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by

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*ABSTRACT*

The paper presents a general stability proof for continuous time adaptive control, with very general assumptions on the identifier and controller. Applications of the proof to pole placement design and design based on the factorization approach are discussed.

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# Global Stability Proofs for Continuous Time Indirect Adaptive Control Schemes

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## 1 Introduction

A popular technique of adaptive control is the so-called indirect technique: a non-adaptive controller is designed parametrically i.e. the controller parameters are written as a function of plant parameters. This scheme is made adaptive by replacing the plant parameters in the design calculation by their estimates at time t, obtained from an on-line identifier. Reasons for the popularity of indirect adaptive controllers stem from the considerable flexibility in choice of both the controller and identifier. Global stability of indirect schemes have been shown in the discrete time case (Goodwin&Sin [5], Anderson&Johnstone [1] Polak,Salcudean&Mayne [11]) but less so in the continuous time context. A recent paper of Elliot et al [4] uses random sampling to establish convergence results in the continuous time case. Other papers have assumed that the plant parameters lie in a convex set in which no unstable pole-zero cancellations occur.

In this paper (section 2), we discuss a general, indirect adaptive control scheme for SISO continuous time systems using an identifier in conjunction with a stabilizing controller. We show that when the reference input to the closed loop system is rich enough (in the sense of having sufficient frequency content) then the signal input to the identifier is persistently exciting so as to cause parameter convergence. In turn the controller is updated only when adequate information has been obtained for a 'meaningful' update. Thus, roughly speaking, the adaptive system consists of a fast parameter identification loop and a slow controller update loop. A sufficient richness condition on the exogenous reference input is used to give an insightful global stability proof with no restrictions the

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on parameter estimate lying in a convex set or lack of unstable pole-zero cancellation in the identifier.

In section 3, we show the specialization of our general scheme to a pole placement type adaptive controller.

The second contribution of this paper is the application of our techniques (in section 4 ) to the adaptive stabilization of a SISO system using the factorization approach (factorization over the ring of stable, proper rational functions) that has proved to be a useful and elegant tool (see [12], [13]) for the study of robust multivariable design. Since it is known [12] that when the stable coprime factorization approach is used, a plant with unstable unmodeled dynamics is really no different from a plant with stable unmodeled dynamics as far as the effect of the unmodeled dynamics of the robustness of the system concerned. We feel that our techniques lay the groundwork for obtaining an adaptive version of $H^\infty$ optimal controller design by the factorization approach. In this context our work has contact with a recent paper of Ma&Vidyasagar [9]. In this paper, we only discuss SISO continuous time case, the extension to the discrete time case is trivial. We feel that our result could be extended to MIMO case as well, if a good MIMO identifier structure is obtained.

## 2 Basic Structure of the Identifier and Controller

The basic structure of the adaptive controller is shown in Fig.2.1

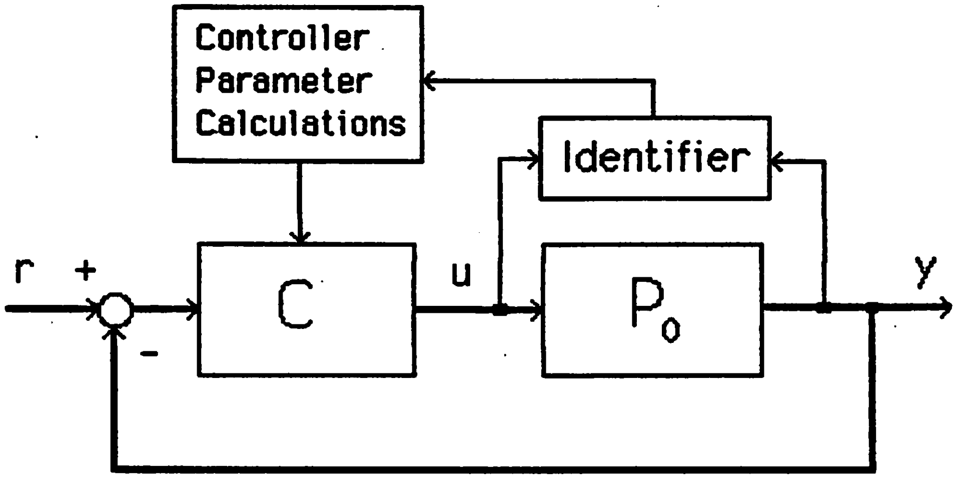


Fig. 2-1

**2.1 Basic Indirect Adaptive Controller Structure**

The unknown plant is assumed to be described by

$$P_0(s)=n_p(s)/\,d_p(s)=\frac{\alpha_1 s^{n-1}+\ldots+\alpha_n}{s^n+\beta_1 s^{n-1}+\ldots+\beta_n} \tag{2.1}$$

where $P_0 \in R(s)$ is a strictly proper transfer function. The proper nth order compensator is defined by

$$C(s)=n_c(s)/\,d_c(s)=\frac{a_0 s^m+\ldots+a_m}{b_0 s^m+\ldots+b_m} \tag{2.2}$$

The adaptive scheme proceeds as follows: the identifier gets an estimate of the plant parameters. The compensator design (pole placement. model reference,...) is performed assuming that the plant parameter estimate corresponds to the true parameter value. We will assume that there exists a unique choice of compensator $\hat{C}(s)$ of the form (2.2) for the estimate plant $\hat{P}_0$ . The hope is that as $t \to \infty$ the identifier identifies the plant correctly and that the compensator converges asymptotically to the desired one. In this section, we discuss indirect adaptive control abstractly without restricting attention to any specific control scheme—pole placement, model reference, etc. In later sections, we specialize to a pole-placement type controller and a controller derived using the factorization approach.

Basically the most important element of the adaptive loop is the convergence of the identifier. We design an identifier which uses the input and output of the not necessarily stable plant as follows: the equation (2.1) relating the transform of the input and output of the plant can be written (with initial condition terms unspecified) as

$$s^n y(t)=\theta^{*T} v(t) \tag{2.3}$$

where s denotes the differentiation and

$$\theta^{*T}=(-\beta_1,...-\beta_n,\alpha_1,\ldots,\alpha_n)$$

$$v^T(t)=(s^{n-1}y(t),...,y(t),s^{n-1}u(t),...,u(t))$$

Since the signal v(t) involves differentiation of the input and output of the plant, we filter both side of (2.3) by the transfer function $1/(s+\alpha)^n$, $\alpha>0$, to get

$$\frac{s^n}{(s+\alpha)^n}y(t)=\theta^{*T}w(t) \tag{2.4}$$

where

$$w^T(t)=(\frac{s^{n-1}}{(s+\alpha)^n}y(t),....,\frac{1}{(s+\alpha)^n}y(t),\frac{s^{n-1}}{(s+\alpha)^n}u(t),....,\frac{1}{(s+\alpha)^n}u(t))$$

Note that the signal vector w(t) may be obtained by proper, stable filtering of the input and output of the plant. The equation error for identification of $\theta^*$ is developed as follows: let $\hat{\theta}(t)$ be the estimate of the parameter $\theta^*$ at time t. Then, define the equation error to be

$$e(t)=\hat{\theta}(t)^T w(t)-\frac{s^n}{(s+\alpha)^n}y(t) \tag{2.5}$$

If $\phi(t)$ denotes the parameter error $(\hat{\theta}(t)-\theta^*)$, then it follows that, upto exponentially decaying terms, we have

$$e(t)=\phi^T(t)w(t) \tag{2.6}$$

As is standard in the literature, we will in future drop the exponentially decaying terms. The interested reader may wish to confirm that the presence of such terms do not change any of the proofs (or conclusions) that follow.

The identification technique used is of the least squares type with resetting, given by

$$\dot{\hat{\theta}}(t)=\dot{\phi}(t)=-P(t)w(t)e(t) \tag{2.7a}$$

$$\dot{P}(t)=-P(t)w(t)w^T(t)P(t) \quad P(t_i)=\beta I>0 \tag{2.7b}$$

where $\{t_i\}=\{0,t_1,...\}$ will be specified shortly. It is easy to verify (using the Lyapunov function $\phi^T P^{-1}\phi$ ) that the parameter error $\phi$ is bounded even though y(t) may not be and further $\phi(t)\to 0$ asymptotically, if w(t) is persistently exciting, i.e. there exist $\alpha,\delta>0$ such that

$$\int_t^{t+\delta} w(\tau)w^T(\tau)d\tau \geqslant \alpha I \quad \text{for } all\ t$$

It has further been shown [2] that under the condition that $n_p$ , $d_p$ are coprime polynomials, that w is persistent exciting if u is rich enough i.e. the support of the spectrum of u has greater than 2n points ( assuming that u(t) is stationary).

The design of the compensator is based on the plant parameter estimate namely $\hat{\theta}(t)$. It is would appear to be intuitive that if as $t\to\infty$, $\hat{\theta}(t)\to\theta^*$ that the time varying compensator would converge to the true compensator and that the closed loop system would be asymptotically stable. In this section, we do not deal with a specific compensator design, however the system of Fig.2-1 can be understood to be a time varying linear system which is asymptotically time invariant and stable. Such systems are themselves stable; more precisely, we have (using standard Lyapunov function arguments).

**Lemma 2.1** Consider the time varying system

$$\dot{x}=(A+\Delta A(t))x \tag{2.8}$$

where A is a constant matrix and $\Delta A(t)$ is time varying. Assume that $||\Delta A(t)||$ is bounded and converges to a sufficient small ball as $t\to\infty$ Suppose that $\sigma(A)\subset C^0_-$, then (2.8) is asymptotically stable. Furthermore , there exist $T,M,\lambda>0$ such that the state transition matrix $\Phi(t,\tau)$ of the equation (2.8) satisfies

$$||\Phi(t,\tau)||\leqslant M\exp(-\lambda(t-\tau)) \quad \text{for } all\ t>\tau>T$$

## 2.2 Update Law

Though the update law (2.7a) (2.7b) for the identifier is easily shown to be asymptotically convergent when w is persistent exciting, it is of practical importance to limit the update of the controller to instants when sufficient new information has been obtained. The amount of information is measured through the 'information matrix'

$$\int_t^{t+\delta} w(\tau)w^T(\tau)d\tau$$

Thus given $\gamma > 0$ , we choose update time $\{t_i\}$, a sequence by $t_0=0$ and $t_{i+1}=t_i+\delta_i$, where $\delta_i$ satisfies

$$\delta_i := \underset{\Delta}{argmin} \int_{t_i}^{t_i+\Delta} ww^T d\tau \geqslant \gamma I \tag{2.9}$$

The compensator $\hat{C}$ is constant between $t_i$ and $t_{i+1}$. Further, we assume that the compensator parameters are continuous function of $\theta^*$.

Remark: (1) The idea of updating the controller only when new data becomes available was first proposed by [11] for the discrete time case. A similar idea was proposed by Elliot, et al [4], but they use a sequence of independent random variables to generate the update sequence.

(2) The update times are based on a monitoring of the excitation contained in the signal w.

We may state the following lemma relating the richness of the reference signal r(t) in the scheme of Fig.2-1 to the convergence of the identifier.

**Lemma 2.2 (Convergence of The Identifier)**

Consider the system of Fig. 2-1 with identifier described in equation (2.7) and resetting times $\{t_i\}$ given by (2.9). Further assume that there is a unique choice of controller for each estimate of the plant and that the controller is updated only at $\{t_i\}$. If the input r(t) is bounded and stationary and the supports of the spectrum of r has greater than 3n+m points, then the identifier parameter error converges to zero exponentially as $t \to \infty$. More precisely, there exists $0 < \rho < 1$ such that

$$||\phi(t_i)|| \leqslant \rho^i ||\phi(0)|| \tag{2.10}$$

and $\{\delta_i = t_{i+1} - t_i\}$ is a bounded sequence.

**Proof:** By lemma A3, it is enough to show that $\{\delta_i\}$ is an bounded sequence. Suppose, for the sake of contradiction that $\{\delta_i\}$ is an unbounded sequence, then one of the two following possibilities occurs:

(i) There exist $i < \infty$ such that $\delta_i = \infty$, or

(ii) $\{\delta_i\} \to \infty$ as $i \to \infty$.

Consider the scenario of (i) first. If (i) happens, then the system becomes time invariant after time $t_i$, since the controller is not updated. Consequently one can get the transfer function (not necessarily stable) from r to u to be

$$H_{ur} = \frac{\hat{d}_c(t_i) n_p}{n_p \hat{n}_c(t_i) + d_p d_c(t_i)} = n / d \tag{2.12}$$

where $\hat{d}_c(t_i)$ and $\hat{n}_c(t_i)$ are numerator and denominator of controller at time $t_i$. Using (2.12), we may write the transfer function from r to w to be

$$H_{wr}(s) = \frac{n}{(s+\alpha)^n d_p d}(s^{n-1} n_p, \ldots, n_p, s^{n-1} d^p, \ldots, d_p)^T$$

Since the degree of $n$ is (n+m), so that roughly speaking, no more than (n+m) of the spectral lines of the input can correspond to zeros of the numerator polynomial. Assuming that (n+m) of the special lines do, in fact, coincide with the zeros of n, we can see that under the assumption of $n_p$, $d_p$ being coprime, w is persistently exciting. The proof of this for the stable case was given by Boyd and Sastry [3]. For the unstable case, the idea is that we have a minimal state space realization of $H_{wr}(s)$ as

$$\dot{x} = Ax + br$$

$$w = cx$$

where $A \in R^{k \times k}$ ($k \leq 3n + m$). Then, the persistency of excitation of x(t) follows from the hypothesis of the input r(t) and the fact that (A,b) is controllable (see, Nordstrom [10]). Further, notice that the rows of $H_{wr}(s)$ are linearly independent and

$$H_{wr}(s) = c(sI - A)^{-1} b$$

we see that c has full row rank i.e.

$$cc^T \geq \alpha I \qquad \text{for } some\ \alpha > 0$$

Thus,

$$\int ww^T dt = c \int xx^T c^T dt \geq \gamma cc^T \geq \gamma \alpha I$$

where

$$\int xx^T dt \geq \gamma I$$

This implies that w(t) is persistently exciting. This fact however contradicts the assumption that $\delta_i = \infty$.

Now consider scenario (ii). First notice that when the plant parameters are known, then the closed loop system is time invariant and stable, so that we may write the following equation relating input r(t) to signal $w_0(t)$ ($w_0(t)$ means w(t) in the case of $\phi(t) \equiv 0$).

$$\dot{z}_0 = Az_0 + br$$

$$w_0 = cz_0$$

where A is a constant stable matrix. For the adaptive control situation, the plant parameters are unknown, i.e. parameter error $\phi(t) \neq 0$. However, we may write the following equation relating r(t) to w(t)

$$\dot{z}(t) = (A + \Delta A(t))z(t) + (b + \Delta b(t))r$$

$$w = (c + \Delta c(t))z$$

where $\Delta A(t)$, $\Delta c(t)$ and $\Delta b(t)$ are continuous functions of $\phi(t)$ and $\Delta A(t)$, $\Delta b(t)$ and $\Delta c(t) \to 0$ as $\phi(t) \to 0$. Now if scenario (ii) happens, we still have that $\phi(t) \to 0$ as $i \to \infty$ from lemma A3. It follows from lemma A1 that $w_0(t)$ and $w(t)$ are arbitrarily close when t is large enough. Then the persistency of excitation of w(t) follows as a consequence of the result of lemma A2 and the fact the $w_0(t)$ is persistently exciting. This however contradicts the assumption that $\delta_i \to \infty$ as $i \to \infty$.

We are now in a position to prove the following theorem.

**Theorem 2.1 (Stability of the Closed Loop System)**

Consider the system of Fig. 2-1. Assume that the plant and compensator are described as in lemma 2.2. Suppose that input r(t) is bounded and stationary and the support of its spectrum has greater than 3n+m points, then the overall system is asymptotically time invariant and stable.

**Proof:** Follows from lemma 2.1 and lemma 2.2.

## 3 Adaptive Pole Placement

In this section, we consider an indirect, adaptive pole placement scheme. Pole placement is easily described in the context of the Fig.2-1. Given a plant $P_0$ of the form $n_p / d_p$ as in (2.2), find a compensator C so that the closed loop poles lie at the zeros of a given characteristic polynomial $d^*(s)$ of order (2n-1), i.e. find $n_c$, $d_c$ to satisfy

$$n_c n_p + d_c d_p = d^* \tag{3.1}$$

Where the plant $P_0$ is unknown, the 'adaptive' pole placement scheme is mechanised by using the estimates $\hat{n}_p(t_i)$ and $\hat{d}_p(t_i)$ of the numerator and denominator polynomials respectively. It is easy to verify (see lemma A4) that if $\hat{n}_p(t_i)$ and $\hat{d}_p(t_i)$ are coprime then there exist $\hat{n}_c(t_i)$ and $\hat{d}_c(t_i)$ of the order n-1 such that

$$\hat{n}_c(t_i)\hat{n}_p(t_i)+\hat{d}_c(t_i)\hat{d}_p(t_i)=d^* \tag{3.2}$$

The estimates for $\hat{n}_p(t_i)$ and $\hat{d}_p(t_i)$ follow from the plant parameter estimates $\hat{\theta}(t)$ of section 2 (the estimates of the coefficients of the denominator followed by those of the numerator). In analogy to the plant parameter vector $\theta^*$, we have the parameter vector of the compensator

$$\theta_c = (b_0, \ldots, b_{n-1}, a_0, \ldots, a_{n-1}) \tag{3.3}$$

Recall from equation (2.2) that the compensator is given by

$$C = \frac{a_0 s^{n-1} + \ldots + a_{n-1}}{b_0 s^{n-1} + \ldots + b_{n-1}}$$

Further, to guarantee that $\hat{n}_p(t_i)$ and $\hat{d}_p(t_i)$ are coprime at $t_i$, we need to modify the definition of the update times as follows:

$$t_{i+1}=t_i+\tau \tag{3.4}$$

where $\tau$ is the smallest real number satisfying

$$(i) \quad \int_{t_i}^{t_i+\tau} w(t)w^T(t)dt \geq \gamma I \tag{3.5}$$

and

$$(ii) \quad \hat{n}_p(t_i+\tau) \text{ and } \hat{d}_p(t_i+\tau) \text{ are coprime.} \tag{3.6}$$

More precisely so that the smallest singular value of the matrix (A.11) (see appendix) measuring the extent of coprimeness exceeds a number $\sigma > 0$. Then, we have

**Theorem 3.1 (Convergence of the Pole Placement Scheme)**

Consider the adaptive pole placement law (3.2) applied to the system of (2.2), along with the least squares identifier of (2.7) and the update sequence $t_i$ defined by (3.4-3.6). Now, if the input r(t) is stationary with spectral support not concentrated on less than 4n-1 points. then all signals in the loop are bounded and the characteristic polynomial of the closed loop system tends to $d^*(s)$. Moreover

$$||\hat{\theta}_c(t_i)-\theta_c|| \rightarrow 0 \quad \textit{exponentially}$$

**Proof:** The first half of the theorem is a direct consequence of lemmas 2.1 and 2.2. For the second half, note from (A.11) that

$$A(\hat{\theta}(t_i))\hat{\theta}_c(t_i)=d_* \tag{3.7}$$

with $d_*$ the vector of coefficients of $d^*$.

It is easy to see from (A.11) that there is an $M_1>0$ such that

$$||A(\hat{\theta}(t_i))-A(\theta^*)|| \leq M_1||\hat{\theta}(t_i)-\theta^*|| \tag{3.8}$$

Now,

$$A(\theta^*)\theta_c=d_* \tag{3.9}$$

Subtracting (3.9) from (3.7) we get

$$-(A(\hat{\theta}(t_i))-A(\theta^*))\hat{\theta}_c(t_i)=A(\theta^*)(\hat{\theta}_c(t_i)-\theta_c)$$

Using the estimate

$$||\hat{\theta}_c(t_i)-\theta_c|| \leqslant ||A^{-1}(\hat{\theta}(t_i))|| \, ||A(\hat{\theta}(t_i))-A(\theta^*)|| \, ||\hat{\theta}_c(t_i)||$$

Noting that $\hat{\theta}_c(t_i)$ is bounded (see equation (3.6) and the remark following it), we get

$$||\hat{\theta}_c(t_i)-\theta_c|| \leqslant M_2 ||\hat{\theta}(t_i)-\theta^*|| \quad \text{for } some \ M_2>0 \tag{3.10}$$

Since $\hat{\theta}(t_i)$ converges to $\theta^*$ exponentially, it follows that $\hat{\theta}_c(t_i) \to \theta_c$ exponentially.

# 4 Adaptive Stabilization Using The Factorization Approach to Controller Design

## 4.1 The Factorization Approach to Controller Design--the Non-adaptive Version

We consider the linear time-invariant system shown in Fig 4-1

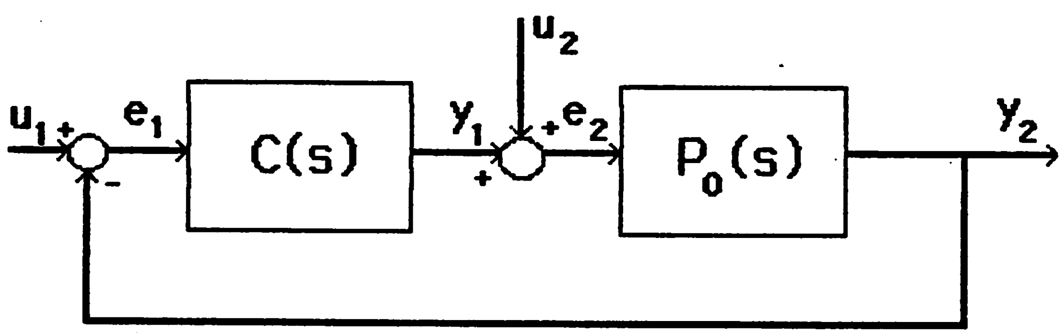


Fig. 4-1

The plant $P_0(s)$ is defined as in equation (2.1) and the compensator C(s) as in (2.2) with m=n. The equations relating $e_1, e_2$ to $u_1, u_2$ are

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \frac{1}{1+P_0C} \begin{bmatrix} 1 & -P_0C \\ C & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{4.1}$$

The system (4.1) is BIBO stable if and only if each of the four elements in (4.1) is stable, i.e. belongs to **R** the ring of proper, stable rational functions. The ring **R** is a more convenient ring than the ring of polynomials for the study of robust control systems. since a plant with unstable unmodeled dynamics is really no different from a plant with stable unmodeled dynamics. Thus, we assume that $P_0$ and $C$ are factored coprimely in **R** (not uniquely!).

$$P_0(s)=d_p^{-1}(s)n_p(s)$$

$$C_0(s)=d_c^{-1}(s)n_c(s) \tag{4.2}$$

From (4.1) it follows that (for details see [12] ) the system of Figure 4.1 is BIBO stable if and only if $(n_p n_c+d_p d_c)^{-1}\in$ **R**, or equivalently $n_p n_c+d_p d_c$ is a unimodular element of the ring **R**. Without loss of generality, then, we can state that a compensator stabilizes the system of Figure 4.1 if and only if

$$n_p n_c+d_p d_c=1 \tag{4.3}$$

Equation 4.3 parametrizes all stabilizing compensators. Let $(A, B, C)$ be a controllable canonical realization of $P_o$ i.e

$$P_0(s)=c(sI-A)^{-1}b \tag{4.4}$$

with

$$A=\begin{bmatrix} 0 & 1 & & 0 \\ 0 & 0 & & . \\ . & . & & . \\ . & . & & 1 \\ -\alpha_1 & -\alpha_2 & & -\alpha_n \end{bmatrix} \quad b=\begin{bmatrix} 0 \\ . \\ . \\ 0 \\ 1 \end{bmatrix} \tag{4.5}$$

$$c=(\beta_1, \ldots, \beta_m)$$

If $f^T\in R^n$ and $l\in R^n$ are chosen so that $A_f=A-bf$ and $A-lc=A_l$ are stable( such a choice is possible by the minimality of the realization of (4.4) and (4.5) ), then it is shown [12, pg.83] that all the solutions of ( 4.3 ) can be written in the form

$$n_p=c(sI-A_l)^{-1}b \tag{4.6}$$

$$d_p=1-c(sI-A_l)^{-1}l \tag{4.7}$$

$$d_c=1+c(sI-A_f)^{-1}l-q(s)c(sI-A_f)^{-1}b \tag{4.8}$$

$$n_c=f(sI-A_f)^{-1}l+q(s)(1-f(sI-A_f)^{-1}b) \tag{4.9}$$

with $q(s)$ an arbitrary element of **R** which is chosen to meet other performance criteria (for instance, minimization of the disturbance to output map , obtaining the desired closed loop transfer function, optimal desensitization to unmodeled dynamics, etc...).

The optimal choice of $q(s)$ depends on the plant parameters. However, such a choice of q(s) may not be unique or depend continuously on plant parameters. This may give rise to difficulties in applying the method discussed in section 2, since the design of the compensator may not be unique as required by the assumptions of the scheme. We defer this to further investigation. However, if our only concern is the problem of adaptive stabilization of the unknown plant, then any fixed $q(s) \in \mathbf{R}$ will do. For simplicity, we fix $q(s)=0$ in what follows.

**4.2 Adaptive Stabilization Using Factorization Approach**

Given a plant $P_0$ with unknown parameters as described in equation (2.1), and a feedback controller configuration as shown in Fig. 4.1. The objective is to design a compensator C adaptively, i.e. based on the estimate $\hat{\theta}$ of plant parameters, using the factorization approach, so that the closed loop system is asymptotically stable with all signals are bounded. In what follows, we assume that $u_2(t) \equiv 0$.

The identifier and compensator update time $\{t_i\}$ are defined as in (2.7 ) and (3.4-3.6) respectively. The first difficulty in choosing the compensator is the choice of $l(t_i)$ and $f(t_i)$ at time $t_i$ ( see equations (4.8) and (4.9) ). From linear system theory, we have that for the controllable canonical realization of plant $P_0(s)$.

$$\dot{x} = Ax + bu$$

$$y = cx$$

there is a nonsingular matrix $M^{-1}$, such that by the coordinate change $\bar{x} = Mx$ , we get the observable canonical form of $P_0(s)$. i.e

$$\dot{\bar{x}} = MAM^{-1}\bar{x} + Mbu = \bar{A}\bar{x} + \bar{b}u$$

$$y = cM^{-1}\bar{x} = \overline{c}\overline{x} \qquad (4.10)$$

with

$$\bar{A}=\begin{bmatrix} 0 & & 0 & -\alpha_1 \\ 1 & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & 0 & \cdot \\ 0 & & 1 & -\alpha_n \end{bmatrix} \quad \bar{b}=\begin{bmatrix} \beta_1 \\ \cdot \\ \cdot \\ \cdot \\ \beta_n \end{bmatrix}$$

$$\bar{c}=(0,\ldots,0,1)$$

Then for (any) given Hurwitz polynomial

$$p(s)=s^n+p_1 s^{n-1}+\cdots+p_n \qquad (4.11)$$

there exists a vector

$$\bar{l}^T=(\bar{l}_1,\ldots,\bar{l}_n)=(p_1,\ldots,p_n)-(\alpha_1,\ldots,\alpha_n)$$

Such that the matrix

$$\bar{A}-\bar{l}\bar{c}=\begin{bmatrix} 0 & & 0 & -p_1 \\ 1 & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ 0 & & 1 & -p_n \end{bmatrix}$$

is stable and has a characteristic polynomial p(s). Define

$$l=M^{-1}\bar{l}$$

With this definition, it is easy to see that (A-lc) is stable and has characteristic polynomial p(s).

Now the controller design procedure can be stated as follow:

(Step1)

At time $t_i$, the parameter estimate $\hat{\theta}(t_i)$ generated by identifier is used to obtain the estimates $A(t_i)$,$b(t_i)$, and $c(t_i)$.

(Step2)

By calculation, we obtain $M^{-1}(t_i)$ as described in (4.10). Define

$$f(t_i)=\bar{l}^T(t_i)=(p_1,\ldots,p_n)-(\alpha_1(t_i),\ldots,\alpha_n(t_i)) \qquad (4.12)$$

with $(p_1,\ldots,p_n)$ as defined in (4.11) and

$$l(t_i)=M^{-1}(t_i)\bar{l}(t_i) \qquad (4.13)$$

We see now that the matrices

$$A_l(t_i)=A(t_i)-l(t_i)c(t_i)$$

and

$$A_f(t_i)=A(t_i)-b(t_i)f(t_i)$$

are stable with characteristic polynomial p(s). Furthermore, $f(t_i)$ and $l(t_i)$ converge to some constant vectors as $i \to \infty$.

(Step3)

Choose the compensator $C(t_i)=n_c(t_i)d_c^{-1}(t_i)$ as follows

$$n_c(t_i)=f(t_i)(sI-A_f(t_i))^{-1}l(t_i) \tag{4.14}$$

$$d_c(t_i)=1+c(t_i)(sI-A_f(t_i))^{-1}l(t_i) \tag{4.15}$$

This compensator can be easily implemented. Then, as expected, we have

**Theorem 4.1 (Convergence of the Overall System)**

Assume that the identifier and controller update described above are applied to the plant $P_0(s)$. Suppose that the input r(t) is stationary and bounded and that the spectral support of r(t) is not concentrated on $k \leqslant 4n$ points. Then, the closed loop system is asymptotically stable and all signals are bounded.

## 5 Concluding Remarks

This paper has presented a proof of global stability for indirect adaptive control. In the paper, only two applications (pole placement and factorization approach ) have been discussed, however the results are applicable to several kinds of controller design methodologies. The key assumption is a richness condition on the reference input. To our knowledge, this is the first verification of the persistency of excitation of the regressor signal in the closed loop (which is time varying) without using artificial random sampling signal (see [4]) for the continuous time case. We show persistence of excitation without preassuming the boundedness of the signals. Boundedness of all signals and the convergence of the compensator in turn follow from the convergence of the identifier, which is a direct consequence of the persistency of excitation of the signal in identification loop.

The scheme presented here offers a great deal of flexibility in the controller design and allows for very general richness conditions on the reference input. The results of this paper are easily extended to the discrete time case.

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## 7 Appendix

In this appendix, we prove some lemmas of use in the main body of the paper.

### Lemma A1

Consider the following linear systems

$$\dot{z}_0 = Az_0 + br \qquad \text{(A1)}$$

$$\dot{z} = (A + \Delta A(t))z + (b + \Delta b(t))r \qquad \text{(A2)}$$

with A stable and $\Delta A$ , $\Delta b$ both bounded and converging to zero as $t \to \infty$. Assume that the input r(t) is bounded. Then given $\epsilon > 0$, there exists k > 0 (k is independent of the choice $\epsilon$) and a $T(\epsilon)$ such that

$$||z(t) - z_0(t)|| \leqslant \epsilon k \quad \text{for } all\ t \geqslant T \qquad \text{(A3)}$$

**Proof:** From lemma 2.1, it follows that (A2) is asymptotically stable and that there exists $T_1$ such that the state transition matrix of (A2) satisfies

$$||\Phi(t,\tau)|| \leqslant M \exp - \lambda(t - \tau)$$

for some $M, \lambda > 0$ and for all $t > \tau > T_1$. Using this estimate it is easy to show that z(t) is bounded. Defining the error $e(t) := z(t) - z_0(t)$ we see that

$$\dot{e} = Ae + \Delta Az + \Delta br$$

Using the facts that $\Delta A$ , $\Delta b \to 0$ as $t \to \infty$; that z,r are bounded and A is stable, it is easy to establish (A3).

### Lemma A2

Suppose that $w_0(t) \in R^n$ is persistently exciting,i.e there exist $\delta, \alpha > 0$ such that

$$\int_s^{s+\delta} w_0 w_0^T \, dt \geqslant \alpha I$$

Then any signal $w \in R^n$ satisfying

$$||w(t) - w_0(t)|| < \alpha / (\delta)^{1/2}$$

is also persistently exciting.

**Proof:** Can be found in [2].

**Lemma A3**

Consider the least squares identification algorithm described by (2.7) with resetting sequence $\{0, t_1, t_2, \ldots\}$, that is

$$\dot{\phi} = -Pww^T\phi \tag{A4}$$

and

$$\dot{P}^{-1} = ww^T \qquad t \neq 0, t_1, t_2, \ldots \tag{A5}$$

$$P^{-1}(t_i^+) = \alpha I \qquad t = 0, t_1, t_2, \ldots \tag{A6}$$

If w is persistently exciting, that is

$$\int_{t_i}^{t_{i+1}} ww^T dt \geqslant \gamma I \quad \text{for } all \ t_i \tag{A7}$$

Then, there exist $1 > \rho > 0$ such that

$$||\phi(t_i)|| \leqslant \rho^i ||\phi(0)|| \tag{A8}$$

**Proof:** Note that for $t \neq \{0, t_1, \ldots\}$,

$$\frac{d}{dt} P^{-1}\phi = 0$$

Thus

$$P^{-1}(t_i^-)\phi(t_i) = P^{-1}(t_{i-1}^+)\phi(t_{i-1})$$

so that

$$\phi(t_i) = \alpha P(t_i^-)\phi(t_{i-1})$$

and we get

$$||\phi(t_i)|| \leqslant \left(\frac{\alpha}{\alpha+\gamma}\right) ||\phi(t_{i-1})|| \tag{A9}$$

In last step we use equation (A7). Recursion on (A9) yields the conclusion (A8).

**Lemma A4**

Consider two coprime polynomials $d_p$ monic of order n and $n_p$ of order n-1. Then given an arbitrary polynomial $d^*$ of order 2n-1, there exist unique polynomials $n_c$ and $d_c$ of order n-1 so that

$$n_c n_p + d_c d_p = d^* \tag{A10}$$

**Proof:** Is a standard result from algebra (see [5] ). It is useful for the proof of theorem 3.1 to note that if

$$d_p = s^n + \beta_1 s^{n-1} + \ldots + \beta_n$$

$$n_p = \alpha_1 s^{n-1} + \ldots + \alpha_n$$

$$n_c = a_1 s^{n-1} + \ldots + a_n$$

$$d_c = b_1 s^{n-1} + \ldots + b_n$$

Then, the linear equation relating the coefficients of $n_c$ , $d_c$ to those of $d^*$ is

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \beta_1 & 1 & 0 & \cdots & \cdot & \alpha_1 & 0 & \cdot \\ \beta_2 & \beta_1 & 1 & \cdots & \cdot & \alpha_2 & \alpha_1 & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & 1 & \cdot & \cdot & \cdot \\ \beta_n & \beta_{n-1} & \cdot & \cdots & \beta_1 & \alpha_n & \alpha_{n-1} & \alpha_1 \\ 0 & \beta_n & \cdot & \cdots & \cdot & 0 & \alpha_n & \cdot \\ 0 & 0 & \beta_n & \cdots & \cdot & 0 & 0 & \cdot \\ \cdot & \cdot & 0 & \cdots & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \beta_n & 0 & 0 & \alpha_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_n \\ a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_n \end{bmatrix} = \begin{bmatrix} d_1^* \\ d_2^* \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ d_{2n}^* \end{bmatrix} \tag{A11}$$

i.e.

$$A(\theta^*)\theta_c = d^*$$