GENERALIZED GRADIENT CONTROL SCHEME FOR ROBOT MANIPULATORS

by

B. E. Paden and S. S. Sastry

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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*Brad E. Paden and Shankar S. Sastry**

Department of Electrical Engineering and Computer Science
Electronics Research Laboratory
University of California, Berkeley 94720

ABSTRACT

We present a variable structure control scheme for the tracking and compliance control of rigid-link robot manipulators. A simple analysis of this scheme is achieved by using Clarke's generalized gradient [1] and Filippov's solution concept for differential equations with discontinuous right-hand side [2]. The technique developed is quite general and may be applied to many variable structure control schemes described by nonsmooth gradient systems.

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1. Introduction.

The theory of *variable structure systems* (VSS) as described in [3, 4], and [5] has been applied to the control of manipulators by Young [6] and Slotine and Sastry [7]. Both of these schemes decouple the manipulator dynamics by introducing one hyperplane of control discontinuity for each joint of the manipulator via feedback control. In [6] a hierarchical method (see [4]) is used to move the manipulator state to the hyperplanes of control discontinuity sequentially, whereas in [7] the manipulator state moves to all the hyperplanes simultaneously.

The qualitative properties of a VSS are shown in Figure 1. Figure 1(a) depicts a phase diagram for a hypothetical VSS with control discontinuities at S_1 and S_2 . Trajectories for the flow in Figure 1(a) move to, and then slide along the switching surface S_1. This motion of the state along the control discontinuity motivates the nomenclatures sliding mode and sliding surface. Although there is a control discontinuity across S_2, no sliding mode exists there. Figure 1(b) represents a disturbance which is added to the original flow in Figure 1(a). The robust nature of the sliding surface is demonstrated in the resulting flow shown in figure 1(c). The flows have changed but a sliding mode still exists along S_1. The primary reason that sliding modes are introduced into dynamical systems is this robustness to disturbances.

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The VSS control scheme proposed here for robot manipulators is a multivariable design which produces a sliding mode on the *intersection* of several switching surfaces but does not necessarily generate sliding modes on the switching surfaces independently. This type of sliding mode is mentioned in [3] and is analyzed for the first time here. Figure 2 is a phase diagram of this type of VSS having a sliding mode at the origin. The techniques used in this paper to analyze this type of sliding mode are new. Essential to the analysis is the use of Clarke's generalized gradient [1] and Filippov's solution concept for differential equations with discontinuous right-hand side [2]. A simple relationship between these two ideas is proved in Theorem 1 part (6). As is common in VSS, saturating switching controls are used in our scheme. In addition to providing robust tracking, there is a natural force limiting provided by these saturating controls which allow the manipulator to "give" when a slight misalignment in an assembly operation requires the manipulator to deviate from its nominal trajectory. We show that there is bounded set of forces that the manipulator can apply at its gripper without deviating from the nominal trajectory. The size and shape of this set can be be varied by adjusting the gains in the VSS controller. Thus, the apparent stiffness of the gripper can be varied making the manipulator suitable for compliant assembly.

Existing compliance control formulations which are important for comparison are due to Salisbury, and Raibert and Craig. Salisbury [8] varies the servo stiffness of a linear controller to control the stiffness of the manipulators gripper. Our approach is similar in that we use the natural stiffness properties of the control scheme to control compliance at the gripper. The resulting compliance forces of the two schemes is quite different however. In [9] Raibert and Craig switch various degrees of freedom of the gripper from position to force control to allow compliant motion. The VSS control scheme presented here switches *implicitly* to force control when the manipulator is perturbed from its nominal trajectory. This is a result of our choice of discontinuous control.

Other control schemes using continuous control laws are closely related to the VSS scheme contained in this paper. Corless and Leitmann [10] develop a robust controller based on a Lyapunov design. Their controller becomes a VSS controller in the limit as their saturation function parameter β tends to zero. Related work appears in [11] where Ha and Gilbert use the same saturation function to achieve disturbance rejection.

It is important to point out that the direct application of discontinuous control in mechanical systems is almost always impractical since the effects of switching forces on actuators and gear trains can be destructive. Thus, in real systems the control discontinuity is smoothed [7] so that the system trajectory moves to a neighborhood of the approximate discontinuity. The study of the idealized discontinuous control scheme, however, gives a clear picture of the salient properties of the system dynamics. Nonidealities other than smoothed discontinuities such as small delays and hysteresis produce chattering along sliding surfaces rather the ideal sliding described above. Descriptions of the ideal behavior as a limit of these nonideal motions are contained in[2, 4] and [3] and provide additional motivation for studying VSS.

The format of this paper is as follows. Section 2 contains the non-standard mathematical framework used in the analysis of the control scheme. Section 3 presents the manipulator dynamics and formulates the tracking problem. The control scheme is developed in section 4 and a design example is worked through in section 5. The effects of a linear coordinate transformation of the joint coordinates is discussed in section 6. Compliance properties are analyzed in section 7 and section 8 contains our conclusions.

2. Mathematical Framework.

2.1 Notation.

We adopt the following notation throughout this paper.

$\mathbb{N}$	natural numbers, $\{1,2,3,...\}$
$\mathbb{R}$	real numbers
C^r	continuously differentiable r times
$\sigma_{\min}, \sigma_{\max}$	minimum and maximum singular values
$\lVert\cdot\rVert$	usual 2-norm of a vector
$\lVert\cdot\rVert_P$	2-norm induced by the positive definite matrix P
$\lVert\cdot\rVert_1$	usual 1-norm of a vector
$2^{\mathbb{R}^n}$	the collection of subsets of $\mathbb{R}^n$
$sgn(\cdot)$	sign function $sgn(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -1 \text{ if } x > 0 \end{cases}$
$SGN(\cdot)$	set-valued sign function $sgn(x) = \begin{cases} \{1\} \text{ if } x > 0 \\ [-1,1] \text{ if } x = 0 \\ \{-1\} \text{ if } x > 0 \end{cases}$
co	"convex hull"
$\overline{co}$	"convex closure"
$argmin$	"the argument which minimizes"
S^C	denotes the complement of the set S
μ	Lebesgue measure
$a.e.$	almost everywhere with respect to Lebesgue measure.
∂f	generalized gradient of f
A^T	the transpose of A
$\sigma(A)$	the spectrum of A
$\mathbb{C}_+^o$	the open right half complex plane
$B(x,\delta)$	the open ball of radius δ centered at x
$f\mid_U$	f restricted to U

2.2 Differential Equations with Discontinuous RHS and Nonsmooth Potential Functions.

Since we will be considering control laws which are discontinuous and potential functions which are not differentiable everywhere, the associated (non-standard) mathematical framework is developed in this section. We begin by defining a solution to differential equations with discontinuous right-hand side. A solution concept for such differential equations has been developed by Filippov and is used here. Other solution concepts are discussed and compared with Filippov's in [12].

Consider the vector differential equation

$$\dot{x} = f(x,t) \tag{2.1}$$

where $f:\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ satisfies the following condition [2].

Condition B : f is defined almost everywhere and measurable in an open region $Q \subset \mathbb{R}^{n+1}$. Further, $\forall$ compact $D \subset Q$ $\exists$ integrable $A(t)$ such that $\|f(x,t)\| \leq A(t)$ *a.e.* in D.

Definition [Filippov] A vector function $x(\cdot)$ is called a solution of (2.1) on $[t_0,t_1]$ if $x(\cdot)$ is absolutely continuous on $[t_0,t_1]$ and for almost all $t \in [t_0,t_1]$

$$\dot{x} \in K[f](x) \tag{2.2}$$

where

$$K[f](x) \equiv \bigcap_{\delta>0} \bigcap_{\mu N=0} \overline{co} f(B(x,\delta) - N, t)$$

and $\bigcap_{\mu N=0}$ denotes the intersection over all sets N of Lebesgue measure zero. The time dependence in $K[f](x)$ is dropped for economy - all results in this paper that pertain to $K[\cdot]$ hold with time dependence since t can be viewed as a parameter in the definition. Note that the definition of $K[f]$ makes sense for $f:\mathbb{R}^m \to \mathbb{R}^n$; this is a minor generalization, but it is useful in theorem 1. We will *assume* throughout that all functions are defined *a.e.* and Lebesgue measurable.

The definition of K in (2.2) is quite cumbersome to use in applications so that the set of properties summarized in Theorem 1 is useful. Before proceeding with the theorem we need to introduce Clarke's generalized gradient.

Definition: Let $V:\mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz continuous and define ∂V, the *generalized gradient* of V, by

$$\partial V(x) \equiv co\{\lim \nabla V(x_i) \mid x_i \to x, x_i \notin \Omega_V \cup N\}$$

where Ω_V is the set of Lesbegue measure zero where ∇V does not exist and N is an arbitrary set of zero measure.

Theorem 1. (Properties of $K[f]$) The map $K:\{f \mid f:\mathbb{R}^m \to \mathbb{R}^n\} \to \{g \mid g:\mathbb{R}^m \to 2^{\mathbb{R}^n}\}$ has the following properties.

(1) Assume that $f:\mathbb{R}^m \to \mathbb{R}^n$ is locally bounded. Then $\exists\ N_f \subset \mathbb{R}^m, \mu N_f = 0$ such that $\forall\ N \subset \mathbb{R}^m$, $\mu N = 0$

$$K[f](x) = co\{\lim f(x_i) \mid x_i \to x,\ x_i \notin N_f \cup N\} \tag{2.3}$$

(2) Assume that $f, g:\mathbb{R}^m \to \mathbb{R}^n$ are locally bounded then

$$K[f + g](x) \subset K[f](x) + K[g](x) \tag{2.4}$$

(3) Assume that $f_j:\mathbb{R}^m \to \mathbb{R}^{n_j}$, $j \in \{1,2,\ldots,N\}$ are locally bounded, then

$$K[\mathop{\times}_{j=1}^{N} f_j](x) \subset \mathop{\times}_{j=1}^{N} K[f_j](x)^{\ddagger} \tag{2.5}$$

(4) Let $f:\mathbb{R}^m \to \mathbb{R}^n$ be C^1, $rank\ Df(x) = n$, and $f:\mathbb{R}^n \to \mathbb{R}^p$ be locally bounded, then

‡ Cartesian product notation and column vector notation are used interchangeably.

$$K[f \circ g](x) = K[f](g(x)) \tag{2.6}$$

(5) (equivalent control [4]) Let $g:\mathbb{R}^m \to \mathbb{R}^{p \times n}$ (i.e. matrix valued) be C^0 and $f:\mathbb{R}^m \to \mathbb{R}^n$ be locally bounded, then

$$K[gf](x) = g(x)K[f](x) \tag{2.7}$$

where $gf(x) \equiv g(x)f(x) \in \mathbb{R}^p$.

(6) Let $V:\mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz continuous, then

$$K[\nabla V](x) = \partial V(x) \tag{2.8}$$

(7) Let $f:\mathbb{R}^m \to \mathbb{R}^n$ be continuous, then

$$K[f](x) = \{f(x)\} \tag{2.9}$$

Proof: See appendix A. ∎

The manipulator dynamics together with our proposed control law is best described as a nonsmooth gradient system, i.e. a gradient system whose potential function is not differentiable everywhere. The following definition and theorem provide the formalism necessary to calculate certain time derivatives associated with nonsmooth gradient systems.

Definition: $V:\mathbb{R}^m \to \mathbb{R}$ is called a *max function* if $V(x) = \max_{j \in Y} f_j(x)$ where $f_j:\mathbb{R}^m \to \mathbb{R}$ are C^1 and Y is a finite index set.

Theorem 2.

Let $V:\mathbb{R}^m \to \mathbb{R}$ be a max function and $x:\mathbb{R} \to \mathbb{R}^m$ be differentiable at t. If $\frac{d}{dt}[V(x(t))]$ exists, then

$$\frac{d}{dt}[V(x(t))] = \xi^T \dot{x} \qquad \forall\, \xi \in \partial V(x)$$

Proof: See Appendix A. ∎

3. Manipulator Dynamics and Problem Formulation.

The dynamics of an n-joint rigid-link manipulator may be described by the equation

$$M(\vartheta)\ddot{\vartheta} + C(\vartheta,\dot{\vartheta}) + G(\vartheta) + D(\vartheta,\dot{\vartheta},t) = F \tag{3.1}$$

where

ϑ	is the n × 1 vector of joint coordinates
$M(\vartheta)$	is the n × n inertia matrix
$C(\vartheta,\dot{\vartheta})$	is the n × 1 vector of Coriolis and centrifugal forces[1]
$G(\vartheta)$	is the n × 1 vector of gravitational forces
$D(\vartheta,\dot{\vartheta},t)$	is the n × 1 vector of disturbances
F	is the n × 1 vector of generalized forces applied by the actuators at the joints of the manipulator.

and (3.1) has the following properties.

(P1) $M(\vartheta)$ is symmetric and positive definite. $M(\cdot)$, $C(\cdot,\cdot)$, and $G(\cdot)$ are C^1 functions of the manipulator state $[\vartheta,\dot{\vartheta}]^T$. $D(\cdot,\cdot,\cdot)$ is locally bounded.

The positive definiteness of $M(\vartheta)$ is an important property of the manipulator dynamics as it is essential to the stability analysis of the proposed sliding mode control scheme. This property is exploited in [6] and [7] to guarantee the invertibility of $M(\vartheta)$. Another important feature of the dynamics for earth-bound manipulators is the gravitational force $G(\vartheta)$ which is usually large. To

[1] "Forces" and "generalized forces" will be used interchangeably throughout.

accommodate this fact the joint forces will include a compensation[2] term for the gravitational forces. For the sake of generality we allow for the compensation of other forces as well. We therefore write the joint forces applied by the actuators in the form

$$F = \hat{C}(\vartheta,\dot{\vartheta}) + \hat{G}(\vartheta) + \hat{D}(\vartheta,\dot{\vartheta},t) + u \tag{3.2}$$

where the hatted terms are estimates of the corresponding unhatted objects and satisfy the following assumption.

(A1) $\hat{C}$, $\hat{G}$, and $\hat{D}$ are locally bounded. (Note that no continuity assumption is made so that discontinuous models of friction may be used in $\hat{D}$.)

The vector u is the additional joint force beyond the compensation forces and will be referred to as the *control*. The expression of the dynamics described by (3.1) and (3.2) is simplified by defining the "disturbance" vector

$$\tilde{D}(\vartheta,\dot{\vartheta},t) \equiv \hat{G}(\vartheta) - G(\vartheta) + \hat{C}(\vartheta,\dot{\vartheta}) - C(\vartheta,\dot{\vartheta}) + \hat{D}(\vartheta,\dot{\vartheta},t) - D(\vartheta,\dot{\vartheta},t). \tag{3.3}$$

Using (3.1),(3.2), and (3.3) the manipulator dynamics become

$$M(\vartheta)\ddot{\vartheta} = u + \tilde{D}(\vartheta,\dot{\vartheta},t). \tag{3.4}$$

Dropping the functional dependencies, the state equation form of (3.4) is

$$\begin{bmatrix} \dot{\vartheta} \\ \ddot{\vartheta} \end{bmatrix} = \begin{bmatrix} \dot{\vartheta} \\ M^{-1}(u + \tilde{D}) \end{bmatrix}. \tag{3.5}$$

Let $[\vartheta_d,\dot{\vartheta}_d]^T$ be the desired state trajectory that we would like the manipulator to follow. Further, let it satisfy

(A2) $[\vartheta_d,\dot{\vartheta}_d]^T$ is C^1 on $[t_0,\infty)$.

[2] Any or all of the compensation terms may be set to zero.

Now define the tracking error by

$$\begin{bmatrix} e \\ \dot{e} \end{bmatrix} \equiv \begin{bmatrix} \vartheta \\ \dot{\vartheta} \end{bmatrix} - \begin{bmatrix} \vartheta_d \\ \dot{\vartheta}_d \end{bmatrix}. \tag{3.6}$$

In terms of (3.5) and (3.6) the *tracking problem* is the following:

Find a feedback control u such that for any given initial state $[\vartheta, \dot{\vartheta}](t_0) = [\vartheta_0, \dot{\vartheta}_0]^T$, $[e(t), \dot{e}(t)]^T \to 0$ as $t \to \infty$.

Once u is chosen to achieve accurate tracking, the usefulness of the feedback control scheme for compliant motion is considered. The restoring forces exerted by the manipulator when it is perturbed from a nominal trajectory determine the suitability of the control scheme for tasks that require compliance. These forces are calculated in section 7.

4. The Control Scheme.

Choose $B \in R^{n \times n}$ such that $\sigma(B) \subset \mathbb{C}_+^o$.

Define the "switching" vector

$$s := [B \quad I] \begin{bmatrix} e \\ \dot{e} \end{bmatrix} \tag{4.1}$$

and the control u by

$$u = -k(\vartheta, \dot{\vartheta}, \vartheta_d, \dot{\vartheta}_d, \ddot{\vartheta}_d) \nabla V(s) \quad ^3 \tag{4.2}$$

$$V(s) := \|s\|_1 = \sum_{i=1}^{n} |s_i|. \tag{4.3}$$

where the gain k satisfies

(A3) $k : \mathbb{R}^{5n} \to \mathbb{R}$ is C^0.

[3] ∇V is not defined on a set of Lebesgue measure zero. The analysis to follow takes this into account.

Clearly, if $s \equiv 0$ then

$$\dot{e} = -Be \tag{4.4}$$

and it follows that $[e\ \dot{e}]^T \to 0$ exponentially for arbitrary initial conditions. Our goal, then, is to choose k such that s becomes zero in finite time. To find such a k a Lyapunov based design approach is used with the obvious choice of Lyapunov function, $V(s)$.

We begin by computing $\dot{V}$ and then choosing k such that $\dot{V}$ is bounded below zero (i.e. $\dot{V}(t) \le -\varepsilon\ \forall\ t \ge t_0$) whenever $s \ne 0$.

This will guarantee that $s \to 0$ in finite time. Lyapunov theory as developed say in [13] holds for differential equations with continuous right hand side. However, the nondifferentiability of $V(s)$ and the discontinuous nature of the control pose some technical problems. Using the results of section 2 we can compute an upper bound for $\dot{V}$.

Theorem 3 Let the manipulator dynamics and control be described by (3.5) and (4.1-4.3). Assume that P1, A2 and A3 are satisfied. If $[\vartheta, \dot{\vartheta}]$ is a solution to (3.5) on $[t_0, \infty)$ in the sense of Fillipov then

(i) $V(s(t))$ is the Lesbegue integral of its derivative

(ii) $\exists\ \delta \in K[\tilde{D}]$ such that

$$\dot{V} \le -k\xi^T M^{-1}\xi + \xi^T(M^{-1}\delta + B\dot{e} - \dot{\vartheta}_d) \qquad a.e.$$

where $\xi = \mathit{argmin}\{\|\eta\|_{M^{-1}} \mid \eta \in \partial V(s)\}$.

Proof: From (3.5), (4.2) and the fact that $[\vartheta, \dot{\vartheta}]$ is a solution to (3.5) on $[t_0, \infty)$ we have the following $a.e.$ in $[t_0, \infty)$.

$$\begin{bmatrix} \dot{\vartheta} \\ \ddot{\vartheta} \end{bmatrix} \in K\begin{bmatrix} \dot{\vartheta} \\ -kM^{-1}\nabla V(s) + M^{-1}\tilde{D} \end{bmatrix} \tag{4.5}$$

Now by Thm 1, property (3),

$$\begin{bmatrix}\dot{\vartheta}\\ \ddot{\vartheta}\end{bmatrix} \in \begin{bmatrix} K[\dot{\vartheta}] \\ K[-kM^{-1}\nabla V(s) + M^{-1}\tilde{D}]\end{bmatrix} \tag{4.6}$$

Next, by Properties (2),(5), and (7),

$$\begin{bmatrix}\dot{\vartheta}\\ \ddot{\vartheta}\end{bmatrix} \in \begin{bmatrix} \dot{\vartheta} \\ -kM^{-1}K[\nabla V(s)] + M^{-1}K[\tilde{D}]\end{bmatrix} \tag{4.7}$$

(3.6),(4.1),(4.7), and property (6) yield

$$\dot{s} \in -kM^{-1}\partial V(s) + B\dot{e} + M^{-1}K[\tilde{D}] - \ddot{\vartheta}_d \tag{4.8}$$

The absolute continuity of the solution $[\vartheta,\dot{\vartheta}]^T$ on compact intervals , and the continuous differentiability of $[\vartheta_d,\dot{\vartheta}_d]^T$ imply s is absolutely continuous on compact intervals. This, in turn, implies the absolute continuity of V on compact intervals. Thus, $\dot{V}$ exists almost everywhere, V is the Lesbegue integral of its derivative and (i) holds.

From (4.3) we have

$$V(s) = \sum_{i=1}^{n} max(-s_i, s_i). \tag{4.9}$$

Since the finite sum of max functions is a max function we have by Theorem 2, and the absolute continuity of V and s that

$$\dot{V} = \xi^T \dot{s} \qquad a.e. \tag{4.10}$$

$$\forall \ \xi \in \partial V(s)$$

From (4.8) and (4.10)

$$\dot{V} = -k\xi^T M^{-1}\beta + \xi^T[M^{-1}\delta + B\dot{e} - \ddot{\vartheta}_d] \qquad a.e. \tag{4.11}$$

$$\forall \ \xi \in \partial V(s)\text{, some } \beta \in \partial V(s)\text{, and some } \delta \in K[\tilde{D}].$$

Choose

$$\xi = argmin\{\|\eta\|_{M^{-1}} \mid \eta \in \partial V(s)\} \tag{4.12}$$

Then, from the convexity of the set $\partial V(s)$,

$$\dot{V} \leq -k\xi^T M^{-1}\xi + \xi^T[M^{-1}\delta + B\dot{e} - \ddot{\vartheta}_d] \quad a.e. \tag{4.13}$$

Whence we have (ii). ∎

Part (i) of theorem 3 tells us that we can ignore the set of measure zero where $\dot{V}$ does not exist and obtain an upper bound on V by integrating the bound on $\dot{V}$ in part (ii). The following corollary uses this fact to determine k such that $s \to 0$ in finite time.

Corollary Let (3.5) satisfy the conditions of theorem 3, and let $[\vartheta, \dot{\vartheta}]^T$ be a solution of (3.5) on $[t_0, \infty)$. If k satisfies

$$k \geq \sigma_{\max} M(\varepsilon + \frac{\|\tilde{D}\|}{\sigma_{\min} M} + \|B\dot{e}\| + \|\ddot{\vartheta}_d\|) \tag{4.14}$$

for some constant $\varepsilon > 0$, then $\exists\ T \in \mathbb{R}$ such that

$$s = 0 \ \forall \ t > T. \tag{4.15}$$

Proof: From Theorem 3 we have

$$\dot{V} \leq -k\|\xi\|^2 \sigma_{\min} M^{-1} + \|\xi\|(\sigma_{\max} M^{-1}\|\tilde{D}\| + \|B\dot{e}\| + \|\ddot{\vartheta}_d\|) \quad a.e. \tag{4.16}$$

where $\|\tilde{D}\| \equiv \sup\{\|\delta\| \mid \delta \in K[\tilde{D}]\}$

The assumption (4.14) on k yields

$$\dot{V} \leq (\|\xi\| - \|\xi\|^2)(\sigma_{\max} M^{-1}\|\tilde{D}\| + \|B\dot{e}\| + \|\ddot{\vartheta}\|) - \|\xi\|^2\varepsilon \quad a.e. \text{ on } [t_0, \infty) \tag{4.17}$$

Since $\partial V(0) = [-1,1]^n$, the unit cube in $\mathbb{R}^n$, we have by the convexity of the function V, $\partial V(s) \cap (-1,1)^n = \phi \ \forall s \neq 0$ (see[1] proposition 2.2.9). Thus, by definition of ξ, $\|\xi\| \geq 1 \ \forall \ s \neq 0$ and from (4.17) we have $\dot{V} \leq -\varepsilon \ \forall \ s \neq 0 \ a.e.$ on$[t_0, \infty)$. Thus, since $V \geq 0$ and $V = 0 \Leftrightarrow s = 0$ we have $s = 0 \ \forall t \geq T \equiv t_0 + V(t_0)/\varepsilon$. ∎

In order to use the corollary we must show that a Filippov solution to (3.5) exists on $[t_0, \infty)$. In appendix B it is proved, under the assumptions of the corollary and (A2), that a solution exists. Thus, under these assumptions, $s \to 0$ in finite time and by the definition of s, $[e, \dot{e}]^T \to 0$ exponentially. These results justify the following design procedure for a manipulator controller. The procedure generates a control law that solves the *tracking problem.*

Design Procedure

Data:

Manipulator dynamics of the form (3.1) satisfying (P1) and a class of desired trajectories satisfying (A2).

Step 1:

Choose $\hat{C}, \hat{G}, \hat{D}$ satisfying (A1).

Step 2:

Choose $B \in \mathbb{R}^{n \times n}$ such that $\sigma(B) \subset \mathbb{C}_+^o$.

Step 3:

Choose k satisfying (A3) and (4.14).

Step 4:

Choose actuator forces according to (3.2) and (4.1-4.3).

In practice a large value of k may excite unmodeled dynamics. Thus, in order to minimize the required gain the estimates in step 1 of the procedure should be as close to the true values as possible. Also, if the eigenvalues of B are large, the gain k may be large due to (4.14); this should be considered in step 2.

Note that the only information necessary to design a controller satisfying (4.14) is bounds on $\sigma_{\min} M$, $\sigma_{\max} M$, and $\tilde{D}$. Thus, variation of M and $\tilde{D}$ within these bounds will not affect the tracking performance of the controller. This robustness to parameter variations and disturbances is common in VSS controllers.

5. Design Example

Consider the two-degrees-of-freedom manipulator shown in Figure 3. Each link has unit mass concentrated at its endpoint, unit length, and the acceleration of gravity is taken to be one. Each joint actuator has unit inertia also. Given these parameters the dynamics are [14]

$$\begin{bmatrix} 4 + 2\cos(\vartheta_2) & 1 + \cos(\vartheta_2) \\ 1 + \cos(\vartheta_2) & 2 \end{bmatrix} \begin{bmatrix} \ddot{\vartheta}_1 \\ \ddot{\vartheta}_2 \end{bmatrix} +$$

$$\begin{bmatrix} 2\dot{\vartheta}_1\dot{\vartheta}_2\sin(\vartheta_2) + \dot{\vartheta}_2^2\sin(\vartheta_2) \\ 2\dot{\vartheta}_1\dot{\vartheta}_2\sin(\vartheta_2) + \dot{\vartheta}_1^2\sin(\vartheta_2) \end{bmatrix} + \begin{bmatrix} \sin(\vartheta_1) + \sin(\vartheta_1 + \vartheta_2) \\ \sin(\vartheta_1 + \vartheta_2) \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}. \tag{5.1}$$

Equation (5.1) satisfies (P1) and has the form of (3.1) where the disturbance term is equal to zero. The only contribution to $\tilde{D}$ will be from the error in estimating $C(\vartheta,\dot{\vartheta})$ and $G(\vartheta)$. A standard practice, which will be followed here, is to estimate $G(\vartheta)$ and to approximate the Coriolis and centrifugal terms by zero. With the simplifying assumption that the estimate of the gravitational forces is *exact* it follows that

$$\tilde{D} = C(\vartheta,\dot{\vartheta}) = \begin{bmatrix} 2\dot{\vartheta}_1\dot{\vartheta}_2\sin(\vartheta_2) + \dot{\vartheta}_2^2\sin(\vartheta_2) \\ 2\dot{\vartheta}_1\dot{\vartheta}_2\sin(\vartheta_2) + \dot{\vartheta}_1^2\sin(\vartheta_2) \end{bmatrix} \tag{5.2}$$

and (A1) is satisfied. To simplify the form of the gain k the following bound for $\tilde{D}$ will be used.

$$\|\tilde{D}\| \le 2(\dot{\vartheta}_1 + \dot{\vartheta}_2)^2. \tag{5.3}$$

We begin by choosing the matrix B diagonal;

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.4}$$

Next, for concreteness, set $\varepsilon = 1$, and from (5.1), and a simple calculation, it follows that

$$\sigma_{\max} M < 7 \tag{5.5}$$

$$\sigma_{\min} M > 1.$$

Now verify that

$$k = 7(1 + 2(\dot{\vartheta}_1 + \dot{\vartheta}_2)^2 + \|\dot{\vartheta} - \dot{\vartheta}_d\| + \|\ddot{\vartheta}_d\|) \tag{5.6}$$

satisfies (A3) and (4.14). Putting together (3.2),(4.1-4.3),(5.4), and (5.6) yields

$$F = -\hat{g}(\vartheta) + u \tag{5.7}$$

$$= \begin{bmatrix} \sin(\vartheta_1) + \sin(\vartheta_1 + \vartheta_2) \\ \sin(\vartheta_1 + \vartheta_2) \end{bmatrix}$$

$$-(7 + 14(\dot{\vartheta}_1 + \dot{\vartheta}_2)^2 + 7\|\dot{\vartheta} - \dot{\vartheta}_d\| + 7\|\ddot{\vartheta}_d\|) \begin{bmatrix} sgn(\vartheta_{d1} - \vartheta_1 + \dot{\vartheta}_{d1} - \dot{\vartheta}_1) \\ sgn(\vartheta_{d2} - \vartheta_2 + \dot{\vartheta}_{d2} - \dot{\vartheta}_2) \end{bmatrix} \tag{5.8}$$

where ϑ_d is any trajectory satisfying (A2). The choice of joint forces in (5.8) will move the switching vector s to zero in finite time. Thus, by our choice of B, the tracking error tends to zero exponentially.

There are many possible variations in deriving a gain that satisfies (4.14); in practice all bounds used should be made as tight as possible without violating constraints on computation time for the joint forces. The next section discusses a method for reducing the required gain k by scaling.

6. Linear Coordinate Transformation.

In the design example of the last section the link masses and lengths were the same so that $\sigma_{\max} M / \sigma_{\min} M$ was not excessively large for any configuration. However, this is not the case for most manipulators as their link masses and lengths vary widely. From equation (4.14) it is clear that a large value of $\sigma_{\max} M / \sigma_{\min} M$ will cause the gain k to be large. Also, equation (4.2) suggests that all joint forces are approximately the same modulo the gravity compensation. This is not appropriate for a manipulator with differing link sizes. The natural modification to the "normalized" control (4.2) is a scaling. This is accomplished by making a linear transformation of the joint coordinates.

Choose nonsingular $A \in \mathbb{R}^{n \times n}$ and define transformed coordinates and forces by

$$q := A^{-1}\vartheta \tag{6.1}$$

and

$$f := A^T F \tag{6.2}$$

Multiplying equation (3.1) on the left by A^T yields

$$m(q)\ddot{q} + c(q,\dot{q}) + g(q) + d(q,\dot{q},t) = f \tag{6.3}$$

where

$$m(q) = A^T M(Aq)A$$

$$c(\dot{q},q) = A^T C(Aq,A\dot{q})$$

$$g(q) = A^T G(Aq)$$

$$d(\dot{q},q,t) = A^T D(Aq,A\dot{q},t)$$

This equation in the transformed variable q has the same form as (3.1) and satisfies (P1). The design approach, therefore, works on these transformed dynamics as well. The advantage of allowing this transformation is that we may choose A t minimize $\sigma_{\max} m / \sigma_{\min} m$, fit the joint forces to match the actuators more closely, or achieve some compromise between the two.

The force transformation (6.2), and equation (4.2) suggest that a good choice for A might be a diagonal matrix with A_{ii} equal to the inverse of the ith actuator force rating. A nonlinear transformation may be desirable to achieve a particular dynamic behavior [15] but the discussion here will consider linear transformations only.

7. Compliance.

In assembly operations requiring compliance, the forces that are generated when the manipulator moves one workpiece into contact with another must be controlled. For example, consider the peg insertion task depicted in Figure 4; in order to execute this task with the proposed VSS control scheme a nominal trajectory must be specified for the manipulator to follow. The manipulator follows this trajectory until some misalignment of the peg or hole causes the manipulator to deviate from the nominal trajectory. If the resulting forces do not cause binding or excessive friction, the manipulator will follow a path close to the nominal path and complete the task.

We use the approach of [8] and describe the compliance of the control scheme by the restoring forces [1] generated by the control when the manipulator is forced from the nominal trajectory. To study the performance of the proposed control scheme in compliant motion it is assumed that the motion is *quasi−static*. That is, all time derivatives of the manipulator state and the desired trajectory are approximated by zero. This approximation is reasonable for most assembly operations requiring programmed compliance [16].

With this assumption the force exerted by the manipulator on its environment is calculated. Let F_C be the force that the manipulator exerts at its gripper in some set of workspace oriented coordinates. This force is translated into joint forces by the usual Jacobian transformation [14] and is equal to $J^T(\vartheta)F_C$ where $J(\vartheta)$ is the Jacobian of the workspace oriented coordinates with respect to the joint coordinates. With this added force, which is not accounted for in the design procedure, equation (3.1) becomes

$$M(\vartheta)\ddot{\vartheta} + C(\vartheta,\dot{\vartheta}) + G(\vartheta) + D(\vartheta,\dot{\vartheta},t) = F + J^T(\vartheta)F_C. \qquad (7.1)$$

Making the linear transformation described in the last section yields

$$m(q)\ddot{q} + c(q,\dot{q}) + g(q) + d(q,\dot{q},t) = f + A^T J^T(A^{-1}q)F_C \qquad (7.2)$$

Applying the design procedure to (7.2) with $F_C \equiv 0$, we obtain

$$f = \hat{c}(q,\dot{q}) + \hat{g}(q) + \hat{d}(q,\dot{q},t) + u \tag{7.3}$$

where

$$u = -k\nabla V(s)$$

$$V(s) = \|s\|_1$$

$$k \geq \sigma_{\max} m(\varepsilon + \frac{\|\tilde{d}\|}{\sigma_{\min} m} + \|B\dot{e}\| + \|\ddot{\vartheta}_d\|), \tag{7.4}$$

and

$$s = B(q_d - q) + (\dot{q}_d - \dot{q})$$

where q_d is the desired trajectory and $\tilde{d}$ is defined in (3.3) with upper-case characters replaced by lower-case. The dynamics are then

$$m(q)\ddot{q} = u + \tilde{d}(q,\dot{q},t) + A^T J^T(A^{-1}q)F_C. \tag{7.5}$$

Choosing B to be the identity matrix and applying the quasi-static assumption we have

$$k \geq \sigma_{\max} m(\varepsilon + \frac{\|\tilde{d}\|}{\sigma_{\min} m}), \tag{7.6}$$

$$s = (q_d - q) \tag{7.7}$$

Given the control specified by (7.3) - (7.7) the compliance question is: what force does the manipulator apply at its gripper when the manipulator is perturbed slightly to $q_c \neq q_d$? Using the quasi-static assumption again we set $\ddot{q} = 0$ in equation (7.5) and from the Filippov definition of solution of section 2, it follows that

$$[A^T J^T]F_C \in -K[u + \tilde{d}]. \tag{7.8}$$

Equation (7.8) defines the compliance of the control scheme. We restrict our attention now to manipulators with six degrees-of-freedom. Let ϑ_0 be the approximate configuration of the manipulator for an assembly task and assume $J(\vartheta_0)$ is nonsingular. Choosing $A \equiv J(\vartheta_0)^{-1}$ for the transformation will simplify the compliance of the control scheme since (7.8) becomes

$$F_C \in [k\partial V(s) - K[\tilde{d}]]. \tag{7.9}$$

Define

$$\begin{aligned}\Delta q &= q_d - q_c \\ &= A^{-1}(\vartheta_d - \vartheta_c) \\ &= J(\vartheta_d - \vartheta_c) \\ &\approx \Delta x\end{aligned} \tag{7.10}$$

where Δx represents a small change in the gripper coordinates by definition of the Jacobian.

Then we have (approximately)

$$F_C \in k\,\partial V(s) - K[\tilde{d}] = k\begin{bmatrix} SGN\,\Delta x_1 \\ SGN\,\Delta x_2 \\ \vdots \\ SGN\,\Delta x_6 \end{bmatrix} - K[\tilde{d}] \tag{7.11}$$

Here the compliance behavior of the quasi-static manipulator is apparent. When $q = q_d$ the manipulator can apply at its gripper any reaction force in $[-k, k]^6$ modulo disturbances and modeling errors. Once the manipulator is forced from the desired position the control applies an approximately constant restoring force. The gain k is a stiffness parameter which may be used to control the compliance behavior of the manipulator. Note that equation (7.6) puts a lower bound on the stiffness and that this bound is given primarily by the magnitude of disturbances that must be rejected. In words, the stiffness of the manipulator may be controlled but the manipulator can only be as compliant as modeling errors, joint friction, and other disturbances allow.

Various choices of A and B will give different stiffness behavior to the manipulator. For example, stiffnesses along different axes may be controlled independently by suitable choice of these matrices.

8. Conclusions.

We have shown by using a multivariable approach to VSS that an extremely simple controller can be designed for robot manipulators. The control scheme developed provides for robust tracking and for compliance control and the compliance behavior is such that when the manipulator is forced from a nominal trajectory the control switches implicitly to force control.

The techniques used for proving stability are new for VSS and should be useful for the analysis of a wide variety of VSS described by nonsmooth gradient systems.

Appendix A

Proof of Theorem 1:

(1) To prove this property we first need two lemmas.

Lemma (1.1)

Let $\{E_m\}$ be a sequence of compact subsets of $\mathbb{R}^n$ such that $E_{m+1} \subset E_m$. Then

$$\bigcap_{m \in \mathbf{N}} co\, E_m = co \bigcap_{m \in \mathbf{N}} E_m. \tag{A.1}$$

Proof: This is a simple application of Caratheodory's theorem for convex sets [17]. ∎

Lemma (1.2)

Let f be defined almost everywhere and measurable on a set E, $\mu E \neq 0$. Then

$\exists\ N_f$ of measure zero such that

$$\bigcap_{\mu N = 0} \overline{co}\, f(E - N) = \overline{co}\, f(E - N_f). \tag{A.2}$$

Proof: See [2]. ∎

Henceforth, N subscripted with a function will be interpreted in terms of this lemma.

Proceeding with the proof of the property.

$$K[f](x) = \bigcap_{\delta>0} \bigcap_{\mu N=0} \overline{co} f(B(x,\delta) - N) \quad \text{(A.3)}$$

$$= \bigcap_{m \in \mathbf{N}} \bigcap_{\mu N=0} \overline{co} f(B(x,1/m) - N) \quad \text{(A.4)}$$

Now, from lemma (1.2) we have

$$K[f](x) = \bigcap_{m \in \mathbf{N}} \overline{co} f(B(x,1/m) - N_{f,m}) \quad \text{(A.5)}$$

Define

$$N_f = \bigcup_{m \in \mathbf{N}} N_{f,m} \quad \text{(A.6)}$$

then

$$K[f](x) = \bigcap_{m \in \mathbf{N}} \overline{co} f(B(x,1/m) - N_f) \quad \text{(A.7)}$$

since $N_{f,m}$ can be enlarged by a set of measure zero in (A.2).

Now f is locally bounded $\Rightarrow$

$$K[f(x)] = \bigcap_{m \in \mathbf{N}} co\, \overline{f(B(x,1/m) - N_f)}. \quad \text{(A.8)}$$

$$= \bigcap_{m \in \mathbf{N}} co\, \{\lim f(x_i) \mid x_i \in B(x,1/m) - N_f\} \quad \text{(A.9)}$$

By lemma 1.1

$$K[f(x)] = co \bigcap_{m \in \mathbf{N}} \{\lim f(x_i) \mid x_i \in B(x,1/m) - N_f\} \quad \text{(A.10)}$$

$$= co\, \{\lim f(x_i) \mid x_i \to x, x_i \not\in N_f\} \quad \text{(A.11)}$$

Finally, by noting that N_f can be enlarged by any set of measure zero in (A.2) the result follows.

(2) By property (1)

$$K[f+g](x) = co\{\lim(f+g)(x_i) \mid x_i \to x,\, x_i \not\in N_{f+g} \cup N_f \cup N_g\} \quad \text{(A.12)}$$

Since f and g are locally bounded, for each sequence $x_i \to x$ such that the limit in (A.12) exists, $\exists$ a subsequence (we do not reindex) $x_i \to x$ such that $\lim f(x_i)$ and $\lim g(x_i)$ exist and $\lim f(x_i) + \lim g(x_i) = \lim (f + g)(x_i)$ Thus,

$$K[f+g](x) = co\{\lim f(x_i) + \lim g(x_i) \mid x_i \to x,\ x_i \notin N_{f+g} \cup N_f \cup N_g\}$$

$$\subset co\{\lim g(x_i) \mid x_i \to x,\ x_i \notin N_{f+g} \cup N_f \cup N_g\}$$

$$+ co\{\lim f(x_i) \mid x_i \to x,\ x_i \notin N_{f+g} \cup N_f \cup N_g\}$$

$$= K[f](x) + K[g](x). \tag{A.13}$$

(3)

Define

$$g(x) \equiv \mathop{\times}_{j=1}^{N} f_j(x) \tag{A.14}$$

then by property (1)

$$K[\mathop{\times}_{j=1}^{N} f_j](x) = co\{\lim \mathop{\times}_{j=1}^{N} f_j(x_i) \mid x_i \to x,\ x_i \notin \bigcup_{j=1}^{N} N_{f_j} \cup N_g\}$$

$$\subset co \mathop{\times}_{j=1}^{N} \{\lim f_j(x_i) \mid x_i \to x,\ x_i \notin \bigcup_{j=1}^{N} N_{f_j} \cup N_g\}$$

$$= \mathop{\times}_{j=1}^{N} co\{\lim f_j(x_i) \mid x_i \to x,\ x_i \notin \bigcup_{j=1}^{N} N_{f_j} \cup N_g\}$$

$$= \mathop{\times}_{j=1}^{N} K[f_j](x) \tag{A.15}$$

(4)

We begin the proof of this property with a lemma.

Lemma (4.1)

Let $f:\mathbb{R}^{m+n} \to \mathbb{R}^n$ be C^1 and $x \in \mathbb{R}^{m+n}$ be such that $rank(Df(x)) = n$. Then $\exists$ neighborhoods U of x and W of $f(x)$, such that $\forall\ M \subset \mathbb{R}^{m+n}, N \subset \mathbb{R}^n$, with $\mu M = \mu N = 0$, we have

$$\mu\{[f(U \cap M^C)]^C \cap W\} = 0$$

$$\mu\{[f^{-1}(W \cap N^C)]^C \cap U\} = 0 \tag{A.16}$$

Proof: $rank(Df(x)) = n \Rightarrow$ we can choose(WLOG) a partition (x_1,x_2) of $\mathbb{R}^{m+n}$ with $x_1 \in \mathbb{R}^m, x_2 \in \mathbb{R}^n$ so that $D_2 f(x_1,x_2)$ is nonsingular. By the implicit function theorem, $\exists$ a C^1 function $g: \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, and neighborhoods U_1 containing x_1, U_2 containing x_2, and W containing $f(x_1,x_2)$ such that

$$f(x_1, g(x_1,w)) = w \quad \forall \; x_1 \in U_1, \; w \in W, \tag{A.17}$$

and Φ defined by $\Phi(x_1,w) = (x_1, g(x_1,w))$ is a C^1 diffeomorphism of $U_1 \times W$ onto $U \equiv U_1 \times U_2$. By continuity of Φ it follows (see [18] pg. 551) that $\Phi|_{U_1 \times W}$ maps null (zero-measure) sets to null sets and similarly for $\Phi^{-1}|_U$. It is therefore sufficient to prove the result for $f \circ \Phi|_{U_1 \times W}$ which is simply a projection. This is straight forward and is left to the reader. ∎

Now the proof of the property: By lemma (4.1), $\exists$ neighborhoods U of x and W of $f(x)$ such that $[g^{-1}(W \cap N_f^C)]^C \cap U$ and $[g(U \cap N_{f \circ g}^C)]^C \cap W$ are null sets. Next, by property (1), and the fact that $K[f](x)$ depends on f only near x we obtain

$$K[f \circ g](x) = co\, L[f \circ g](x) \tag{A.18}$$

where

$$L[f \circ g](x) \equiv \{\lim f \circ g(x_i) \mid x_i \to x, x_i \in U \cap N_{f \circ g}^C \cap g^{-1}(W \cap N_f^C)\}$$

and

$$K[f](g(x)) = co\, L[f](g(x)) \tag{A.19}$$

where

$$L[f](g(x)) \equiv \{\lim f(y_i) \mid y_i \to g(x), y_i \in W \cap g(U \cap N_{f \circ g}^C) \cap N_f^C\}$$

For every $z \in L[f \circ g](x)$, $\exists \; x_i \to x$ such that $x_i \in U \cap N_{f \circ g}^C \cap g^{-1}(W \cap N_f^C)$, and $f(g(x_i)) \to z$.

Now, let $y_i = g(x_i)$, then $y_i \in g(U \cap N^c_{f \circ g}) \cap W \cap N^c_f$ and $y_i \to g(x)$ since g is continuous. Now $f(y_i) \to z \implies$

$$L[f \circ g](x) \subset L[f](g(x)). \tag{A.20}$$

For the reverse inclusion, let $z \in L[f]g(x)$ then $\exists\, y_i \to g(x)$ such that $y_i \in W \cap g(U \cap N^c_{f \circ g}) \cap N^c_f$ and $f(y_i) \to z$. By the rank condition on $Dg(x)$, g is locally surjective (see [18] page 108) so $\exists$ a subsequence of $\{y_i\}$ (we do not index) such that $y_i \in W \cap g(U \cap N^c_{f \circ g} \cap B(x, 2^{-i})) \cap N^c_f$. Thus,

$$\exists\, x_i \to x, x_i \in U \cap N^c_{f \circ g} \cap g^{-1}(W \cap N^c_f) \text{ such that } y_i = g(x_i) \text{ and } f(g(x_i)) \to z$$

$$\implies z \in L[f \circ g](x)$$

Thus we have $L[f \circ g](x) = L[f](g(x))$ and the result follows by taking the convex hull of both sides.

(5)

By proposition 1, we obtain

$$K[gf](x) = co\,\{\lim g(x_i) f(x_i) \mid x_i \to x,\ x_i \notin N_{gf} \cup N_f\} \tag{A.21}$$

Since g is continuous in its first argument and f is locally bounded

$$K[gf](x) = co\,\{g(x) \lim f(x_i) \mid y_i \to x,\ x \notin N_{gf} \cup N_f\}.$$

$$= g(x)\, K[f](x). \tag{A.22}$$

since co commutes with linear maps.

(6) Since V is locally Lipschitz, ∇V is defined almost everywhere and is locally bounded. Therefore by (1) we have

$$K[\nabla V](x) = co\,\{\lim \nabla V(x_i) \mid x_i \to x,\ x_i \notin N_{\nabla V}\}$$

$$= \partial V(x) \tag{A.23}$$

(7)

This is a corollary of (4) obtained by taking f to be the identity map. ∎

Proof of Thm 2: First, we have by definition

$$V(x(t)) = \max_{j \in Y} f_j(x(t)) \tag{A.24}$$

Computing left and right derivatives we obtain

$$\frac{d}{dt}[V(x(t))] \exists \iff \max_{j \in Y^*(x)} \nabla f_j^T(x)\dot{x}(t) = \min_{j \in Y^*(x)} \nabla f_j^T(x)\dot{x}(t) \tag{A.25}$$

where $Y^*(x) = \{j \mid f_j(x) = V(x)\}$

Thus, the existence of $\dot{V} \Rightarrow$

$$\frac{d}{dt}[V(x(t))] = \nabla f_j^T(x)\dot{x}(t) \quad \forall\, j \in Y^*(x) \tag{A.26}$$

$\Rightarrow$

$$\frac{d}{dt}[V(x(t))] = [\sum_{j \in Y^*(x)} \lambda_j \nabla f_j^T(x)]\,\dot{x}(t)$$

$$\forall\, \{\lambda_j\} \text{ such that } \sum_{j \in Y^*(x)} \lambda_j = 1 \tag{A.27}$$

$\Rightarrow$

$$\frac{d}{dt}[V(x(t))] = \xi^T\dot{x}(t) \quad \forall\, \xi \in co\{\nabla f_j \mid j \in Y^*(x)\} \tag{A.28}$$

and $co\{\nabla f_j \mid j \in Y^*(x)\} = \partial V(x)$. ∎

Appendix B

Here we prove the existence and continuation of a Filippov solution to (3.5).

Theorem 4 Let u be defined by (4.1-4.3) and (4.14). If A1,A2,A3, and P1 are satisfied, then, for any initial condition $[\vartheta,\dot{\vartheta}]^T(t_0) = [\vartheta_0,\dot{\vartheta}_0]^T$, (3.5) has a solution continuable on $[t_0,\infty)$.

Proof: Let $Q = \mathbb{R}^{2n} \times \mathbb{R}$ and let D be an arbitrary compact set in Q. By A1,A2,A3, and P1 we have that $\dot{\vartheta}$, M^{-1}, $\tilde{D}$, and k are bounded on D. Also, ∇V is defined *a.e.* and bounded. Thus, RHS of (3.5) is bounded by, say, L on D. Choose $A(t) = L$ which is integrable on D. The RHS of (3.5) is measurable and defined *a.e.* in Q. Thus, the RHS of 3.5 satisfies condition B. Now by theorem 4 of [2] we have the local existence of a solution to (3.5).

By theorem 5 of [2] any solution of (3.5) is continuable on $[t_0, t_1)$ where $t_1 = \infty$ or $\|[\vartheta, \dot{\vartheta}]^T\| \to \infty$. By (4.15) we have for any solution of (3.5) that s is bounded $\Rightarrow$ $[e, \dot{e}]^T$ is bounded by (4.1) $\Rightarrow$ $[\vartheta, \dot{\vartheta}]^T$ is bounded on bounded sets by A1. Therefore, there exists a solution continuable on $[t_0, \infty)$. ∎

We do not prove uniqueness. The interested reader is referred to [19].

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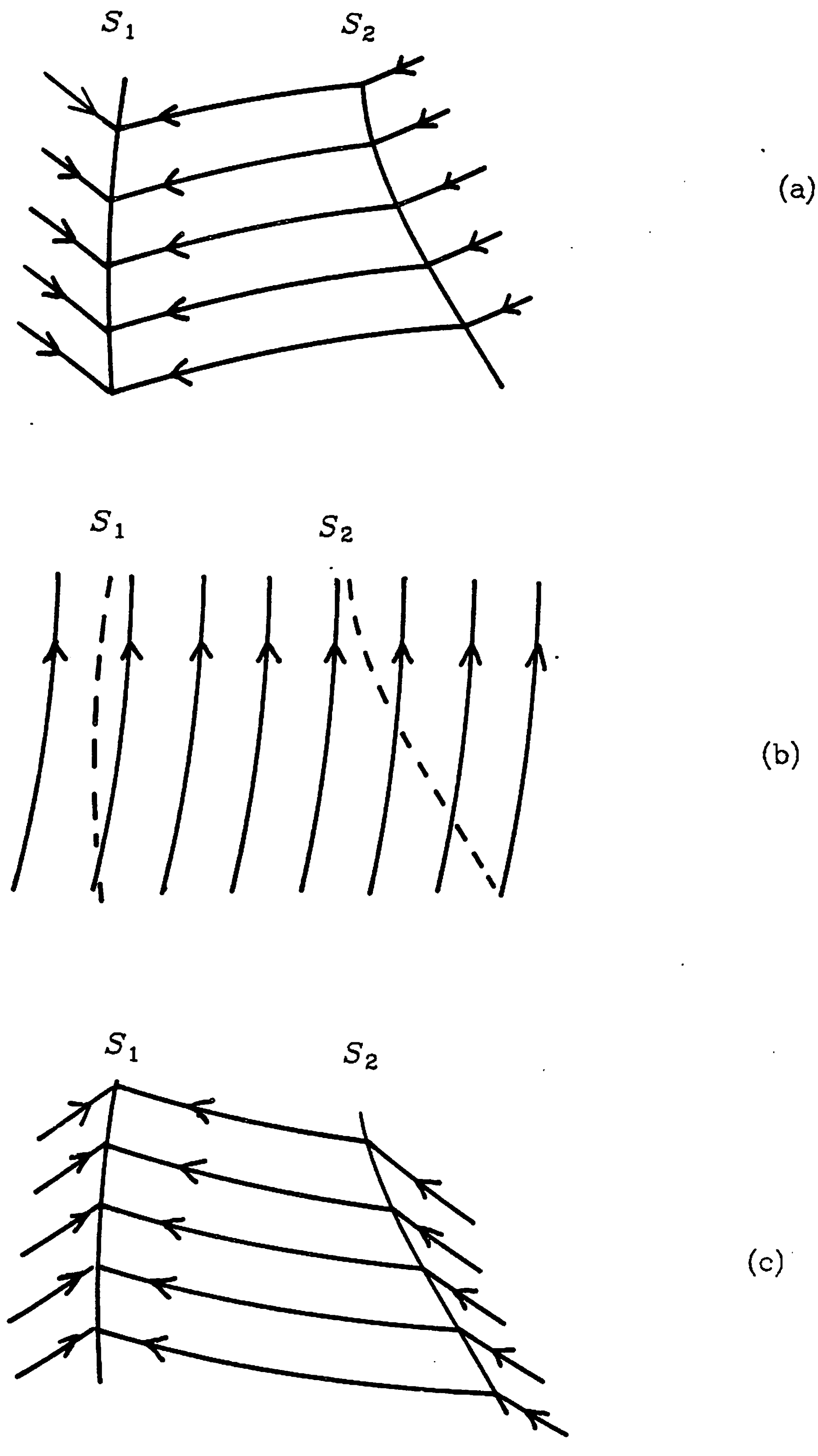

Fig. 1 Phase Portrait of Hypothetical VSS

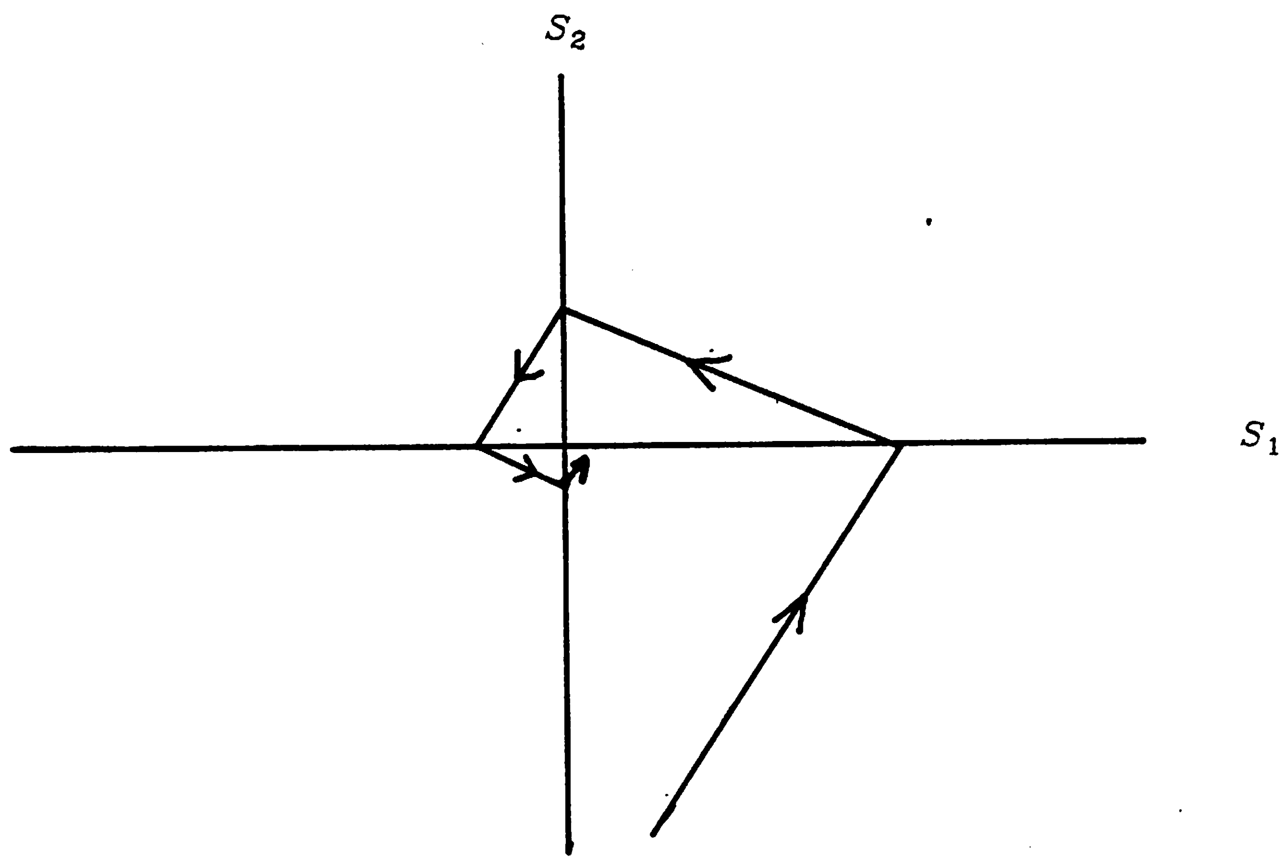

Fig. 2 Phase Portrait of Multivariable VSS

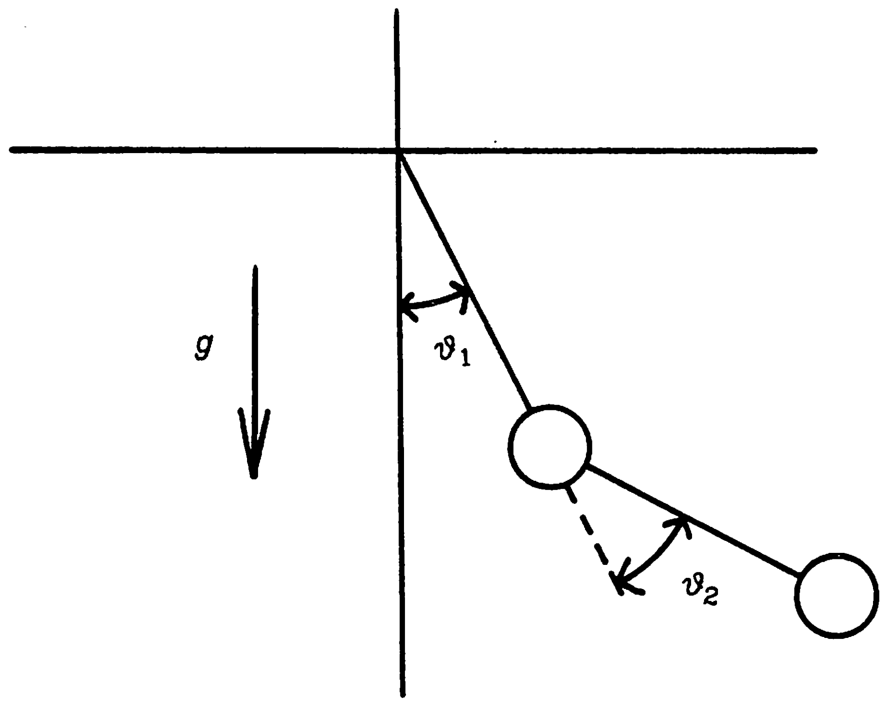

Fig. 3 2 Degree–of–Freedom Manipulator

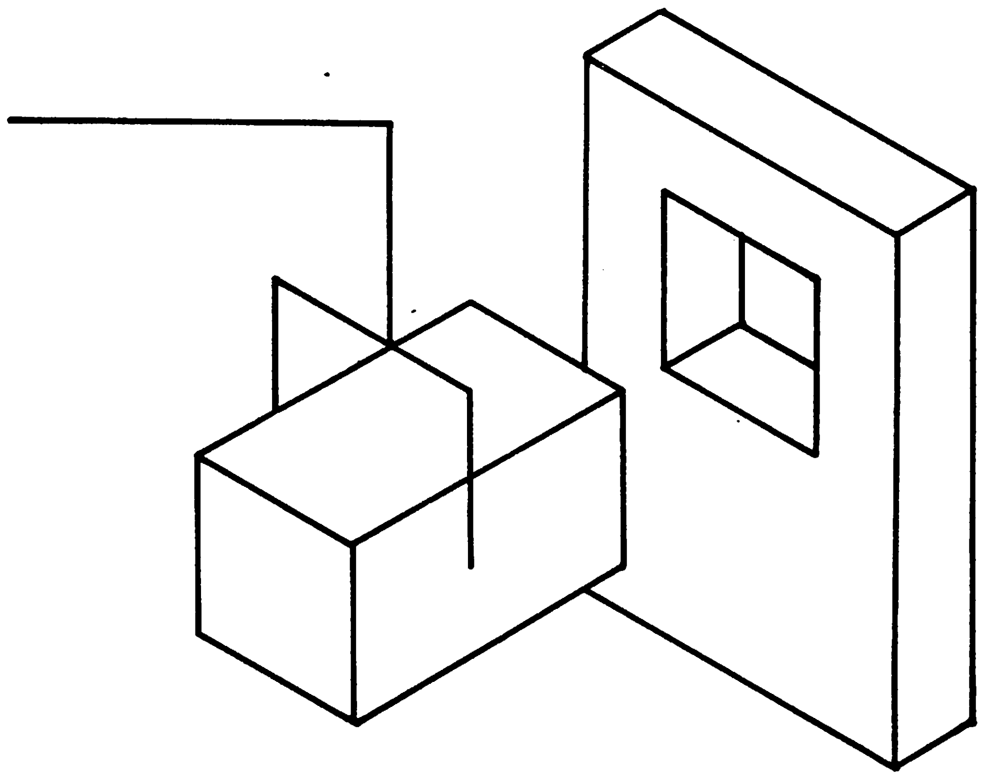

Fig. 4 Peg–in–Hole Task