SLOW DRIFT INSTABILITY IN MODEL REFERENCE ADAPTIVE SYSTEM--AN AVERAGING ANALYSIS

by

Li-Chen Fu and Shankar Sastry

Memorandum No. UCB/ERL M86/25

20 March 1986

1. Introduction:

The primary purpose of this paper is to propose mathematically rigorous tools for analyzing the slow-drift instability in Model Reference Adaptive Control systems using averaging analysis. There has been a great deal of excitement in the adaptive community ever since Rohrs et al [18] (conference versions of this work have appeared since 1981) announced the extreme sensitivity of stability proofs for Model Reference Adaptive Control schemes to the following assumptions:

(i) Plant order known.

(ii) Relative degree of the plant known.

While their work was extremely stimulating to the field, their analysis of the instability mechanism based either on an "Infinite-Gain" operator or on a "Linearized-Root Locus type" analysis was not sufficiently rigorous or insightful. The first breakthrough in this regard came in the work of Krause et al [9] (in very preliminary fashion) and Astrom [1] (a conference version of his work appeared in 1984), who explained that the cause of instability was a lack of sufficiently rich input signals to (a) allow for parameter convergence of the 'nominal' system; and (b) prevent drift of the parameters from a neighborhood of the 'true' values due to unmodeled dynamics/output disturbance. In particular, Astrom introduced techniques of averaging to the study of the evolution of the adaptive systems, by slowing down the parameter update law (to a time scale slower than the plant dynamics). The following work was a outgrowth of the results of: (1) Fu et al [5], Bodson et al [2] and Kosut [11], used averaging to get estimates of parameter convergence for the nominal system; and (2) Kosut et al [12], Riedle and Kokotovic [16], studied instability and stability boundaries for the disturbance free adaptive system. Their work led to the conclusion that reference input signal should have energy concentrated in a frequency range where the closed loop transfer function $H_{\theta_0}(p,s)$ (will be defined in the sequel) is like SPR so as to assure stability.

In this paper, we take the analysis of [16] one step further. We develop general stability theorems for averaging in one and two-time scale system. We use them to give a slightly more general instability theorem for (nonlinear) averaging which take into account output disturbance terms. Moreover, we discuss the concept of "tuned plant parameter values" and characterize it in terms of the frequency content of the reference input. Finally, we apply the results in detail to several specific cases of stability/instability presented by Rohrs and others in simulation form. We would like to emphasize that we view our contribution as largely tutorial and expository along with

SLOW DRIFT INSTABILITY IN MODEL REFERENCE ADAPTIVE

SYSTEM--AN AVERAGING ANALYSIS

by

Li-Chen Fu and Shankar Sastry

Memorandum No. UCB/ERL M86/25

20 March 1986

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

1. Introduction:

The primary purpose of this paper is to propose mathematically rigorous tools for analyzing the slow-drift instability in Model Reference Adaptive Control systems using averaging analysis. There has been a great deal of excitement in the adaptive community ever since Rohrs et al [18] (conference versions of this work have appeared since 1981) announced the extreme sensitivity of stability proofs for Model Reference Adaptive Control schemes to the following assumptions:

(i) Plant order known.

(ii) Relative degree of the plant known.

While their work was extremely stimulating to the field, their analysis of the instability mechanism based either on an "Infinite-Gain" operator or on a "Linearized-Root Locus type" analysis was not sufficiently rigorous or insightful. The first breakthrough in this regard came in the work of Krause et al [9] (in very preliminary fashion) and Astrom [1] (a conference version of his work appeared in 1984), who explained that the cause of instability was a lack of sufficiently rich input signals to (a) allow for parameter convergence of the 'nominal' system; and (b) prevent drift of the parameters from a neighborhood of the 'true' values due to unmodeled dynamics/output disturbance. In particular, Astrom introduced techniques of averaging to the study of the evolution of the adaptive systems, by slowing down the parameter update law (to a time scale slower than the plant dynamics). The following work was a outgrowth of the results of: (1) Fu et al [5], Bodson et al [2] and Kosut [11], used averaging to get estimates of parameter convergence for the nominal system; and (2) Kosut et al [12], Riedle and Kokotovic [16], studied instability and stability boundaries for the disturbance free adaptive system. Their work led to the conclusion that reference input signal should have energy concentrated in a frequency range where the closed loop transfer function $H_{\theta_0}(p, s)$ (will be defined in the sequel) is like SPR so as to assure stability.

In this paper, we take the analysis of [16] one step further. We develop general stability theorems for averaging in one and two-time scale system. We use them to give a slightly more general instability theorem for (nonlinear) averaging which take into account output disturbance terms. Moreover, we discuss the concept of "tuned plant parameter values" and characterize it in terms of the frequency content of the reference input. Finally, we apply the results in detail to several specific cases of stability/instability presented by Rohrs and others in simulation form. We would like to emphasize that we view our contribution as largely tutorial and expository along with

SLOW DRIFT INSTABILITY IN MODEL REFERENCE ADAPTIVE SYSTEM--AN AVERAGING ANALYSIS

by

Li-Chen Fu and Shankar Sastry

Memorandum No. UCB/ERL M86/25

20 March 1986

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

1. Introduction:

The primary purpose of this paper is to propose mathematically rigorous tools for analyzing the slow-drift instability in Model Reference Adaptive Control systems using averaging analysis. There has been a great deal of excitement in the adaptive community ever since Rohrs et al [18] (conference versions of this work have appeared since 1981) announced the extreme sensitivity of stability proofs for Model Reference Adaptive Control schemes to the following assumptions:

(i) Plant order known.

(ii) Relative degree of the plant known.

While their work was extremely stimulating to the field, their analysis of the instability mechanism based either on an "Infinite-Gain" operator or on a "Linearized-Root Locus type" analysis was not sufficiently rigorous or insightful. The first breakthrough in this regard came in the work of Krause et al [9] (in very preliminary fashion) and Astrom [1] (a conference version of his work appeared in 1984), who explained that the cause of instability was a lack of sufficiently rich input signals to (a) allow for parameter convergence of the 'nominal' system; and (b) prevent drift of the parameters from a neighborhood of the 'true' values due to unmodeled dynamics/output disturbance. In particular, Astrom introduced techniques of averaging to the study of the evolution of the adaptive systems, by slowing down the parameter update law (to a time scale slower than the plant dynamics). The following work was a outgrowth of the results of: (1) Fu et al [5], Bodson et al [2] and Kosut [11], used averaging to get estimates of parameter convergence for the nominal system; and (2) Kosut et al [12], Riedle and Kokotovic [16], studied instability and stability boundaries for the disturbance free adaptive system. Their work led to the conclusion that reference input signal should have energy concentrated in a frequency range where the closed loop transfer function $H_{\theta_0}(p,s)$ (will be defined in the sequel) is like SPR so as to assure stability.

In this paper, we take the analysis of [16] one step further. We develop general stability theorems for averaging in one and two-time scale system. We use them to give a slightly more general instability theorem for (nonlinear) averaging which take into account output disturbance terms. Moreover, we discuss the concept of "tuned plant parameter values" and characterize it in terms of the frequency content of the reference input. Finally, we apply the results in detail to several specific cases of stability/instability presented by Rohrs and others in simulation form. We would like to emphasize that we view our contribution as largely tutorial and expository along with

Slow Drift Instability in Model Reference Adaptive System — an Averaging Analysis

Li-Chen Fu and Shankar Sastry

Electronics Research Laboratory
Department of Electrical Engineering & Computer Science
University of California
Berkeley, CA 94720, U.S.A.

ABSTRACT

The paper presents instability theorems for one and two-time scale time-varying nonlinear systems using averaging theory. These theorems are then applied to the Model Reference Adaptive Control system with unmodelled dynamics and output disturbance to analyze the mechanism of slow-drift instability.

March 19, 1986

Keywords: Adaptive Control, Model Reference, Averaging, Instability Analysis, Tuned Parameter, Slow-Drift Instability, Robustness

Research supported by Army Research Office Grant DAAG-29-85-K-0072

1. Introduction:

The primary purpose of this paper is to propose mathematically rigorous tools for analyzing the slow-drift instability in Model Reference Adaptive Control systems using averaging analysis. There has been a great deal of excitement in the adaptive community ever since Rohrs et al [18] (conference versions of this work have appeared since 1981) announced the extreme sensitivity of stability proofs for Model Reference Adaptive Control schemes to the following assumptions:

(i) Plant order known.

(ii) Relative degree of the plant known.

While their work was extremely stimulating to the field, their analysis of the instability mechanism based either on an "Infinite-Gain" operator or on a "Linearized-Root Locus type" analysis was not sufficiently rigorous or insightful. The first breakthrough in this regard came in the work of Krause et al [9] (in very preliminary fashion) and Astrom [1] (a conference version of his work appeared in 1984), who explained that the cause of instability was a lack of sufficiently rich input signals to (a) allow for parameter convergence of the 'nominal' system; and (b) prevent drift of the parameters from a neighborhood of the 'true' values due to unmodeled dynamics/output disturbance. In particular, Astrom introduced techniques of averaging to the study of the evolution of the adaptive systems, by slowing down the parameter update law (to a time scale slower than the plant dynamics). The following work was a outgrowth of the results of: (1) Fu et al [5], Bodson et al [2] and Kosut [11], used averaging to get estimates of parameter convergence for the nominal system; and (2) Kosut et al [12], Riedle and Kokotovic [16], studied instability and stability boundaries for the disturbance free adaptive system. Their work led to the conclusion that reference input signal should have energy concentrated in a frequency range where the closed loop transfer function $H_{\theta_0}(p,s)$ (will be defined in the sequel) is like SPR so as to assure stability.

In this paper, we take the analysis of [16] one step further. We develop general stability theorems for averaging in one and two-time scale system. We use them to give a slightly more general instability theorem for (nonlinear) averaging which take into account output disturbance terms. Moreover, we discuss the concept of "tuned plant parameter values" and characterize it in terms of the frequency content of the reference input. Finally, we apply the results in detail to several specific cases of stability/instability presented by Rohrs and others in simulation form. We would like to emphasize that we view our contribution as largely tutorial and expository along with

some new results.

The paper is organized in the following way: in sec2 and sec3, nonlinear version instability theorems for one-time scale and two-time scale systems respectively are proposed; in sec4, conditions for instability of the Model Reference Adaptive System is derived; in sec5, robustness issues is discussed.

2. Instability Theorem For One Time Scale Systems

In this section, we consider differential equations of the form:

$$\dot{x} = \epsilon f(t, x, \epsilon) \qquad x(0) = x_0 \tag{2.1}$$

where $x \in R^n$, $t \geq 0$, $0 \leq \epsilon \leq \epsilon_0$ and f piecewise continuous with respect to time. For small ϵ, the variation of x with time is slow, as compared to the rate of time variation of f. The following definition will be useful in the sequel:

Definition: Average value of a function, Convergence function.

The function $f(t, x, 0)$ is said to have average value $f_{av}(x)$ if there exists a continuous, strictly decreasing function $\gamma(T): R_+ \to R_+$ such that $\gamma(T) \to 0$ as $T \to \infty$ and

$$\left|\left| \frac{1}{T} \int_t^{t+T} f(\tau, x, 0)\, d\tau - f_{av}(x) \right|\right| \leq \gamma(T) \tag{2.2}$$

for all $t, T \geq 0$, $x \in B_h$.

The function $\gamma(T)$ is called the convergence function, and the system

$$\dot{x}_{av} = \epsilon f_{av}(x_{av}) \qquad x_{av}(0) = x_0 \tag{2.3}$$

is called the averaged system corresponding to (2.1).

Now we will make the following assumptions: let B_h be a closed ball of radius h in R^n.

(A1) $x = 0$ is an equilibrium point of system (2.1), i.e. $f(t, 0, 0) = 0$ for all $t \geq 0$. $f(t, x, \epsilon)$ is Lipschitz in x, i.e.

$$|| f(t, x_1, \epsilon) - f(t, x_2, \epsilon) || \leq l_1 || x_1 - x_2 || \tag{2.4}$$

for all $t \geq 0$, $x_1, x_2 \in B_h$ and $\epsilon \leq \epsilon_0$.

(A2) $f(t, x, \epsilon)$ is Lipschitz in ϵ, linearly in x, i.e.

$$|| f(t, x, \epsilon_1) - f(t, x, \epsilon_2) || \leq l_2 || x || \, | \epsilon_1 - \epsilon_2 | \tag{2.5}$$

for all $t \geq 0$, $x \in B_h$ and $\epsilon_1, \epsilon_2 \leq \epsilon_0$.

(A3) $f_{av}(0) = 0$ and $f_{av}(x)$ is Lipschitz in x, i.e.

$$|| f_{av}(x_1) - f_{av}(x_2) || \leq l_{av} || x_1 - x_2 || \tag{2.6}$$

for all $x_1, x_2 \in B_h$.

(A4) Let $d(t, x) = f(t, x, 0) - f_{av}(x)$, so that $d(t, x)$ has zero average value. Assume that the convergence function can be written as $\gamma(T) || x ||$, and that $\dfrac{\partial d(t, x)}{\partial x}$ has zero average value with convergence function $\gamma(T)$.

(A5) For some $h' \leqslant h$, $||x_{av}(t)|| \leqslant h'$ on the time interval considered.

Theorem: (Instability of an Unaveraged One-Time Scale System)

If: the original system (2.1) and the averaged system (2.3) satisfy assumptions (A1)-(A5) along with the additional assumption that there exists a continuously differentiable decrescent function $v(t,x)$ such that

(i) $v(t,0) = 0$.

(ii) $v(t,x) > 0$ for some x arbitrarily close to the origin.

(iii) $|| \dfrac{\partial v(t,x)}{\partial x} || \leqslant k_1 ||x||$ for some $k_1 > 0$.

(iv) the derivative of $v(t,x)$ along the trajectory (2.3) satisfies

$$\dot{v}_{(2.3)}(t,x) \geqslant \epsilon k_2 ||x||^2 \tag{2.7}$$

for some $k_2 > 0$.

Then: the unaveraged system (2.1) is unstable provided $\epsilon \leqslant \epsilon_0$ for some $\epsilon_0 > 0$.

Proof: It was shown in (F.B.S. [5]) that, under assumptions (A1)-(A5), there exists a change of coordinates

$$x = z + \epsilon \omega_\epsilon(t,z) \tag{2.8}$$

such that

$$||\epsilon \omega_\epsilon(t,z)|| \leqslant \xi(\epsilon) ||z|| \quad and \quad ||\epsilon \frac{\partial \omega_\epsilon(t,z)}{\partial z}|| \leqslant \xi(\epsilon) \tag{2.9}$$

for some $\xi(\epsilon) \in$ class K. Under this change of coordinates, it was shown that the system (2.1) becomes

$$\dot{z} = \epsilon f_{av}(z) + \epsilon p(t,z,\epsilon) \tag{2.10}$$

where $p(t,z,\epsilon)$ satisfies

$$|| p(t,z,\epsilon) || \leqslant \psi(\epsilon) ||z|| \tag{2.11}$$

for some $\psi(\epsilon) \in$ class k, $\epsilon_1 > 0$ and for all $\epsilon \leqslant \epsilon_1$.

By an Instability Theorem of Lyapunov [20], the additional assumptions (i)-(iv) guarantees that the averaged system is unstable, i.e. there exists an initial condition close to the origin such that the state vector, starting from it, will be expelled from a nbhd of the origin. The function v is now used to study the instability of the perturbed system (2.10). The derivative of $v(t,z)$ along the trajectory (2.10) is given by

$$\dot{v}_{(2.10)}(t,z) = \dot{v}_{(2.3)}(t,z) + \frac{\partial v}{\partial z}(\epsilon p(t,z,\epsilon)) \tag{2.12}$$

and using inequalities (2.7), (2.11)

$$\dot{v}_{(2.10)}(t, z) \geqslant \epsilon k_2 \,||z||^2 - \epsilon\psi(\epsilon) k_1 \,||z||^2$$
$$:= \epsilon\alpha(\epsilon)\,||z||^2 \tag{2.13}$$

where $\alpha(\epsilon) = k_2 - \psi(\epsilon) k_1$. Let $\epsilon_0 \leqslant \epsilon_1$ be such that $\alpha(\epsilon) > 0$ for all $\epsilon \leqslant \epsilon_0$, then, again by Lyapunov Instability Theorem, the system (2.10), and hence the unaveraged system, are unstable.

Q.E.D.

Comment: The continuously differentiable, decrescent function v that the *Theorem* requires can be found, for example, if the averaged system (2.3) has the form

$$\dot{x}_{av} = \epsilon A x_{av} \tag{2.14}$$

where $A \in R^{n \times n}$, $\sigma(A) \bigcap \mathbb{C}_+^o \neq \emptyset$ and for all $\lambda \in \sigma(A)$, $Re(\lambda) \neq 0$. In this case, the function v can be chosen as

$$v(x) = x^T P x \tag{2.15}$$

where P, Q satisfy the Lyapunov equation

$$A^T P + P A = Q > 0 \tag{2.16}$$

It was shown in [15] that at least one eigenvalue of P is > 0.

3. Instability Theorem of Two-Time Scale System

The system of the form (2.1) studied is to be thought of as a one-time scale system in that the entire state variable x is varying slowly in comparison with the rate of time variation of the right hand side of the differential equation. In this section, we will study averaging for the case when only some of the state variables are slowly varying.

Consider the system:

$$\dot{x} = \epsilon f(t,x,y) \qquad x(0) = x_0 \tag{3.1}$$

$$\dot{y} = A\,y + \epsilon g(t,x,y) \qquad y(0) = y_0 \tag{3.2}$$

where $x \in R^n$ is called the slow state, $y \in R^m$ is called the fast state. $A \in R^{m \times m}$. Then the averaged system of the slow state is

$$\dot{x}_{av} = \epsilon f_{av}(x_{av}) \qquad x_{av}(0) = x_0 \tag{3.3}$$

where f_{av} is defined as in section 2.

To validate the following derivation, we make the following assumptions:

(B1) The functions f and g are piecewise continuous functions of time and continuous functions of x and y. Moreover, $f(t,0,0) = 0$, $g(t,0,0) = 0$ for all $t \geqslant 0$ and for some $l_3, l_4, l_5, l_6 \geqslant 0$.

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \leqslant l_3\|x_1 - x_2\| + l_4\|y_1 - y_2\| \tag{3.4}$$

$$\|g(t,x_1,y_1) - g(t,x_2,y_2)\| \leqslant l_5\|x_1 - x_2\| + l_6\|y_1 - y_2\| \tag{3.5}$$

for all $t \geqslant 0$, $x_1, x_2 \in B_h$, $y_1, y_2 \in B_h$. Also assume that $f(t,x,0)$ has continuous and bounded partial derivatives with respect to x for all $t \geqslant 0$ and $x \in B_h$.

(B2) $f_{av}(0)$, and f_{av} has continuous and bounded partial derivatives with respect to x for all $x \in B_h$ so that for some $l_{av} \geqslant 0$

$$\|f_{av}(x_1) - f_{av}(x_2)\| \leqslant l_{av}\|x_1 - x_2\| \tag{3.6}$$

for all $x_1, x_2 \in B_h$.

(B3) Let $d(t,x) = f(t,x,0) - f_{av}(x)$ satisfy the assumption (A4) in section 2.

(B4) A is Hurwitz.

(B5) For some $h' < h$, $\|x_{av}(t)\| \leqslant h'$ on the time interval considered.

The *Theorem* stated in the following concerns the instability of the system (3.1), (3.2). It provides a sufficient condition under which the instability of the averaged system (3.3) can imply that of the original system (3.1), (3.2).

Theorem: (Instability of an Unaveraged Two-Time Scale System)

If: the original system (3.1), (3.2) and the averaged system (3.3) satisfy the assumptions (B1)-(B5) along with the assumption that there exists a continuously differentiable, decrescent function $v(t,x)$ such that

(i) $v(t,0) = 0$.

(ii) $v(t,x) > 0$ for some x arbitrarily close to the origin.

(iii) $\left|\left|\frac{\partial v(t,x)}{\partial x}\right|\right| \leqslant k_3 ||x||$ for some $k_3 > 0$.

(iv) The derivative of $v(t,x)$ along the trajectory (3.3) satisfies

$$\dot{v}_{(3.3)}(t,x) \geqslant \epsilon k_4 ||x||^2 \tag{3.7}$$

for some $k_4 > 0$.

Then: the unaveraged system (3.1), (3.2) is unstable provided $\epsilon \leqslant \epsilon_0$ for some $\epsilon_0 > 0$.

Proof: With the assumptions (i)-(iv) in effect, Lyapunov Instability Theorem shows that the averaged system (3.3) is unstable. To study the instability of the system (3.1), (3.2), we need to construct another decrescent function $\tilde{v}$:

$$\tilde{v}(t,x,y) = v(t,x) - k_5 y^T P y \tag{3.8}$$

where P satisfies

$$A^T P + P A = -Q < 0 \tag{3.9}$$

and $q_1 I \leqslant Q \leqslant q_2 I$, $q_1, q_2 > 0$.

In (F.B.S. [5]), it was shown that, under assumptions (B1)-(B5), there exists a change of coordinates

$$x = z + \epsilon \omega_\epsilon(t,z) \tag{3.10}$$

such that the original system (3.1), (3.2) is transformed into

$$\begin{aligned} \dot{z} &= \epsilon f_{av}(z) + \epsilon p_1(t,z,\epsilon) + \epsilon p_2(t,z,y,\epsilon) \\ \dot{y} &= A\, y + \epsilon g(t,x(z),y) \end{aligned} \tag{3.11}$$

where $p_1(t,z,\epsilon)$ and $p_2(t,z,y,\epsilon)$ satisfy

$$||p_1(t,z,\epsilon)|| \leqslant \xi(\epsilon) k_6 ||z|| \tag{3.12}$$

and

$$||p_2(t,z,y,\epsilon)|| \leqslant k_7 ||y|| \tag{3.13}$$

for some $\xi(\epsilon) \in$ class k and $k_6, k_7 > 0$.

Clearly, $\tilde{v}(t,z,y) > 0$ for some (z,y) arbitrarily close to the origin (i.e. let $y \equiv 0$ and use assumption (ii)). Now, the differentiation of $\tilde{v}(t,z,y)$ with respect to time along the trajectories of the perturbed system (3.11) can be shown to be bounded below by using the assumption (iv) and the previous results similar to the derivation in (F.B.S), i.e.

$$\tilde{v}_{(3.11)}(t,z,y) = \dot{v}_{(3.11)}(t,z) + k_5 y^T Q\, y - 2\epsilon k_5 y^T P\, g(t,z,y)$$

$$\geqslant \epsilon\Big(k_4 - \xi(\epsilon)k_3k_6 - \frac{\epsilon^{\frac{1}{3}}k_3k_7}{2} - \epsilon^{\frac{4}{3}} l_5 k_5 p_2\Big)\,||z||^2$$

$$+\Big(k_5q_2 - 2\epsilon k_5 l_6 p_2 - \frac{\epsilon^{\frac{2}{3}}k_3k_7}{2} - \epsilon^{\frac{2}{3}}k_5 l_5 p_2\Big)\,||y||^2$$

$$:= \epsilon\tilde{\alpha}(\epsilon)\,||z||^2 + q(\epsilon)\,||y||^2 \tag{3.14}$$

During the derivation above, we use the fact that

$$||\frac{\partial z}{\partial x}|| \leqslant 1 + \xi(\epsilon) < 2 \tag{3.15}$$

Note that $\tilde{\alpha}(\epsilon) \to k_4$ and $q(\epsilon) \to k_5q_2$ as $\epsilon \to 0$. Then, using a Lyapunov Instability Theorem as before, we prove that the perturbed system (3.11) is unstable. Hence one can easily prove that the original system (3.1) is unstable from (F.B.S. [5]).

Mixed-Time Scale System:

In adaptive control, a frequently encountered Two-Time Scale system has the following form:

$$\dot{x} = \epsilon f'(t,x,y') \tag{3.16}$$

$$\dot{y}' = A\,y' + h(t,x) + \epsilon g(t,x,y') \tag{3.17}$$

As shown in (F.B.S. [5]), the system (3.16), (3.17) can be transformed into the system (3.1), (3.2) through the use of the coordinate change

$$y = y' - v'(t,x) \tag{3.18}$$

where $v'(t,x)$ is defined to be

$$v'(t,x) = \int_0^t e^{A(t-\tau)}h(\tau,x)\,d\tau \tag{3.19}$$

The averaged system of (3.16), (3.17) will exist if the following limits exist uniformly in

t and x, i.e.

$$f_{av}(x) = \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} f'(\tau, x, v'(\tau, x))\, d\tau \tag{3.20}$$

The *Theorem* above is applicable to this case with one more condition:

(B6) $h(t, 0) = 0$ for all $t \geq 0$ and $\| \frac{\partial h(t, x)}{\partial x} \| \leq k$ for all $t \geq 0$, $x \in B_h$ and some finite positive k.

This new assumption implies that $v'(t, 0) = 0$ and

$$\| \frac{\partial v(t, x)}{\partial x} \| \leq k' \tag{3.22}$$

for all $t \geq 0$, $x \in B_h$ and some $k' > 0$.

4. Application to Model Reference Adaptive Control in The Presence of Unmodelled Dynamics and Output Disturbances

We apply the averaging result of section 3 to the Model Reference Adaptive Control System for the relative degree one case where the plant has unmodelled dynamics as well as output disturbance. Before we proceed further, we intend to concretize the tuned model concept of the reference [10].

In the sequel, we will assume that the plant has stable multiplicative unmodelled dynamics which are described by

$$p = p^{*}(1 + L) \tag{4.1}$$

where p^{*} is the nominal plant transfer function and L is a stable (perturbation) transfer function. If the adjustable parameters are frozen, then the adaptive system is merely a LTI system which is characterized by model transfer function M and the closed loop plant transfer function $H_\theta(p, s)$. For the Narendra-Valavani Scheme [14], it was shown that there exists a $\theta^* \in R^{2n}$ such that

$$H_{\theta^*}(p^*, s) = M(s) \tag{4.2}$$

We now introduce some useful notation:

$$M(s) = k_m \frac{n_m(s)}{d_m(s)} \qquad p^*(s) = k_p \frac{n_p(s)}{d_p(s)} \tag{4.3}$$

$$\theta^* = [c_0^*, \underline{c}^{*T}, d_0^*, \underline{d}^{*T}]^T \tag{4.4}$$

and

$$\hat{C}^*(s) = s^{n-1} + c_1^* s^{n-2} + \cdots + c_{n-1}^* \tag{4.5}$$

$$\hat{D}^*(s) = s^{n-1} + d_1^* s^{n-2} + \cdots + d_{n-1}^* + d_0^* n_m(s) \tag{4.6}$$

Here $\underline{c}^{*T}$ stands for $[c_1^*, \ldots, c_{n-1}^*]$ while $\underline{d}^{*T}$ stands for $[d_1^*, \ldots, d_{n-1}^*]$.

Using this notation, we express $H_\theta(p, s)$ in terms of the model transfer function $M(s)$ and the true parameters:

$$H_\theta(p, s) = M(s) \frac{c_0}{c_0^*} \left[\frac{1 + L(s)}{1 - \Delta(s)} \right] \tag{4.7}$$

where

$$\Delta(s) = L(s) \left[\frac{d_p(s) - d_m(s)}{d_m(s)} \right] + k_p \frac{d_p(s)}{n_p(s)} \frac{\Delta\hat{C}(s)}{d_m(s)} + k_p (1 + L(s)) \frac{\Delta\hat{D}(s)}{d_m(s)} \tag{4.8}$$

and

$$\Delta\hat{C}(s) = \hat{C}(s) - \hat{C}^*(s) \qquad \Delta\hat{D}(s) = \hat{D}(s) - \hat{D}^*(s) \tag{4.9}$$

Suppose that L satisfies

$$| 1 + L(j\omega) | < l \tag{4.10}$$

$$| L(j\omega) | < \frac{1}{3} \left| \frac{d_m(j\omega)}{d_p(j\omega) - d_m(j\omega)} \right| \tag{4.11}$$

for all ω, then there exists a subset $\pi(\theta^*)$ contained in R^{2n}, such that

$$|\Delta\hat{C}(j\omega)| < \frac{1}{3} \left| \frac{d_m(j\omega) n_p(j\omega)}{k_p d_p(j\omega)} \right| \tag{4.12}$$

$$|\Delta\hat{D}(j\omega)| < \frac{1}{3l} \left| \frac{d_m(j\omega)}{k_p} \right| \tag{4.13}$$

for all $\omega \in R$ and the closed loop plant transfer function $H_\theta(p, s)$ remains stable whenever $\theta \in \pi(\theta^*)$ ($\pi(\theta^*) \neq \emptyset$ since $\theta^* \in \pi(\theta^*)$). Moreover, the difference between the two outputs y_θ and y_{θ^*} ($= y_M$), namely $e_\theta(t)$, can be evaluated through the difference of transfer functions, i.e.

$$H_\theta(p,s) - H_{\theta^*}(p,s) = M(s)E(s) \tag{4.14}$$

where

$$E(s) = \frac{c_0}{c_0^*} L(s) + \left(\frac{\Delta c_0}{c_0^*} + \Delta(s) \right) \tag{4.15}$$

Thus

$$\int_{-\infty}^{\infty} S_{e_\theta}(d\nu) = \int_{-\infty}^{\infty} \| M(j\nu)E(j\nu) \|^2 S_r(d\nu) \tag{4.17}$$

where $S_{e_\theta}(d\nu)$ denotes the spectral density function of $e_\theta(t)$. Due to the fact that

$$\lim_{\nu \to 0} L(j\nu) = 0 \tag{4.18}$$

, one can show that, given $\rho > 0$ and frequency range of the reference inputs, e.g. $[-\omega_0, \omega_0]$, there exists a reference input with appropriate support at frequency within that range such that

$$\int_{-\infty}^{\infty} S_{e_\theta}(j\nu) \leq \rho \int_{-\infty}^{\infty} S_r(d\nu) \tag{4.19}$$

and with $\theta \in \pi(\theta^*)$ (i.e. $H_\theta(p,s)$ remains stable). The collection of such θ, corresponding to such reference input, will then be called the tuned parameter set as defined in Kosut & Johnson.

Let θ_0 be chosen, such that (4.19) is satisfied, as a tuned value, then $H_{\theta_0}(p,s)$ will be defined as the tuned plant transfer function (Riedle & Kokotovic [16]). This will replace the role of the model in the following way: we rewrite the output error $e_1(t)$ by

$$e_1(t) := \tilde{e}_1(t) + e_{\theta_0}(t) + d(t) \tag{4.20}$$

where

$$\tilde{e}_1(t) = y_p(t) - y_{\theta_0}(t) \tag{4.21}$$

and $d(t)$ is an output disturbance.

By applying Narendra-Valavani Scheme to this tuned system, the equations describing the system can be shown to be:

$$\dot{e} = (A_{\theta_0} + b\phi^T Q)e + b\tilde{W}^T{}_{\theta_0}\phi \tag{4.22}$$

$$\dot{\phi} = -\epsilon(\tilde{W}_{\theta_0} h^T + (e_{\theta_0} + d)Q)e - \epsilon Q e h^T e - \epsilon(e_{\theta_0} + d)\tilde{W}_{\theta_0} \tag{4.23}$$

where $\tilde{W}_{\theta_0}$ is the regressor signal plus the output disturbance term, i.e.

$$\tilde{W}_{\theta_0} = [r(t), V^{(1)T}, y_{\theta_0}, V^{(2)T}]^T + [0, \underline{0}, 1, \underline{0}]^T d$$

$$:= W_{\theta_0} + \tilde{b}\, d \tag{4.24}$$

and

$$\phi = \theta - \theta_0 \qquad \theta \in \pi(\theta^*) \tag{4.25}$$

Assume that $(d + e_{\theta_0})$ is relatively small compared with the regressor signals, e.g. d is chosen to be so and conditions described in (4.19) are satisfied, then the local stability of the system (4.22), (4.23) around its zero solution is determined by the linearized system

$$\dot{e} = A_{\theta_0} e + b\tilde{W}^T{}_{\theta_0}\phi \tag{4.26}$$

$$\dot{\phi} = -\epsilon(\tilde{W}_{\theta_0} h^T)\, e \tag{4.27}$$

Now that the system described above has the same form as described in (3.16), (3.17), the averaging result may be applied. If the matrix R_{θ_0} defined by

$$R_{\theta_0} = -\lim_{T\to\infty} \frac{1}{T} \int_s^{s+T} \tilde{W}_{\theta_0}(t)\, \hat{W}^T{}_{\theta_0}(t)\, dt \tag{4.28}$$

,where

$$\hat{W}_{\theta_0}(t) = \int_0^t h^T e^{A_{\theta_0}(t-\tau)} b\tilde{W}_{\theta_0}(\tau)\, d\tau \tag{4.29}$$

contains eigenvalues with positive real parts and if for all eigenvalues $\lambda \in \sigma(R_{\theta_0})$, $Re(\lambda) \neq 0$, then the original system (4.22), (4.23) is unstable. This result matches that obtained by Riedle & Kokotovic if one applies a stationary reference input $r(t)$.

5. Robustness and Persistent Excitation:

In this section, we are interested in examining cases where the systems described in (4.22), (4.23) and (4.26), (4.27) are unstable, in the event that the input has only finitely many spectral lines. Recall that, when the reference input has more than $2n$ spectral lines (is rich enough), it yields exponential stability for the ideal system [3].

When $r(t)$, reference input, has finitely many spectral lines, the matrix R_{θ_0} in equation (4.28) can be written as:

$$R_{\theta_0} = -\frac{1}{c_0}\sum_{i=1}^{k}\left[\hat{n}(j\nu_i)\hat{n}^*(j\nu_i)H_{\theta_0}(p,j\nu_i)\frac{r_i^{\,2}}{4}\right] \tag{5.1}$$

where $\hat{n}(s)$ is the transfer function matrix of $\tilde{W}_{\theta_0}$ from the input $r(t)$ and $d(t)$ and r_i is the magnitude of the input spectral line at frequency ν_i (Here we use r_i to represent both magnitudes of $r(t)$ and $d(t)$).

Since $H_{\theta_0}(p,s)$ may no longer be SPR for high frequency inputs, the phase of $H_{\theta_0}(p,s)$ may be less than -90^0 at those frequencies. The following theorem will provide a sufficient condition under which the input with such frequencies can destroy the stability of the system. For the sake of simplicity, we put down the following definitions which will be frequently used:

Definition: Good Signals, Bad Signals.

A stationary signal is said to be **good** if its spectral support $\subset \{\nu \mid -90^0 < \angle H_{\theta_0}(p,j\nu) < +90^0\}$. A stationary signal is called **bad** if the spectral support $\subset \{\nu \mid \angle H_{\theta_0}(p,j\nu) > +90^0 \text{ or } \angle H_{\theta_0}(p,j\nu) < -90^0\}$.

Theorem: Suppose the unforced linearized system described by (4.26), (4.27) is not persistently excited by good signals, then a bad signal with either small enough or large magnitude will result in the instability of the adaptive system.

Before we prove the *Theorem*, we state a *Lemma* whose proof is in the Appendix.

Lemma: Given a block diagonal matrix of the form:

$$A = P\begin{bmatrix} a_0 r_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_1 r_1 & b_1 r_1 & \cdots & 0 & 0 \\ 0 & -b_1 r_1 & a_1 r_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & a_k r_k & b_k r_k \\ \vdots & \vdots & \vdots & \cdots & -b_k r_k & a_k r_k \end{bmatrix}P^T \qquad r_i > 0,\ i = 0,1,\ldots,k$$

where $P \in R^{2n \times (2k+1)}$, $k \leq n$, and $P = \left[u_0, u_1, v_1, \cdots, u_k, v_k \right]$, $u_i, v_i \in R^{2n}$, is of full column rank. Further, $a_i \neq 0$, $i = 0,1,2, \cdots, k$. If there exists an $a_j > 0$, $j \neq 0$, then there exist $\xi_0, \xi_1 > 0$, $\xi_0 < \xi_1$, such that $\sigma(A) \bigcap \mathbb{C}_+^o \neq \emptyset$ and for all $\lambda \in \sigma(A)$, $Re(\lambda) \neq 0$ when either $r_j \leq \xi_0$ or $r_j \geq \xi_1$.

Proof of Theorem:

In (5.1), denote $\hat{n}(j\nu_k)$ and $H_{\theta_0}(p, j\nu_k)$ by the following:

$$\hat{n}(j\nu_k) = u_k + jv_k \qquad \textit{and} \qquad H_{\theta_0}(p, j\nu_k) = a_k + jb_k \tag{5.2}$$

where $u_k, v_k \in R^{2n}$ and $a_k, b_k \in R$. Note that a sinusoidal input with frequency ν, in fact, will yield two spectral lines ν and $-\nu$ in the frequency spectrum. Hence, R_{θ_0} in (5.1) can be rewritten, in terms of u_k, v_k, a_k, b_k as:

$$R_{\theta_0} = -a_0 r_0^2 u_0 u_0^T - \sum_{j=1}^{k} \left\{ a_j (u_j u_j^T + v_j v_j^T) + b_j (u_j v_j^T - v_j u_j^T) \right\} \frac{r_j^2}{2} \tag{5.3}$$

Grouping u_k, v_k into P, i.e.

$$P = [u_0, u_1, v_1, \cdots, u_k, v_k] \tag{5.4}$$

P is of full column rank from [19]. Substitution of P into equation (5.3) yields the following expression:

$$R_{\theta_0} = P \begin{bmatrix} -\frac{a_0 r_0^2}{2} & \begin{matrix} 0 \\ -\frac{a_1 r_1^2}{2} \end{matrix} & \begin{matrix} 0 \\ -\frac{b_1 r_1^2}{2} \end{matrix} & \cdots & 0 & 0 \\ 0 & \frac{b_1 r_1^2}{2} & -\frac{a_1 r_1^2}{2} & \cdots & 0 & 0 \\ 0 & & & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & -\frac{a_k r_k^2}{2} & -\frac{b_k r_k^2}{2} \\ & & & & \frac{b_k r_k^2}{2} & -\frac{a_k r_k^2}{2} \end{bmatrix} P^T \tag{5.5}$$

By assumption that there exists a bad signal which yields an $a_j < 0$, $j \neq 0$ in the matrix of equation (5.5). Consequently, the *Lemma* shows that $\sigma(R_{\theta_0}) \cap \mathbb{C}_+^o \neq \emptyset$ and for all $\lambda \in \sigma(R_{\theta_0})$, $Re(\lambda) \neq 0$ for sufficiently large or small r_j. Therefore, applying *Theorem* in section 4, the results then follows.

Q.E.D.

Comments:

(i) This *Theorem* points out the importance of sufficiently rich good signals for maintaining the stability of the adaptive system in the presence of unmodeled dynamics and output disturbances.

(ii) The *Theorem* does not guarantee that, the system will become unstable for any magnitude of bad signal. However, heuristically speaking, the closer the phase of $H_{\theta_0}(p, j\nu)$ is to -180° (when ν is the frequency of a bad signal), the wider will be the dynamic range of the bad signal giving rise to instability in the system (4.22), (4.23).

(iii) Unstable behavior of the system (4.22), (4.23) will arise even in the presence of persistent excitation of good signals if the magnitude of the bad signal is sufficiently large, due to the fact that

$$\sum \lambda_l = tr(R_{\theta_0}) = -a_0 r_0^2 \, ||u_0||^2 - \sum_{i=1}^{k} \left\{ \frac{a_i r_i^2}{2} (\, ||u_i||^2 + ||v_i||^2 \,) \right\} \tag{5.6}$$

where $\sum \lambda_l$ is the sum of all the eigenvalues of R_{θ_0}.

(iv) On the other hand, in the paper by Bodson et al [2], one can directly obtain the conclusion that dominantly rich good signals can guarantee that the parameter will stay in the nbhd of the tuned parameter value θ_0 provided that the initial guess is close enough to θ_0 by visualizing the system (4.22), (4.23).

Beside the instability which appears in the *Gradient Type* adaptation algorithm, the *Least Squares Type* algorithm also possess the similar property. The following *Corollary* will be stated with proof in the Appendix 2.

Corollary: If all conditions in the *Theorem* are satisfied with the adaptation law changed to *Least Squares Type* with forgetting factor plus the total spectral lines due to either input or output disturbance are more than $2n$, then a bad signal with sufficiently small magnitude will result in the instability of the adaptive system.

The possible slow drift of the parameters to the extent that the closed loop plant transfer function becomes unstable results from the fact that the quadratic term of equation (4.23) is of the form

$$-\epsilon Q \, eh^T e = -\epsilon [\, 0 \, , F_1^T(t) \, , (h^T e)^2 \, , F_2^T(t) \,]^T \tag{5.7}$$

where $F_1(t)$ and $F_2(t)$ are internal signal from the filter block. Note that there is a constant sign term $-\epsilon(h^T e)^2$ which corresponds to the rate change of parameter d_0. Hence, if Δd_0 starts from a negative value and drifts away from the origin, then d_0 could drift to minus infinity provided other parameters don't retard its drift. Eventually, when d_0 drifts to a critical value, the closed loop transfer function becomes unstable which causes the system to fall apart. On the other hand, if these conditions don't hold, then the system parameters could drift to the region where oscillations occurs.

We illustrate the results on Rohrs' examples. In his 1st example, the simulation was generated using a nominally 1st order plant with a pair of complex but highly damped poles, described by

$$y_p(t) = \frac{2}{(s+1)} \frac{229}{(s^2+30s+229)} [u(t)] \tag{5.8}$$

and a reference model

$$y_M(t) = \frac{3}{(s+3)} [r(t)] \tag{5.9}$$

In Rohrs' 2nd example, he used the same nominal plant but with less-well damped unmodeled dynamics at a somewhat lower frequency, namely:

$$y_p(t) = \frac{2}{(s+1)} \frac{100}{(s^2+8s+100)} [u(t)] \tag{5.10}$$

The following table summarize the simulation for different examples with several different reference inputs and/or output disturbances.

Example 1.					
Case:	Tuned Value θ_0	r(t)	d(t)	e.v. of R_{θ_0}	Fig.
1	0.51 , -0.01	2+0.5*sin(16.1*t)	0	-15.68 , 0.0085	2
2	0.51 , -0.01	2	0.5*sin(16.1*t)	-15.68 , 0.0075	3
3	0.51 , -0.01	2+0.3*sin(8*t)	0	-15.68 , 0.0075	4
4	1.28 , -5.11	0.3+3*sin(8*t)	0	-0.556 ±0.663j	5

Example 2.					
Case:	Tuned Value θ_0	r(t)	d(t)	e.v. of R_{θ_0}	Fig.
1	0.51 , -0.01	2+0.1*sin(10*t)	0	-15.69 , 0.00013	6
2	0.51 , -0.01	2	0.02*sin(10*t)	-15.69 , 0.00005	7
3	1.50 , -1.00	0	3	00.000 , -6.000	8

Remarks for Example 1:

(1) The instability behavior of the adaptive system in the first three cases can be predicted in light of the fact that one eigenvalue of R_{θ_0} is positive.

(2) In the cases where the output disturbance $d \equiv 0$, the quadratic term in equation (5.7) is simply

$$-\epsilon Q\, eh^T e = -\epsilon \begin{bmatrix} 0 \\ (h^T e)^2 \end{bmatrix} \tag{5.11}$$

which explains the drift of the parameters d_0, and hence c_0, as shown in Fig 2 and Fig 4.

(3) In case 2, the bad signal resides in the output disturbance term rather than in the reference input term, which yields the similar destabilizing effect since the regressor signal $\tilde{W}_{\theta_0}$ contains the output disturbance.

(4) Cases 3 and 4 show the contrast between stability and instability of the adaptive system. The sinusoid, $sin(8t)$, which appear in both cases is a bad signal in case 3 whereas a good signal in case 4 since the tuned values θ_0 are different. This simulation result serves as counterexamples to Rohrs et al [18] and Chen & Cook [4].

Remarks for Example 2:

(1) The simulation shown in case 2 counteract the claim by Chen & Cook in the fact that the slow drift instability occurs regardless of the relatively small magnitude of the output disturbance. Due to long time elapse for this simulation, we simply choose the ratio between magnitudes of good and bad signals to be 100.

(2) In case 3, the system is driven by zero reference input along with the constant output disturbance. The slow drift of parameters when d is large enough become obvious if one rewrite the expression of (4.23) in this case as:

$$\dot{\phi} = -\epsilon \begin{bmatrix} 0 \\ 2d \end{bmatrix} h^T e - \epsilon \begin{bmatrix} 0 \\ (h^T e)^2 \end{bmatrix} - \epsilon \begin{bmatrix} 0 \\ d^2 \end{bmatrix} \tag{5.12}$$

presuming all the stable initial conditions are zero.

The last example, still using the same plant and model, illustrates the instability of the Least Squares Type algorithm. Due to the similarity to the previous examples using a Gradient Type algorithm, only a single result is provided here and is shown in Fig. 9.

6. Concluding Remarks:

In this paper, we presented instability theorems for averaging analysis of one and two time scale systems. These techniques were then applied to the model reference adaptive control system of relative degree one using either *Gradient* or *Lease Squares* type adaptation algorithms to explain the slow drift instability due to unmodeled dynamics and output disturbance existing in the slow adaptation case. The importance of persistent excitation of *good signals* was well stressed, which was not directly shown in the previous work [1], [8], [16], [17], [18].

The remedy to this kind of instability can be either changing the adaptive law as suggested in [7] or making the reference input dominantly rich in the right frequence content. On the other hand, this analysis also facilitates one to see that certain extent of robustness of such system may still be achieved.

[1] K. J. Astrom. "Interactions Between Excitation and Unmodeled Dynamics." IEEE Trans. on AC. Volume AC-30. No. 9. September 1985. pp 889-891

[2] Marc Bodson. S. Sastry. B.D.O. Anderson. I. Mareels and R. Bitmead. "Nonlinear Averaging Theorems. and the determination of Parameter Convergence Rates in Adaptive Control." Memorandum No. UCB/ERL 85/92. Electronics Research Laboratory. University of California. Berkeley. November. 1985

[3] S. Boyd and S. Sastry. "Necessary and Sufficient Conditions for Parameter Convergence in Adaptive Control." Proc. of the Ames-Berkeley Conf. on Non-Linear Dynamics and Control. Math-Sci Press. Brookline. Massachusetts. 1984. pp 81-101

[4] Z. J. Chen and P. A. Cook. "Robustness of Model-Reference Adaptive Control Systems with Unmodeled Dynamics." Int. J. Control. Volume 39. No. 1. 1984. pp 201-214

[5] Li-Chen Fu. M. Bodson and S. Sastry. "New Stability Theorems for Averaging and Their Application to Convergence Analysis of Adaptive Identification and Control Schemes." Proc. 24th IEEE Conf. on Decision and Control. Ft. Lauderdale. FL. 1985. pp 473-477

[6] J. K. Hale. "Ordinary Differential Equations." Kreiger. Molaban (Florida). 1980

[7] P. A. Ioannou and P. V. Kokotovic. "Instability Analysis and Improvement of Robusteness of Adaptive Control." IFAC 1984. pp 583-594

[8] James Michael Krause. "Adaptive Control In the Presence of High Frequency Modelling Errors." Master Thesis. Massachusetts Institute of Technology. 1983

[9] J. Krause. M. Athans. S. Sastry and L. Valavani. "Robustness Studies in Adaptive Control." Proc. 22nd IEEE Conf. on Decision and Control. San Antonio. Texas. 1983. pp 977-981

[10] R. L. Kosut and C. R. Johnson. "An Input-Output View of Robustness in Adaptive Control." IFAC 1984. pp 569-581

[11] Robert L. Kosut. "Methods of Averaging for Adaptive Systems." Research Paper. 1985

[12] R. Kosut. B. D. O. Anderson and I. Mareels. "Stability Theory for Adaptive Systems: Method of Averaging and Persistency of Excitation." to appear in IEEE Trans on AC. 1986

[13] T. Kato. "Perturbation Theory for Linear Operator."

[14] K. Narendra and L. Valavani, "Stable Adaptive Controller Design- Direct Control." IEEE Trans. on AC, Volume AC-23, 1978, pp 570-583

[15] Alexander Ostrowski and H. Schneider, "Some Theorems on the Inertia of general Matrices." Journal of Mathematical Analysis and Applications 4, 1962, pp 72-84

[16] Bradley D. Riedle and P. V. Kokotovic, "A stability-Instability Boundary for Disturbance-Free Slow Adaptation and Unmodeled Dynamics." IEEE Trans. on AC, Volume AC-30, No. 10, October 1985, pp 1027-1030

[17] Bradley D. Riedle and P. V. Kokotovic, "Stability Analysis of An Adaptive System with Unmodeled Dynamics." Int. J. Control, Volume 41, No. 2, 1985, pp 389-402

[18] Charles E. Rohrs, L. Valavani, M. Athans and G. Stein, "Robustness of Continuous-Time Adaptive Control Algorithms in the Presence of Unmodeled Dynamics." IEEE Trans. on AC, Volume AC-30, No. 9, September 1985, pp 881-889

[19] S. S. Sastry, "Model Reference Adaptive Control-Stability, Parameter Convergence, and Robustness." IMA Journal of Mathematical Control & Information, Volume 1, 1984, pp 27-66

[20] M. Vidyasagar, "Nonlinear Systems Analysis," Prentice-Hall, 1978

Appendix 1.

Proof of Lemma:

Without loss of generality we, assume $a_k > 0$ and decompose A into two parts:

$$A = P_0 B_0 P_0^T + P_1 B_P P_1^T$$
$$:= A_0 + r_k A_p \tag{a.1}$$

where

$$P_0 = [u_0, u_1, v_1, \cdots, u_{(k-1)}, v_{(k-1)}] \qquad P_1 = [u_k, v_k] \tag{a.2}$$

and

$$B_0 = \begin{bmatrix} a_0 r_0 & 0 & 0 & \cdot\,\cdot & 0 & 0 \\ 0 & a_1 r_1 & b_1 r_1 & \cdot\,\cdot & 0 & 0 \\ 0 & -b_1 r_1 & a_1 r_1 & \cdot\,\cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot\,\cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\,\cdot & a_{k-1} r_{k-1} & b_{k-1} r_{k-1} \\ \cdot & \cdot & \cdot & \cdot\,\cdot & -b_{k-1} r_{k-1} & a_{k-1} r_{k-1} \end{bmatrix} \qquad B_p = \begin{bmatrix} a_k & b_k \\ -b_k & a_k \end{bmatrix} \tag{a.3}$$

Define

$$A(t) = A_0 + t A_p \qquad t \geqslant 0 \tag{a.4}$$

Clearly, A_0 is a singular matrix. To show 0 is a semisimple eigenvalue of A_0, it suffices to show that: for all $m > 0$ and some $x \in R^{2n}$ such that $A_0^m x = 0$, then $A_0 x = 0$.

Since

$$A_0^m = P_0 B_0 (P_0^T P_0) B_0 \cdots B_0 (p_0^T P_0) B_0 P_0^T \tag{a.5}$$

and $B_0, (P_0^T P_0)$ are all nonsingular matrices, $A_0^m x = 0$ if and only if $A_0 x = 0$, which shows the proof. Moreover, all the (right) eigenvectors of A_0 associated with 0 eigenvalue coincide with the left eigenvectors of A_0 associated with the same eigenvalue. Consequently, the orthogonal projection denoted by P_{A_0} associated with the 0 eigenvalue is symmetric.

In $(a.4)$, when t is small, the perturbed 0 eigenvalue of A_0 can be approximated to the 1st order of t by: (Kato [13])

$$\lambda(t) \approx t\,\sigma \qquad t > 0 \tag{a.6}$$

where σ is the nonzero eigenvalue of the matrix $P_{A_0} A_p P_{A_0}$. From the fact that the real part of $P_{A_0} A_p P_{A_0}$ is positive semidefinite and $a_k > 0$, one can show that

$$Re(\lambda(t)) \approx t\, Re(\sigma) > 0 \tag{a.7}$$

Since $(a.6)$ holds for small t, hence there exists $\xi_0 > 0$ such that $\sigma(A(t)) \cap \mathbb{C}_+^o \neq \emptyset$ for all

$0 < t \leqslant \xi_0$.

On the other hand, for all eigenvalues $\lambda_i(t) \in \sigma(A(t))$

$$\sum \lambda_i(t) = tr(A(t))$$

$$= \sum_{i=0}^{k-1} \left\{ a_i r_i \left(\|u_i\|^2 + \|v_i\|^2 \right) \right\} + t \left\{ a_k \left(\|u_k\|^2 + \|v_k\|^2 \right) \right\} \quad \text{(a.8)}$$

which shows that $\sigma(A(t)) \cap \mathbb{C}_+^o \neq \emptyset$ provided t is large enough, i.e. there exists $\xi_1 > 0$, $t \geqslant \xi_1$.

Finally, suppose there exists an eigenvalue $\lambda \in \sigma(A_0)$ where $Re(\lambda) = 0$, then the perturbed $\lambda(t)$ satisfies $Re(\lambda(t)) \neq 0$ for t small or large enough ([13]). This completes the proof.

Q.E.D.

Appendix 2.

Proof of Corollary:

For Least Squares Type Algorithm with forgetting factor, the equations describing the system can be shown to be:

$$\dot{e} = (A_{\theta_0} + b\phi^T Q)e + b\tilde{W}^T{}_{\theta_0}\phi \quad \text{(a.8)}$$

$$\dot{\phi} = -\epsilon \tilde{P}^{-1}(\tilde{W}_{\theta_0} h^T + (e_{\theta_0} + d)Q)e - \epsilon \tilde{P}^{-1} Q e h^T e - \epsilon \tilde{P}^{-1}(e_{\theta_0} + d)\tilde{W}_{\theta_0} \quad \text{(a.9)}$$

$$\dot{\tilde{P}} = -\epsilon\alpha\tilde{P} + \epsilon \tilde{W}_{\theta_0}\tilde{W}^T{}_{\theta_0} + \epsilon \left\{ \tilde{W}_{\theta_0}(Qe)^T + (Qe)\tilde{W}^T{}_{\theta_0} + (Qe)(Qe)^T \right\} \quad \text{(a.10)}$$

Consider the nbhd of $(e, \phi) = (0,0)$, we can approximate the system by:

$$\dot{e} = A_{\theta_0} e + b\tilde{W}^T{}_{\theta_0}\phi \quad \text{(a.11)}$$

$$\dot{\phi} = -\epsilon \hat{P}^{-1} \tilde{W}_{\theta_0} h^T e - \epsilon \hat{P}^{-1}(e_{\theta_0} + d)\tilde{W}_{\theta_0} \quad \text{(a.12)}$$

$$\dot{\hat{P}} = -\epsilon\alpha\hat{P} + \epsilon \tilde{W}_{\theta_0}\tilde{W}^T{}_{\theta_0} \quad \text{(a.13)}$$

Now, $\hat{P}$ is simply a time varying function independent of e and ϕ and is a bounded, positive definite matrix for all time t if $\hat{P}(0) = \tilde{P}(0) = I$. Then, assume ϵ is sufficiently small, we can apply the averaging results to the system described above, i.e. the averaged system for the slow variable ϕ is

$$\dot{\phi}_{av} = \epsilon R(0)\,\phi_{av} \tag{a.14}$$

where

$$R(0) = -\lim_{T\to\infty}\frac{1}{T}\int_s^{s+T} P_{av}^{-1}(t)\tilde{W}_{\theta_0}(t)\hat{W}_{\theta_0}(t)dt \tag{a.15}$$

and

$$\dot{P}_{av} = -\epsilon\alpha P_{av} + \epsilon\tilde{R}_{\theta_0} \tag{a.16}$$

where

$$\tilde{R}_{\theta_0} = \lim_{T\to\infty}\frac{1}{T}\int_s^{s+T}\tilde{W}_{\theta_0}(t)\tilde{W}^T_{\theta_0}(t)dt \tag{a.17}$$

Since $P_{av} \to \frac{1}{\alpha}\tilde{R}_{\theta_0}$ as $t \to \infty$, $R(0)$ then will converge to $\alpha\tilde{R}^{-1}_{\theta_0}R_{\theta_0}$, where by assumption $\tilde{R}_{\theta_0}$ is positive definite. Using results in the *Lemma*, one can show that: by small perturbation theory, for all sufficiently small r_j in *Theorem* of section 5, $\sigma(A) \bigcap \mathbb{C}_+^o \neq \varnothing$ Hence the similar results follows.

Q.E.D.

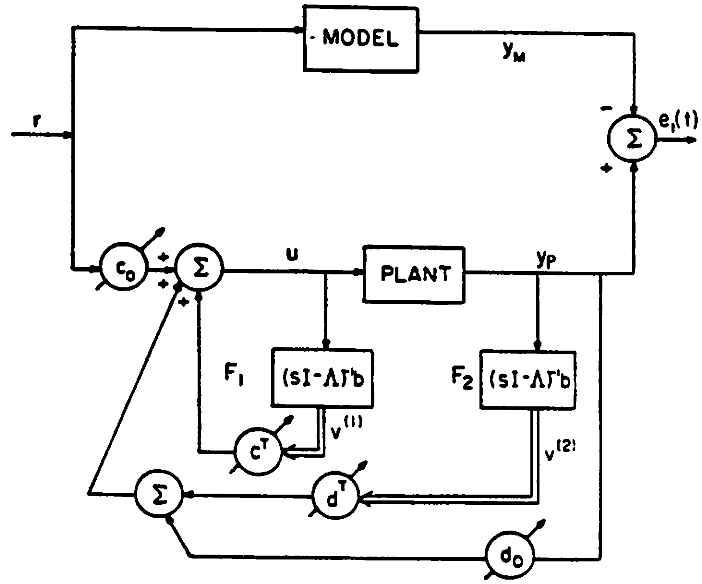

Fig. 1 MRAC System

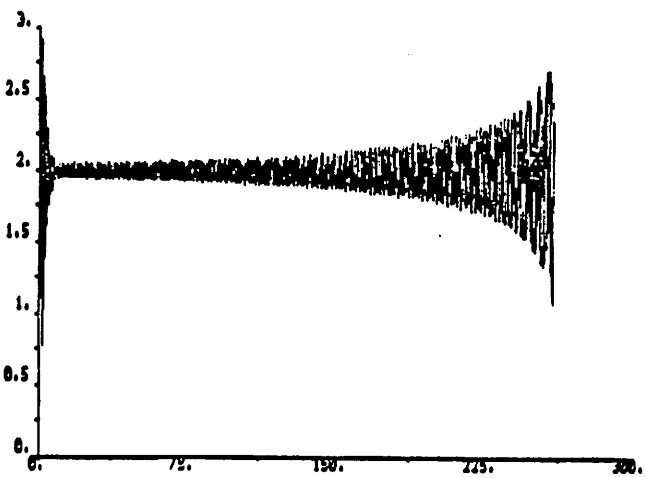

Fig. 2(a) $e_1(t)$

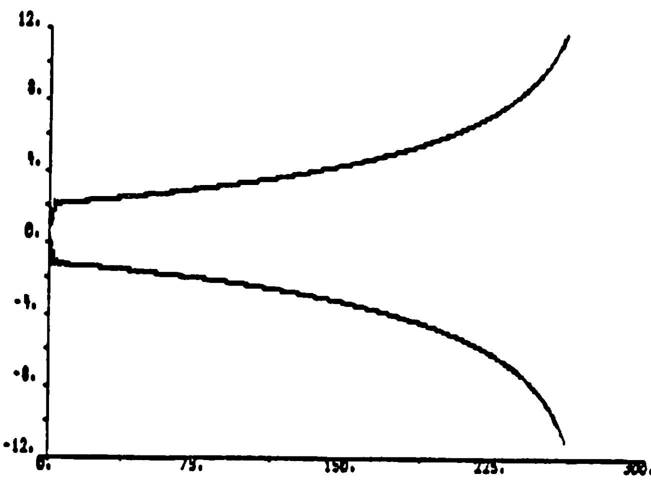

Fig. 2(b) d_0, c_0

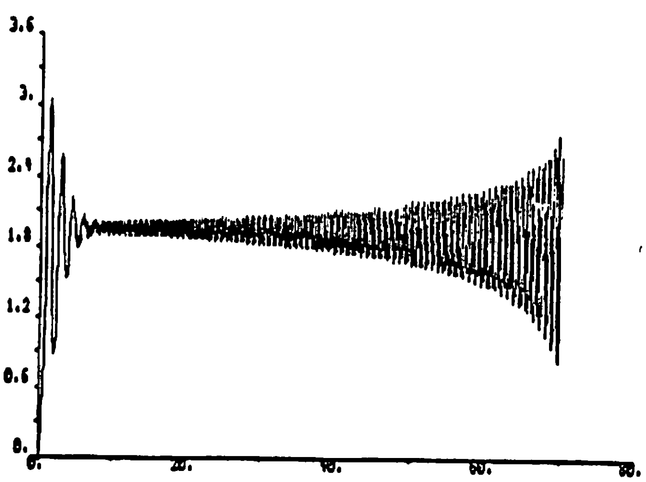

Fig. 3(a) $e_1(t)$

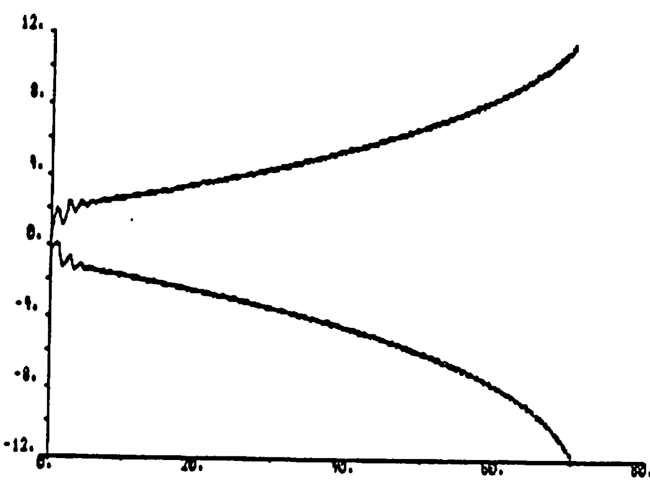

Fig. 3(b) d_0, c_0

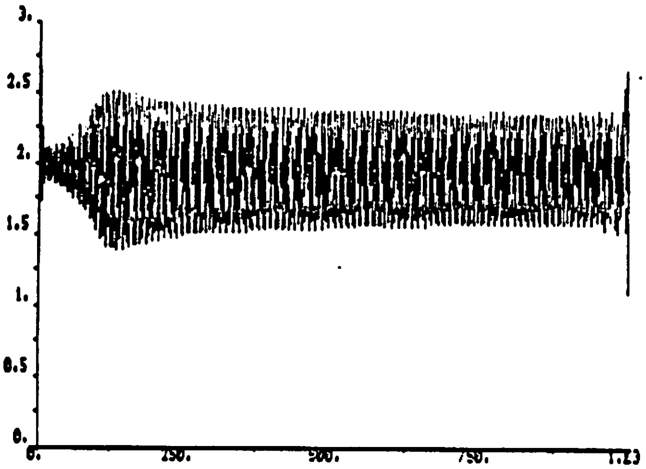

Fig. 4(a) $e_1(t)$

Fig. 4(b) $d_0. c_0$

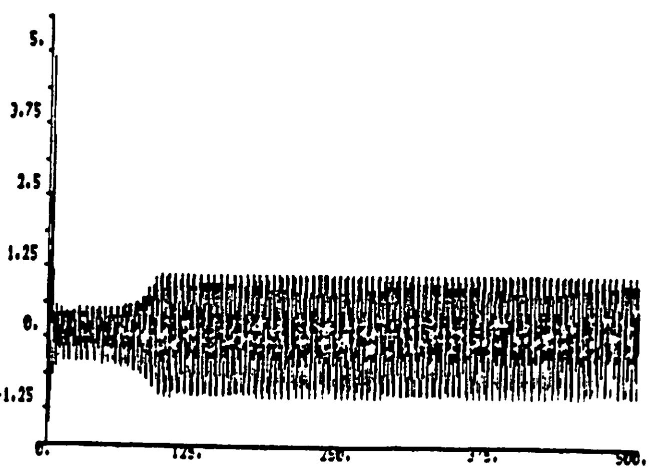

Fig. 5(a) $e_1(t)$

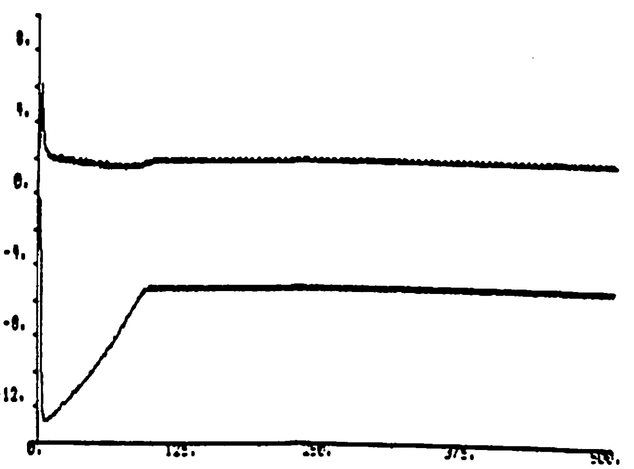

Fig. 5(b) $d_0. c_0$

Fig. 6(a) $e_1(t)$

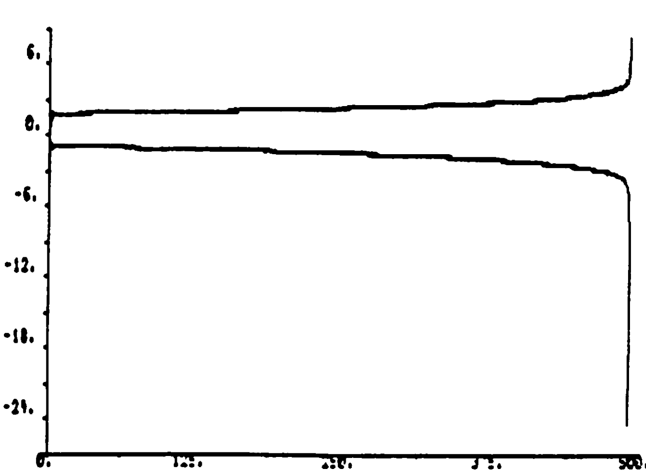

Fig. 6(b) $d_0. c_0$

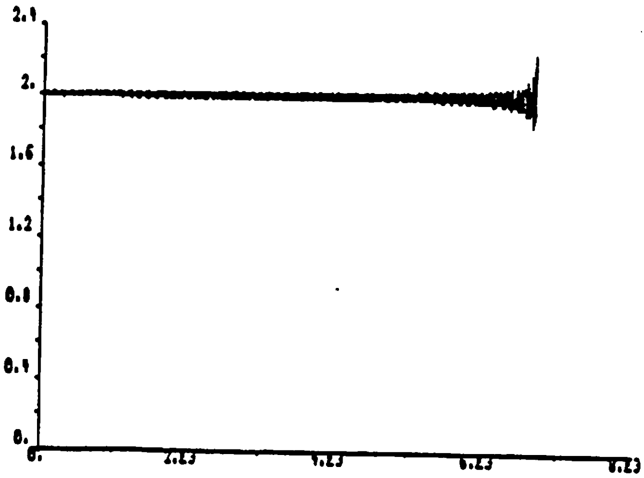

Fig. 7(a) $e_1(t)$

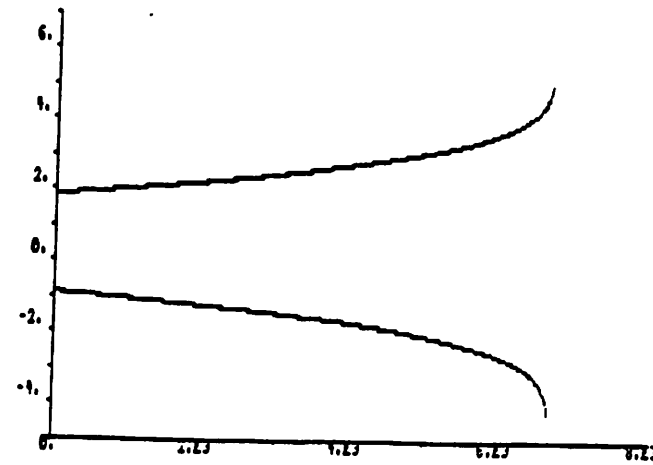

Fig. 7(b) d_0, c_0

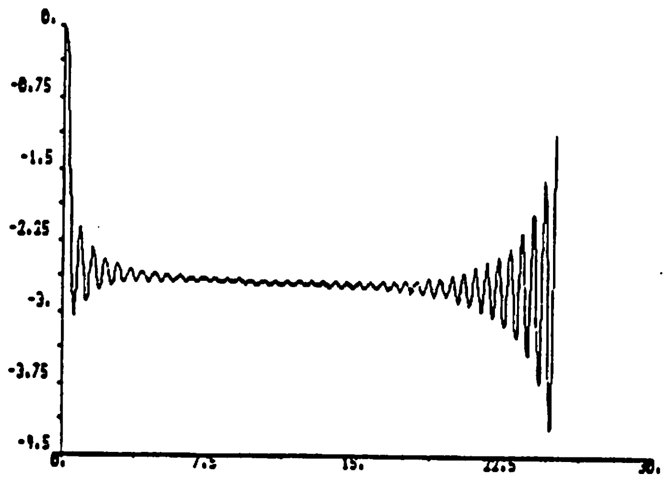

Fig. 8(a) $e_1(t)$

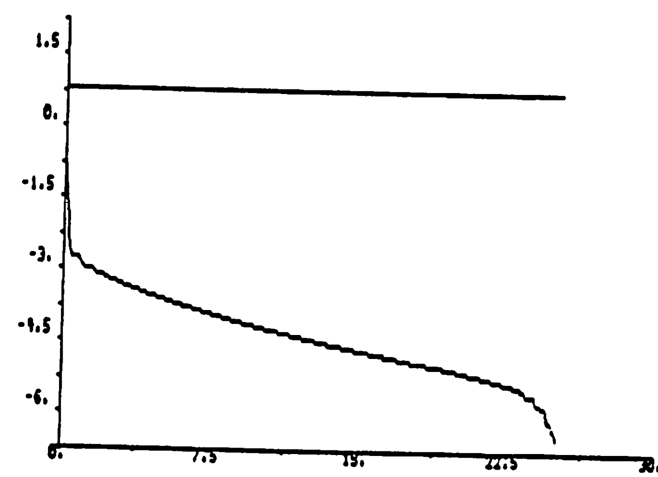

Fig. 8(b) d_0, c_0

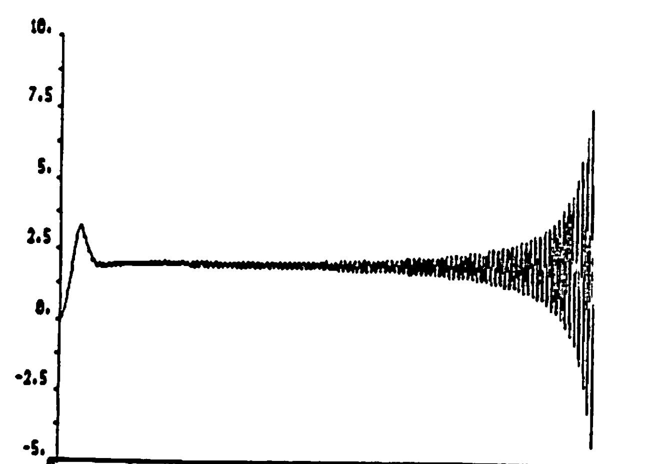

Fig. 9(a) $e_1(t)$

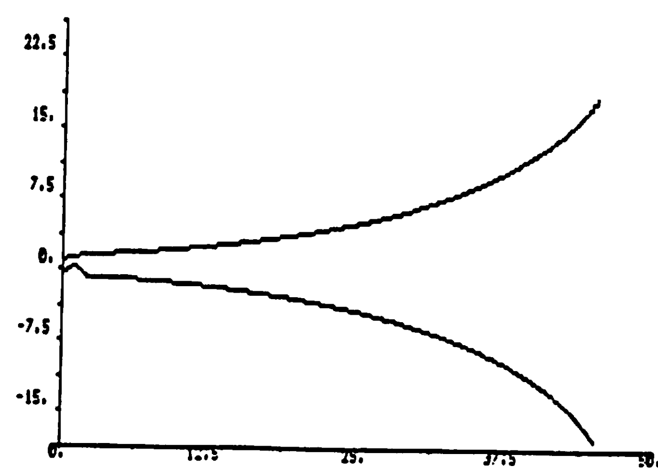

Fig. 9(b) d_0, c_0