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G. M. Bernstein and L. O. Chua‡

ABSTRACT

The method of averaging has been used for years to prove the existence of oscillations in nonlinear circuits. In the past the application of averaging has tended to be ad hoc rather than systematic. In addition the validity of the method was not well established. The purpose of this paper is to rigorize and systematize the analysis of weakly nonlinear oscillator circuits via the method of averaging. In particular this paper will put on a rigorous foundation the work of Kuramitsu et. al.[1, 2, 3] on the "Averaged Potential" and the work of T. Endo and others on the oscillatory modes of coupled oscillator circuits. Furthermore we give a novel way of simplifying the calculation of averages when we have a potential function representation.

1. Introduction

The method of averaging has been widely known to physicists and engineers for many years, with its origins stemming from Van der Pol, Krylov and Bogoliubov. Averaging in the past has been treated primarily as a perturbation technique with various higher order terms being thrown out of equations to simplify the analysis. This has made averaging difficult to apply and the results of averaging somewhat questionable. The modern approach is to treat averaging as a change of coordinates and then use it in conjunction with another mathematical theory.

Results in averaging tend to fall into two different categories according to time scale, that is finite time and infinite time. With finite time averaging one gets an approximation to trajectories of a system over finite time by the trajectories of a simpler averaged system. The most widely known application is in approximating the orbits of the planets in the solar system when the gravitational interaction between planets is taken into account. See [4] for details and mathematical justification of this approach.

Infinite time averaging relates certain general features of the solution of a system to those of a simpler averaged system. Infinite time averaging can be used to prove the existence of a host of interesting nonlinear phenomena such as: oscillations, frequency entrainment and subharmonic solutions. The mathematical justification of these results in the most general case, stems from applying the method of averaging, as a transformation technique, and combining it with the theory of integral manifolds. See [5] for a tutorial on the theory of integral manifolds with circuit applications. When dealing with simple forced circuits it becomes possible to combine the method of averaging with the theory of noncritical perturbations of linear systems. This is a simpler theory than that of integral manifolds and allows us to obtain estimates of the parameter ranges for which averaging holds. This will be reported upon in a later paper.

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‡ The authors are with the University of California, Berkeley, CA 94720

In this paper we will deal with infinite time averaging only and apply it to prove the existence of oscillations in nonlinear circuits. We will proceed as follows: First, we define the class of circuits for which the theory of averaging is applicable. Second, we will transform the state equations of these circuits into a form suitable for averaging. Third, we will review the mathematical justification of Averaging/Integral manifolds along with some novel simplifications that we can make when we have a potential formulation and independent frequencies. Finally, we present a systematic analysis algorithm for applying the previous theory and illustrate it by proving rigorously several classic results.

2. Equation formulation

Let N be a circuit containing passive linear capacitors (may be coupled), passive linear inductors (may be coupled), two terminal linear and nonlinear resistors, independent voltage sources and independent current sources. We require that N satisfy the following topological conditions:*

(T1) The capacitors and voltage sources form no loops,

(T2) The inductors and current sources form no cutsets,

(T3) Each current controlled resistor must form a cutset with inductors and/or current sources,

(T4) Each voltage controlled resistor must form a loop with capacitors and/or voltage sources.

Label and order the circuit elements as follows: inductors (L's), current sources (J's), voltage controlled resistors (G's), capacitors (C's), voltage sources (E's) and current controlled resistors (R's). Let n_L, n_J, n_G, n_C, n_E, and n_R denote the number of elements in each of the above sets respectively. The branch voltage and branch current vectors can be partitioned in accordance with the labeling convention:

$$v = \left[v_L^T,\ v_J^T,\ v_G^T,\ v_C^T,\ v_E^T,\ v_R^T\right]^T \tag{2.1}$$

and

$$i = \left[i_L^T,\ i_J^T,\ i_G^T,\ i_C^T,\ i_E^T,\ i_R^T\right]^T \tag{2.2}$$

Note: v_E and i_J are constants of the independent voltage and current sources respectively.

The topological hypotheses (T1) - (T4) and the colored branch theorem [7] imply the existence of a tree containing all of the capacitors, independent voltage sources and current controlled resistors, i.e. {C's, E's, R's}, and a cotree containing all of the inductors, independent current sources and voltage controlled resistors, i.e. {L's, J's, G's }, such that the fundamental loop matrix has the following form:

$$\mathbf{B} = \begin{bmatrix} I_{n_L} & 0 & 0 & \mathbf{B}_{LC} & \mathbf{B}_{LE} & \mathbf{B}_{LR} \\ 0 & I_{n_J} & 0 & \mathbf{B}_{JC} & \mathbf{B}_{JE} & \mathbf{B}_{JR} \\ 0 & 0 & I_{n_G} & \mathbf{B}_{GC} & \mathbf{B}_{GE} & 0 \end{bmatrix} \tag{2.3}$$

Under the labeling convention KVL, $\mathbf{B}v = 0$, may be written as follows:

$$v_L = -\mathbf{B}_{LC} v_C - \mathbf{B}_{LE} v_E - \mathbf{B}_{LR} v_R \tag{2.4}$$

$$v_J = -\mathbf{B}_{JC} v_C - \mathbf{B}_{JE} v_E - \mathbf{B}_{JR} v_R \tag{2.5}$$

$$v_G = -\mathbf{B}_{GC} v_C - \mathbf{B}_{GE} v_E \tag{2.6}$$

* In [6] it is shown how these topological conditions can be somewhat relaxed with an increase in the complexity in the equation formulation process. If a circuit contains a capacitor only loop (inductor only cutset) then this loop (cutset) may be replaced by a set of mutually coupled capacitors (inductors) which no longer form a loop (cutset). See [6], pages 434-436.

Similarly, the KCL equations, $i = \mathbf{B}^T i_{links}$, can be written as:

$$i_C = \mathbf{B}_{LC}^T i_L + \mathbf{B}_{JC}^T i_J + \mathbf{B}_{GC}^T i_G \tag{2.7}$$

$$i_E = \mathbf{B}_{LE}^T i_L + \mathbf{B}_{JE}^T i_J + \mathbf{B}_{GE}^T i_G \tag{2.8}$$

$$i_R = \mathbf{B}_{LR}^T i_L + \mathbf{B}_{JR}^T i_J \tag{2.9}$$

The elements have the following constitutive relations:

Capacitors:

$$i_C = \mathbf{C}\frac{dv_C}{dt} \tag{2.10}$$

where

$$\mathbf{C} = diag[C_1, C_2, \ldots, C_{n_C}], \quad C_i > 0, \quad i = 1, \ldots, n_C$$

Inductors:

$$v_L = \mathbf{L}\frac{di_L}{dt} \tag{2.11}$$

where $\mathbf{L}$ is an n_L by n_L positive definite matrix.

Current controlled resistors:

$$v_R = \mathbf{r}(i_R) \tag{2.12}$$

where

$$\mathbf{r}(i_R) = [r_1(i_{R_1}), r_2(i_{R_2}), \ldots, r_{n_r}(i_{R_{n_R}})]^T$$

Voltage controlled resistors:

$$i_G = \mathbf{g}(v_G) \tag{2.13}$$

where

$$\mathbf{g}(v_G) = [g_1(v_{G_1}), g_2(v_{G_2}), \ldots, g_{n_G}(v_{G_{n_G}})]^T$$

Combining KCL, KVL and the element relations we obtain the following state equations for N:

$$\mathbf{C}\frac{dv_C}{dt} = \mathbf{B}_{LC}^T i_L + \mathbf{B}_{JC}^T i_J + \mathbf{B}_{GC}^T \mathbf{g}(-\mathbf{B}_{GC} v_C - \mathbf{B}_{GE} v_E) \tag{2.14a}$$

$$\mathbf{L}\frac{di_L}{dt} = -\mathbf{B}_{LC} v_C - \mathbf{B}_{LE} v_E - \mathbf{B}_{LR} \mathbf{r}(\mathbf{B}_{LR}^T i_L + \mathbf{B}_{JR}^T i_J) \tag{2.14b}$$

Note that all other circuit variables can be obtained from v_C and i_L via KVL, KCL and the element relations.

The above formulation can be easily generalized when there is coupling between the nonlinear resistors. However, when there is no coupling or the multi-terminal resistors are all reciprocal, the circuit equations can be written in terms of a potential function as follows:

Let

$$\hat{G}(v_G) = \sum_{j=1}^{n_G} \left[\int_0^{v_{G_i}} g_j(u)\,du \right] \tag{2.15}$$

be the co-content and

$$G(i_R) = \sum_{j=1}^{n_R} \left[\int_0^{i_{R_j}} r_j(u)du \right] \tag{2.16}$$

be the content.

Define

$$H(v_C, i_L) = i_L^T \mathbf{B}_{LC} v_C + i_J^T v_C - \hat{G}(-\mathbf{B}_{GC} v_C - \mathbf{B}_{GE} v_E)$$
$$+ i_L^T \mathbf{B}_{LV} v_E + G(\mathbf{B}_{LR}^T i_L + \mathbf{B}_{JR}^T i_J) \tag{2.17}$$

With $H(v_C, i_L)$ defined as above the state equations can be written as:

$$\mathbf{C}\frac{dv_C}{dt} = \left[\frac{\partial H(v_C, i_L)}{\partial v_C}\right]^T \tag{2.18a}$$

$$-\mathbf{L}\frac{di_L}{dt} = \left[\frac{\partial H(v_C, i_L)}{\partial i_L}\right]^T \tag{2.18b}$$

3. Analysis of the linear lossless circuit

Let N be a circuit consisting of passive linear inductors and capacitors which may be coupled. Assuming N satisfies the topological conditions (T1) and (T2), the state equations can be written as:

$$\mathbf{C}\frac{dv_C}{dt} = \mathbf{B}_{LC}^T i_L \tag{3.1a}$$

$$\mathbf{L}\frac{di_L}{dt} = -\mathbf{B}_{LC} v_C \tag{3.1b}$$

Since C and L are symmetric and positive definite, we can write C and L as follows [8]:

$$\mathbf{C} = \mathbf{C}^{\frac{1}{2}T}\mathbf{C}^{\frac{1}{2}} \tag{3.2}$$

$$\mathbf{L} = \mathbf{L}^{\frac{1}{2}T}\mathbf{L}^{\frac{1}{2}} \tag{3.3}$$

Let $\mathbf{C}^{-\frac{1}{2}}$ and $\mathbf{L}^{-\frac{1}{2}}$ denote the inverses of $\mathbf{C}^{\frac{1}{2}}$ and $\mathbf{L}^{\frac{1}{2}}$.

Theorem 3.1 (Existence of a decoupling transformation)

Under the above conditions there exist orthogonal matrices $\mathbf{P} \in \mathbf{R}^{n_C \times n_C}$ and $\mathbf{Q} \in \mathbf{R}^{n_L \times n_L}$ such that the change of variables

$$v_C = \mathbf{C}^{-\frac{1}{2}}\mathbf{P}x, \quad x \in \mathbf{R}^{n_C} \tag{3.4a}$$

$$i_L = \mathbf{L}^{-\frac{1}{2}}\mathbf{Q}y, \quad y \in \mathbf{R}^{n_L} \tag{3.4b}$$

transforms the state equations (3.1) into the form:

$$\frac{dx}{dt} = \mathbf{\Omega}^T y \tag{3.5a}$$

and

$$\frac{dy}{dt} = -\mathbf{\Omega} x \tag{3.5b}$$

where

$$\Omega = \begin{bmatrix} \Sigma & 0_{m \times (n_L - m)} \\ 0_{(n_C - m) \times m} & 0_{(n_C - m) \times (n_L - m)} \end{bmatrix} \tag{3.6}$$

with

$$\Sigma = diag\left[\omega_1, \omega_2, \ldots, \omega_m\right] \tag{3.7}$$

$\omega_1 \geq \omega_2 \geq \cdots \geq \omega_m > 0$, and $0_{k \times l}$ is a k by l matrix of zeros. The ω_i's are called the *natural* or *mode* frequencies and m is said to be the *number of degrees of freedom* or *modes* of the oscillator circuit.

Circuit Theoretic Interpretation

Let N, Figure 1, satisfy the previous hypotheses. Theorem 1 tells us that N is equivalent to $\bar{N}$, Figure 2, with

$$\bar{x}_i = \sqrt{\bar{C}_i}\, \bar{v}_{C_i}, \quad i = 1, 2, \ldots, n_C \tag{3.8a}$$

and

$$\bar{y}_j = \sqrt{\bar{L}_j}\, \bar{i}_{L_j}, \quad j = 1, 2, \ldots, n_L \tag{3.8b}$$

where

$$\omega_k = \frac{1}{\sqrt{\bar{L}_k \bar{C}_k}}, \quad k = 1, 2, \ldots, m \tag{3.9}$$

In essence, Theorem 3.1 allows us to decouple the circuit into separate tank circuits.

Proof of Theorem 3.1

Let

$$v_C = \mathbf{C}^{-1/2} w, \quad w \in \mathbf{R}^{n_C} \tag{3.10a}$$

$$i_L = \mathbf{L}^{-1/2} z, \quad z \in \mathbf{R}^{n_L} \tag{3.10b}$$

Under the above change of variables we get

$$\frac{dw}{dt} = \mathbf{A}^T z \tag{3.11a}$$

$$\frac{dz}{dt} = -\mathbf{A} w \tag{3.11b}$$

where

$$\mathbf{A} = \mathbf{L}^{-1/2T} \mathbf{B}_{LC} \mathbf{C}^{-1/2} \in \mathbf{R}^{n_L \times n_C} \tag{3.12}$$

The singular value decomposition of the matrix A, see [8], gives orthogonal matrices **P** and **Q** such that

$$\mathbf{Q}^T \mathbf{A} \mathbf{P} = \Omega \tag{3.13}$$

Letting

$$w = \mathbf{P} x \quad and \quad z = \mathbf{Q} y$$

then

$$\frac{dx}{dt} = \mathbf{P}^T \mathbf{A}^T \mathbf{Q} y = \left[\mathbf{Q}^T \mathbf{A} \mathbf{P}\right]^T y = \Omega^T y$$

and

$$\frac{dy}{dt} = -\mathbf{Q}^T \mathbf{A} \mathbf{P} x = -\mathbf{\Omega} x \quad \blacksquare$$

4. Transformation to a form suitable for averaging

Let N be a circuit satisfying the hypotheses of Section 2. If we open circuit the voltage controlled resistors and short circuit the current controlled resistors, we obtain a lossless circuit that satisfies the conditions of section 3. The circuit obtained in this manner is sometimes called the *generating circuit.* Applying the decoupling transformation to the state equation for the original circuit gives us the following state equations:

$$\frac{dx}{dt} = \mathbf{\Omega}^T y + x_s + \tilde{g}(x) \tag{4.1a}$$

$$\frac{dy}{dt} = -\mathbf{\Omega} x - y_s - \tilde{r}(y) \tag{4.1b}$$

where

$$\tilde{g}(x) = \mathbf{P}^T \mathbf{C}^{-\frac{1}{2}T} \mathbf{B}_{GC}^T g(-\mathbf{B}_{GC} \mathbf{C}^{-\frac{1}{2}} \mathbf{P} x - \mathbf{B}_{GE} v_E) \tag{4.1c}$$

$$\tilde{r}(y) = \mathbf{Q}^T \mathbf{L}^{-\frac{1}{2}T} \mathbf{B}_{LR}\, r(\mathbf{B}_{LR}^T \mathbf{L}^{-\frac{1}{2}} \mathbf{Q} y + \mathbf{B}_{JR}^T i_J) \tag{4.1d}$$

$$x_s = \mathbf{P}^T \mathbf{C}^{-\frac{1}{2}T} \mathbf{B}_{JC}^T i_J \tag{4.1e}$$

$$y_s = \mathbf{Q}^T \mathbf{L}^{-\frac{1}{2}T} \mathbf{B}_{LE} v_E \tag{4.1f}$$

We now make a *generalized* Van der Pol or polar coordinate transformation as follows. Let

$$x_i = \rho_i \cos\theta_i \ \text{ and } \ y_i = -\rho_i \sin\theta_i \quad 1 \le i \le m \tag{4.2a}$$

$$x_i = \rho_i \quad m < i \le n_C \tag{4.2b}$$

$$y_i = \rho_{n_C - m + i} \quad m < i \le n_L \tag{4.2c}$$

The state equations now become:

$$\frac{d\rho_i}{dt} = \cos\theta_i [x_{s_i} + \tilde{g}_i(x)] + \sin\theta_i [y_{s_i} + \tilde{r}_i(y)] \tag{4.3a}$$

$$\frac{d\theta_i}{dt} = \omega_i - \frac{\sin\theta_i}{\rho_i}[x_{s_i} + \tilde{g}_i(x)] + \frac{\cos\theta_i}{\rho_i}[y_{s_i} + \tilde{r}_i(y)] \tag{4.3b}$$

for $1 \le i \le m$

$$\frac{d\rho_i}{dt} = x_{s_i} + \tilde{g}_i(x) \tag{4.3c}$$

for $m < i \le n_C$ and

$$\frac{d\rho_{n_C - m + i}}{dt} = -y_{s_i} - \tilde{r}_i(y) \tag{4.3d}$$

for $m < i \le n_L$. From the above we can see that the state equations now have the general form:

$$\frac{d\theta}{dt} = \omega + \Theta(\theta, \rho), \quad \theta \in \mathbf{R}^m \tag{4.4a}$$

$$\frac{d\rho}{dt} = R(\theta, \rho), \quad \rho \in \mathbf{R}^{n_C + n_L - m} \tag{4.4b}$$

The Transformed Potential

When the differential equations of a circuit admit a potential function formulation, as in Section 2, equation (2.18), the analysis of the ordinary differential equation can take this *structure* into account to reveal additional information about the behavior of the circuit. One would hope that even after all the transformations we have performed on the circuit we can still find a suitable potential function. This turns out to be the case and will become the basis for the *averaged* potential.

Define

$$\begin{aligned} H(\theta, \rho) = {} & x_s^T x - \hat{G}(-\mathbf{B}_{GC}\mathbf{C}^{-1/2}Px - \mathbf{B}_{GE}v_E) \\ & - y^T y_s - G(\mathbf{B}_{LR}^T \mathbf{L}^{-1/2}Qy + \mathbf{B}_{JR}^T i_J) \end{aligned} \tag{4.5}$$

where $\hat{G}(\cdot)$ and $G(\cdot)$ are defined in equations (2.15) and (2.16) respectively.

Applying the chain rule, see Appendix I, we see that

$$\frac{d\rho}{dt} = \left[\frac{\partial H(\theta, \rho)}{\partial \rho}\right]^T \tag{4.6a}$$

and

$$\frac{d\theta_i}{dt} = \omega_i + \frac{1}{\rho_i^2}\left[\frac{\partial H(\theta, \rho)}{\partial \theta_i}\right]^T \tag{4.6b}$$

for $1 \le i \le m$.

5. Applying the theory of Averaging/Integral Manifolds

In this section we show how the theory of Averaging/Integral Manifolds allows us to analyze the solutions of the differential equation (4.4) via a *simpler* averaged equation. Note, we say Averaging/Integral Manifolds instead of just averaging because we are using the *method of averaging* combined with the *theory of integral manifolds.*

The equation to be studied is of the form

$$\dot{\theta} = \omega + \varepsilon\Theta(\theta, \rho) \tag{5.1a}$$

$$\dot{\rho} = \varepsilon R(\theta, \rho) \tag{5.1b}$$

where $\varepsilon \in \mathbf{R}$, $\theta \in \mathbf{R}^m$, $\rho \in \mathbf{R}^n$, $n = n_C + n_L - m$, $\omega = (\omega_1, \cdots, \omega_m)^T$, $\omega_j > 0$, $j = 1, \cdots, m$, the functions Ω, R are periodic of period 2π in each component of the vector θ, and are continuous with all derivatives up through the second order.

Note: all the above conditions are satisfied for the transformed equations, (4.4), for our circuit with the resistor characteristics twice continuously differentiable and multiplied, that is scaled, by ε.

The parameter ε in the above equations puts a mathematical meaning behind the idea that the nonlinearity is weak. One can also think of this as small *damping*, that is the resistance of the current controlled resistors and the conductance of the voltage controlled resistors are small. The theorems that follow will state various results under the condition that ε is sufficiently small.

The basic object that we will analyze in the differential equation (5.1) is an *integral manifold.* To motivate the definition and use of integral manifolds, consider a 2-dimensional autonomous oscillator, such as the classic Van der Pol oscillator, whose defining differential equation depends continuously on some parameter. As we vary that parameter a little bit the limit cycles will change a small amount and so will the frequency. Since the frequencies of the two systems are different, trajectories of the two systems starting close will eventually separate, even though their limit cycles as curves in the plane are close. It is for this reason we don't study the trajectories of the two systems but their limit cycles as invariant surfaces.

Definition 5.1. (Integral Manifolds)
Given a differential equation $\dot{z} = Z(t, z)$, $z \in \mathbf{R}^n$, a surface S in (t, z) space is said to be an *integral manifold* if for any point P in S, the solution $z(t)$ of the differential equation through P is such that (t, z(t)) is in S for all t in the domain of definition of the solution.

Define

$$\Theta_0(\theta, \rho) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \Theta(\theta + \omega t, \rho) dt \tag{5.2a}$$

$$R_0(\theta, \rho) = \lim_{T \to \infty} \frac{1}{T} \int_0^T R(\theta + \omega t, \rho) dt \tag{5.2b}$$

With the above definition of Θ_0 and R_0, we define the averaged system as:

$$\dot{\theta} = \omega + \varepsilon \Theta_0(\theta, \rho) \tag{5.3a}$$

$$\dot{\rho} = \varepsilon R_0(\theta, \rho) \tag{5.3b}$$

The above averages are usually referred to as the *time* average. As will be seen shortly, when Θ_0 and R_0 are independent of θ, the analysis is greatly simplified. If this is not the case, a meaningful analysis usually cannot be carried out. The following results tell us about the existence of the time average and its dependence on θ. In addition, these results can be used to simplify its computation.

Let $f(\cdot)$ be a real valued function on $\mathbf{R}^m$ such that $f(\cdot)$ is 2π periodic in each component of θ. We usually call $f(\cdot)$ a *function on the Torus* $\mathbf{T}^m$.

Definition 5.2
The *time average* of the function $f(\cdot)$ on the torus $\mathbf{T}^m$ with respect to the frequency vector ω is the function

$$f_{time}(\theta) = \lim_{T \to \infty} \int_0^T f(\theta + \omega t) dt \tag{5.4}$$

when the limit exists.

Definition 5.3
The *space average* of a function $f(\cdot)$ on the torus $\mathbf{T}^m$ is the number

$$f_{space} = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\theta) d\theta_1 \cdots d\theta_m \tag{5.5}$$

Definition 5.4
The frequencies $\omega_1, \omega_2, \ldots, \omega_m$ are said to be *incommensurable* or *independent* if

$$k_1\omega_1 + k_2\omega_2 + \cdots k_m\omega_m = 0$$

for $k_1, k_2, \cdots k_m \in \mathbf{Z}$ implies

$$k_1 = k_2 = \cdots k_m = 0$$

Theorem 5.1 (Theorem on averages [9])
(i) The *time average* exists everywhere.

(ii) If $f(\cdot)$ is continuous and the frequencies $\omega_1 \cdots \omega_m$ are incommensurable then the *time average* is equal to the *space average*.

Note: this is a sufficient condition for the time average to be independent of θ. Depending on the nonlinearity the condition for independence can be greatly relaxed, as the examples will show.

Fundamental results of Averaging/Integral Manifolds

Theorem 5.2 [10, 11]

Given that the averaged system (5.3) is independent of θ, that is,

$$\dot{\theta} = \omega + \varepsilon \Theta_0(\rho) \tag{5.6a}$$

$$\dot{\rho} = \varepsilon R_0(\rho) \tag{5.6b}$$

Suppose there exist a ρ_0 such that $R_0(\rho_0) = 0$ and the real parts of the eigenvalues of

$$A = \frac{\partial R_0(\rho)}{\partial \rho}\Big|_{\rho = \rho_0} \tag{5.7}$$

are nonzero, then there exists $\varepsilon_1 > 0$, a continuous function $D(\varepsilon)$, $0 < \varepsilon < \varepsilon_1$, approaching zero as $\varepsilon \to 0$ and a function $f(\theta, \varepsilon)$ in $\mathbf{R}^n$ which is continuous in $\mathbf{R}^m \times [0, \varepsilon_1]$ and satisfying

$$|f(\theta, \varepsilon) - \rho_0| < D(\varepsilon)$$

such that $f(\theta, \varepsilon)$ is 2π periodic in each component of θ and the set

$$S_\varepsilon = \left\{ (\theta, \rho) : \rho = f(\theta, \varepsilon),\ \theta \in \mathbf{R}^m \right\}$$

is an integral manifold of (5.1) with the manifold being stable if all eigenvalues of A have negative real parts and unstable if one eigenvalue has a positive real part.

When Θ_0 and R_0 depend on θ, the results get more specialized and complicated. One of the simplest is

Theorem 5.3 [10]

Given the averaged system (5.3). Suppose there exists a ρ_0 such that $R_0(\theta, \rho_0) = 0$ for all θ,

$$A = \frac{\partial R_0(\theta, \rho)}{\partial \rho}\Big|_{\rho = \rho_0}$$

and $\Theta_0(\theta, \rho_0)$ are independent of θ. Then the conclusions of Theorem 5.2 remain valid if the real part of the eigenvalues of A are non-zero.

The Averaged Potential

Suppose we have the following potential formulation

$$\dot{\rho} = \varepsilon \left[\frac{\partial H(\theta, \rho)}{\partial \rho} \right]^T \tag{5.8a}$$

$$\dot{\theta}_i = \omega_i + \frac{\varepsilon}{\rho_i^2} \left[\frac{\partial H(\theta, \rho)}{\partial \theta_i} \right] \tag{5.8b}$$

which can be obtained from the circuit equations as in Section 4.

Definition 5.5

Define

$$H_{av}(\theta, \rho) = \lim_{T \to \infty} \frac{1}{T} \int_0^T H(\theta + \omega t, \rho) dt \tag{5.9}$$

We call $H_{av}(\theta, \rho)$ the *averaged potential* [1].

The following theorems show the implications of the potential formulation on the averaged equations.

Theorem 5.4
Given the system (5.8) with ρ lying in a closed ball of finite radius. Then the averaged system is

$$\dot{\theta}_i = \omega_i + \frac{\varepsilon}{\rho_i^2}\frac{\partial H_{av}(\theta, \rho)}{\theta_i} \tag{5.10a}$$

$$\dot{\rho} = \varepsilon \left[\frac{\partial H_{av}(\rho)}{\partial \rho}\right]^T \tag{5.10b}$$

Proof
It will suffice to establish the interchange of averaging and differentiation which will also establish the differentiability of the averaged potential. We prove the partial derivative with respect to θ_i-case. The ρ_i-case is analogous. Since $H(\theta, \rho)$ is 2π-periodic in each θ_i and continuously differentiable, $\partial H(\theta, \rho)/\partial\theta_i$ is uniformly continuous in θ_i. Given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that

$$\left| \frac{\partial H(\theta + \omega t, \rho)}{\partial \theta_i} - \frac{\partial H(\theta_0 + \omega t, \rho)}{\partial \theta_i} \right| < \varepsilon \tag{5.11}$$

where

$$\theta_0 = [\theta_1, \ldots, \theta_{i-1}, \theta_{i0}, \theta_{i+1}, \ldots, \theta_m]$$

if

$$0 < |\theta_i - \theta_{i0}| < \delta(\varepsilon)$$

for all t and ρ. The mean value theorem gives:

$$H(\theta + \omega t, \rho) - H(\theta_0 + \omega t, \rho) = (\theta_i - \theta_{i0})\frac{\partial H(\theta_a + \omega t, \rho)}{\partial \theta_i} \tag{5.12}$$

where

$$\theta_a = [\theta_1, \ldots, \theta_{i-1}, \theta_{ia}, \theta_{i+1}, \ldots, \theta_m]^T$$

and θ_{ia} is between θ_i and θ_{i0}. Equations (5.11) and (5.12) imply:

$$\left| \frac{H(\theta + \omega t, \rho) - H(\theta_0 + \omega t, \rho)}{\theta_i - \theta_{i0}} - \frac{\partial H(\theta + \omega t, \rho)}{\partial \theta_i} \right| < \varepsilon \tag{5.13}$$

Finally we see that

$$\left| \frac{H_{av}(\theta, \rho) - H_{av}(\theta_0, \rho)}{\theta_i - \theta_{i0}} - \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{\partial H(\theta + \omega t, \rho)}{\partial \theta_i} \right|$$

$$\leq \lim_{T\to\infty} \frac{1}{T}\int_0^T \left| \frac{H(\theta + \omega t, \rho) - H(\theta_0 + \omega t, \rho)}{\theta_i - \theta_{i0}} - \frac{\partial H(\theta + \omega t, \rho)}{\partial \theta_i} \right|$$

$$\leq \lim_{T\to\infty} \frac{1}{T}(\varepsilon T) = \varepsilon \quad \blacksquare$$

Theorem 5.5
Given the system (5.8). Suppose the frequencies $\omega_1, \omega_2, \ldots \omega_m$ are incommensurable. Then the averaged system is

$$\dot{\theta} = \omega \tag{5.14a}$$

$$\dot{\rho} = \varepsilon \left[\frac{\partial H_{av}(\rho)}{\partial \rho} \right]^T \tag{5.14b}$$

and

$$H_{av}(\rho) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} H(\theta, \rho) d\theta_1 \cdots d\theta_m \tag{5.15}$$

Proof

Since the frequencies are incommensurable, we can replace the time averages with space averages.

$$\Theta_{0_i}(\theta, \rho) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{1}{\rho_i^2} \frac{\partial H(\theta, \rho)}{\partial \theta_i} d\theta_1 \cdots d\theta_m \tag{5.16}$$

This is easily seen to be zero by first applying the fundamental theorem of calculus to the θ_i integral and then using the fact that $H(\theta, \rho)$ is 2π periodic in each component of θ.

$$R_0(\theta, \rho) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{\partial H(\theta, \rho)}{\partial \rho} d\theta_1 \cdots d\theta_m \tag{5.17}$$

From real analysis[12] we can interchange the order of differentiation and integration to get

$$R_0(\theta, \rho) = \frac{\partial H_{av}(\rho)}{\partial \rho} \quad \blacksquare \tag{5.18}$$

The differential equation (5.8a) for ρ is known as a *gradient* system[13]. Consequently, if ρ_0 is an isolated maximum of $H_{av}(\rho)$, then ρ_0 is an asymptotically stable equilibrium of (5.8a). If ρ_0 is an asymptotically stable equilibrium of (5.8a), then the eigenvalues of

$$\mathbf{A} = \left. \frac{\partial^2 H_{av}(\rho)}{\partial \rho^2} \right|_{\rho = \rho_0} \tag{5.19}$$

are real and less than or equal to zero. Hence, once we have an isolated maximum of $H_{av}(\rho)$ all we need to do is check that $\mathbf{A}$ is non-singular and then the conditions of Theorem 5.2 are satisfied with the integral manifold stable.

6. Analysis algorithm and examples

In the previous sections we have given the necessary circuit theoretic and mathematical background needed for the analysis of weakly nonlinear oscillator circuits. Here we summarize the steps of the analysis.

Algorithm

0. Check that the circuit satisfies the topological conditions (T1)-(T4).
1. For the graph of the circuit let the tree contain all the capacitors, independent voltage sources and current controlled resistors, i.e., {C's, E's, R's} and the cotree contain all the inductors, independent current sources and voltage controlled resistors, i.e., {L's, J's, G's}. Label and order the circuit variables as in (2.1) and (2.2). Find the fundamental loop matrix and partition it as in equation (2.3).
2. Factor the C and L matrices as in (3.2) and (3.3). Note: both the $\mathbf{C}^{1/2}$ matrix and the $\mathbf{L}^{1/2}$ matrix can be obtained using Cholesky decomposition[8] since they are symmetric and positive definite.
3. Form the matrix $\mathbf{A} = \mathbf{L}^{-1/2 T} \mathbf{B}_{LC} \mathbf{C}^{-1/2 T}$ as in equation (3.12), then calculate the singular value decomposition of the matrix $\mathbf{A}$, i.e., obtain the orthogonal matrices $\mathbf{P}$ and $\mathbf{Q}$ such that $\mathbf{Q}^T \mathbf{A} \mathbf{P} = \Omega$ with Ω as in (3.6) and (3.7). The ω_i's will be the mode

frequencies.

4. Use the matrices obtained in steps 2. and 3. to calculate the transformed equations (4.1).
5. Apply the generalized polar coordinate transformation of equation (4.2) to finally arrive at the state equation (4.3). Note: since the resistors are two-terminal, we can use the co-content (2.15) and content (2.16) to form the transformed potential, $H(\theta, \rho)$, as in equation (4.5), and obtain the potential formulation (4.14).
6. Obtain the averaged system. Note: if the frequencies are incommensurable, we can make use of Theorem 5.1 and use space averages. Furthermore, if we have a potential formulation, we can make use of the results of Theorem 5.4 and possibly Theorem 5.5.
7. Finally, apply the main theorems: if the averaged system is independent of θ, use Theorem 5.2, otherwise try Theorem 5.3.

Remarks:

1. If multi-terminal resistors are present then we can still form the state equations[6]. However, for general multi-terminal resistors there will be no potential formulation.
2. If the multi-terminal resistors are *reciprocal,* then we can obtain a potential formulation.
3. If the circuit contains capacitor loops or inductor cutsets, we can transform the circuit via the techniques of [6] and proceed with the analysis.

Example 1 (Endo and Mori[14])

The circuit shown in Figure 3 is an inductively coupled pair of Van der Pol oscillators. The resistor nonlinearity is $g(v) = -g_1 v + g_3 v^3$, with $g_1, g_3 > 0$. For our analysis we will take $C_1 = C_2 = C$, $L_1 = L_2 = L$ and $L_3 = L_0$. For illustrative purposes, we will go through all the steps of the analysis algorithm in this example.

Step 1) We pick $\{C_1, C_2\}$ as the tree for the graph and $\{L_1, L_2, L_3, G_1, G_2\}$ as the subtree. With this we obtain the crucial components of the fundamental loop matrix $\mathbf{B}_{LC}$ and $\mathbf{B}_{GC}$.

$$\mathbf{B}_{LC} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} \tag{6.1a}$$

and

$$\mathbf{B}_{GC} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{6.1b}$$

Step 2) Since $\mathbf{C}$ and $\mathbf{L}$ are both diagonal, obtaining $\mathbf{C}^{1/2}$ and $\mathbf{L}^{1/2}$ is trivial, i.e.,

$$\mathbf{C}^{1/2} = \begin{bmatrix} \sqrt{C} & 0 \\ 0 & \sqrt{C} \end{bmatrix} \tag{6.2a}$$

and

$$\mathbf{L}^{1/2} = \begin{bmatrix} \sqrt{L} & 0 & 0 \\ 0 & \sqrt{L} & 0 \\ 0 & 0 & \sqrt{L_0} \end{bmatrix} \tag{6.2b}$$

Step 3)

$$\mathbf{A} = L^{-1/2^T}\mathbf{B}_{LC}\mathbf{C}^{-1/2^T} = \begin{bmatrix} -\frac{1}{\sqrt{LC}} & 0 \\ 0 & -\frac{1}{\sqrt{LC}} \\ -\frac{1}{\sqrt{L_0 C}} & \frac{1}{\sqrt{L_0 C}} \end{bmatrix} \tag{6.3}$$

Calculating the singular value decomposition of A gives:

$$\mathbf{Q}^T\mathbf{AP} = \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \\ 0 & 0 \end{bmatrix} \tag{6.4a}$$

where

$$\omega_1 = \frac{1}{\sqrt{LC}} \tag{6.4b}$$

$$\omega_2 = \sqrt{\frac{1}{C}\left(\frac{1}{L} + \frac{2}{L_0}\right)} \tag{6.4c}$$

$$\mathbf{P} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{6.4d}$$

and

$$\mathbf{Q} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\left[\frac{L_0}{2(L_0+2L)}\right]^{1/2} & -\left[\frac{L}{(L_0+2L)}\right]^{1/2} \\ -\frac{1}{\sqrt{2}} & \left[\frac{L_0}{2(L_0+2L)}\right]^{1/2} & \left[\frac{L}{(L_0+2L)}\right]^{1/2} \\ 0 & -\left[\frac{2L}{(L_0+2L)}\right]^{1/2} & \left[\frac{L_0}{(L_0+2L)}\right]^{1/2} \end{bmatrix} \tag{6.4e}$$

Step 4) We now use the matrices **P** and **Q** to obtain the transformed state equations.

$$\frac{dx}{dt} = \begin{bmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \end{bmatrix} y + \frac{g_1}{C}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \frac{g_3}{2C^2}\begin{bmatrix} x_1(x_1^2 + 3x_2^2) \\ x_2(x_2^2 + 3x_1^2) \end{bmatrix} \tag{6.5a}$$

$$\frac{dy}{dt} = -\begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \\ 0 & 0 \end{bmatrix} x \tag{6.5b}$$

Step 5) We now apply the *generalized* polar coordinate transformation to the system (6.5) to obtain:

$$\dot{\rho}_1 = \frac{g_1}{C}\rho_1\cos^2\theta_1 - \frac{g_3}{2C^2}\rho_1\cos^2\theta_1[\rho_1^2\cos^2\theta_1 + 3\rho_2^2\cos^2\theta_2] \tag{6.6a}$$

$$\dot{\rho}_2 = \frac{g_1}{C}\rho_2\cos^2\theta_2 - \frac{g_3}{2C^2}\rho_2\cos^2\theta_2[\rho_2^2\cos^2\theta_2 + 3\rho_1^2\cos^2\theta_1] \tag{6.6b}$$

$$\dot{\rho}_3 = 0 \tag{6.6c}$$

$$\dot{\theta}_1 = \omega_1 - \frac{g_1}{C}\sin\theta_1\cos\theta_1 + \frac{g_3}{2C^2}\sin\theta_1\cos\theta_1[\rho_1^2\cos^2\theta_1 + 3\rho_2^2\cos^2\theta_2] \tag{6.6d}$$

$$\dot{\theta}_2 = \omega_2 - \frac{g_1}{C}\sin\theta_2\cos\theta_2 + \frac{g_3}{2C^2}\sin\theta_2\cos\theta_2[\rho_2^2\cos^2\theta_2 + 3\rho_1^2\cos^2\theta_1] \tag{6.6d}$$

The transformed potential can be calculated to be:

$$H(\theta, \rho) = \frac{1}{2C}(\rho_1^2\cos^2\theta_1 + \rho_2^2\cos^2\theta_2)$$
$$- \frac{g_3}{8C^2}(\rho_1^4\cos^4\theta_1 + 6\rho_1^2\rho_2^2\cos^2\theta_1\cos^2\theta_2 + \rho_2^4\cos^4\theta_2) \tag{6.7}$$

Note, it is usually easier to calculate $H(\theta, \rho)$ first and then use it to obtain the new state equations. Furthermore, we usually don't need to bother calculating the transformed equations, just the averaged equations.

Step 6) To obtain the averaged system we can calculate directly from equations (6.6) or use the averaged potential. If $\omega_1 \neq \omega_2$ then

$$H_{av}(\rho) = \frac{g_1}{4C}(\rho_1^2 + \rho_2^2) - \frac{3g_3}{64C^2}(\rho_1^4 + 4\rho_1^2\rho_2^2 + \rho_2^4) \tag{6.8}$$

Using $H_{av}(\rho)$ we obtain the averaged equations.

$$\dot{\rho}_1 = \frac{g_1}{2C}\rho_1 - \frac{3g_3}{16C^2}\rho_1(\rho_1^2 + 2\rho_2^2) \tag{6.9a}$$

$$\dot{\rho}_2 = \frac{g_1}{2C}\rho_2 - \frac{3g_3}{16C^2}\rho_2(\rho_2^2 + 2\rho_1^2) \tag{6.9b}$$

$$\dot{\theta}_1 = \omega_1, \quad \dot{\theta}_2 = \omega_2 \tag{6.9c}$$

Step 7) We will use Theorem 5.2 to tell us of the existence of an integral manifold and hence oscillations for the circuit. We need to look at the equilibrium points of (6.9a) and (6.9b) and calculate their stability. Note:

$$\frac{\partial R_0}{\partial \rho} = \begin{bmatrix} \frac{g_1}{2C} - \frac{3g_3}{16C^2}(3\rho_1^2 + 2\rho_2^2) & \frac{-3g_3}{4C^2}\rho_1\rho_2 \\ \frac{-3g_3}{4C^2}\rho_1\rho_2 & \frac{g_1}{2C} - \frac{3g_3}{16C^2}(3\rho_2^2 + 2\rho_1^2) \end{bmatrix} \tag{6.10}$$

Case (i):

$$\rho_1 = 0, \quad \rho_2 = 0$$

The eigenvalues of $\partial R_0/\partial\rho$ at this equilibrium point are

$$\lambda_1 = \frac{g_1}{2C}, \quad \lambda_2 = \frac{g_1}{2C}$$

Both are positive. Hence, this integral manifold is (completely) unstable. This is the non-oscillatory case.

Case (ii):

$$\rho_1 = 0, \quad \rho_2 = \left[\frac{8g_1C}{3g_3}\right]^{1/2}$$

The eigenvalues of $\partial R_0/\partial\rho$ at this equilibrium point are

$$\lambda_1 = \frac{-g_1}{2C}, \quad \lambda_2 = \frac{-g_1}{C}$$

Since the eigenvalues are both negative, the integral manifold is stable. In this case we have a stable single-mode oscillation.

Case (iii):

$$\rho_2 = 0, \quad \rho_1 = \left[\frac{8g_1C}{3g_3}\right]^{1/2}$$

The eigenvalues of $\partial R_0/\partial\rho$ at this equilibrium point are

$$\lambda_1 = \frac{-g_1}{2C}, \quad \lambda_2 = \frac{-g_1}{C}$$

Since the eigenvalues are both negative, the integral manifold is stable. In this case we have a stable single-mode oscillation.

Case (iv):

$$\rho_1 = \left[\frac{8g_1C}{9}g_3\right]^{1/2}, \quad \rho_2 = \left[\frac{8g_1C}{9}g_3\right]^{1/2}$$

The eigenvalues of $\partial R_0/\partial\rho$ are

$$\lambda_1 = \frac{-g_1}{C}, \quad \lambda_2 = \frac{g_1}{3C}$$

One eigenvalue is positive the other negative. Hence, the integral manifold is unstable (saddle like). In this case we say that we have an unstable double-mode oscillation.

Example 2

The circuit shown in Figure 4 is a double tank circuit with a single nonlinear resistor. This circuit was first studied by Van der Pol and later by many others, notably Bruyland[15]. We will study this circuit for the case $C_1 = C_2 = C$ and resistor nonlinearity $g(v) = g_1v + g_3v^3 + g_5v^5$.

Picking $\{C_1, C_2\}$ for the tree and $\{L_1, L_2, g\}$ for the cotree we get:

$$\mathbf{B}_{LC} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{B}_{GC} = [-1, -1] \tag{6.11}$$

The A matrix in this case is very simple, namely

$$\mathbf{A} = \begin{bmatrix} \frac{-1}{\sqrt{L_1C}} & 0 \\ 0 & \frac{-1}{\sqrt{L_2C}} \end{bmatrix} \tag{6.12a}$$

The singular value decomposition is trivial, i.e.,

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6.12b}$$

$$\mathbf{Q} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{6.12c}$$

The linearly transformed state equations are:

$$\dot{x} = \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix} y - \frac{1}{\sqrt{C}} g\left(\frac{1}{\sqrt{C}}x_1 + \frac{1}{\sqrt{C}}x_2\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{6.13a}$$

$$\dot{y} = \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix} x \tag{6.13b}$$

After applying the generalized polar coordinate transformation, the state equation above becomes somewhat messy and not very useful. So we will use the potential formulation to simplify matters. The transformed potential is:

$$H(\theta, \rho) = \frac{-g_1}{2C}(x_1^2 + 2x_1x_2 + x_2^2) - \frac{g_3}{4C^2}(x_1^4 + 4x_1^3x_2 + 6x_1^2x_2^2 + x_2^4)$$
$$- \frac{g_5}{6C^3}(x_1^6 + 6x_1^5x_2 + 15x_1^4x_2^2 + 20x_1^3x_2^3 + 15x_1^2x_2^4 + 6x_1x_2^5 + x_2^6) \quad (6.14)$$

where $x_1 = \rho_1\cos\theta_1$ and $x_2 = \rho_2\cos\theta_2$.

The averaged potential in the non-resonant case, i.e., $\omega_i \neq \omega_j$, $\omega_i \neq 2\omega_j$, $\omega_i \neq 3\omega_j$ and $\omega_i \neq 5\omega_j$ where $(i, j) = (1, 2)$ or $(2, 1)$, is:

$$H_{av}(\rho) = \frac{-g_1}{4C}(\rho_1^2 + \rho_2^2) - \frac{g_3}{32C^2}(3\rho_1^4 + 12\rho_1^2\rho_2^2 + 3\rho_2^4)$$
$$- \frac{g_5}{96C^3}(5\rho_1^6 + 45\rho_1^4\rho_2^2 + 45\rho_1^2\rho_2^4 + 5\rho_2^6) \quad (6.15)$$

Using the averaged potential we can find the averaged equations:

$$\dot{\rho}_1 = -\rho_1\left[\frac{g_1}{2C} + \frac{g_3}{8C^2}(3\rho_1^2 + 6\rho_2^2) + \frac{g_5}{16C^3}(5\rho_1^4 + 30\rho_1^2\rho_2^2 + 15\rho_2^4)\right] \quad (6.16a)$$

$$\dot{\rho}_2 = -\rho_2\left[\frac{g_1}{2C} + \frac{g_3}{8C^2}(3\rho_2^2 + 6\rho_1^2) + \frac{g_5}{16C^3}(5\rho_2^4 + 30\rho_1^2\rho_2^2 + 15\rho_1^4)\right] \quad (6.16b)$$

We are interested in the existence of double mode oscillations. So we will study the equilibrium points of (6.16) and there stability when ρ_1 and ρ_2 are strictly greater than zero. The analysis is broken into two cases.

Case (i): Third order eventually passive nonlinearity, i.e., $g_5 = 0$ and $g_3 > 0$.

Assuming $\rho_1 > 0$ and $\rho_2 > 0$, we get the following equilibrium point for (6.16):

$$\rho_1^2 = \rho_2^2 = \frac{-4g_1C}{9g_3} \quad (6.17a)$$

Note that this equation requires $g_1 < 0$ for the equilibrium point to exist. Calculating the eigenvalues of $\partial R_0/\partial\rho$ at this equilibrium point gives:

$$\lambda_1 = \frac{g_1}{C} < 0, \quad \lambda_2 = \frac{-g_1}{3C} > 0 \quad (6.17b)$$

Thus the integral manifold is saddle-like and for the third power nonlinearity there is no stable double mode oscillation. This is the classic result of Van der Pol.

Case (ii): Fifth power eventually passive nonlinearity, i.e., $g_5 > 0$. In solving (6.16) with $\rho_1 > 0$ and $\rho_2 > 0$ we find that $\rho_1 = \rho_2$. Define $\rho = \rho_1 = \rho_2$. The equilibrium point equation becomes:

$$25g_5\rho^4 + 9g_3C\rho^2 + 4g_1C^2 = 0 \quad (6.18)$$

Define $\hat{\lambda}_1 = 8C^3\lambda_1$ and $\hat{\lambda}_2 = 8C^3\lambda_2$, where λ_1 and λ_2 are eigenvalues of $\partial R_0/\partial\rho$ at ρ. Since $C > 0$, the signs of $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are the same as λ_1 and λ_2. Requiring the equilibrium point to be stable gives the following inequalities:

$$\hat{\lambda}_1 = -125g_5\rho^4 - 27g_3C\rho^2 - 4g_1C^2 < 0 \quad (6.19a)$$

$$\hat{\lambda}_2 = -5g_5\rho^4 - 3g_3C\rho^2 - 4g_1C^2 < 0 \quad (6.19b)$$

From these equations we arrive at the following conditions for the existence of a stable double mode oscillation:

$$g_1 > 0 \quad \text{and} \quad g_3 < 0 \tag{6.20a}$$

$$g_1 < \frac{81}{400}\frac{g_3^2}{g_5} \tag{6.20b}$$

$$g_1 > \frac{45}{400}\frac{g_3^2}{g_5} \tag{6.20c}$$

with the equilibrium at

$$\rho^2 = \left[\frac{-9g_3 + \sqrt{81g_3^2 - 400g_1g_5}}{50g_5}\right]C \tag{6.21}$$

These relations are plotted in the $g_1 - g_3$ plane for a fixed g_5 in Figure 5. Hence, we have just proven the existence of a stable double mode oscillation in this circuit for the parameter range specified above.

7. Conclusion

In this paper we have shown how to systematically analyze weakly nonlinear oscillator circuits via averaging and put this analysis on a firm mathematical foundation. In a later paper we will look more closely at the resonance cases, phase locking and frequecy entrainment.

Appendix I

Let $g(x)$, $h(y)$, $u(x)$ and $v(y)$ be continuously differentiable. Then from the chain rule we easily see that

$$\frac{\partial g(x)}{\partial \rho_i} = \frac{\partial g(x)}{\partial x_i}\begin{Bmatrix}\cos\theta_i, & 1 \le i \le m \\ 1, & otherwise\end{Bmatrix} \tag{A.1}$$

$$\frac{\partial h(y)}{\partial \rho_i} = \frac{\partial h(y)}{\partial y_i}\begin{Bmatrix}-\sin\theta_i, & 1 \le i \le m \\ 1, & otherwise\end{Bmatrix} \tag{A.2}$$

and

$$\frac{\partial u(x)}{\partial \theta_i} = \frac{\partial u(x)}{\partial x_i}(-\rho_i \sin\theta_i) \tag{A.3}$$

$$\frac{\partial v(y)}{\partial \theta_i} = \frac{\partial v(y)}{\partial y_i}(-\rho_i \cos\theta_i) \tag{A.4}$$

for $1 \le i \le m$. Applying these to $H(\theta, \rho)$ we get

$$\frac{\partial H(\theta, \rho)}{\partial \rho_i} = \cos\theta_i(x_{s_i} + \tilde{g}_i(x)) + \sin\theta_i(y_{s_i} + \tilde{f}_i(y)) \tag{A.5}$$

for $1 \le i \le m$

$$\frac{\partial H(\theta, \rho)}{\partial \rho_i} = x_{s_i} + \tilde{g}_i(x) \tag{A.6}$$

for $m < i \le n_C$

$$\frac{\partial H(\theta, \rho)}{\partial \rho_{n_C - m + i}} = y_{s_i} + \tilde{f}_i(y) \tag{A.7}$$

for $m < i \le n_L$ and

$$\frac{\partial H(\theta, \rho)}{\partial \theta_i} = -\rho_i \sin\theta_i (x_{s_i} + \tilde{g}_i(x)) + \rho_i \cos\theta_i (y_{s_i} + \tilde{f}_i(y)) \tag{A.8}$$

for $1 \le i \le m$. Comparing with the transformed equation (4.3) we see that

$$\frac{d\rho}{dt} = \left[\frac{\partial H(\theta, \rho)}{\partial \rho}\right]^T \tag{A.9a}$$

and

$$\frac{d\theta_i}{dt} = \omega_i + \frac{1}{\rho_i^2}\left[\frac{\partial H(\theta, \rho)}{\partial \theta_i}\right]^T \tag{A.9b}$$

for $1 \le i \le m$.

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Figure captions

Fig. 1
An arbitrary LC circuit, N, satisfying (T1) and (T2).

Fig. 2
$\bar{N}$ the decoupled equivalent circuit to N.

Fig. 3
Inductively coupled pair of Van der Pol oscillators.

Fig. 4
Double tank circuit of Van der Pol.

Fig. 5
Plot of parameter region where a stable double-mode oscillation occurs for the double tank circuit.

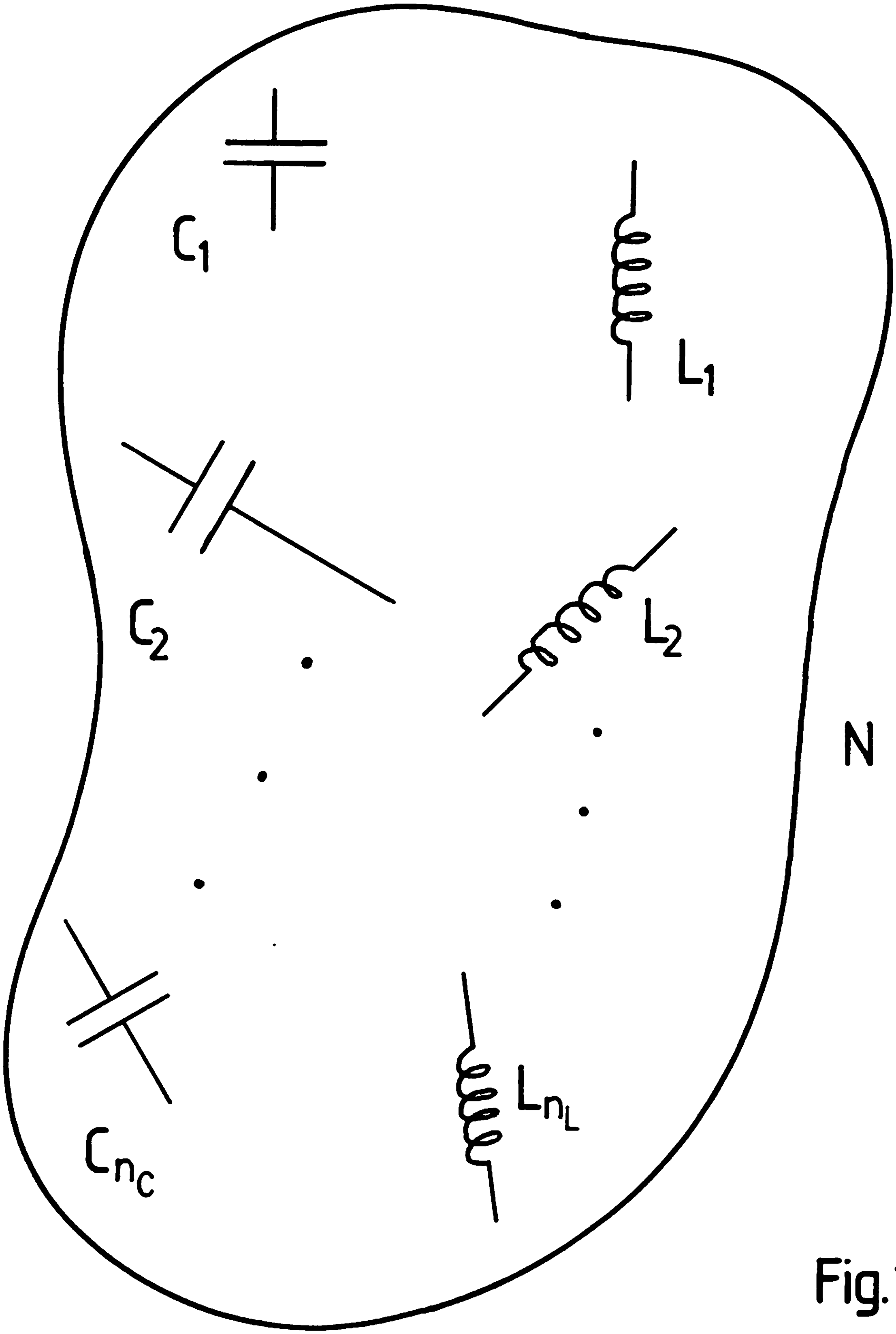
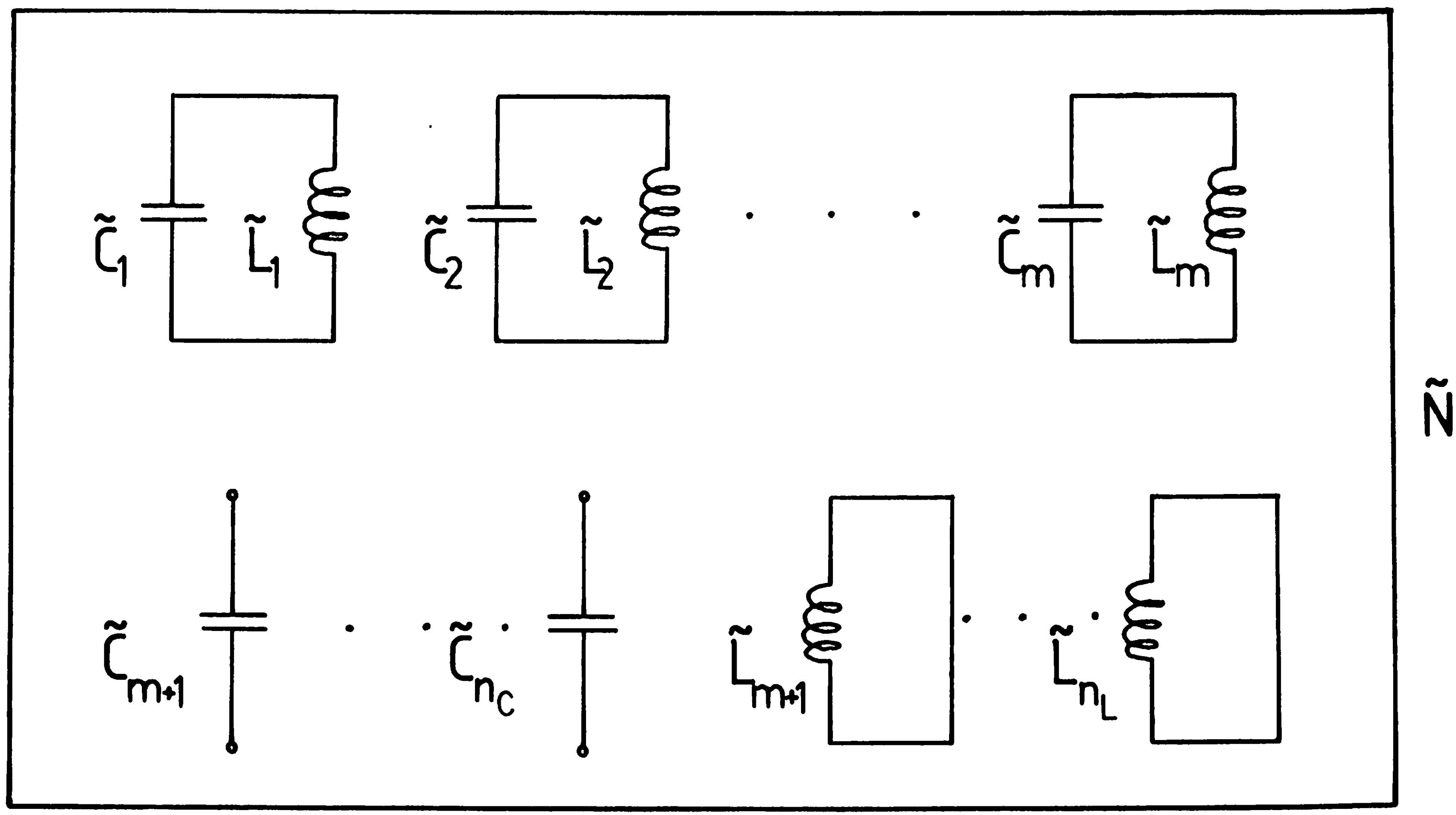


Fig.1



$\tilde{C}_1$
$\tilde{L}_1$
$\tilde{C}_2$
$\tilde{L}_2$
$\tilde{C}_m$
$\tilde{L}_m$
$\tilde{N}$
$\tilde{C}_{m+1}$
$\tilde{C}_{n_C}$
$\tilde{L}_{m+1}$
$\tilde{L}_{n_L}$

Fig. 2

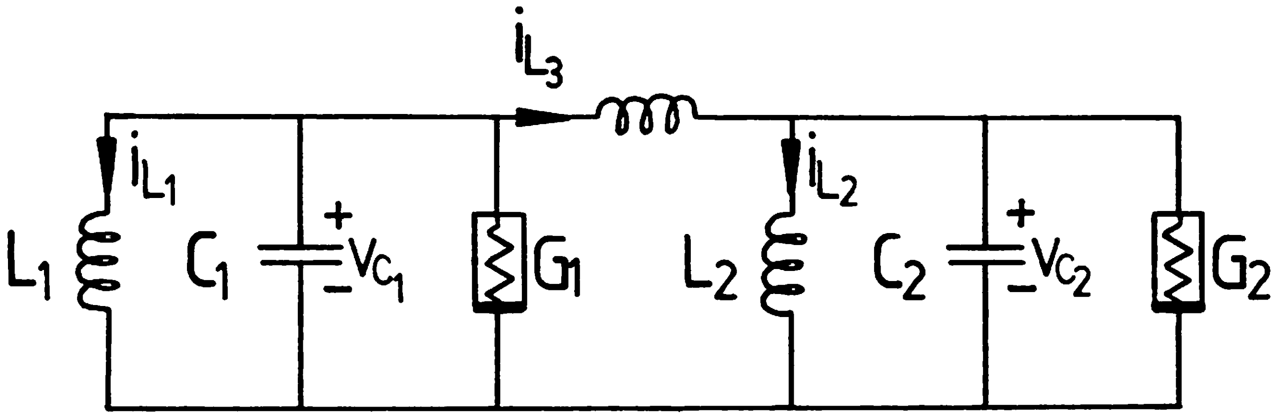


Fig. 3

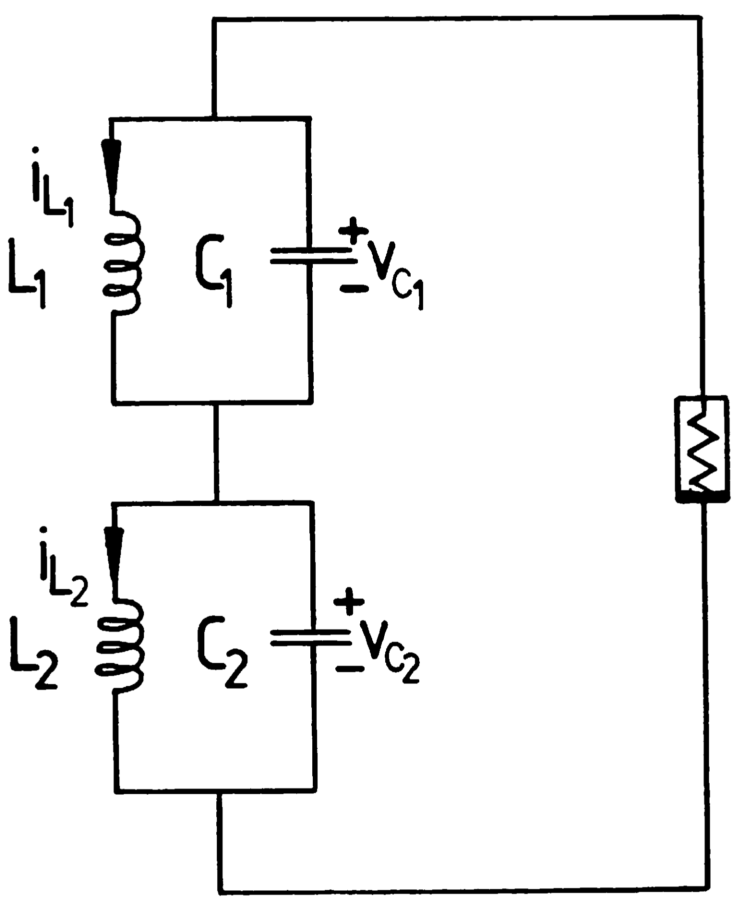


Fig. 4

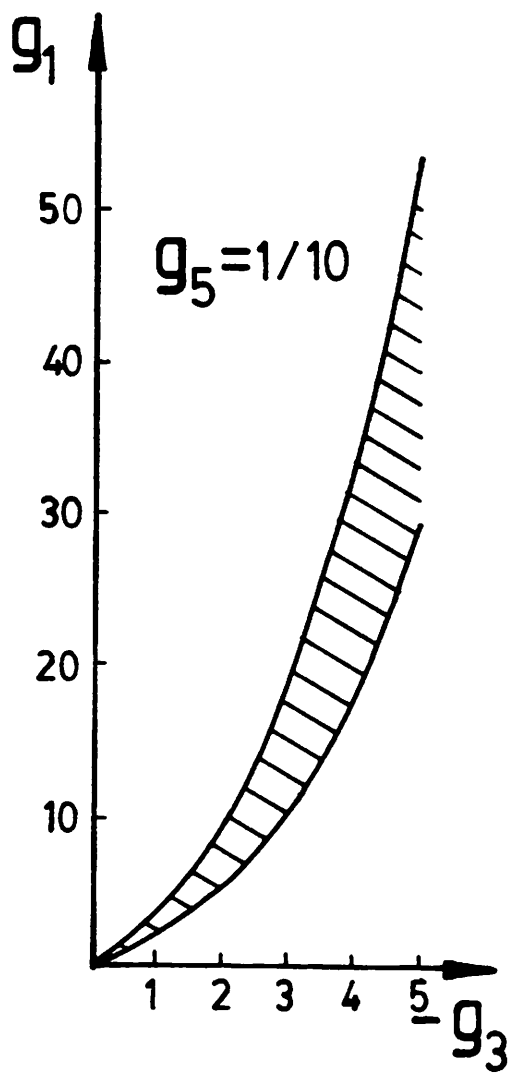


Fig. 5