)-x_{av}(t)\| \leq \psi(\epsilon)\, b_T \tag{4.24}$$

By assumption, $\|x_{av}(t)\| \leq h' < h$. Let ϵ_T $(0 < \epsilon_T \leq \epsilon_3)$ such that $\psi(\epsilon_T)b_T < h-h'$. Further, let y_0, and ϵ_T sufficiently small that, by (4.22), $y(t) \in B_h$, for all $t \in [0, T/\epsilon]$. It follows, from a simple contradiction argument, that the estimate in (4.24) is valid for all $t \in [0, T/\epsilon]$, whenever $\epsilon \leq \epsilon_T$.

Theorem 4.2 Exponential Stability Theorem for Two-Time Scale Systems

If: The original system (4.1), (4.2), and the averaged system (4.3) satisfy assumptions (B1)-(B4), the function $f_{av}(x)$ has continuous and bounded first partial derivatives in x, and $x=0$ is an exponentially stable equilibrium point of the averaged system,

Then: There exists $\epsilon_4 > 0$ such that the equilibrium point $x=0$ of the original system is exponentially stable for all $\epsilon \leq \epsilon_4$.

Proof: Since $x_{av}=0$ is an exponentially stable equilibrium point of the averaged system, there exists a function $v(x_{av})$ satisfying (2.26)-(2.28). On the other hand, since A is Hurwitz, there exist matrices $P, Q > 0$, such that $A^TP + PA = -Q$. Denote by p_1, p_2, q_1, q_2 the minimum and maximum eigenvalues of the P and Q matrices. We now study the stability of the system (4.11), (4.2), and consider the following Lyapunov function:

$$v_1(z,y) = v(z) + \frac{\alpha_2}{p_2} y^T P y \tag{4.25}$$

so that:

$$\alpha'_1 (\|z\|^2 + \|y\|^2) \leq v_1(z,y) \leq \alpha_2 (\|z\|^2 + \|y\|^2) \tag{4.26}$$

where $\alpha'_1 = \min(\alpha_1, \frac{\alpha_2}{p_2} p_1)$. The derivative of v_1 along the trajectories of (4.11), (4.2) can be estimated, using the previous results:

$$\dot{v}_1(z,y) \leq -\epsilon\alpha_3\|z\|^2 + \epsilon k_1 \xi(\epsilon)\alpha_4\|z\|^2$$

$$+ \epsilon k_2 \alpha_4 \|z\|\|y\| - \frac{\alpha_2}{p_2} q_1 \|y\|^2$$

$$+ 4\epsilon l_3\alpha_2\|z\|\|y\| + 2\epsilon l_4\alpha_2\|y\|^2 \tag{4.27}$$

for $\epsilon \le \epsilon_1$ (so that, in particular, $\|z\| \le 2\|z\|$). Note that since $ab \le (a^2 + b^2)/2$ for all $a,b \in R$, we have:

$$\epsilon\|z\|\|y\| \le \frac{1}{2}\left(\epsilon^{4/3}\|z\|^2 + \epsilon^{2/3}\|y\|^2\right) \tag{4.28}$$

so that:

$$\dot v_1(z,y) \le -\epsilon\left(\alpha_3 - \xi(\epsilon)k_1\alpha_4 - \epsilon^{1/3}\frac{k_2\alpha_4}{2} - 2\epsilon^{1/3}l_3\alpha_2\right)\|z\|^2$$

$$-\left(\frac{\alpha_2}{p_2}q_1 - 2\epsilon l_4\alpha_2 - \epsilon^{2/3}\frac{k_2\alpha_4}{2} - 2\epsilon^{2/3}l_3\alpha_2\right)\|y\|^2$$

$$:= -2\epsilon\alpha_2\,\alpha(\epsilon)\|z\|^2 - q(\epsilon)\|y\|^2 \tag{4.29}$$

Note that, with this definition, $\alpha(\epsilon) \to \frac{1}{2}\frac{\alpha_3}{\alpha_2}$ as $\epsilon \to 0$.

Let $\epsilon_4 (0 < \epsilon_4 \le \epsilon_1)$ be sufficiently small that $\alpha(\epsilon) > 0$, $q(\epsilon) > 0$, and $2\epsilon\alpha_2\,\alpha(\epsilon) \le q(\epsilon)$ whenever $\epsilon \le \epsilon_4$. Consequently:

$$\dot v_1(z,y) \le -2\epsilon\alpha(\epsilon)v_1(z,y) \tag{4.30}$$

and:

$$v_1(z,y) \le v_1(z(t_0),y(t_0))\, e^{-2\epsilon\alpha(\epsilon)(t-t_0)} \tag{4.31}$$

As in *Theorem 2.5,* this implies the exponential convergence of the original system, with rate of convergence $\epsilon\alpha(\epsilon)$. Also, for $z(t_0)$, $y(t_0)$ sufficiently small, all signals are guaranteed to remain in B_h, so that all assumptions are applicable.

Comments

The comments of *Theorem 2.5* apply similarly to *Theorem 4.2.* In particular, the proof gives a useful bound on the rate of convergence of the original system, and this bound again tends to the bound on the rate of convergence of the averaged system.

Mixed Time Scales

We now discuss a more general class of two-time scale systems, arising in adaptive control:

$$\dot x = \epsilon f'(t,x,y') \tag{4.32}$$

$$\dot y' = Ay' + h(t,x) + \epsilon g'(t,x,y') \tag{4.33}$$

We will show that system (4.32)-(4.33) can be transformed into the system described in the previous section. In this case, x is a slow variable, but y' has both a fast, and a slow component.

The averaged system corresponding to (4.32), (4.33) is obtained as follows. Define the function:

$$v(t,x)=\int_0^t e^{A(t-\tau)}h(\tau,x)d\tau \tag{4.34}$$

and assume that the following limit exists uniformly in t and x:

$$f_{av}(x)=\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T} f'(\tau,x,v(\tau,x))d\tau \tag{4.35}$$

Intuitively, $v(t,x)$ represents the steady-state value of the variable y with x frozen and $\epsilon=0$ in (4.33).*

To show that the averaged system of (4.35) is the right one, we transform the system (4.32), (4.33) to the form (4.1), (4.2), using the transformation:

$$y=y'-v(t,x) \tag{4.36}$$

From (4.34), $v(t,x)$ satisfies:

$$\frac{\partial}{\partial t}v(t,x)=Av(t,x)+h(t,x) \qquad v(t,0)=0 \tag{4.37}$$

Differentiating (4.36), we have that:

$$\dot y=Ay+\epsilon\left(-\frac{\partial v(t,x)}{\partial x}f'(t,x,y+v(t,x))+g'(t,x,y+v(t,x))\right) \tag{4.38}$$

so that system (4.32), (4.33), is of the form (4.1), (4.2), with:

$$f(t,x,y)=f'(t,x,y+v(t,x)) \tag{4.39}$$

$$g(t,x,y)=-\frac{\partial v(t,x)}{\partial x}f'(t,x,y+v(t,x))+g'(t,x,y+v(t,x)) \tag{4.40}$$

The averaged system is obtained by averaging the right-hand side of (4.39) with $y=0$, so that the definitions (4.4), and (4.35) agree.

To apply *Theorem 4.1,* and *Theorem 4.2,* we require that assumptions (B1)-(B5) be satisfied. In particular, we assume similar Lipschitz conditions on f', g', and the following assumption on $h(t,x)$:

(B6) $h(t,0)=0$ for all $t\geq 0$, and:

*This choice of transformation was pointed out to us by B. Riedle & P. Kokotovic.

$$\left\|\frac{\partial h(t,x)}{\partial x}\right\| \leq k \tag{4.41}$$

for all $t \geq 0$, $x \in B_h$.

This new assumption implies that $v(t,0)=0$, and:

$$\left\|\frac{\partial v(t,x)}{\partial x}\right\| \leq k' \tag{4.42}$$

for all $t \geq 0$, $x \in B_h$.

This condition is sufficient to guarantee Lipschitz conditions for the system (4.1), (4.2), given Lipschitz conditions for the system (4.32), (4.33). The theory developed earlier in this section can therefore be directly applied to systems of the form (4.32), (4.33). The key to the preceding transformation is the fact that the new state variable y is truly a *fast* variable, so that the two time scales have been separated.

5. Two-Time Scale Averaging Applied to Model Reference Adaptive Controller

To apply the theory of Section 4 to model reference adaptive controllers (our results here are a small extension of those of Riedle, Kokotovic [11]) , we review the model reference adaptive system of Narendra, Valavani [18] for the relative degree 1 case (our notation is however consistent with Sastry [19]). Consider a plant with transfer function

$$\hat{p}\,(s) = k_p\,\frac{\hat{n}_p(s)}{\hat{d}_p(s)} = c_p(sI-A)^{-1}b_p \tag{5.1}$$

where $\hat{n}_p$, $\hat{d}_p$ are relatively prime monic polynomials of degree $n-1$, n respectively and k_p is a scalar (the representation (5.1) is assumed minimal). The following are assumed to be known about the plant transfer function:

(C1) The degrees of the polynomials $\hat{d}_p$, $\hat{n}_p$ are known.

(C2) The sign of k_p is known (say $k_p > 0$).

(C3) The plant transfer function is assumed to be minimum phase.

The objective is to build a compensator so that the plant output asymptotically matches that of a stable reference model $\hat{m}\,(s)$ with input $r(t)$, output $y_m(t)$ and transfer function

$$\hat{m}\,(s) = k_m\,\frac{\hat{n}_m(s)}{\hat{d}_m(s)}$$

where $k_m > 0$ and $\hat{n}_m$, $\hat{d}_m$ are monic polynomials of degree $n-1$, n respectively (not necessarily relatively prime but both Hurwitz). If we denote the input and output of the plant $u(t)$ and $y_p(t)$ respectively, the objective may be stated as: find $u(t)$ so that $y_p(t) - y_m(t) \rightarrow 0$ as $t \rightarrow \infty$. By using suitable prefiltering of the reference signal if necessary, we may assume that the model $\hat{m}\,(s)$ is strictly positive real.

The scheme is shown in Figure 5.1. The dynamical compensator blocks F_1 and F_2 (reminiscent of those in Section 3) are identical one input, n-1 output systems, each with transfer function $(sI-\Lambda)^{-1}b$; $\Lambda \in R^{n-1\times n-1}$, $b \in R^{n-1}$ where Λ is chosen so that its eigenvalues are the zeros of $\hat{n}_m$. The pair Λ, b is assumed controllable and, for ease of book-keeping (in the algorithm proof alone), we assume that they are in controllable form so that

$$(sI-\Lambda)^{-1} = \frac{1}{\hat{n}_m(s)}\begin{bmatrix} 1 \\ s \\ \cdot \\ \cdot \\ s^{n-2} \end{bmatrix}$$

The parameters $c \in R^{n-1}$ in the precompensator block serve to tune the closed loop plant zeros; $d \in R^{n-1}$, $d_0 \in R$ in the feedback compensator assign the closed loop plant poles. The parameter

c_0 adjusts the overall gain of the closed loop plant. Thus, the vector of 2n adjustable parameters denoted θ is

$$\theta^T = [\, c_0 \,, c^T \,, d_0 \,, d^T \,]$$

with the signal vector $w \in R^{2n}$ defined by

$$w^T = [\, r \,, v^{(1)T} \,, y_p \,, v^{(2)T} \,]$$

The input to the plant is seen to be

$$u = \theta^T w$$

and the state equations of the plant loop are given by

$$\begin{bmatrix} \dot{x}_p \\ \dot{v}^{(1)} \\ \dot{v}^{(2)} \end{bmatrix} = \begin{bmatrix} A_p & 0 & 0 \\ 0 & \Lambda & 0 \\ bc^T{}_p & 0 & \Lambda \end{bmatrix} \begin{bmatrix} x_p \\ v^{(1)} \\ v^{(2)} \end{bmatrix} + \begin{bmatrix} b_p \\ b \\ 0 \end{bmatrix} \theta^T w \tag{5.2}$$

It may be verified that there is a unique constant $\theta^* \in R^{2n}$ such that, when $\theta = \theta^*$, the transfer function of the plant plus controller equals $\hat{m}(s)$. It can also be shown [18] that when r is bounded and the parameter update law is given by

$$\dot{\theta} = -\Gamma \, e_1 \, w = -\Gamma \, (\, y_p - y_m \,) \, w \tag{5.3}$$

with $\Gamma \in R^{2n \times 2n}$, a positive definite matrix, all signals in the loop, i.e. $u, v, v^{(1)}, v^{(2)}, y_p, y_m$ are bounded. In addition, $\lim_{t \to \infty} e_1(t) = 0$ so that asymptotically $y_p(t)$ approaches $y_m(t)$. The proof of this fact used the following procedure: represent the model (in non-minimal form) as the plant loop with θ set equal to θ^*. The state equations for the model loop are given by

$$\begin{bmatrix} \dot{x}_m \\ \dot{v}_m^{(1)} \\ \dot{v}_m^{(2)} \end{bmatrix} = \begin{bmatrix} A_p + b_p d_0^* c_p^T & b_p c^{*T} & b_p d^{*T} \\ b d_0^* c_p^T & \Lambda + b c^{*T} & b d^{*T} \\ b c_p^T & 0 & \Lambda \end{bmatrix} \begin{bmatrix} x_m \\ v_m^{(1)} \\ v_m^{(2)} \end{bmatrix} + \begin{bmatrix} b_p \\ b \\ 0 \end{bmatrix} c_0^* \, r \tag{5.4}$$

The 3n-2 $\times$ 3n-2 matrix in (5.4) is henceforth referred to as $\bar{A}$, and the 3n-2 vector in (5.4) as $\bar{b}$. Then, subtracting (5.4) from (5.2) with

$$e^T = [\, x_p^T, v^{(1)T}, v^{(2)T} \,] - [\, x_m^T, v_m^{(1)T}, v_m^{(2)T} \,]$$

we have that

$$\dot{e} = \bar{A} \, e + \bar{b} \, \phi^T w \tag{5.5}$$

and

$$e_1 := y_p - y_m = [\, c_p^T, 0, 0 \,] \, e =: \bar{c}^T e \tag{5.6}$$

where ϕ, the parameter error $:= \theta - \theta^*$. Note from (5.4) that $\bar{c}^T (sI - \bar{A})^{-1} \bar{b} c_0^*$ is equal to the model transfer function and that $c_0^* = k_p/k_m$ is the ratio of the high frequency gain (positive by assumption). Now the update law (5.3) is

$$\dot{\theta} = \dot{\phi} = -\Gamma w(t) \tilde{c}^T e \tag{5.7}$$

To apply averaging, we consider slow adaptation, i.e. $\Gamma = \epsilon I$ resulting in

$$\dot{e} = \tilde{A}\, e + \tilde{b}\, w^T \phi \tag{5.5}$$

and

$$\dot{\phi} = -\epsilon\, w\, \tilde{c}^T\, e \tag{5.8}$$

Equations (5.5) and (5.8) are superficially of the form (4.32), (4.33) with $h(t,\phi) := bw^T(t)\phi$ and $f'(t,\phi,e) = -w(t)ce$. Difficulty, however, arises from the fact that $w(t)$ in (5.5), (5.8) is not independently and exogenously specified, but in fact depends on e. To show this dependence explicitly, we set

$$w_m^T := [\, r, v_m^{(1)T}, y_m, v_m^{(2)T}]$$

an exogenously defined 3n-2 dimensional vector that can be obtained either from $r(t)$ alone or as linear combinations of the state variables of (5.4), and rewrite

$$w = w_m + Qe \tag{5.9}$$

where

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ c_p^T & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

Using (5.9), the equations (5.3) and (5.8) are

$$\dot{e} = \tilde{A}e + \tilde{b}\, w_m^T \phi + \tilde{b}\, e^T Q^T \phi \tag{5.10}$$

$$\dot{\phi} = -\epsilon w_m\, \tilde{c}^T\, e - \epsilon Q\, e\, \tilde{c}^T\, e \tag{5.11}$$

With the exception of the last terms (quadratic in e and ϕ), equations (5.10), (5.11) are linear time varying equations describing the linearized adaptive control system, around the equilibrium $e = 0$, $\phi = 0$. In this section, we apply averaging to the linearized equations (5.10), (5.11) corresponding to small e and ϕ. Averaging of the full non-linear equations (5.10), (5.11) is more subtle and is not considered here. We consider

$$\dot{e} = A^{\sim} e + \tilde{b}\, w_m^T \phi \tag{5.12}$$

$$\dot{\phi} = -\epsilon\, w_m\, \tilde{c}^T\, e \tag{5.13}$$

Since r is bounded and $\tilde{A}$ is stable (its eigenvalues are the union of the zeros of $\hat{d}_m$, $\hat{n}_p$ and the eigenvalues of Λ), w_m is bounded. Hence it is easy to see that the equations (5.12), (5.13) are of the form of (4.33), (4.32) with the functions f' and h satisfying the conditions of Section 4.

To establish the averaging results, we assume that r is stationary. This implies, as has been shown in Boyd and Sastry [14], that w_m is stationary. Its spectral measure is related to that of r by

$$S_{w_m}(d\nu) = \hat{n}(j\nu)\hat{n}^*(j\nu)s_r(d\nu) \tag{5.14}$$

with

$$\hat{n}(s) := \begin{bmatrix} 1 \\ \hat{m}\hat{p}^{-1}(sI-\Lambda)^{-1}b \\ \hat{m} \\ \hat{m}(sI-\Lambda)^{-1}b \end{bmatrix}$$

an exponentially stable transfer function.

The function $v(t,\phi)$ of Section 4 for the systems (5.12), (5.13) is

$$v(t,\phi) := [\int_0^t e^{\bar{A}(t-\tau)}\bar{b}w_m^T(\tau)\,d\tau]\,\phi$$

and the averaged f is given by

$$f_{av}(\phi) = -\lim_{T\to\infty}\frac{1}{T}\int_s^{s+T} w_m(t)\bar{c}^T[\int_0^t e^{\bar{A}(t-\tau)}\bar{b}w_m^T(\tau)\,d\tau]\,dt\ \phi \tag{5.15}$$

Since w_m is stationary, the limit in (5.15) may be shown to exist as follows. Define a filtered version of w_m to be

$$w_{mf}(t) = \int_0^t \bar{c}^T e^{\bar{A}(t-\tau)}\bar{b}w_m(\tau)\,d\tau \tag{5.16}$$

Since $\bar{c}^T(sI-\bar{A})^{-1}\bar{b} = \frac{1}{c_0^*}\hat{m}(s)$ is stable so that it follows that $w_{mf}(t)$ is also stationary. The quantity inside the square brackets in (5.15) is

$$\lim_{T\to\infty}\frac{1}{T}\int_s^{s+T} w_m(t)w_{mf}^T(t)\,dt = R_{w_m w_{mf}}(0) \tag{5.17}$$

i.e. the cross correlation between w_m and w_{mf} evaluated at 0. Consequently, we may use (5.14) and (5.16) to obtain a formula for $R_{w_m w_{mf}}(0)$ as

$$R_{w_m w_{mf}}(0) = \frac{1}{c_0^*}\int_{-\infty}^{\infty}\hat{n}(j\nu)\hat{n}^*(j\nu)\hat{m}(j\nu)s_r(d\nu) \tag{5.18}$$

and the averaged system is a linear time-invariant system

$$\dot{\phi}_{av} = -\epsilon R_{v_m v_{mf}}(0)\,\phi_{av} \tag{5.19}$$

Since $\hat{m}(0)$ is strictly positive real, the matrix $R_{v_m v_{mf}}(0)$ is a positive definite matrix. Unlike the matrix $R_v(0)$ of Section 3, $R_{v_m v_{mf}}(0)$ need not be symmetric, and its eigenvalues need not be real. However, the real parts are guaranteed to be positive. When the reference input r is almost periodic, i.e.

$$r(t) \sim \sum_k r_k e^{j\nu_k t}$$

a simple formula for $R_{v_m v_{mf}}(0)$ is given by

$$\frac{1}{c_0^*}\sum_k \hat{n}\,(j\nu_k)\hat{n}^*(j\nu_k)\hat{m}\,(j\nu_k) r_k \tag{5.20}$$

As an illustration of the preceding results, we consider the following example of a first order plant with an unknown pole and an unknown gain :

$$\hat{p}\,(s) = \frac{k_p}{s + a_p}$$

The adaptive process is to adjust the feedforward gain c_0 and the feedback gain d_0 so as to make the closed loop transfer function match the model transfer function

$$\hat{m}\,(s) = \frac{k_m}{s + a_m}$$

To guarantee persistency of excitation, we use a sinusoidal input signal of the form

$$r(t) = a\ sin(\omega t)$$

Thus, equations (5.5), (5.7) become

$$\begin{aligned} \dot{e} &= -a_m e + k_p\,(\phi_1 r + \phi_2 y_m) \\ \dot{\phi}_1 &= -\epsilon\, e\, r \\ \dot{\phi}_2 &= -\epsilon\, e\, y_m \end{aligned}$$

where

$$\phi_1 = c_0 - c_0^* \quad , \quad \phi_2 = d_0 - d_0^*$$

Consequently, the averaged system defined in (5.19) now is

$$\begin{pmatrix} \dot{\phi}_{av1} \\ \dot{\phi}_{av2} \end{pmatrix} = -\frac{\epsilon\, a^2}{4}\frac{k_p}{k_m}\begin{pmatrix} \dfrac{18}{(9+\omega^2)} & \dfrac{18(9-\omega^2)}{(9+\omega^2)^2} \\ \dfrac{18}{(9+\omega^2)} & \dfrac{162}{(9+\omega^2)^2} \end{pmatrix}\begin{pmatrix} \phi_{av1} \\ \phi_{av2} \end{pmatrix}$$

using equation (5.18). With $a_m=3$, $k_m=3$, $a_p=1$, $k_p=2$, $a=3$, $w=2$, the two eigenvalues of the averaged system are computed to be -3.10ϵ and -0.43ϵ, both real negative. Fig 5.2, 5.3 show the plots of the parameter errors of c_0 and d_0 for the original and averaged system, with three different adaptation gains. Fig. 4 corresponds to a higher frequency input signal $\omega=4$ such that the eigenvalues of the matrix $R_{v_m v_{mf}}(0)$ are complex $(-0.49\pm 0.30i)\epsilon$, and explains the oscillatory behavior of the original and averaged systems.

Using the results of Boyd-Sastry [14], it is easy to verify the following facts

(i) $R_{v_m v_{mf}}(0)$ is singular unless $R_v(0)>0$, i.e. $w(t)$ is persistently exciting. Thus persistent excitation of w is a necessary condition for stability of (5.19).

(ii) If $\hat{m}(s)$ is strictly positive real and $w(t)$ is persistently exciting, then $R_{v_m v_{mf}}(0)$ is Hurwitz. Hence $\hat{m}(s)$ being strictly positive real is a sufficient condition for stability of (5.19), given that $w(t)$ is persistently exciting.

It is intuitive that if w is persistently exciting and $\hat{m}(s)$ is close in some sense to being strictly positive real that $R_{v_m v_{mf}}(0)$ will be Hurwitz (in particular, this is the case if $\mathrm{Re}\,\hat{m}(j\nu)$ fails to be positive at frequencies where $\hat{n}(j\nu)$ is small enough.). More specific results in this context are in [11,17].

6. Concluding Remarks

We have presented in this paper new stability theorems for averaging analysis of one and two time scale systems . We have applied these techniques to obtain bounds on the rates of convergence of adaptive identifiers and controllers of relative degree 1.

We feel that the techniques presented here can be extended to obtain instability theorems for averaging. Such theorems could be used to study the mechanism of slow drift instability in adaptive schemes in the presence of unmodelled dynamics, in a framework resembling that of [10]. Also, our analysis of the use of averaging in the study of adaptive control required that the scheme be linearized in a certain sense made precise in Section 5. In order to relax this requirement, averaging techniques for a wider class of two-time scale systems than that discussed in Section 4 will be needed.

References

[1] Bogoliuboff, N. N. and Yu. A. Mitropolskii, "Asysmptotic Methods in the Theory of Non-Linear Oscillators," Gordon & Breach, New York, 1961.

[2] Hale, J. K., "Oscillations in Non-linear Systems," McGraw-Hill, New York, 1963.

[3] Hale, J. K., "Ordinary Differential Equations," Krieger, Molaban (Florida), 1980.

[4] Sethna, P. R., "Method of Averaging for Systems Bounded for Positive Time," Journal of Math Anal and Applications, Vol. 41 (1973), pp 621-631.

[5] Balachandra, M. and P. R. Sethna, "A Generalization of the Method of Averaging for Systems with Two-Time Scales," Archive for Rational Mechanics and Analysis, Vol. 58, 1975, pp 261-283.

[6] Arnold, V. I., "Geometric Methods in the Theory of Ordinary Differential Equations," Springer Verlag, 1982.

[7] Guckenheimer, J. and P. Holmes, "Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields," Springer Verlag, New York, 1983.

[8] Hoppensteadt, F. C., "Singular Perturbations on the Infinite Time Interval," Trans. American Math Society, Vol. 123 (1966), pp 521-535.

[9] Krause, J., M. Athans, S. Sastry and L. Valavani, "Robustness Studies in Adaptive control," Proc. 22nd IEEE Conf. on Decision and Control, San Antonio, Texas (1983), pp 977-981.

[10] Astrom, K. J., "Interactions between Excitation and Unmodelled Dynamics in Adaptive Control," Proc. 23rd IEEE Conf. on Decision and Control, Las Vegas, Nevada (1984), pp 1276-1281.

[11] Riedle, B. and P. V. Kokotovic, "A Stability-Instability Boundary for Disturbance Free Slow Adaptation and Unmodelled Dynamics," Proc. 23rd IEEE Conf. on Decision and Control, Las Vegas, Nevada (1984), pp 998-1101.

[12] Ljung, L., "On positive Real Transfer Functions and the Convergence of Some Recursions," IEEE Trans. on Automatic Control, Vol. AC-22 (1977), pp 539-557.

[13] Rohrs, C. E., L. Valavani, M. Athans and G. Stein, "Analytical Verification of Undesirable Properties of Direct Model Reference Adaptive Control Algorithms," Proc. 20th IEEE Conf. on Decision and Control, San Diego, California (1981).

[14] Boyd, S. and S. Sastry, "Necessary and Sufficient Conditions for Parameter Convergence in Adaptive Control," Proc. of the Ames- Berkeley Conf. on Non-Linear Dynamics and Control, Math-Sci Press, Brookline, Massachusetts, 1984, pp 81-101.

[15] Bodson, M. and S. Sastry, "Small Signal I/O Stability of Nonlinear Control Systems: Application to the Robustness of a MRAC Scheme," Memorandum No. UCB/ERL M84/70, 17 Sep. 1984, Electronics Research Laboratory, University of California, Berkeley.

[16] Riedle, B. and P. Kokotovic, "Stability Analysis of Adaptive System with Unmodelled Dynamics," To appear in Int. J. of Control, Vol. 41, 1985.

[17] Kosut, R. and B.D.O.Anderson, "Stability Theory for Adaptive Systems: Method of Averaging and Persistency of Excitation," Preprint, Feb. 1985.

[18] Narendra, K. and L. Valavani, "Stable Adaptive Controller Design-Direct Control," IEEE Trans. on Automatic Control. Vol. AC-23 (1978), pp 570-583.

[19] Sastry, S., "Model Reference Adaptive Control--Stability, Parameter Convergence, and Robustness," IMA Journal of Mathematical Control & Information, Vol. 1 (1984), pp 27-66.

[20] Hahn, W., "Stability of Motion," Springer Verlag, Berlin, 1967.

[21] Luders, G. and K.S. Narendra, "An Adaptive Observer and Identifier for a Linear System," IEEE Trans. on Automatic Control, Vol. AC-18 (1973), pp. 496-499.

[22] Kreisselmeier, G. "Adaptive Observers with Exponential Rate of Convergence," IEEE Trans. on Automatic Control, Vol. AC-22 (1977), pp. 2-8.

[23] Goodwin, G. C. and R. L. Payne, "Dynamic System Identification," Academic Press, New York, 1977.

APPENDIX

Proof of Lemma 2.1

Define:

$$w_\epsilon(t,x)=\int_0^t d(\tau,x)e^{-\epsilon(t-\tau)}d\tau \tag{L1}$$

and:

$$w_0(t,x)=\int_0^t d(\tau,x)d\tau \tag{L2}$$

From the assumptions:

$$\|w_0(t+t_0,x)-w_0(t_0,x)\|\leq\gamma(t).t \tag{L3}$$

for all $t,t_0\geq 0$, $x\in B_h$. Integrating (L1) by parts:

$$w_\epsilon(t,x)=w_0(t,x)-\epsilon\int_0^t e^{-\epsilon(t-\tau)}w_0(\tau,x)d\tau \tag{L4}$$

Using the fact that:

$$\epsilon\int_0^t e^{-\epsilon(t-\tau)}w_0(t,x)d\tau=w_0(t,x)-w_0(t,x)e^{-\epsilon t} \tag{L5}$$

(L4) can be rewritten as:

$$w_\epsilon(t,x)=w_0(t,x)e^{-\epsilon t}+\epsilon\int_0^t e^{-\epsilon(t-\tau)}(w_0(t,x)-w_0(\tau,x))d\tau \tag{L6}$$

and, using (L3):

$$\|w_\epsilon(t,x)\|\leq\gamma(t)\,t\,e^{-\epsilon t}+\epsilon\int_0^t e^{-\epsilon(t-\tau)}(t-\tau)\gamma(t-\tau)d\tau \tag{L7}$$

Consequently,

$$\|\epsilon w_\epsilon(t,x)\|\leq\sup_{t'\geq 0}\gamma\left(\frac{t'}{\epsilon}\right)t'\,e^{-t'}+\int_0^\infty\gamma\left(\frac{\tau'}{\epsilon}\right)\tau'\,e^{-\tau'}\,d\tau' \tag{L8}$$

Since, for some β, $\|d(t,x)\|\leq\beta$, we also have that $\gamma(t)\leq\beta$. Note that, for all $t'\geq 0$, $t'\,e^{-t'}\leq e^{-1}$, and $t'\,e^{-t'}\leq t'$, so that:

$$\|\epsilon w_\epsilon(t,x)\|\leq\sup_{t'\in[0,\sqrt{\epsilon}]}\left[\gamma\left(\frac{t'}{\epsilon}\right)t'\,e^{-t'}\right]+\sup_{t'\geq\sqrt{\epsilon}}\left[\gamma\left(\frac{t'}{\epsilon}\right)t'\,e^{-t'}\right]$$

$$+\int_0^{\sqrt{\epsilon}}\gamma\left(\frac{\tau'}{\epsilon}\right)\tau'\,e^{-\tau'}\,d\tau'+\int_{\sqrt{\epsilon}}^\infty\gamma\left(\frac{\tau'}{\epsilon}\right)\tau'\,e^{-\tau'}\,d\tau' \tag{L9}$$

This, in turn, implies that

$$\|\epsilon w_\epsilon(t,x)\| \leq \beta\sqrt{\epsilon} + \gamma(\frac{1}{\sqrt{\epsilon}})\, e^{-1} + \beta\frac{\epsilon}{2} + \gamma(\frac{1}{\sqrt{\epsilon}})(1+\sqrt{\epsilon})\, e^{-\sqrt{\epsilon}}$$

$$:= \xi(\epsilon) \tag{L10}$$

Clearly $\xi(\epsilon)\in K$. From (L1), it follows that:

$$\frac{\partial w_\epsilon(t,x)}{\partial t} - d(t,x) = -\epsilon\, w_\epsilon(t,x) \tag{L11}$$

so that both (2.6) and (2.7) are satisfied.

If $\gamma(T)=a/T^r$, then the right-hand side of (L8) can be computed explicitly:

$$\sup_{t'\geq 0} a\,\epsilon^r (t')^{1-r} e^{-t'} = a\,\epsilon^r (1-r)^{1-r} e^{r-1} \leq a\,\epsilon^r \tag{L12}$$

and, with Γ denoting the standard gamma function:

$$\int_0^\infty a\,\epsilon^r (\tau')^{1-r} e^{-\tau'}\, d\tau' = a\,\epsilon^r \Gamma(2-r) \leq a\,\epsilon^r \tag{L13}$$

Defining $\xi(\epsilon)=2a\,\epsilon^r$, the second part of the lemma is verified.

Proof of Lemma 2.2

Define $w_\epsilon(t,x)$ as in *Lemma 2.1*. Consequently,

$$\frac{\partial w_\epsilon(t,x)}{\partial x} = \frac{\partial}{\partial x}\left[\int_0^t d(\tau,x) e^{-\epsilon(t-\tau)} d\tau\right] = \int_0^t \left[\frac{\partial}{\partial x} d(\tau,x)\right] e^{-\epsilon(t-\tau)} d\tau \tag{L14}$$

Since $\frac{\partial d(t,x)}{\partial x}$ is zero mean, and is bounded, *Lemma 2.1* can be applied to $\frac{\partial d(t,x)}{\partial x}$, and inequality (2.6) of *Lemma 2.1* becomes inequality (2.10) of *Lemma 2.2*. Note that since $\frac{\partial d(t,x)}{\partial x}$ is bounded, and $d(t,0)=0$ for all $t\geq 0$, $d(t,x)$ is Lipschitz. Since $d(t,x)$ is zero mean, with convergence function $\gamma(T)\|x\|$, the proof of *Lemma 2.1* can be extended, with an additional factor $\|x\|$. This leads directly to (2.8) and (2.9) (although the function $\xi(\epsilon)$ may be different from that obtained with $\frac{\partial d(t,x)}{\partial x}$, these functions can be replaced by a single $\xi(\epsilon)$).

Proof of Lemma 2.3

The proof proceeds in two steps.

Step 1: for ϵ sufficiently small, and for t fixed, the transformation (2.15) is a homeomorphism.

Apply *Lemma 2.2,* and let ϵ_1 such that $\xi(\epsilon_1) < 1$. Given $z \in B_h$, the corresponding x such that:

$$x = z - \epsilon\, w_\epsilon(t,z) \tag{L15}$$

may not belong to B_h. Similarly, given $x \in B_h$, the solution z of (L15) may not exist in B_h. However, for any x,z satisfying (L15), inequality (2.8) implies that:

$$(1-\xi(\epsilon))\,||z|| \le ||x|| \le (1+\xi(\epsilon))\,||z|| \tag{L16}$$

Define:

$$h'(\epsilon) = \min\left[h(1-\xi(\epsilon)), \frac{h}{1+\xi(\epsilon)}\right] \tag{L17}$$

and note that $h'(\epsilon) \to h$ as $\epsilon \to 0$.

We now show that:

- for all $z \in B_{h'}$, there exists a unique $x \in B_h$ such that (L15) is satisfied,
- for all $x \in B_{h'}$, there exists a unique $z \in B_h$ such that (L15) is satisfied.

In both cases, $||x-z|| \le \xi(\epsilon)h$.

The first part follows directly from (L16), (L17). The fact that $||x-z|| \le \xi(\epsilon)h$ also follows from (L16), and implies that, if a solution z exists to (L15), it must lie in the closed ball U of radius $\xi(\epsilon)h$ around x. It can be checked, using (2.10), that the mapping $F_x(z) = x - \epsilon w_\epsilon(t,z)$ is a contraction mapping in U, provided that $\xi(\epsilon) < 1$. Consequently, F has a unique fixed point z in U. This solution is also a solution of (L15), and since it is unique in U, it is also unique in B_h (and actually in R^n). For $x \in B_h$, but outside $B_{h'}$, there is no guarantee that a solution z exists in B_h, but if it exists, it is again unique in B_h. Consequently, the map defined by (L15) is well-defined. From the smoothness of $w_\epsilon(t,z)$ with respect to z, it follows that the map is a homeomorphism.

Step 2: the transformation of variable leads to the differential equation (2.16)

Applying (L15) to the system (2.1):

$$\begin{aligned}(I + \epsilon \frac{\partial w_\epsilon}{\partial z})\,\dot z &= \epsilon f_{av}(z) + \epsilon\big(f(t,z,0) - f_{av}(z) - \frac{\partial w_\epsilon}{\partial z}\big)\\ &\quad + \epsilon\big(f(t,z+\epsilon w_\epsilon,\epsilon) - f(t,z,\epsilon)\big)\\ &\quad + \epsilon\big(f(t,z,\epsilon) - f(t,z,0)\big)\\ &:= \epsilon f_{av}(z) + \epsilon p'(t,x,z,\epsilon)\end{aligned} \tag{L18}$$

where, using the assumptions, and the results of *Lemma 2.2:*

$$\|p'(t,z,\epsilon)\| \leq \xi(\epsilon)\|z\| + \xi(\epsilon)l_1\|z\| + \epsilon l_2\|z\| \tag{L19}$$

For $\epsilon \leq \epsilon_1$, (2.10) implies that $(I + \epsilon \frac{\partial w_\epsilon}{\partial z})$ has a bounded inverse for all $t \geq 0$, $z \in B_h$. Consequently, z satisfies the differential equation:

$$\dot{z} = \left[I + \epsilon \frac{\partial w_\epsilon}{\partial z}\right]^{-1} (\epsilon f_{av}(z) + \epsilon p'(t,z,\epsilon))$$

$$= \epsilon f_{av}(z) + \epsilon p(t,z,\epsilon) \qquad z(0) = z_0 \tag{L20}$$

where:

$$p(t,z,\epsilon) = \left[I + \epsilon \frac{\partial w_\epsilon}{\partial z}\right]^{-1} \left[p'(t,z,\epsilon) - \epsilon \frac{\partial w_\epsilon}{\partial z} f_{av}(z)\right] \tag{L21}$$

and:

$$\|p(t,z,\epsilon)\| \leq \frac{1}{1-\xi(\epsilon_1)} (\xi(\epsilon) + \xi(\epsilon)l_1 + \epsilon l_2 + \xi(\epsilon)l_{av})\|z\|$$

$$:= \psi(\epsilon)\|z\| \tag{L22}$$

for all $t \geq 0$, $\epsilon \leq \epsilon_1$, $z \in B_h$.

Generalized Bellman-Gronwall Lemma (cf. [7], p 169)

If: $x(t)$, $a(t)$, $u(t)$ are positive functions satisfying:

$$x(t) \leq \int_0^t a(\tau)x(\tau)d\tau + u(t) \tag{L23}$$

for all $t \in [0,T]$, and $u(t)$ is differentiable,

Then:

$$x(t) \leq u(0)\, e^{\int_0^t a(\sigma)d\sigma} + \int_0^t \dot{u}(\tau)\, e^{\int_\tau^t a(\sigma)d\sigma} d\tau \tag{L24}$$

for all $t \in [0,T]$.

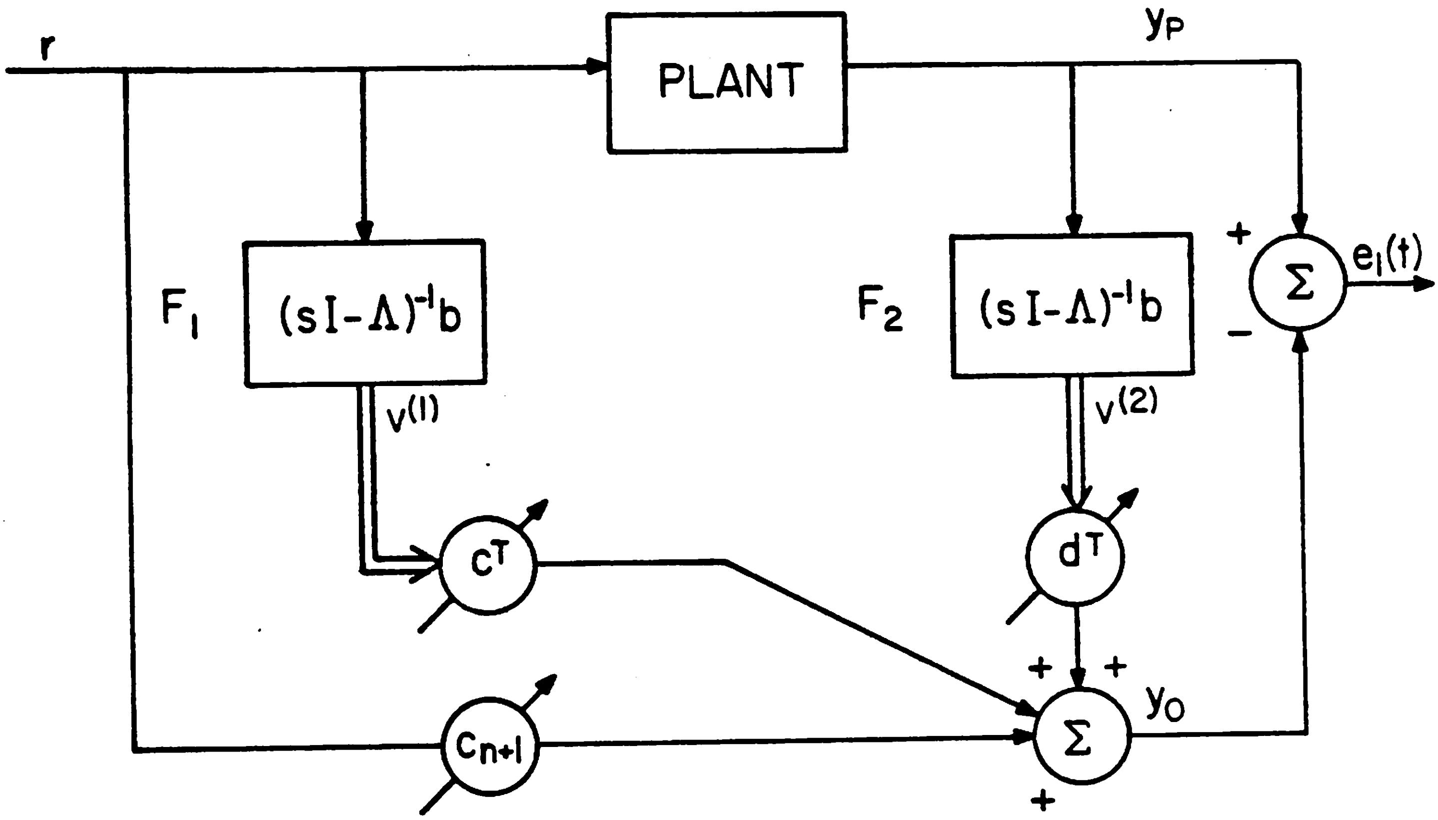


Fig 3.1 Block diagram of adaptive identifier.

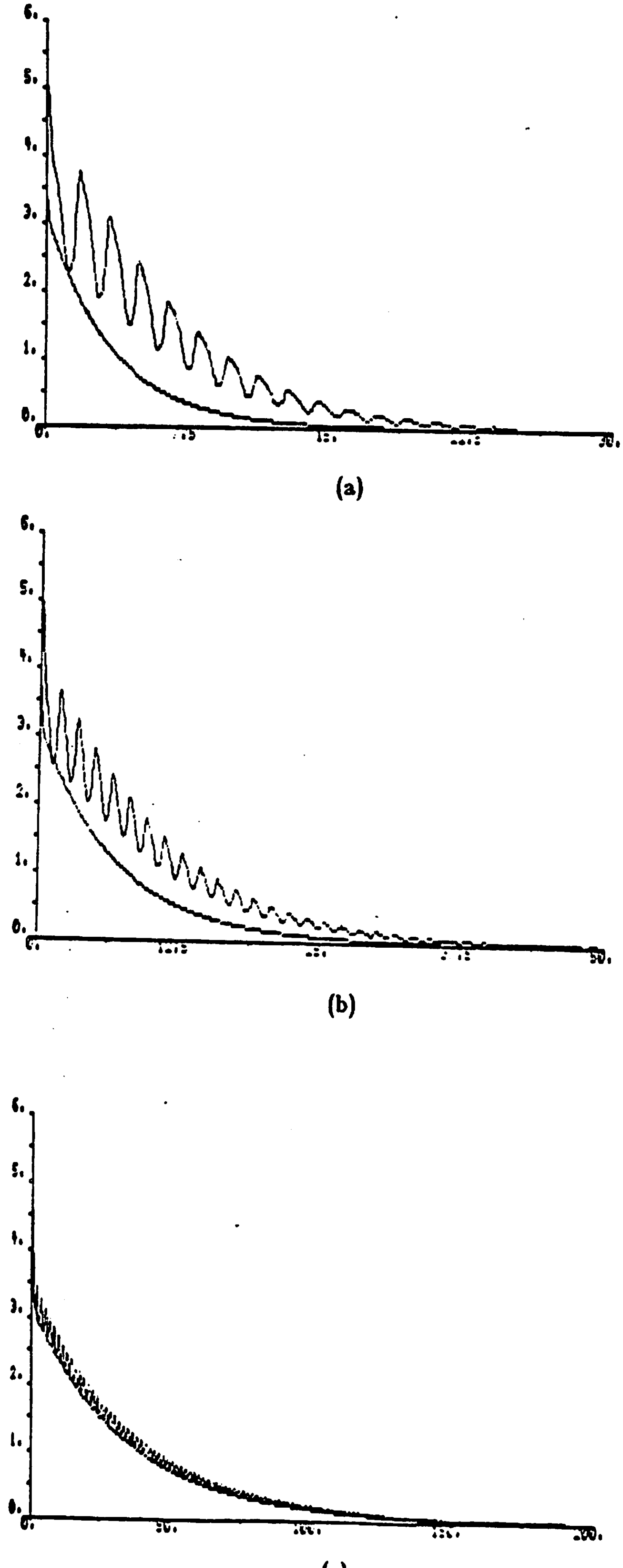


Fig 3.2 Trajectories of parameter error $\phi_1(=c_1-c_1^*)$ and ϕ_{cv1} with three different adaptation gains (a) ϵ=1 (b) ϵ=0.5 (c) ϵ=0.1

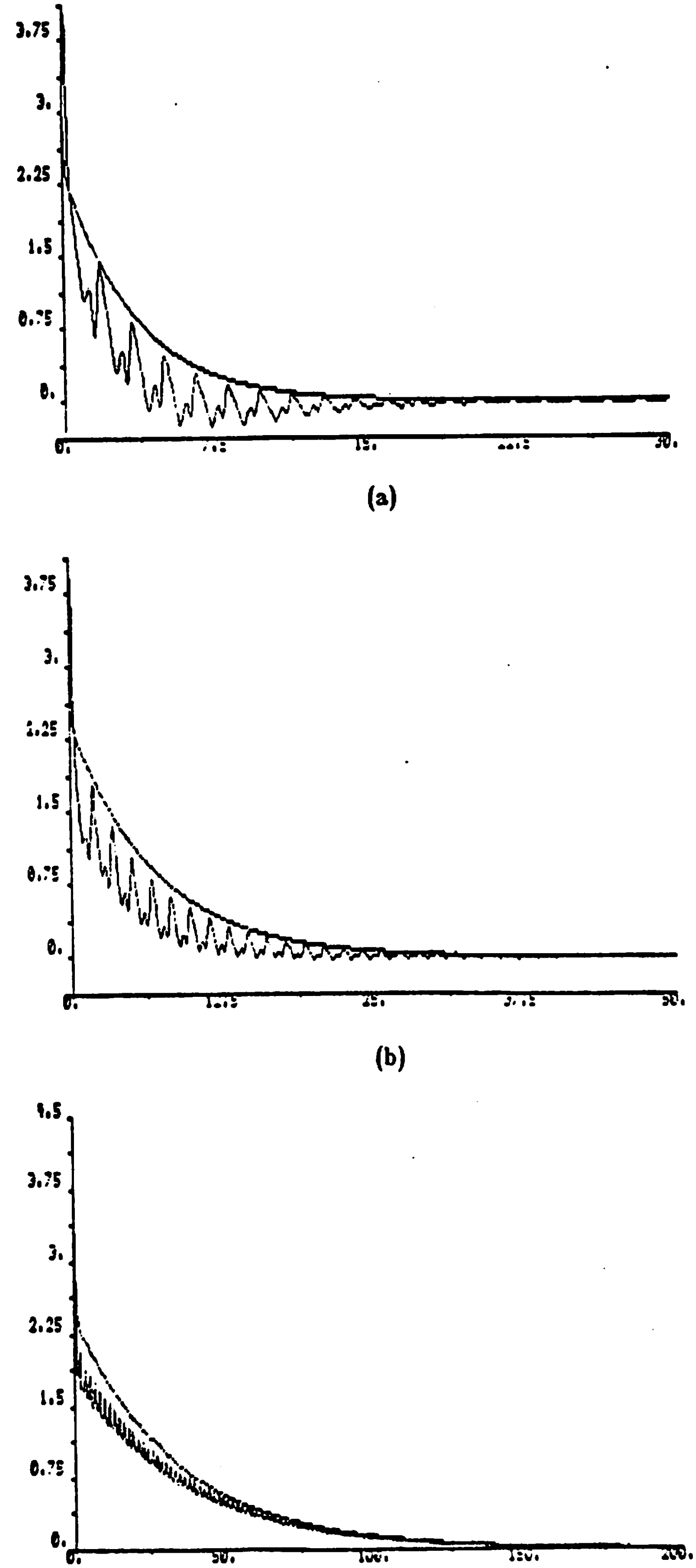
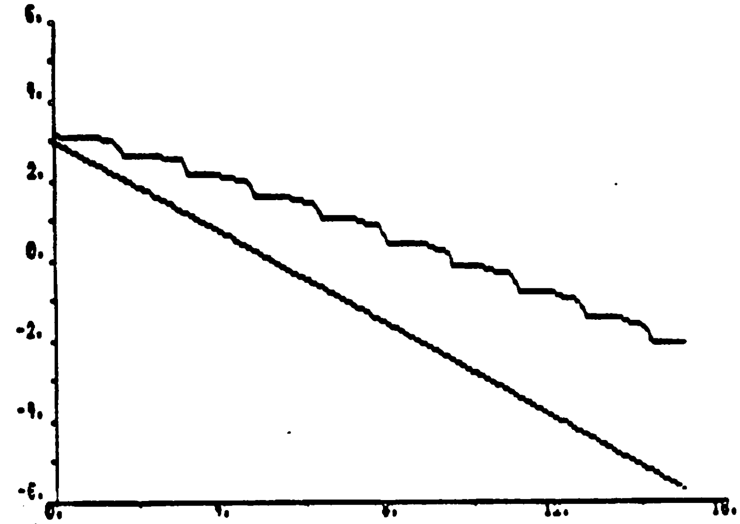
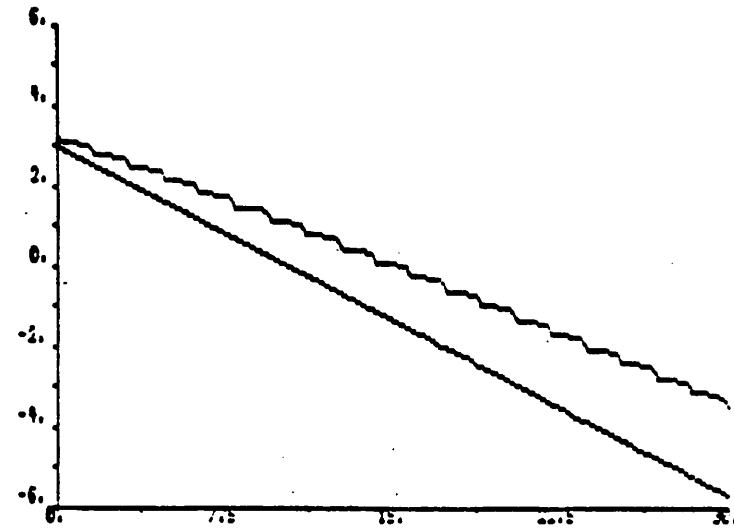


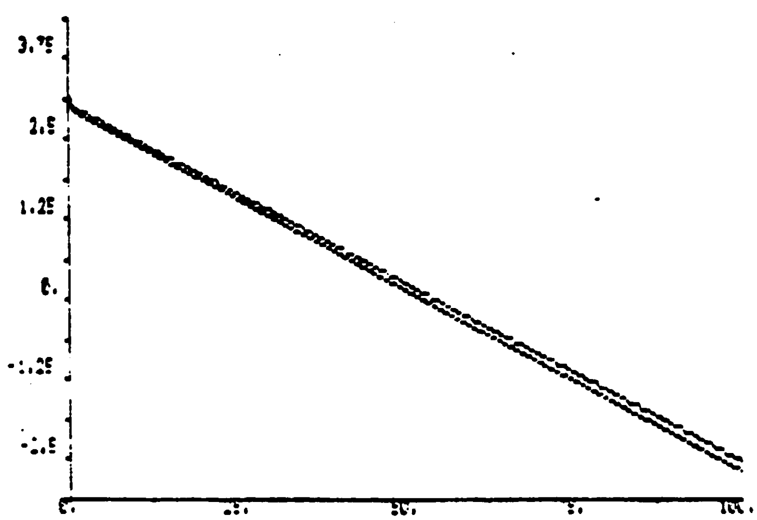
Fig 3.3 Trajectories of parameter error $\phi_2(= d_1 - d_1^*)$ and ϕ_{av2} with three different adaptation gains (a) $\epsilon=1$ (b) $\epsilon=0.5$ (c) $\epsilon=0.1$



(a)



(b)



(c)

Fig 3.4 Trajectories of Lyapunov function $V(\phi)$ and $V(\phi_{av})$ with three adaptation gains (a) ϵ=1 (b) ϵ=0.5 (c) ϵ=0.1 using log scale.

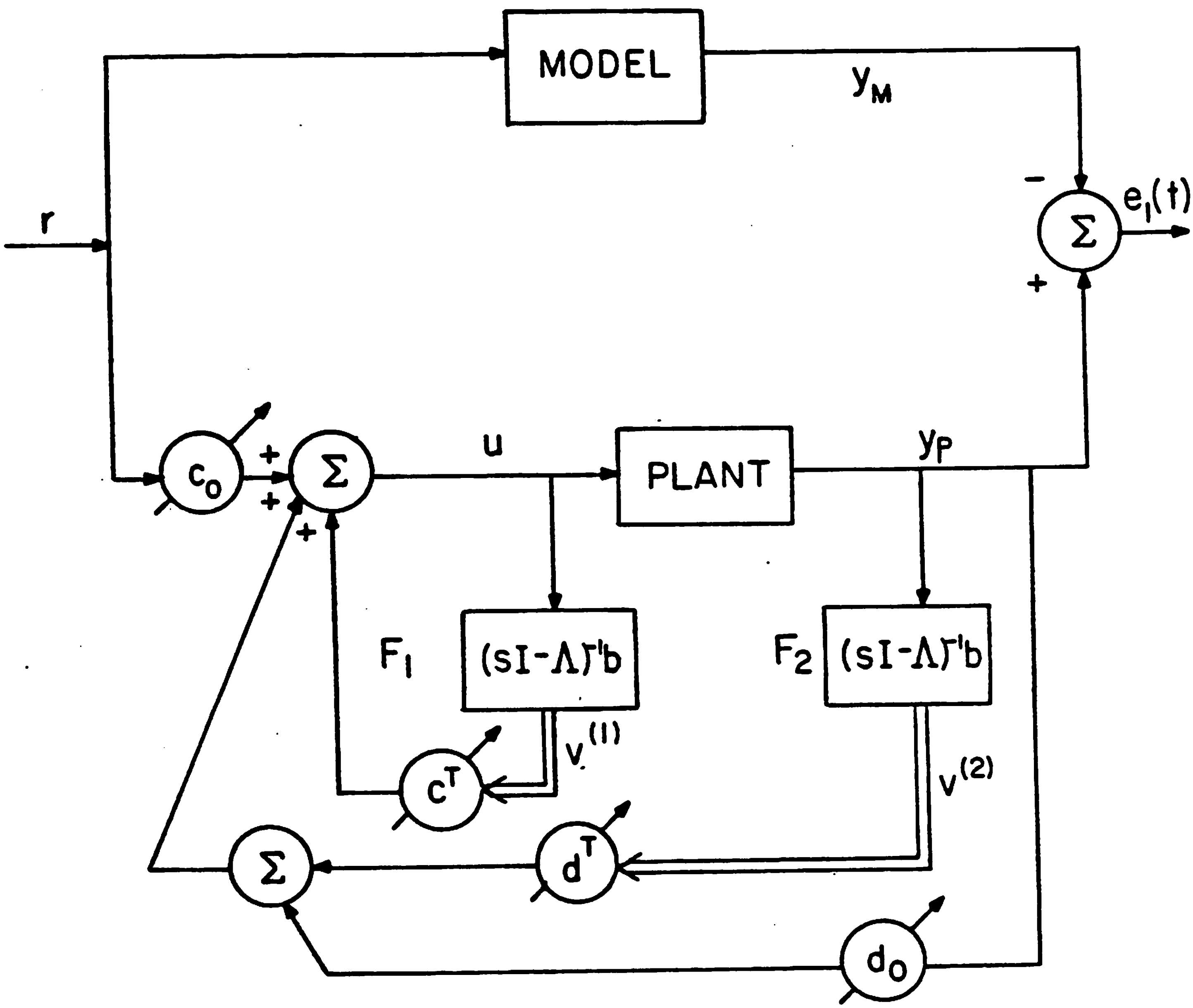
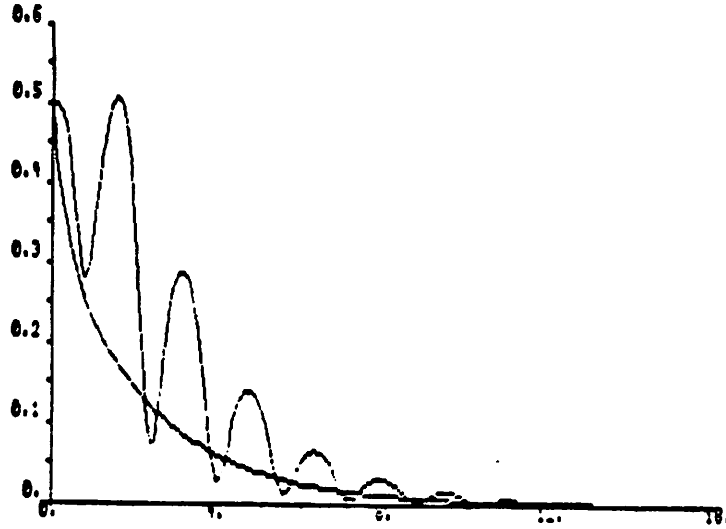
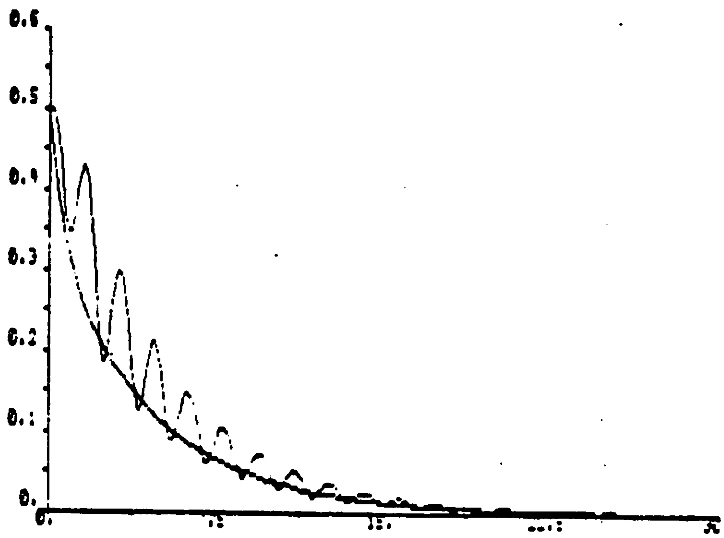


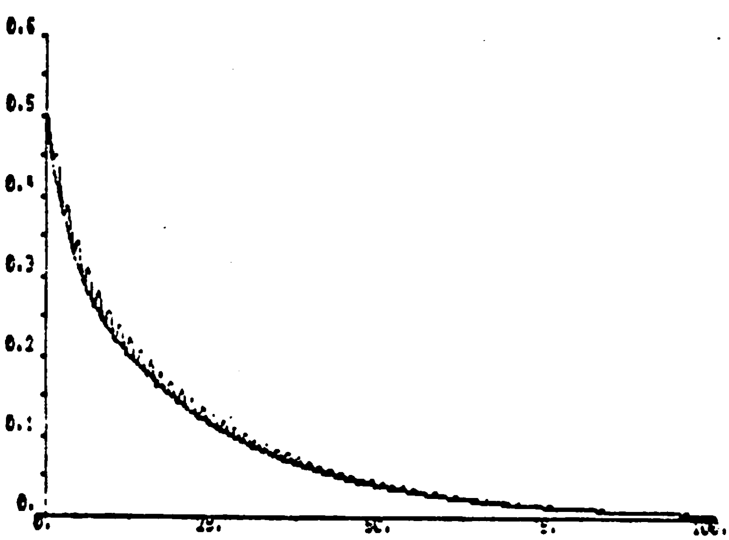
Fig 5.1 Block diagram of model reference adaptive control system



(a)

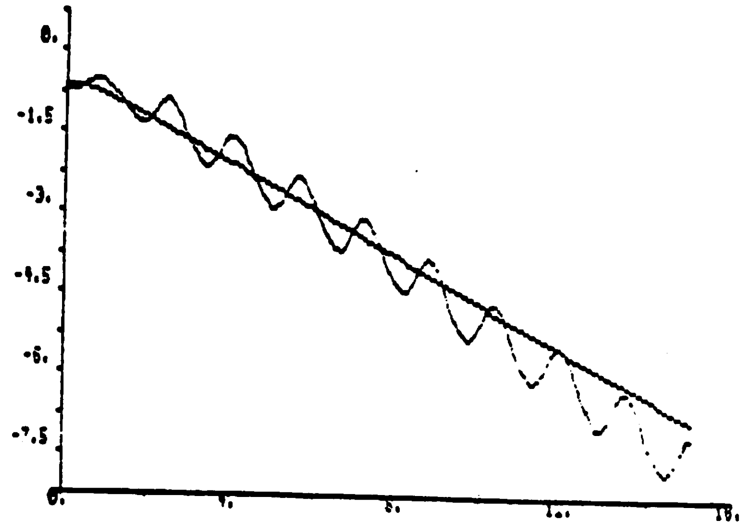


(b)

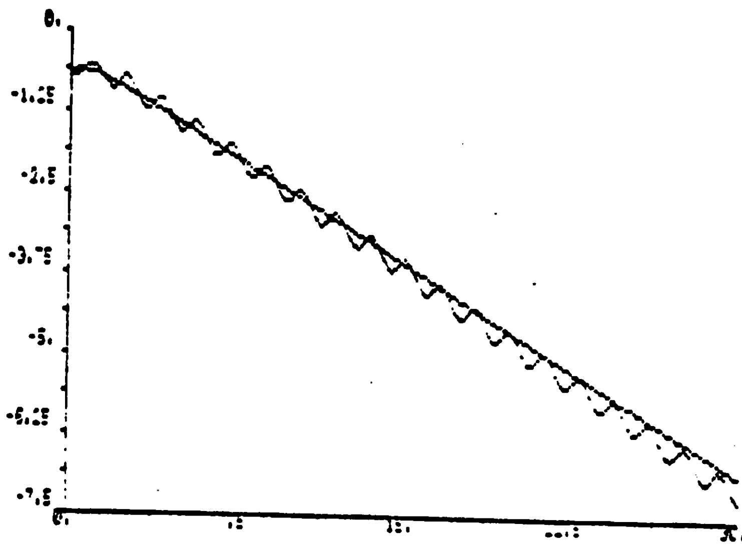


(c)

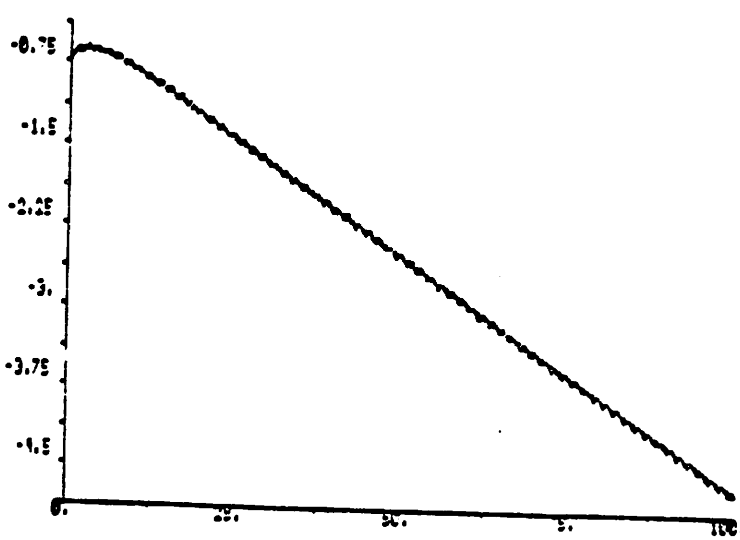
Fig 5.2 Trajectories of parameter error $\phi_1 (= c_0 - c_0^*)$ and ϕ_{av1} with three adaptation gains (a) ϵ=1 (b) ϵ=0.5 (c) ϵ=0.1



(a)

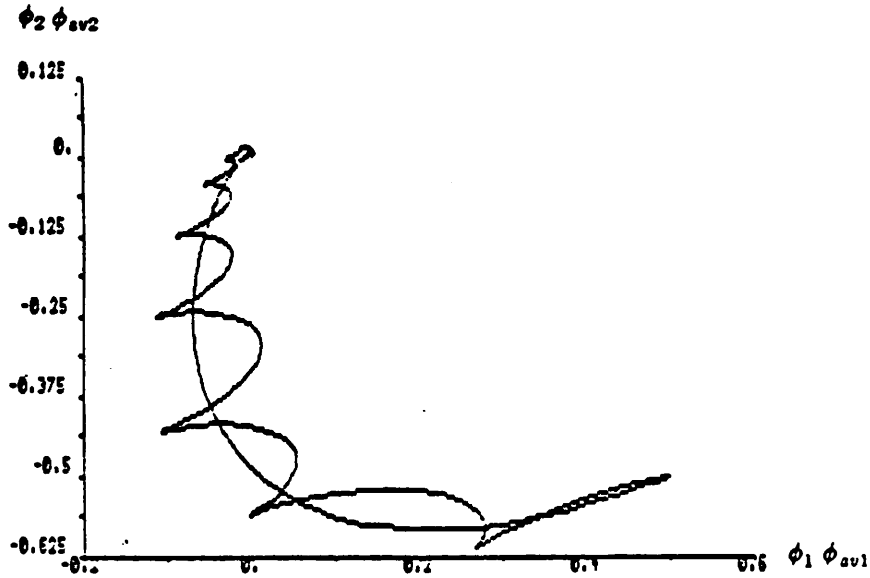


(b)

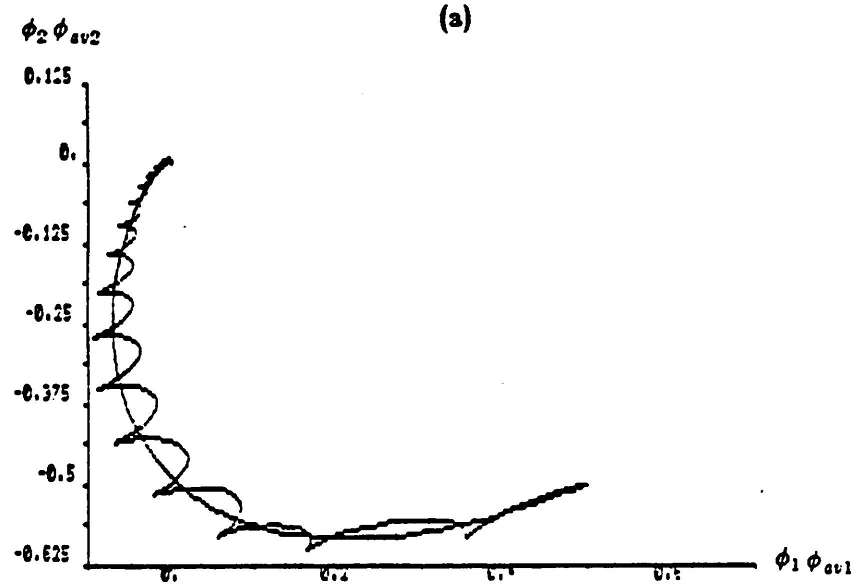


(c)

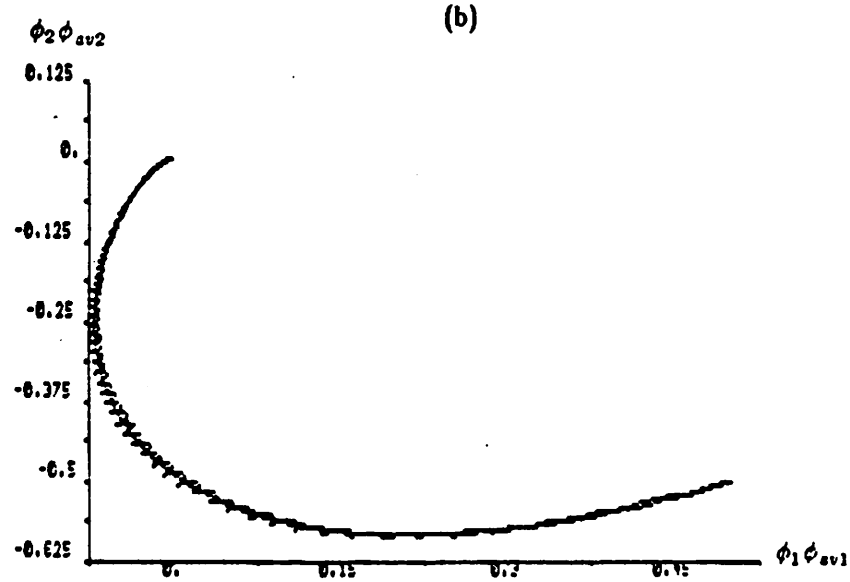
Fig 5.3 Trajectories of parameter error $\phi_2 (= d_0 - d_0^*)$ and ϕ_{av2} with three adaptation gains (a) $\epsilon=1$ (b) $\epsilon=0.5$ (c) $\epsilon=0.1$ using log scale.



(a)



(b)



(c)

Fig 5.4 Phase plot for $\phi_2(\phi_1)$ and $\phi_{av2}(\phi_{av1})$ with three different adaptation gains (a) $\epsilon=1$ (b) $\epsilon=0.5$ (c) $\epsilon=0.1$; and $r=3sin(4t)$.