KINEMATICS AND CONTROL OF ROBOT MANIPULATORS

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Bradley Evan Paden

Memorandum No. UCB/ERL M86/5

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Abstract

This dissertation focuses on the kinematics and control of robot manipulators. The contribution to kinematics is a fundamental theorem on the design of manipulators with six revolute joints. The theorem states, roughly speaking, that manipulators which have six revolute joints and are modeled after the human arm are optimal and essentially unique. In developing the mathematical framework to prove this theorem, we define precisely the notions of length of a manipulator, well-connected-workspace, and work-volume. We contribute to control a set of analysis techniques for the design of variable structure (sliding mode) controllers for manipulators.

The organization of the dissertation is the following. After introductory remarks in chapter one, the group of proper rigid motions, G, is introduced in chapter two. The tangent bundle of G is introduced and it is shown that the velocity of a rigid body can be represented by an element in the Lie algebra of G (commonly called a twist). Further, rigid motions which are exponentials af twists are used to describe four commonly occurring subproblems in robot kinematics. In chapter three, the exponentials of twists are used to write the forward kinematic map of robot manipulators and the subproblems of chapter two are used to solve the Stanford manipulator and an elbow manipulator. Chapter four focuses on manipulator singularities. Twist coordinates are used to find critical points of the forward kinematic map. The contribution to kinematics is contained in chapter five where a mathematical framework for studying the relationship between the design of 6R manipulators and their performance is developed. Chapter seven contains the contribution to control. The work of A. F. Filippov on differential equations with discontinuous right-hand-side and the work of F. H. Clarke on generalized gradients are combined to obtain a calculus for analyzing nonsmooth gradient systems. The techniques developed are applied to design a simple variable structure controller for the nonlinear dynamics of robot manipulators.

Key words: robotics, kinematics, Lie group, screws, sliding mode control, generalized gradient.

S. Shankar Sastry

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Notation

$\mathbf{N}$	Natural numbers, $\{1,2,3,...\}$.
$\mathbf{R}$	Real numbers.
$2^{\mathbf{R}^n}$	The collection of subsets of $\mathbf{R}^n$.
$\lVert \cdot \rVert_1$	1-norm of a vector.
$\lVert \cdot \rVert$	2-norm in $\mathbf{R}^n$.
$\lVert \cdot \rVert_P$	2-norm induced by the positive definite matrix P.
$B(x,\delta)$	The open ball of radius δ centered at x
$\bar{B}(x,\delta)$	The closed ball of radius δ centered at x.
$SO(3)$	The group of proper rotations in $\mathbf{R}^3$. The set of 3×3 orthogonal matrices with determinant one.
G	The group of proper rigid motions in $\mathbf{R}^3$.
G_O	The isotropy group of G at $O \in \mathbf{R}^3$. Those rigid motions which have O as a fixed point.
S^1	The unit circle (i.e., $[0, 2\pi]$ with the endpoints identified).
S^2	The unit 2-sphere.
$\mathbf{T}^n$	The n-torus.
TM	Tangent bundle of the manifold M.
$T_x M$	Tangent space of M at x.
$\sim$	Equivalent to.
$\cong$	"Diffeomorphic to."
$\cong_V$	"Diffeomorphic to" via a volume preserving diffeomorphism.
$\parallel$	"Parallel to."
O, P	Points in $\mathbf{R}^3$.
$\overline{OP}$	The line segment connecting O and P.

$\perp$	"Perpendicular to."
$\bar{\theta}, \bar{\zeta}$	An ordered set or list of indexed objects (usually 6). (e.g. $\bar{\theta}=(\theta_0,\ldots,\theta_5)$).
$\bar{\zeta}^*$	The ordered set or list of objects whose elements are those of $\bar{\zeta}$ with their order reversed.
S^C	Complement of the set S.
$\emptyset$	Empty set.
C^r	Continuously differentiable r times.
$C_{\bar{\xi}}$	The set of linking curves of $\bar{\xi}$.
C_f	The set of critical points of the function f.
$\sigma_{\min}, \sigma_{\max}$	Minimum and maximum singular values.
$sgn(\cdot)$	Sign function $sgn(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -1 \text{ if } x > 0 \end{cases}$.
$SGN(\cdot)$	Set-valued sign function $SGN(x) = \begin{cases} \{1\} \text{ if } x > 0 \\ [-1,1] \text{ if } x = 0. \\ \{-1\} \text{ if } x > 0 \end{cases}$
$ATAN_2$	Two argument arctangent function.
co	"Convex hull."
$\overline{co}$	"Convex closure."
$argmin$	"The argument which minimizes."
μ	Lebesgue measure
$a.e.$	Almost everywhere with respect to Lebesgue measure.
∂f	Generalized gradient of f.
A^T	The transpose of A.
$\sigma(A)$	The spectrum of A.
$\mathbb{C}_+^\circ$	The open right half complex plane.
$f\|_U$	f restricted to U.

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Chapter One

Introduction

This thesis will focus on the kinematics and control of robot manipulators. Robot manipulators are the mechanical arms one sees in automated factories loading and unloading conveyor belts or spot-welding auto bodies. Research in the field of robotics encompasses topics ranging from actuator design to image processing, but the study of robot manipulators constitutes a significant fraction of this research.

Robot manipulators are designed to affect general rigid motions on a tool attached to the end of the manipulator or on objects held in the manipulator's hand. The flexibility of manipulators is what distinguishes them as robotic systems as opposed to a programmable machines (e.g. numerically controlled milling machines etc.). Figure 1.1 shows a schematic diagram of industrial manipulator. This manipulator consists of six joints which allow one degree of freedom each. There are two types of joints on this manipulator. The first type is represented by a cylindrical can and allows only rotation about the axis of the can; this type of joint is called ***revolute*** and is by far the most common joint type. The second type of joint is represented by a square prism and allows only translations along its axis; this type of joint is called prismatic. Essentially all industrial robot joints are either revolute or prismatic. These two types of joints are denoted R and P respectively. With this notation the manipulator in Figure 1.1 is called a PRRRRR or P5R manipulator. The kinematics of robot manipulators deals with the geometry of these machines.

An important map in the study of manipulator kinematics is the ***forward kinematic map***. Referring to Figure 1.1 one can observe that for each set of joint positions we can associate a configuration of the manipulator's hand. This association maps points in the manipulator's jointspace ($\mathbf{R}^1 \times \mathbf{T}^5$ in this case) to points in the configuration space of the manipulator's hand (the group of rigid motions on $\mathbf{R}^3$). The study of this map, its inverse, and its critical points (singularities) are main themes in robot kinematics.

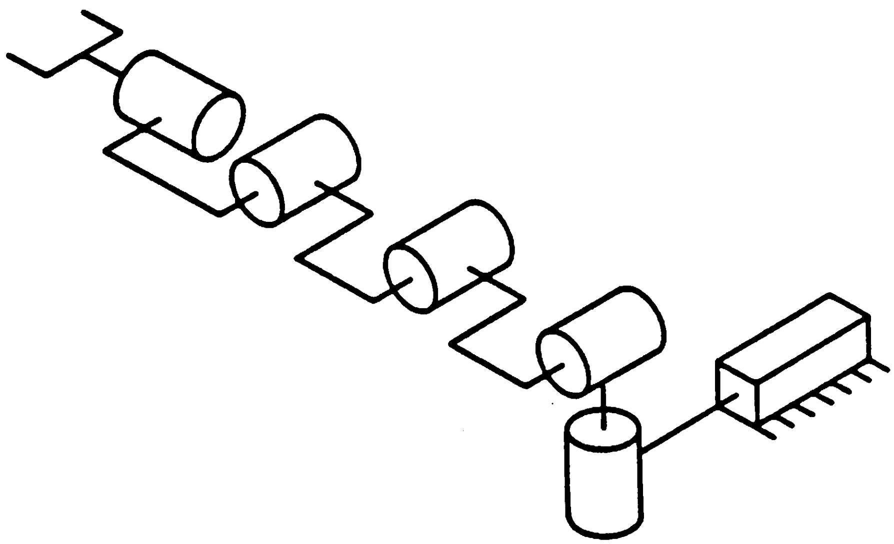

Figure 1.1 Schematic Diagram of a Manipulator with Revolute (R) and Prismatic (P) Joints.

The dynamics and control of manipulators is also a major research area. A manipulator such as the one in Figure 1.1 will have motors at each joint to actuate the manipulator. The first question one considers in the control of a given arm is given a desired trajectory of the manipulator's hand, what control law will give the necessary torques to move the hand along the trajectory? Many control laws have been proposed, most of which can be placed in the following categories. (1) Linear controllers acting independently at each joint. These are typically the PID (proportional, integral, derivative) type. PID controllers are appropriate controllers for manipulators which are either slow or whose inertias are dominated by the actuator inertias (this is not uncommon when gear reductions on the order of 100 exist between motor and joint). (2) Linearizing controllers. These control laws transform, by state feedback, the manipulator's nonlinear dynamics so that the input to state map is linear. Once this is done, a linear control scheme is used to control the manipulator. (3) Lyapunov-designed nonlinear controllers. The design of these controllers is generated from a stability analyzis of the manipulator together with the controller. The advantage of generating control laws in this fashion is that the effects of parameter errors on stability are obvious and these errors can be accommodated. The control scheme we propose in chapter six is in this class. (4) Adaptive controllers. These controllers use information contained in the tracking error to update parameter estimates in the manipulator model. The structure of these controllers can be the same as (1)-(3) with an adaptation law added (see[1, 2, 3]).

This thesis contributes to manipulator kinematics a fundamental result relating the kinematic design and performance of 6R manipulators. In the area of control we develop a calculus for analyzing variable structure control laws which are described by nonsmooth gradient systems. We use this machinery to analyze a variable structure control law for robot manipulators. We begin with a review of related work.

1.1 Kinematics and Manipulator Workspaces.

Several authors have studied the relationship between the kinematic design of manipulators and the resulting set swept out by the manipulators hand. Their work forms the context for chapter five and we review it here. First we discuss the ***jointspace*** and the ***workspace*** of a manipulator.

As we saw with the manipulator diagramed in Figure 1.1, a manipulator defines a map from its configuration space (called *jointspace*) to the configuration space of its hand. For general manipulators with revolute and prismatic joints this is also the case. To each point in the manipulator's jointspace, J, there is a natural assignment of a point in the configuration space of the hand. The configuration space of the hand is almost always the group of rigid motions on $\mathbb{R}^3$, G, so we write $f : J \rightarrow G$ for the forward kinematic map of a manipulator. For a given manipulator, this map depends on which point in jointspace we call zero, but we will work that out later. The image of J, $f(J) \subset G$ is called the ***workspace*** of the manipulator. This is our definition of workspace.

The thrust of the recent studies of manipulator workspaces has been on the generation of projections of $f(J)$ onto $\mathbf{R}^3$ and characterizing this projection. There are two important projections of $f(J)$. They are called the ***reachable workspace*** and ***dextrous workspace*** by Kumar and Waldron [4]. These two projections are obtained as follows. Each $g \in G$ can be written as rotation about a point P followed by a translation. That is, there is a natural diffeomorphism, ϕ_P, of G onto $G_P \times \mathbf{R}^3$ where G_P is the subgroup of G which leaves P fixed. Let $\pi_{\mathbf{R}^3}$ be the natural projection of $G_P \times \mathbf{R}^3$ onto its $\mathbf{R}^3$ component. In terms of $\pi_{\mathbf{R}^3}$ and ϕ_P,

$$W_R(P) \triangleq \pi_{\mathbf{R}^3} \circ \phi_P \circ f(J)$$

is called the ***reachable workspace*** (of P) and

$$W_D(P) \triangleq \left[\pi_{\mathbf{R}^3} \circ \phi_P \left([f(J)]^C \right) \right]^C$$

is called the ***dextrous workspace*** (of P). The point P is chosen to represent some significant point attached to the hand of the manipulator. Often P is chosen to be a point between the fingers of the hand or the point of intersection of the wrist axes. In words, the reachable workspace is the set of points which can be reached by P and the dextrous workspace is the set of points which can be reached by P with arbitrary orientation of the hand.

Bounds on $W_R(P)$ are obtained numerically by Kumar and Waldron in [4]. There it is observed that for manipulators with only revolute joints, the boundary of $W_R(P)$ consists of critical values of $\pi_{\mathbf{R}^3} \circ \phi_P \circ f$. By generating a plot of these critical values Kumar and Waldron were able to obtain graphical bounds. Additional work on the shape of $W_D(P)$ and $W_R(P)$ is contained in [5] where Gupta and Roth classify holes and voids in the workspace projections and conditions for their existence are given. The basic difference between a hole and void can be demonstrated with a bakery doughnut. If the surface of a doughnut represents the reachable workspace of a manipulator, then the doughnut-hole is a hole in this workspace and the doughnut-dough is a void. Gupta and Roth also discuss the behavior of $W_D(P)$ and $W_R(P)$ as a function of P. For a manipulator having its final three axes intersecting at a point P_w, the reachable workspace increases and the dextrous workspace decreases with increasing $\|P - P_w\|$.

Further numerical studies of manipulator workspaces have been carried out by Yang and Lee [6], Tsai and Soni [7], and Hansen *et al* [8]. These studies rely on numerical methods to generate projections of the manipulator workspace. The advantages of these schemes over analytical approaches is that mechanical constraints can be easily included. The disadvantages when compared to analytical techniques are the usual ones.

The approach taken in this thesis to derive relationships between design and performance of manipulators differs in several ways from those mentioned above. (1) Rather than projecting $f(J)$ onto $\mathbf{R}^3$ we consider the volume of $f(J)$ as a subset of the group of rigid motions as a performance measure of a kinematic design. (2) We find designs which optimize our performance measures. (3) The tools used are standard methods of differential geometry which have not been used in the kinematics literature.

1.2 Control

Many controllers have been proposed for robot manipulators and range in complexity from simple linear controllers to linearizing controllers which compute the manipulator dynamics in real time. The PID controllers work well on slow industrial manipulators where a large component of each joint inertia is contributed by the actuator. This is common with high gear-ratio drives. However, when the nonlinear dynamics of the the manipulator become comparable to the linear actuator dynamics, the performance of the PID controllers degrade and it becomes necessary to use more sophisticated control schemes.

To describe some differences between control schemes we need to write a few equations. The ideal dynamics of a rigid-link manipulator are given by

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta}) + G(\theta) = F \qquad (1.2.1)$$

where

θ	is the vector of joint coordinates
$M(\theta)$	is the positive definite inertia matrix
$C(\theta,\dot{\theta})$	is the vector of Coriolis and centrifugal forces
$G(\theta)$	is the vector of gravitational forces
F	is the vector of generalized forces applied by the actuators at the joints of the manipulator.

In terms of equation (1.2.1), linear controllers generate control torques which are linear in the manipulator state $[\theta,\dot{\theta}]^T$ and a desired state trajectory $[\theta_d,\dot{\theta}_d]^T$ ($\ddot{\theta}_d$ is sometimes used as a feed-forward term). Linearizing controllers have a control of the form $F = M(\theta)[\ddot{\theta}_d - k_v(\dot{\theta} - \dot{\theta}_d) - k_p(\theta - \theta_d)] + C(\theta,\dot{\theta}) + G(\theta)$. Plugging this control law into (1.2.1) yields the error dynamics $\ddot{e} + k_v\dot{e} + k_p e = 0$ where $e \triangleq \theta - \theta_d$. Thus, by appropriate choice of feedback gains k_v and k_p the error converges to zero exponentially. Finally, robust nonlinear controllers are designed by Lyapunov methods and have global stability properties. In the analyzis of these schemes one can see explicitly the effects of parameter errors in the model and can increase control gains to compensate for these. Our controller is a member of this class of controllers.

The PID controllers are observed to work in practice when the integral control is omitted or a dead-zone is introduced to eliminate limit cycling due to friction. Linear schemes without integral control have been analyzed by Golla, Garges, and Hughes [9] and Takegaki and Arimoto [10]. Both of these schemes are asymptotically stable for set-point regulation. The later work of [10] uses the Hamiltonian of the manipulator as a natural Lyapunov function to obtain global results whereas the results of [9] are local for essentially the same system. The primary drawback of linear controllers is the lack of global stability results for tracking control. This is overcome easily by using nonlinear controllers.

Nonlinear controllers designed by Lyapunov methods have been proposed by Corless and Leitmann [11], Gilbert and Ha [12], Young [13], Slotine and Sastry [14], and Morgan [15]. These schemes guarantee the convergence of tracking error to a neighborhood of zero in the presence of disturbances and modeling errors. For the variable structure controllers the error converges to zero. The basic idea of these schemes is simple and is summarized by the system described by the following scalar differential equation.

$$\dot{x} = u + d(x,t) \qquad (1.2.2)$$
$$x(0) = x_0$$

where u is the control and $d(x,t)$ is a state dependent disturbance with $\|d(x,t)\| < D$. Suppose that we would like to control x to the origin. Choose the Lyapunov function $V(x) = x^2/2$ and then we have

$$\dot{V}(x) = x(u + d). \qquad (1.2.3)$$

Choosing $u = -(k + D)x$, $k > 0$ we have that

$$\dot{V}(x) \leqslant -kx^2 + D|x| = (-k|x| + D)|x|. \qquad (1.2.4)$$

It follows that x converges to a ball of radius D/k about the origin. By increasing k this ball can be made arbitrarily small. This simple example illustrates the basic idea of the Lyapunov based designs. The robot dynamics together with the control are rewritten as a vector differential equation of the form (1.2.2) and similar arguments are used to derive the controller gains.

When the above ideas are applied to discontinuous control schemes, technical problems arise. In chapter six we work out these problems and describe a simple variable structure controller for robot manipulators. Other variable structure schemes have been proposed for robot manipulators (see [13, 14]) but are not as simple as they could be. This is due to constraints imposed by analyzis techniques rather than the inherent complexity of the problem.

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Chapter 2

Rigid Motions and Twists

The group of proper rigid motions on $\mathbf{R}^3$ plays a central role in the study of rigid-link robot manipulators. It is the configuration space of the links of such manipulators; the most important of which is the final link or hand. This chapter will discuss the representation of rigid motions as exponentials of twists and some commonly occurring problems which arise in the solution of manipulator kinematics.

Representing the rigid motions which are rotations about or translations along joint axes as exponentials of twists is discussed by Brockett in [1]. This approach proves useful for the solution of manipulator kinematic equations in chapter three and as a compact notation with clear geometric interpretation. The exponential of twists notation is also used to express a set of commonly occurring subproblems which arise in the solution of manipulator kinematic equations. These subproblems appear essentially in the form in which they were presented by W. Kahan [2].

First, the notation used for points and vectors in introduced.

2.1 Points and Vectors.

Identify points in physical three-space with points in $\mathbf{R}^3$ via an orthonormal right-handed coordinate system as usual. Thus, a point P can be represented by an element of $\mathbf{R}^3$ written

$$P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{pmatrix} \tag{2.1.1}$$

where the one in the last row allows for the matrix representation of rigid motions by homogeneous transformations in the next section. If O is the origin of the coordinate system then $O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. The four components in the RHS of (2.1.1) do not all have the same units. The first three components have units of length and the "1" is interpreted as a scale factor (particularly in computer graphics) and is therefore unitless. Identifying a point with its coordinates in (2.1.1) should not cause any confusion since only one coordinate system will be used in developing manipulator kinematics. The reader will note that it is never necessary to specify the coordinate system. This is because the objects we consider are intrinsic to three-space and anything said about them is independent of the choice of coordinate system.

Let $P(t)$ be a C^1 curve representing the position of a particle. Then the velocity of the particle at time t is given by

$$\begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \\ 0 \end{pmatrix} = \frac{d}{dt} \begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ 1 \end{pmatrix}. \tag{2.1.2}$$

All column vectors having the form of the left-hand side of (2.1.2) will be called vectors. So for points P and Q, the displacement from Q to P, $P - Q$, is a vector as usual. Also, if v is a vector, $Q + v$, is the point which is Q translated by v. Both points and vectors will be considered as elements in $\mathbf{R}^3$; Their physical interpretations are completely different however. The zero and one which are in the fourth row of vectors and points respectively will remind us of this and enforce a few rules of syntax. For example, it is meaningless to add two points.

The vector cross and dot products are computed in a natural way. If $v = (v_1, v_2, v_3, 0)^T$ and $u = (u_1, u_2, u_3, 0)^T$ then

$$u \cdot v \triangleq u^T v \quad \text{and} \quad u \times v \triangleq \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \\ 0 \end{pmatrix}. \tag{2.1.3}$$

2.2 Proper Rigid Motions

Proper rigid motions are maps from $\mathbf{R}^3$ to $\mathbf{R}^3$ which preserve orientation and distances between points. Formally, $g:\mathbf{R}^3 \rightarrow \mathbf{R}^3$ is a proper rigid motion if $\|gP - gR\| = \|P - R\|$ and $(gR - gQ)^T((gS - gQ) \times (gP - gQ)) = (R - Q)^T((S - Q) \times (P - Q)) \; \forall \; P,Q,R,S \in \mathbf{R}^3$. It is well known that when points are represented as in (2.1.1) such maps can represented by a homogeneous transformation of the form

$$g = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \tag{2.2.1}$$

where $A \in SO(3)$ and $b \in \mathbf{R}^3$. Let G be the group (under matrix multiplication) of all 4×4 matrices of the form (2.2.1). G is called the group of proper rigid motions on $\mathbf{R}^3$. The identity, $I \in G$, is simply the 4×4 identity matrix and

$$\begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} A^T & -A^T b \\ 0 & 1 \end{bmatrix}. \tag{2.2.2}$$

If P is a point, then $g \in G$ acts on P by matrix multiplication.

$$\begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix} = \begin{bmatrix} A \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + b \\ 1 \end{bmatrix}. \tag{2.2.3}$$

The interpretation of (2.2.3) is that P is first rotated about the origin of the coordinate system by A and then translated by b. Observe that the 4×4 identity matrix represents no rotation followed by zero translation.

In a rigid body, the distance between any two points attached to the body and the chirality of any four points attached to the body is constant over the configurations of the rigid body. If we take this as the definition of a rigid body then we have an isomorphism between the configuration space of a rigid body and the group of rigid motions. Identify $I \in G$ with some nominal configuration of the rigid body, then a given configuration of the rigid body is identified with the rigid motion which translates the body from the identity configuration to the given configuration. With this identification, a curve in G is the trajectory of a rigid body such as a moving hand of a manipulator.

If $g(t)$ is a C^1 curve in G representing the trajectory of a rigid body, then the velocity of the rigid body is given by $\frac{d}{dt}g(t) \in T_{g(t)}G$ where $T_h G$ is the tangent space of G at h [3]. The trajectory $g(t)$ can be written

$$g(t) = \left[\begin{array}{c|c} A(t) & P(t) \\ \hline 0 & \end{array}\right] \tag{2.2.4}$$

where $P(t)$ is a C^1 curve of points and $A(t)$ is a C^1 curve in $SO(3)$ which necessarily satisfies

$$A(t)A(t)^T = I \quad \forall\ t \in \mathrm{R}. \tag{2.2.5}$$

Evaluating the derivative of (2.2.4) and (2.2.5) with respect to t at $t = 0$ for an arbitrary curve $g(t)$ through $h = \left[\begin{array}{c|c} A & P \\ \hline 0 & \end{array}\right]$ (at $t = 0$) yields

$$T_h \mathrm{G} = \left\{ \left[\begin{array}{c|c} B & v \\ \hline 0 & \end{array}\right] \;\middle|\; v \text{ a vector}, BA^T + AB^T = 0 \right\}. \tag{2.2.6}$$

There is a isomorphism of $T_g G$ onto $T_I G$ given by right translation by g^{-1}. That is, $vg^{-1} \in T_I G\ \forall\ v \in T_g G$. Thus, the velocity of a rigid body can be represented by an element of $T_I G$. The advantage of doing this is that the element of of $T_I G$ which represents the velocity of a rigid body is independent of the choice of the identity position of the rigid body.

Let $\omega = (\omega_1, \omega_2, \omega_3, 0)^T$ and define the cross product operator constructed from ω by

$$S(\omega) \triangleq \left[\begin{array}{ccc|c} 0 & -\omega_3 & \omega_2 & 0 \\ \omega_3 & 0 & -\omega_1 & 0 \\ -\omega_2 & \omega_1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]. \tag{2.2.7}$$

(Observe that $S(\omega)b = \omega \times b\ \ \forall$ vectors b.)

With this, $\xi \in T_I G$ can be expressed

$$\xi = S(\omega) + \left[\begin{array}{c|c} 0 & v \\ \hline 0 & \end{array}\right] \tag{2.2.8}$$

for some vectors ω and v.

Comment: If ξ, written the form (2.2.8), represents the velocity of a rigid body, then ω is the angular velocity of the rigid body in radians per unit time and v is the linear velocity in length per unit time. The point attached to the rigid body which has velocity v is not the center of mass, but a copy of the origin which is attached to the rigid body at the instant we measure ξ.

Comment: Also, $\xi \in T_I G$ can be interpreted as a vector field. Note that to each point, P, ξ assigns a vector ξP. When ξ represents the velocity of a rigid body, the velocity of a particle, with position P, in the rigid body has velocity ξP. Recall that the units of ξ and P are mixed and note that ξP has the mixed units of a vector; the first three components have units length per unit time and the fourth component is zero (with units of $time^{-1}$).

The vector space $T_I G$ is the Lie algebra of G and has the same dimension (= six) as the manifold G. In the next section, we describe the exponential map from $T_I G$ to G. In chapter four, we study the linear dependence of elements of $T_I G$ when we find singular configurations of manipulators.

2.3 Twists and Their Exponentials

In this section we introduce the notion of a twist [4, 5] and its exponential.

Definition 2.3.1: Elements in $T_I G$ are called ***twists***.

■

It is common practice to identify $T_I G$ with $\mathbf{R}^6$. The six components in $\mathbf{R}^6$ are called twist coordinates and are defined by the map $tc : T_I G \rightarrow \mathbf{R}^6$:

$$\left[\begin{array}{ccc|c} 0 & -\omega_3 & \omega_2 & v_1 \\ \omega_3 & 0 & -\omega_1 & v_2 \\ -\omega_2 & \omega_1 & 0 & v_3 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \overset{tc}{\rightarrow} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}. \tag{2.3.1}$$

For notational convenience a map that extracts the angular velocity and linear velocity vectors from a twist is defined similarly.

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 & v_1 \\ \omega_3 & 0 & -\omega_1 & v_2 \\ -\omega_2 & \omega_1 & 0 & v_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\underline{tc}} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ 0 \\ v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix} = \begin{bmatrix} \omega \\ v \end{bmatrix}. \tag{2.3.2}$$

(Here is a case where we pay a price for using the four component representation of points and vectors.)

A twist is given the attributes of pitch, magnitude, and an axis. These notions have their geometric meaning derived from the screws one finds in a hardware store, and are defined as follows.

Definition 2.3.2: Let $\xi = \underline{tc}^{-1}\begin{bmatrix} \omega \\ v \end{bmatrix}$ then the ***pitch*** of ξ is defined by

$$\text{pitch of } \xi \triangleq \begin{cases} \dfrac{\omega^T v}{\|\omega\|^2} & \text{if } \omega \neq 0 \\ \infty & \text{otherwise} \end{cases} \tag{2.3.3}$$

■

Definition 2.3.3: Let $\xi = \underline{tc}^{-1}\begin{bmatrix} \omega \\ v \end{bmatrix}$ then the ***magnitude*** of ξ is defined by

$$\text{magnitude of } \xi \triangleq \begin{cases} \|\omega\| & \text{if } \omega \neq 0 \\ \|v\| & \text{otherwise} \end{cases} \tag{2.3.4}$$

■

Comment: The magnitude of ξ is not a norm on $T_I G$ but is useful in describing zero and infinite pitch twists particularly.

Definition 2.3.4: An ***axis*** of a twist is a directed line l_u (line l with nonzero direction vector u) with the properties that (i) $u \times \xi Q = 0$ for all $Q \in l_u$ and (ii) u has the same direction* as ω (v resp.) if $\omega \neq 0$ ($\omega = 0$ resp.).

■

The next proposition identifies the relationship between twists and these attributes.

Proposition 2.3.1 (1) Every twist $\xi = \underline{tc}^{-1}\begin{bmatrix} \omega \\ v \end{bmatrix}$ has an axis. If $\omega \neq 0$ then the axis is unique. (2) For each axis, pitch $h \in (-\infty,\infty]$, and magnitude $M \in (-\infty,\infty)$ there exists a unique twist having these attributes.

Proof: (1) If $\omega \neq 0$ then verify that $\{O + \frac{\omega \times v}{\|\omega\|^2} + \lambda\omega \mid \lambda \in R\}_\omega$ is an axis of ξ. If $\omega = 0$, then verify that any directed line with direction v is an axis of ξ.

If $\omega \neq 0$, then, by requirement (ii) above, an axis of ξ must be $\{P + \lambda\omega \mid \lambda \in R\}_\omega$ for some P. Now $\omega \times \xi Q = 0$ for all $Q \in \{P + \lambda\omega \mid \lambda \in R\}_\omega$ implies

$$\omega\times(\omega\times(P + \lambda\omega - O) + v) = 0 \quad \forall\ \lambda \in R$$
$$\Rightarrow \omega\times(\omega\times(P - O)) + \omega\times v) = 0$$

(As $\omega \times \omega = 0$.)

$$\Rightarrow (\omega\omega^T - \|\omega\|^2)(P - O) = -\omega\times v \tag{2.3.5}$$
$$\Rightarrow \left(I - \frac{\omega\omega^T}{\|\omega\|^2}\right)(P - O) = \frac{\omega\times v}{\|\omega\|^2} \tag{2.3.6}$$

Observe that $\left(I - \frac{\omega\omega^T}{\|\omega\|^2}\right)$ is the orthogonal projection map onto the complement of sp$\{\omega\}$, hence the nullspace of $I - \frac{\omega\omega^T}{\|\omega\|^2} = \text{sp}\{\omega\}$. It follows that $(P - O) = \frac{\omega\times v}{\|\omega\|^2} + \lambda\omega$ for some $\lambda \in R$, since $(P - O) = \frac{\omega\times v}{\|\omega\|^2}$ is a particular solution of (2.3.5). This implies $\{O + \frac{\omega\times v}{\|\omega\|^2} + \lambda\omega \mid \lambda \in R\}_\omega$ is the unique axis when $\omega \neq 0$.

(2) Case 1. $h = \infty$ If $M = 0$, then $\xi = 0$ and we are done. If $M \neq 0$ and l_u is the axis, then $v = M\frac{u}{\|u\|}$ and $\xi = \underline{tc}^{-1}\begin{bmatrix}0\\v\end{bmatrix}$. Case 2. h finite. Let l_u be the axis, then $\omega = M\frac{u}{\|u\|}$. Let $P \in l_u$, then

$$\omega\times\xi P = 0 \Rightarrow \omega\times(\omega\times(P - O) + v) = 0$$
$$\Rightarrow (I - \frac{\omega\omega^T}{\|\omega\|^2})(P - O) = \frac{\omega\times v}{\|\omega\|^2} \tag{2.3.7}$$

Premultiplying both sides of (2.3.7) by $S(\omega)$ yields

* The zero vector has arbitrary direction

$$\omega \times (P - O) = -(I - \frac{\omega\omega^T}{\|\omega\|^2})v \tag{2.3.8}$$

Now $\frac{\omega^T v}{\|\omega\|^2}$ is just h, the pitch of ξ. Solving (2.3.8) for v yields $v = \omega \times (O - P) + h\,\omega$.

$$\Rightarrow \xi = \underline{tc}^{-1}(M\frac{u}{\|u\|} \, , \, M\frac{u}{\|u\|} \times (O - P) + hM\frac{u}{\|u\|}).$$

■

Comment: If $\xi = \underline{tc}^{-1}\begin{bmatrix}\omega \\ v\end{bmatrix}$ and $\omega \neq 0$ then ω gives the direction of the axis of ξ and v defines the pitch and position of the the axis of ξ. The projection of v on sp$\{\omega\}$ defines the pitch and the projection of v on the nullspace of ω^T defines the position of the axis.

Comment: If $\xi \in T_I G$ represents the velocity of a rigid body, then the axis of ξ is the instantaneous axis of rotation of the body. The pitch of ξ is the ratio of the translational velocity along the axis to the angular velocity about the axis, and the magnitude of ξ is the magnitude of the angular velocity of the rigid body (translational velocity if the angular velocity if zero).

Comment: A triple consisting of an axis together with a pitch and magnitude is called a *screw*. Proposition 2.3.1 shows that there is nearly an isomorphism between twists and screws and this is why screws are used to represent twists. Screws are drawn to graphically represent twists just as vectors (little arrows) are drawn to represent velocities.

Example: Let an axis be defined by the directed line through Q having unit direction vector z, then the zero-pitch unit screw having this axis is

$$\xi = \underline{tc}^{-1}\begin{bmatrix} z \\ z \times (O - Q) \end{bmatrix}.$$

The infinite-pitch unit screw having this axis is

$$\xi = \underline{tc}^{-1}\begin{bmatrix} 0 \\ z \end{bmatrix}.$$

■

The following proposition allows us to relate the rigid motion of a twist axis and the change in the twist.

Proposition 2.3.2 (Rigid motion of a twist) Let $\xi \in T_I G$ with pitch h, magnitude M, and axis l_u. If $g \in G$, then $g\xi g^{-1} \in T_I G$ with pitch h, magnitude M and axis $(gl)_{gu}$

Proof: Let $\xi = \underline{tc}^{-1}\begin{bmatrix}\omega \\ v\end{bmatrix}$, then verify that $g\xi g^{-1} = \underline{tc}^{-1}\begin{bmatrix} g\omega \\ gv + gS(\omega)g^{-1}O\end{bmatrix}$. It follows that $g\xi g^{-1}$ has the same magnitude and pitch as ξ. Now $u \times \xi Q = 0 \ \forall \ Q \in l_u \implies gu \times g\xi g^{-1}R = 0 \ \forall \ R \in (gl)_{gu}$. Finally, u has direction ω (v resp.) if $\omega \neq 0$ ($\omega = 0$ resp.) implies gu has direction $g\omega$ (gv resp.) if $g\omega \neq 0$ ($g\omega = 0$ resp.)

■

Comment: As we saw earlier, a twist is a vector field in $\mathbf{R}^3$. $g\xi g^{-1}$ is then a vector field defined in terms of g and ξ. To a point P, $g\xi g^{-1}$ assigns a vector as follows. First P is translated by g^{-1} to $g^{-1}P$, then the vector field ξ assigns the vector $\xi g^{-1}P$ which is then translated back by g. Graphically, this is equivalent to shifting the entire vector field ξ by g. Thus, the axis of ξ simply receives a rigid motion by g.

In the following chapters we will see that exponentials of twists are very useful in the study of manipulator kinematics. Twists are square matrices so it makes sense to exponentiate them. There is also an important geometric interpretation given by the following propositions and discussion.

Proposition 2.3.3 Let ω be a vector and define $\hat{\omega} \triangleq \frac{\omega}{\|\omega\|}$ when $\omega \neq 0$, then

$$e^{S(\omega)} = \begin{cases} \hat{\omega}\hat{\omega}^T + \sin(\|\omega\|)S(\hat{\omega}) + \cos(\|\omega\|)\left(1 - \hat{\omega}\hat{\omega}^T\right) & \text{if } \omega \neq 0 \\ I & \text{if } \omega = 0 \end{cases} \tag{2.3.9}$$

Proof: Observe that the spectrum of $S(\omega)$ is $\{0, i\|\omega\|, -i\|\omega\|\}$ and that the minimal polynomial of $S(\omega)$ is $\psi(\lambda) \triangleq \lambda(\lambda + i\|\omega\|)(\lambda - i\|\omega\|)$. Also note that each eigenvalue of $S(\omega)$ is a root of multiplicity one of the minimal polynomial. Thus, we can compute $\exp(S(\omega))$ by finding a polynomial $p(\lambda)$ such that $p(\lambda) = e^{\lambda} \ \forall \ \lambda \in \sigma(S(\omega))$ and then evaluate $p(\lambda)$ at $S(\omega)$. If $\omega = 0$ then $p(\lambda) = 1$ will do and so $e^{S(\omega)} = I$ for this case. If $\omega \neq 0$ then

$$p(\lambda) = 1 + (\sin\|\omega\|)\frac{\lambda}{\|\omega\|} + (1 - \cos\|\omega\|)\left(\frac{\lambda^2}{\|\omega\|^2}\right) \tag{2.3.10}$$

is an appropriate polynomial. Evaluating (2.3.10) at $S(\omega)$ yields

$$e^{S(\omega)} = I + \sin\|\omega\| S(\hat{\omega}) + (1 - \cos\|\omega\|)(S(\hat{\omega}))^2. \tag{2.3.11}$$

Using the identity $\omega \times (\omega \times v) = \omega\omega^T v - \|\omega\|^2 v$, which which is equivalent to $(S(\omega))^2 = \omega\omega^T - \|\omega\|^2 I$, we have

$$e^{S(\omega)} = \hat{\omega}\hat{\omega}^T + \sin\|\omega\| S(\hat{\omega}) + \cos\|\omega\|(I - \hat{\omega}\hat{\omega}^T). \tag{2.3.12}$$

■

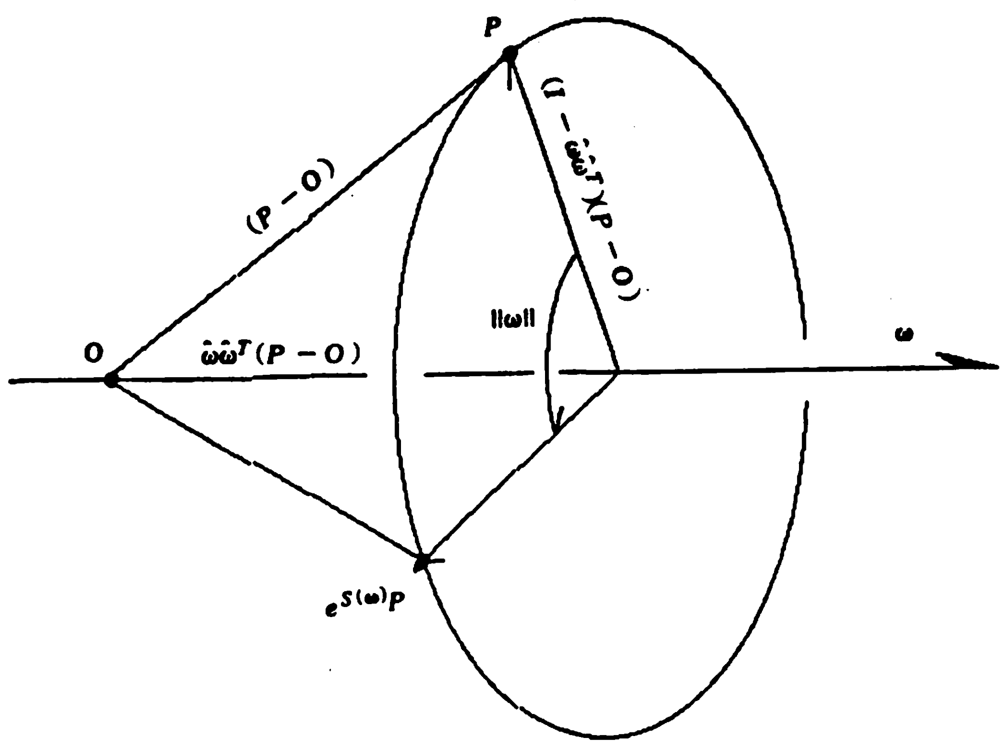

Figure 2.1 Interpretation of $e^{S(\omega)}$.

Figure 2.1 gives the geometric interpretation of $e^{S(\omega)}$ as a rotation by $\|\omega\|$ about the axis through the origin, O, having direction ω.

Proposition 2.3.4 Let $\xi = \underline{tc}^{-1}\begin{bmatrix}\omega\\ v\end{bmatrix}$, Q be a point on an axis of ξ, and P be an arbitrary point. Then

$$e^{\xi}P = \begin{cases} Q + e^{S(\omega)}(P - Q) + \dfrac{\omega\omega^T}{\|\omega\|^2}v & \text{if } \omega \neq 0 \\ P + v & \text{if } \omega = 0 \end{cases} \tag{2.3.13}$$

Proof: If $\omega = 0$, then

$$\xi = \left[\begin{array}{c|c} 0 & v \\ \hline 0 & \end{array}\right] \tag{2.3.14}$$

and $\xi^k = 0 \;\forall\; k > 1$. Thus,

$$e^{\xi} = \sum_{k=0}^{\infty} \frac{\xi^k}{k!} = \sum_{k=0}^{1} \frac{\xi^k}{k!} = I + \left[\begin{array}{c|c} 0 & v \\ \hline 0 & \end{array}\right] \tag{2.3.15}$$

and it follows that $e^{\xi}P = P + v$.

If $\omega \neq 0$, it is convenient to change coordinates to a system whose origin is on the axis of ξ. Since Q is a point on the axis of ξ,

$$g \triangleq \left[\begin{array}{c|c} 0 & Q \\ \hline 0 & \end{array}\right] \tag{2.3.16}$$

is a transformation which takes the origin O to a point on the axis. Note that g is a translation by $Q - O$ and its inverse is a translation by $O - Q$. Now

$$e^{\xi} = gg^{-1}e^{\xi}gg^{-1} = ge^{(g^{-1}\xi g)}g^{-1} \tag{2.3.17}$$

Defining $\xi' \triangleq g^{-1}\xi g$ yields

$$\begin{aligned} \xi' &= \begin{bmatrix} I & Q \\ 0 & \end{bmatrix}^{-1} \left[\begin{array}{ccc|c} 0 & -\omega_3 & \omega_2 & \\ \omega_3 & 0 & -\omega_1 & v \\ -\omega_2 & \omega_1 & 0 & \\ \hline 0 & 0 & 0 & \end{array}\right] \begin{bmatrix} I & Q \\ 0 & \end{bmatrix} \\ &= \begin{bmatrix} I & Q \\ 0 & \end{bmatrix}^{-1} \left[\begin{array}{ccc|c} 0 & -\omega_3 & \omega_2 & \\ \omega_3 & 0 & -\omega_1 & \omega \times (Q - O) + v \\ -\omega_2 & \omega_1 & 0 & \\ \hline 0 & 0 & 0 & \end{array}\right] \end{aligned} \tag{2.3.18}$$

Recall that $Q = O + \dfrac{\omega \times v}{\|\omega\|^2} + \lambda\omega$ for some $\lambda \in \mathbb{R}$ since Q is on the axis of ξ. Therefore, from (2.3.18)

$$\xi' = \begin{bmatrix} I & Q \\ 0 & \end{bmatrix}^{-1} \left[\begin{array}{ccc|c} 0 & -\omega_3 & \omega_2 & \\ \omega_3 & 0 & -\omega_1 & \frac{\omega\omega^T v}{\|\omega\|^2} \\ -\omega_2 & \omega_1 & 0 & \\ \hline 0 & 0 & 0 & \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 0 & -\omega_3 & \omega_2 & \\ \omega_3 & 0 & -\omega_1 & \frac{\omega\omega^T v}{\|\omega\|^2} \\ -\omega_2 & \omega_1 & 0 & \\ \hline 0 & 0 & 0 & \end{array}\right] \tag{2.3.19}$$

It is easy to compute the series of $e^{\xi'}$ at this point. Since $S(\omega)\frac{\omega\omega^T v}{\|\omega\|^2} = 0$, only one term in the taylor series of the exponential depends on v. We have

$$e^{\xi'} = \exp\left[S(\omega) + \left[\begin{array}{c|c} 0 & \frac{\omega\omega^T v}{\|\omega\|^2} \\ \hline 0 & \end{array}\right]\right] = e^{S(\omega)} + \left[\begin{array}{c|c} 0 & \frac{\omega\omega^T v}{\|\omega\|^2} \\ \hline 0 & \end{array}\right] \tag{2.3.20}$$

Moreover,

$$\begin{aligned} g e^{\xi'} g^{-1} P &= g e^{\xi'}[P + (O - Q)] \\ &= g\left[e^{S(\omega)}(P - Q) + e^{S(\omega)}O + \frac{\omega\omega^T v}{\|\omega\|^2}\right] \end{aligned} \tag{2.3.21}$$

Recall that $e^{S(\omega)}$ is a rotation about the origin so $e^{S(\omega)}O = O$. Also, g is a translation by $Q - O$ so we have

$$e^{\xi}P = Q + e^{S(\omega)}(P - Q) + \frac{\omega\omega^T v}{\|\omega\|^2} \tag{2.3.22}$$

■

Figure 2.2 gives the geometric interpretation of e^{ξ} as a rotation about the axis of ξ by $\|\omega\|$ followed by a translation along the axis by the pitch times ω. Note that when the pitch of ξ is zero, e^{ξ} is a pure rotation about the axis of ξ by $\|\omega\|$. When the pitch of ξ is infinity, e^{ξ} is a pure translation by v (see 2.3.13).

2.4 Common Subproblems in Manipulator Kinematics (Kahan [2])

Some geometric subproblems occur frequently in the solution of manipulator kinematics. By identifying the subproblems and solving them as "subroutines", it is easier to conceptualize the solution of many common manipulators. It is important to point out that the following problems are not an exhaustive set for the solution of arbitrary manipulators even though we have found them generally adequate for those commonly encoun-

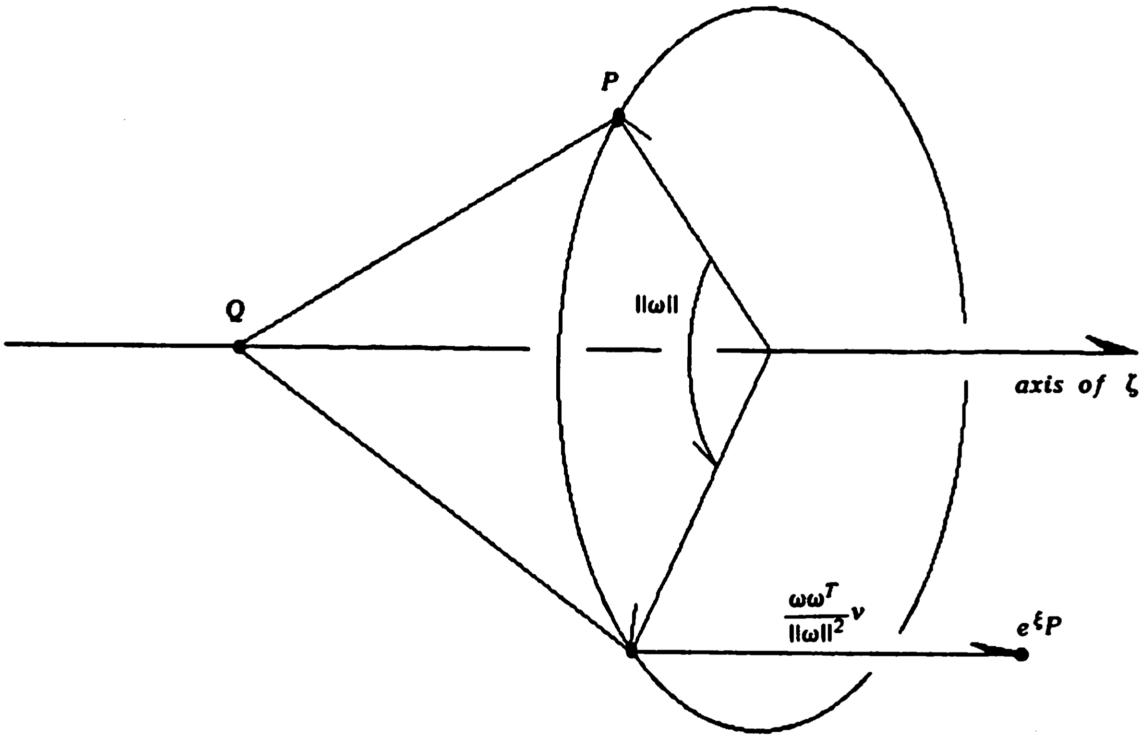

Figure 2.2 Interpretation of e^{ξ}.

tered.

Problem 1 (Fig 2.3) Rotating a point P about an axis by θ until it is coincident with another point R. Formally, let $P, R \in \mathbb{R}^3$, and ζ a *zero pitch* twist with *unit* magnitude. Find θ such that

$$e^{\theta\zeta}P = R. \tag{2.4.1}$$

Solution: Let Q be a point on the axis of $\zeta = \underline{tc}^{-1}\begin{bmatrix}\omega \\ v\end{bmatrix}$. Then

$$e^{\theta\zeta}P = R \tag{2.4.2}$$

$$\Leftrightarrow (e^{\theta\zeta}P) - Q = R - Q \tag{2.4.3}$$

$$\Leftrightarrow e^{\theta\zeta}(P - Q) = (R - Q) \tag{2.4.4}$$

Since $e^{\theta\zeta}Q = Q$ for all Q on the axis of ζ as ζ is zero pitch.

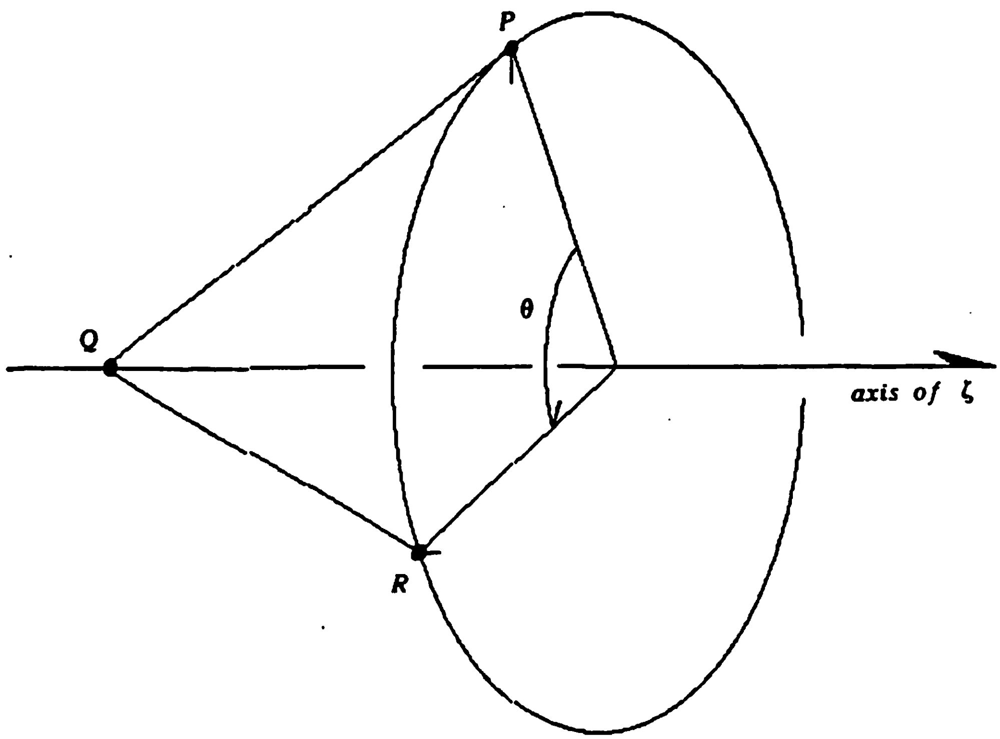

Figure 2.3 Problem 1.

$$\Leftrightarrow e^{\theta S(\omega)} u = y \tag{2.4.5}$$

where $u \triangleq (P - Q)$ and $y \triangleq (R - Q)$. By projecting (2.4.5) onto orthogonal subspaces which are the span of ω and the nullspace of ω^T , we have that (2.4.1) holds

$$\Leftrightarrow \omega^T u = \omega^T y \tag{2.4.6}$$

and

$$e^{\theta S(\omega)}[u - \omega\omega^T u] = [y - \omega\omega^T y] \tag{2.4.7}$$

Define $u' \triangleq [u - \omega\omega^T u]$ and $y' \triangleq [y - \omega\omega^T y]$. Equation (2.4.7) is a rotation in the plane since u' and y' are both orthogonal to the vector of rotation, ω (see Fig. 2.4). The angle θ is obtained from $\sin\theta$ and $\cos\theta$ from the cross and dot products of u' and y'. By expressing the equality in (2.4.7) in polar form we have that (2.4.1) holds

$$\Leftrightarrow \omega^T u = \omega^T y,\ \|u'\| = \|y'\|,$$

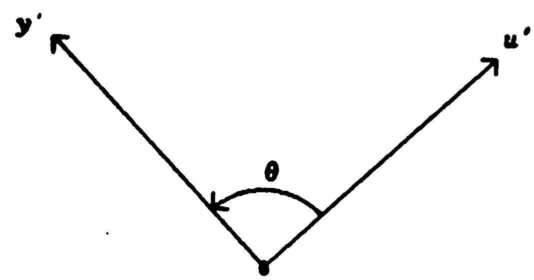

Figure 2.4.

and

$$\text{a) } u' \neq 0 \text{ and } \theta = ATAN_2\left[\frac{\omega^T(u' \times y')}{u'^T y'}\right] \quad (2.4.8a)$$

or

$$\text{b) } u' = 0 \text{ and } \theta \in [0,2\pi] \quad (2.4.8b)$$

■

In b) there are an infinity of solutions since this is the case when $P = R$ and both points lie on the axis of rotation. If (2.4.8) cannot be satisfied there is no solution to (2.4.1).

Problem 2 (Fig. 2.5) Rotating a point P first about one axis by θ_1 and then about a second axis (which intersects the first) by θ_2 so that it is coincident with a point R. To be precise: let $P,R \in \mathbb{R}^3$ and ζ_1,ζ_2 be ***zero-pitch unit*** twists with ***intersecting*** axes. Find θ_1,θ_2 such that

$$e^{\theta_2\zeta_2}e^{\theta_1\zeta_1}P = R. \quad (2.4.9)$$

Solution: Let $\zeta_i = \underline{tc}^{-1}\begin{bmatrix}\omega_i \\ v_i\end{bmatrix}$ for $i \in \{1,2\}$. Since ζ_1 and ζ_2 have intersecting axes, we can find a point Q in the intersection.

(a) In the instance that the twist axes coincide we have that (2.4.9) holds if and only if

$$\omega_1 \times \omega_2 = 0 \text{ and } e^{(\theta_1 + (\omega_1^T\omega_2)\theta_2)\zeta_1}P = R \quad (2.4.10)$$

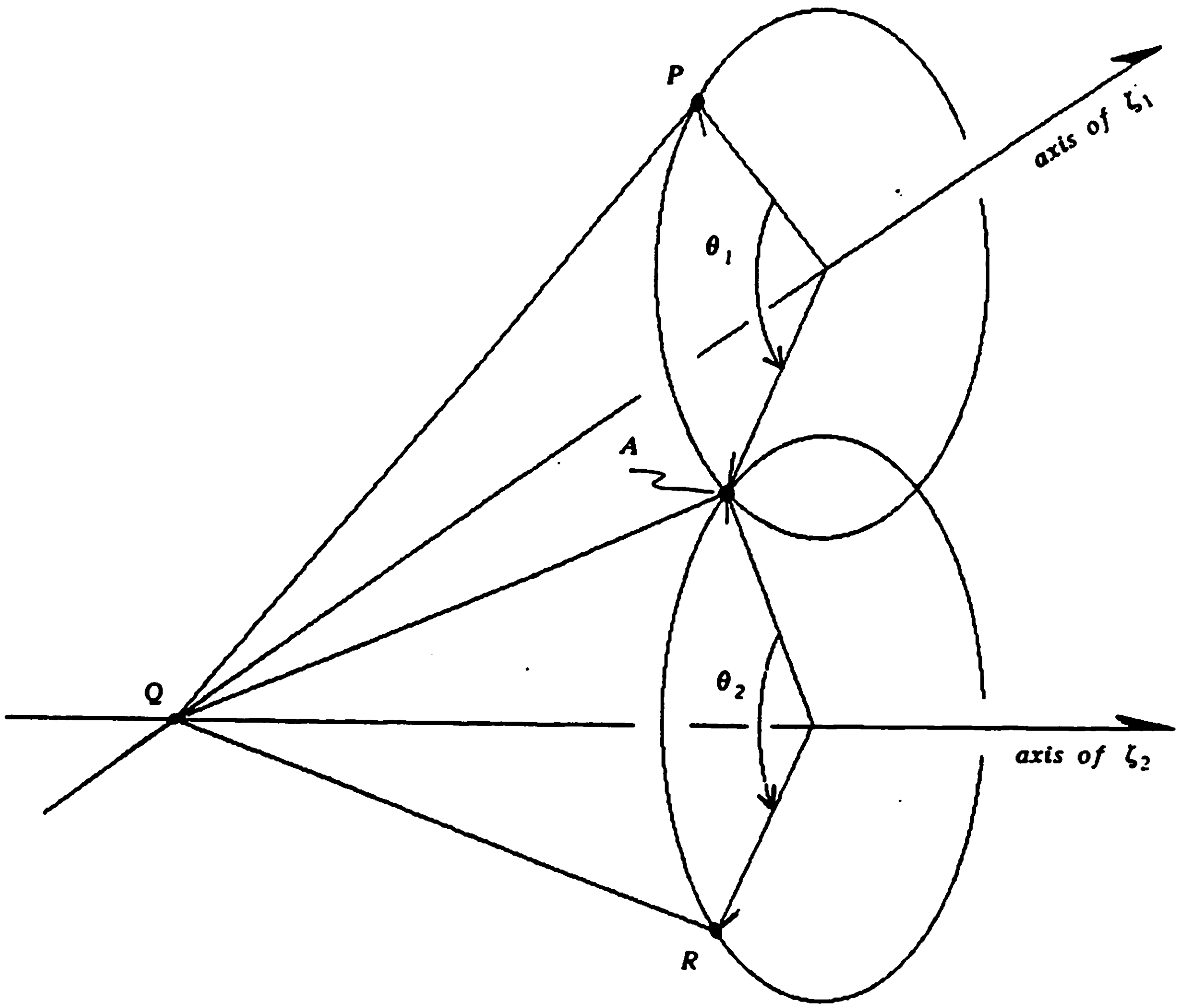

Figure 2.5 Problem 2.

Note that $\omega_1^T\omega_2$ is ± 1 depending on whether the axis directions are parallel or antiparallel. The second part of (2.4.10) can be solved for $(\theta_1 + (\omega_1^T\omega_2)\theta_2)$ using problem 1. There are 0 or infinite solutions in this case where the twist axes are coincident.

(b) If the twist axes are not parallel, then

$$\omega_1 \times \omega_2 \neq 0 \text{ and } e^{\theta_1\hat{\xi}_1}p = e^{\theta_2\hat{\xi}_2}R \qquad (2.4.11)$$

Continuing with the assumption of b) we have (2.4.11) holds

$$\Leftrightarrow \exists \text{ a point } A \text{ such that} \quad (2.4.12)$$
$$e^{\theta_1 \zeta_1} P = A = e^{-\theta_2 \zeta_2} R$$

$$\Leftrightarrow \exists A \text{ such that } e^{\theta_1 \zeta_1}(P - Q) = (A - Q) = e^{-\theta_2 \zeta_2}(R - Q) \quad (2.4.13)$$

(As Q is on the axes of both the zero-pitch screws ζ_1 and ζ_2.)

$$\Leftrightarrow \exists w \text{ such that } e^{\theta_1 S(\omega_1)} u = w = e^{\theta_2 S(\omega_2)} y \quad (2.4.14)$$

where $u \triangleq P - Q$, $y \triangleq R - Q$, $w \triangleq A - Q$.

$$\Leftrightarrow \omega_1^T u = \omega_1^T w \quad (2.4.15a)$$
$$\omega_2^T w = \omega_2^T y \quad (2.4.15b)$$
$$\|u\| = \|w\| = \|y\| \quad (2.4.15c)$$

and

$$e^{\theta_1 S(\omega_1)} u = w \quad (2.4.16a)$$
$$e^{\theta_2 S(\omega_2)} y = w \quad (2.4.16b)$$

Now ω_1, ω_2, and $\omega_1 \times \omega_2$ are linearly independent so w can be expressed

$$w = \alpha\omega_1 + \beta\omega_2 + \lambda(\omega_1 \times \omega_2). \quad (2.4.17a)$$

Plugging (2.4.17a) into (2.4.15) yields (2.4.15) and (2.4.16) hold if and only if

$$\omega_1^T u = \alpha + \beta\, \omega_1^T \omega_2 \quad (2.4.17b)$$
$$\omega_2^T y = \alpha \omega_1^T \omega_2 + \beta \quad (2.4.17c)$$
$$\|u\|^2 = \alpha^2 + \beta^2 + 2\alpha\beta\omega_1^T\omega_2 + \lambda^2\|\omega_1 \times \omega_2\|^2 \quad (2.4.17d)$$

and (2.4.16) holds.

Solving equations (2.4.17) for α, β, and λ, yields that (2.4.17) and (2.4.16) hold if and only if

$$\Leftrightarrow \alpha = \frac{\omega_1^T u - (\omega_1^T \omega_2)(\omega_2^T y)}{1 - (\omega_1^T \omega_2)^2} \quad (2.4.18a)$$

$$\beta = \frac{\omega_2^T y - (\omega_1^T \omega_2)(\omega_1^T u)}{1 - (\omega_1^T \omega_2)^2} \quad (2.4.18b)$$

$$\lambda^2 = \frac{(\|u\|^2 - (\alpha^2 + \beta^2 + 2\alpha\beta(\omega_1^T\omega_2))}{\|\omega_1 \times \omega_2\|^2} \quad (2.4.18c)$$

and (2.4.16) holds.

Depending on the value of λ^2 there will be 0 ($\lambda^2 = 0$), 1 ($\lambda^2 = 0$), or 2 ($\lambda^2 > 0$) values of w and hence A. The values of θ_1 and θ_2 which satisfy (2.4.16a) and (2.4.16b) (if any) are found by applying problem 1 for each possible value of A. Since there are only finitely many values of A, we have infinite solutions to problem 2 only if the application of problem 1 to (2.4.16a), (2.4.16b), or (2.4.10) generates infinite solutions. Thus, a necessary condition for infinite solutions to exist is that P lies on the axis of ζ_1, R lies on the axis of ζ_2 or the axes are coincident (see (a) above).

In summary, problem 2, has a solution if and only if (2.4.18) and (2.4.16) hold. Further, we can get infinite solutions to this problem if P or R lies on the axis of ζ_1 or ζ_2 respectively, or the axes are coincident.

Problem 3 (Fig. 2.6) Rotate a point R about and axis by θ such that it is a given distance from a second point Q. Formally, Let $\zeta = \underline{tc}^{-1}\begin{bmatrix}\omega \\ v\end{bmatrix}$ be a *unit* twist with *zero* pitch, and $P, Q, R \in \mathbb{R}^3$. Find θ such that

$$\|P - e^{\theta\zeta}R\| = d \tag{2.4.19}$$

Solution: Let Q be a point on the axis of ζ. Then equation (2.4.19) holds

$$\Leftrightarrow \|(P - Q) - e^{\theta\zeta}(R - Q)\| = d \tag{2.4.20}$$

$$\Leftrightarrow \|u - e^{\theta S(\omega)}y\| = d \tag{2.4.21}$$

where $u \triangleq (P - Q)$, $y \triangleq (R - Q)$. Projecting $u - e^{\theta S(\omega)}y$ in (2.4.23) onto the orthogonal subspaces sp$\{\omega\}$ and the nullspace of ω^T yields by the Pythagorean theorem that (2.4.19) holds if and only if

$$\|\omega^T(u - y)\|^2 + \|(u - \omega\omega^T u) - e^{\theta S(\omega)}(y - \omega\omega^T y)\|^2 = d^2 \tag{2.4.22}$$

$$\Leftrightarrow \|u' - e^{\theta S(\omega)}y'\|^2 = d^2 - \|\omega^T(u - y)\|^2 \tag{2.4.23}$$

where $u' \triangleq (u - \omega\omega^T u)$ and $y' = (y - \omega\omega^T y)$. Figure 2.7 shows the triangle we must solve to solve (2.4.23). Applying a standard angle formula (see [6], pg. 120) yields that (2.4.23) holds

$$\Leftrightarrow |\theta - \theta_0| = 2\tan^{-1}\left(\sqrt{\frac{(s - \|u'\|)(s - \|y'\|)}{s(s - \sqrt{(d^2 - \|\omega^T(u - y)\|^2)})}}\right) \tag{2.4.24}$$

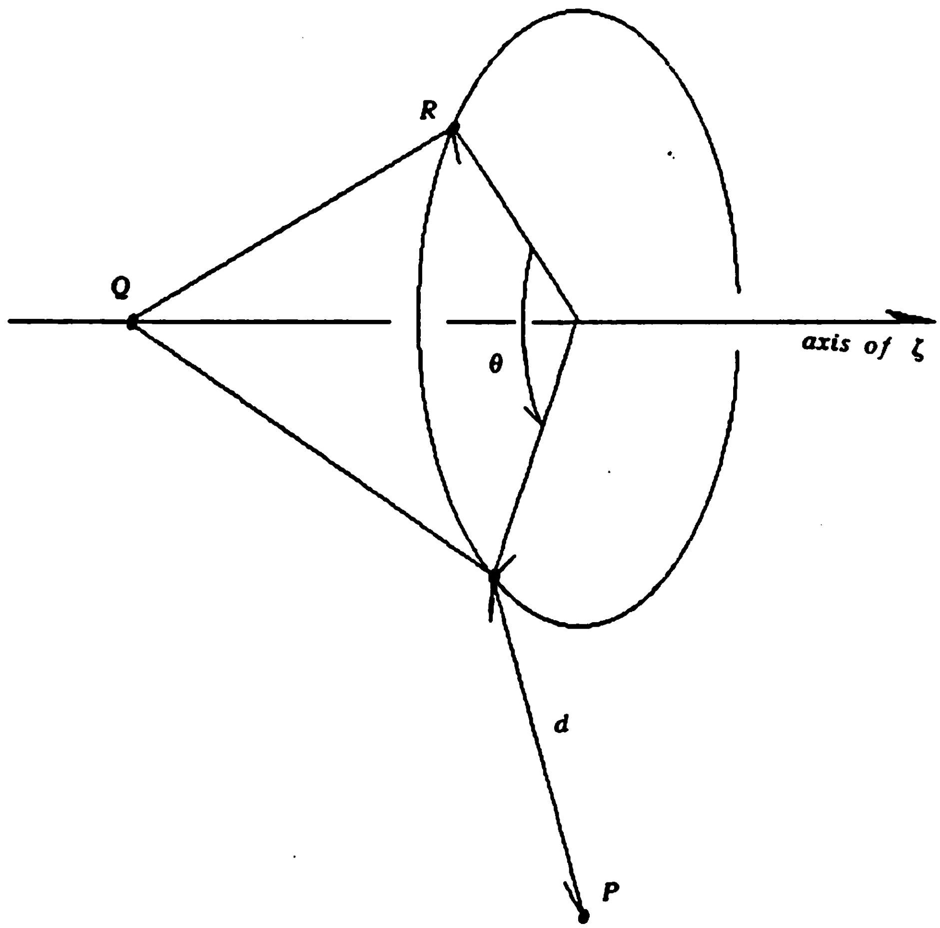

Figure 2.6 Problem 3.

where

$$\theta_0 = \tan^{-1}\left(\frac{\omega^T(y' \times u')}{y'^T u'}\right)$$

$$s = \frac{1}{2}(\ \|u'\| + \|y'\| + \sqrt{d^2 - \|\omega^T(u - y)\|^2})$$

There are solutions to this problem if and only if the quantities under the radicals in (2.4.24) are positive. In this case there are 1 or 2 solutions depending on whether $|\theta - \theta_0|$ is zero or positive. To summarize, problem 3 has a solution if and only if

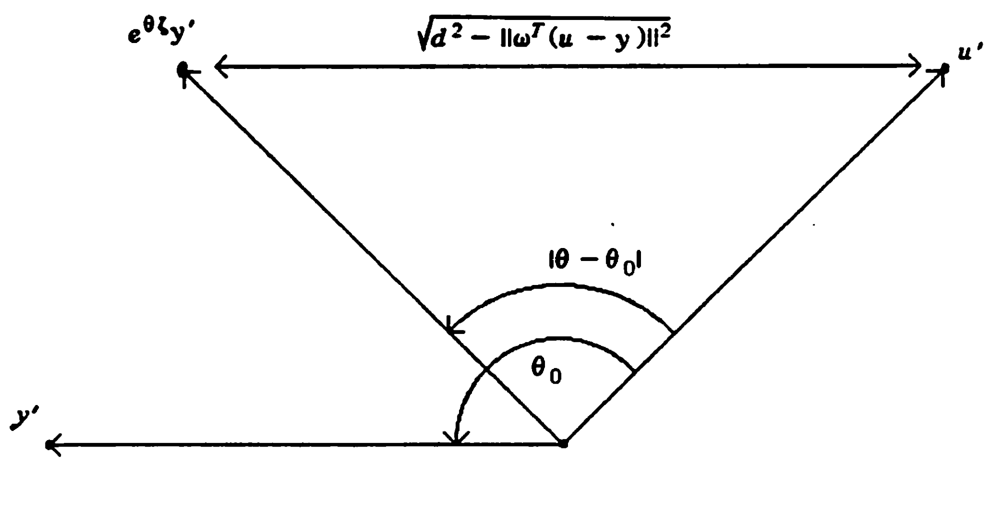

Figure 2.7.

equation (2.4.24) has a solution.

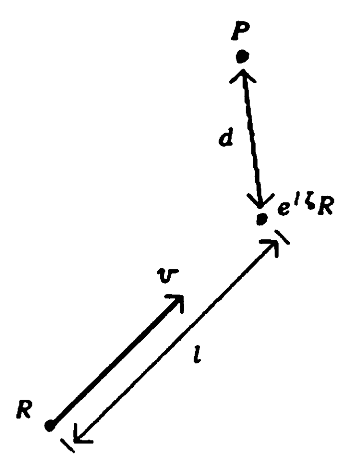

Figure 2.8 Problem 4.

Problem 4 (Fig. 2.8) Translating a point R along a direction v by l so that it is a given distance d from another point P. In out notation, let $\zeta = \underline{tc}^{-1}\begin{bmatrix}0\\v\end{bmatrix}$ be a ***unit twist*** with ***infinite*** pitch. Find l such that

$$\|P - e^{l\zeta}R\| = d \tag{2.4.25}$$

Solution: Equation (2.4.25) holds

$$\Longleftrightarrow \|P - (R + lv)\| = d \tag{2.4.26}$$

$$\Longleftrightarrow \|P - R\|^2 + l^2 - 2lv^T(P - R) = d^2 \tag{2.4.27}$$

$$\Longleftrightarrow l = v^T(P - R) \pm \sqrt{(v^T(P - R))^2 - (\|P - R\|^2 - d^2)} \tag{2.4.28}$$

There are real solutions to (2.4.25) if and only if the quantity under the radical in (2.4.28) is positive.

These problems will be applied to the solution of robot manipulators in the next chapter.

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Chapter 3

Manipulator Kinematics

There are two important problems which arise in applications of robot manipulators. First, given the configuration of the manipulator, find the configuration of the final link. This is called the forward kinematic problem. Second, given the configuration of the final link, find the possible configurations of the manipulator. This, for obvious reasons, is called the inverse kinematic problem.

The forward kinematic problem is relatively simple if a structured approach is used. The basic idea of all solution approaches is to specify what rigid motion each joint affects on those links following it and then compose these motions to obtain the rigid motion of the hand relative to some zero configuration.

The inverse kinematic problem has no closed form solution in general and is therefore more troublesome than the forward kinematic problem. It turns out that many industrial manipulators have closed form solutions which are obtained by repeated application of the subproblems solved in the last chapter. It is a corollary to the results of Chapter five that all "optimal" 6R manipulators have closed form solutions.

3.1 Forward Kinematics

A forward kinematic map of a manipulator assigns to each point in the joint space a point in the configuration space of the last link of the manipulator (the group of rigid motions).

The jointspace of a manipulator is the Cartesian product of the configuration spaces of the joints. The configuration space of a revolute joint is S^1 and the configuration space of a prismatic joint is $\mathbf{R}^1$. If we let J_i (equal to S^1 or $\mathbf{R}^1$) be the configuration of the i-th joint, then the configuration space of the manipulator is $J = \underset{i=0}{\overset{n-1}{\times}} J_i$.

A forward kinematic map for a manipulator is not unique as we may choose any configuration as the nominal "zero" configuration. Choosing a "zero" configuration defines the zero positions for each joint and the hand. In addition to choosing the nominal zero position, the senses for positive rotation and translation must also be chosen.

Figure 3.1 shows a 3R planar manipulator in its zero position. The manipulator has three revolute joints whose axes are perpendicular to the page and are marked by dots. Its joint space is $\mathbf{T}^3$ and its workspace is contained in $\mathbf{R}^2 \times S^1$ (a subgroup of G). To find the forward kinematic map for the manipulator, let ξ_i be zero-pitch unit screws whose axes are coincident with the joint axes when the manipulator is in the zero configuration (and have directions consistent with the desired sense of positive rotation). As long as the axis of joint i is coincident with the axis if ξ_i a rotation of θ_i of joint i effects the rigid motion $e^{\theta_i \xi_i}$ of the hand. Thus, if we move the manipulator from (0,0,0) to $(\theta_0, \theta_1, \theta_2)$ by rotating the joints in reverse order, the joints axes are coincident with axes of the ξ_i whenever we move them. This allows us to write down the resulting rigid motion of the hand directly as $e^{\theta_0 \xi_0} e^{\theta_1 \xi_1} e^{\theta_2 \xi_2}$. Since the configuration of the hand in the zero configuration is the identity, I, in G, the configuration of the hand when the joint angles are $(\theta_0, \theta_1, \theta_2)$ is $e^{\theta_0 \xi_0} e^{\theta_1 \xi_1} e^{\theta_2 \xi_2} I = e^{\theta_0 \xi_0} e^{\theta_1 \xi_1} e^{\theta_2 \xi_2}$. So we have that, the forward kinematic map for this zero configuration is

$$(\theta_0, \theta_1, \theta_2) \xrightarrow{f} e^{\theta_0 \xi_0} e^{\theta_1 \xi_1} e^{\theta_2 \xi_2} \tag{3.1.1}$$

Comment: Clearly, the configuration of the hand is independent of the order in which we rotate the joints. However, if we rotate joint 1 before we rotate joint 2, then the rotation of joint 2 is no longer about the axis of ξ_2 and the order of the rigid motions of the hand due to joint 2 and that due to joint 1 is different. In fact, the rotation of joint 2 is about the axis of $e^{\theta_1 \xi_1} \xi_2 e^{-\theta_1 \xi_1}$. If we keep track of the positions of the axes as we rotate them,

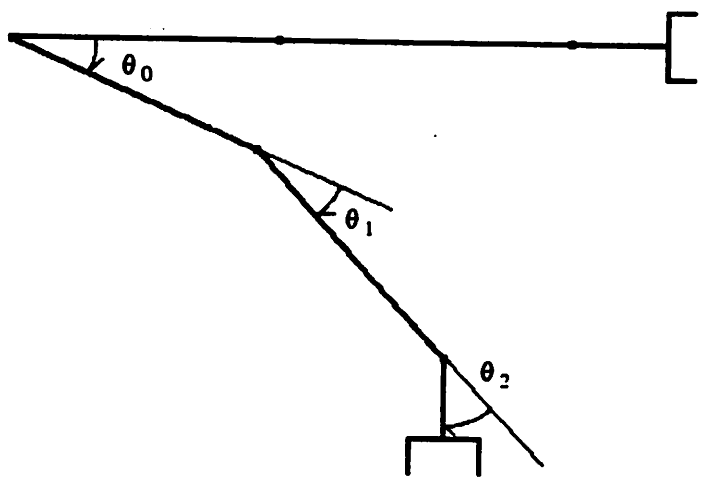

Figure 3.1 3R Planar Manipulator.

we always get the same rigid motion of the hand independent of the order in which we rotate the axes.

The above argument extends to n-joint manipulators with both prismatic and revolute joints. In the case of prismatic joints, we simply replace the zero-pitch unit screws with ∞-pitch unit screws whose axes are parallel to the motion of the prismatic joints. In general, we define a forward kinematic map of an n-joint manipulator as follows. Let $\xi_i, i \in \{0, \cdots, n-1\}$ be zero-pitch (∞-pitch for prismatic joints) unit screws whose axes are coincident with the joint axes of the i-th joint which is revolute. Then

$$f_{\bar{\xi}}(\theta_0, \cdots, \theta_{n-1}) \triangleq \prod_{k=0}^{n-1} e^{\theta_k \xi_k} \tag{3.1.2}$$

is called the forward kinematic map of the manipulator corresponding to $\bar{\xi} \triangleq \{\xi_0, \cdots, \xi_{n-1}\}$.

3.2 Inverse Kinematics

In applications, it is often the case that a desired configuration of the hand is given and the inverse image of this configuration under the the forward kinematic map must be found. In reference to (3.1.2), given $g \in G$, find $f_{\xi}^{-1}(g)$. This problem has some interesting twists. When the joint space has higher dimension than the workspace, $f_{\xi}^{-1}(g)$ is generically an infinite set. An important question for these redundant manipulators is how can the extra degrees of freedom be exploited? Open questions for the case where the dimension of the joint and workspaces have the same dimension are what is the number of solutions for the general problem and how can they be found?

Most industrial manipulators are designed such that $f_{\xi}^{-1}(g)$ can be easily calculated. In Chapter five we will see that nothing is sacrificed for 6R manipulators when it is required that $f_{\xi}^{-1}(g)$ be easily calculable. To give examples of the solution of equation (3.1.2) for a common industrial manipulators, we consider the elbow and stanford manipulators. By abuse of notation, we identify the axes of zero-pitch unit twist and the twists. Thus, we can write $\xi_0 \bigcap \xi_1 \neq \emptyset$ when the axes of ξ_0 and ξ_1 have nonzero intersection.

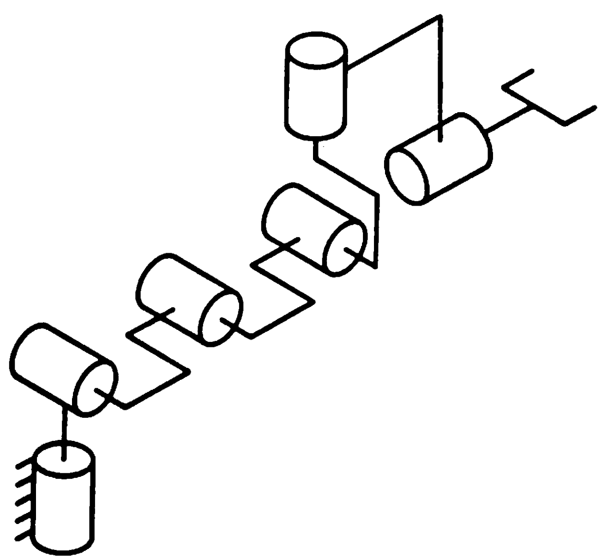

Figure 3.2 An Elbow Manipulator.

3.2.1 Solution of the Elbow Manipulator.

The important features of the elbow manipulator (Fig. 3.2) which allow the simple solution of its kinematic equation are that its first two axes intersect and its last three axes intersect. Let $\bar{\xi} = \{\xi_0, \xi_2, \cdots, \xi_5\}$ be the set of zero-pitch unit screws whose axes are coincident with those of the elbow manipulator for some nominal zero configuration. Define $c_0 \triangleq \xi_0 \bigcap \xi_1$ and $c_5 \triangleq \xi_3 \bigcap \xi_4 \bigcap \xi_5$. And the equation we wish to solve is

$$e^{\theta_0\xi_0} e^{\theta_1\xi_1} e^{\theta_2\xi_2} e^{\theta_3\xi_3} e^{\theta_4\xi_4} e^{\theta_5\xi_5} = g \tag{3.2.1}$$

for given $g \in G$.

Solution: Applying both sides of (3.2.1) to c_5 we have

$$e^{\theta_0\xi_0} e^{\theta_1\xi_1} e^{\theta_2\xi_2} c_5 = g c_5 \tag{3.2.2}$$

since $e^{\theta_i\xi_i} c_5 = c_5 \ \forall i \in \{3,4,5\}$ as $c_5 = \xi_3 \bigcap \xi_4 \bigcap \xi_5$. Next, subtracting c_0 from both sides of (3.2.2) and taking norms yields

$$\| e^{\theta_0\xi_0} e^{\theta_1\xi_1} e^{\theta_2\xi_2} c_5 - c_0 \| = \| g c_5 - c_0 \| \tag{3.2.3}$$

Applying the distance preserving rigid motion $(e^{\theta_0\xi_0} e^{\theta_1\xi_1})^{-1}$ to both terms in the LHS of (3.2.3) leads to

$$\| e^{\theta_2\xi_2} c_5 - c_0 \| = \| g c_5 - c_0 \| \tag{3.2.4}$$

as $c_0 \in \xi_0 \bigcap \xi_1$. Applying the solution procedure of problem 3 in Section 2.4 provides a solution for θ_2. (If there are multiple solutions, we must choose one, and if there are no solutions, then there is no solution to (3.2.1) for the given g.) With θ_2 determined, $e^{\theta_2\xi_2} c_5$ is known and (3.2.2) becomes problem 2 of Section 2.4. Thus, we can obtain θ_0 and θ_1. Next, from (3.2.1) we have

$$e^{\theta_3\xi_3} e^{\theta_4\xi_4} e^{\theta_5\xi_5} = (e^{\theta_0\xi_0} e^{\theta_1\xi_1} e^{\theta_2\xi_2})^{-1} g \tag{3.2.5}$$

Let $O_a \neq c_5$ be a point on ξ_5. Then

$$e^{\theta_3 \xi_3} e^{\theta_4 \xi_4} O_a = (e^{\theta_0 \xi_0} e^{\theta_1 \xi_1} e^{\theta_2 \xi_2})^{-1} g O_a \tag{3.2.6}$$

This is again problem 2 of Section (2.4) and θ_3 and θ_4 can be determined. Only θ_5 remains to be determined. From (3.2.1)

$$e^{\theta_5 \xi_5} = (e^{\theta_0 \xi_0} e^{\theta_1 \xi_1} e^{\theta_2 \xi_2} e^{\theta_3 \xi_3} e^{\theta_4 \xi_4})^{-1} g \tag{3.2.7}$$

Applying both sides of (3.2.7) to $O_a \notin \xi_5$ turns (3.2.7) into problem 1 of Section 2.4 and θ_5 can be determined. Thus, using the subproblems of section 2.4 we have determined all the joint angles of the elbow manipulator.

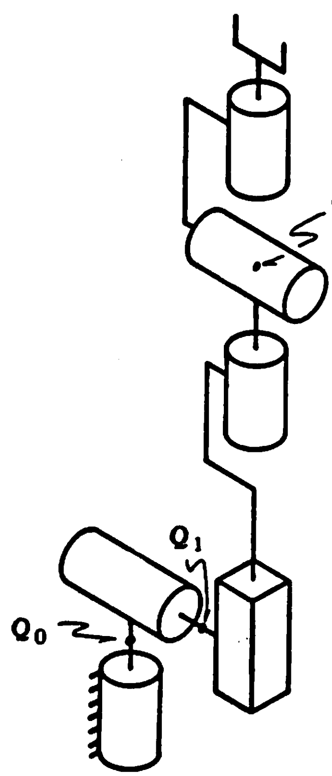

Figure 3.3 Stanford Manipulator.

3.2.2 Stanford Manipulator Solution

The Stanford manipulator drawn in Figure 3.3 is quite similar to the elbow manipulator as is its solution. The only difference is that the Stanford manipulator has a prismatic third joint instead of a revolute joint. Again, the important features of the stanford manipulator which allow the simple solution of its kinematic equation are that its first two axes intersect and its last three axes intersect. Let $\bar{\xi} = \{\xi_0, \xi_2, \cdots, \xi_5\}$ be the set unit screws whose axes are coincident with those of the elbow manipulator for some nominal zero configuration. All of these screws are zero-pitch except for ξ_2 which is ∞–pitch. Define $c_0 \triangleq \xi_0 \cap \xi_1$ and $c_5 \triangleq \xi_3 \cap \xi_4 \cap \xi_5$. Now we solve

$$e^{\theta_0 \xi_0} e^{\theta_1 \xi_1} e^{l_2 \xi_2} e^{\theta_3 \xi_3} e^{\theta_4 \xi_4} e^{\theta_5 \xi_5} = g \tag{3.2.8}$$

for given $g \in G$.

Solution: Applying both sides of (3.2.8) to c_5 we have

$$e^{\theta_0 \xi_0} e^{\theta_1 \xi_1} e^{l_2 \xi_2} c_5 = g c_5 \tag{3.2.9}$$

Next, subtracting c_0 from both sides of (3.2.9) and take norms yields

$$\| e^{\theta_0 \xi_0} e^{\theta_1 \xi_1} e^{l_2 \xi_2} c_5 - c_0 \| = \| g c_5 - c_0 \| \tag{3.2.10}$$

Applying $(e^{\theta_0 \xi_0} e^{\theta_1 \xi_1})^{-1}$ to both terms in the LHS of (3.2.10) yields

$$\| e^{l_2 \xi_2} c_5 - c_0 \| = \| g c_5 - c_0 \|. \tag{3.2.11}$$

Applying the solution procedure to problem 4 in Section 2.4 provides a solution for l_2. With l_2 determined, $e^{l_2 \xi_2} c_5$ is known and (3.2.9) becomes problem 2 of Section 2.4. Thus, we can obtain θ_0 and θ_1. Next, from (3.2.8) we have

$$e^{\theta_3 \xi_3} e^{\theta_4 \xi_4} e^{\theta_5 \xi_5} = (e^{\theta_0 \xi_0} e^{\theta_1 \xi_1} e^{l_2 \xi_2})^{-1} g \tag{3.2.12}$$

Let $O_a \neq c_5$ be a point on ξ_5. Then

$$e^{\theta_3 \xi_3} e^{\theta_4 \xi_4} O_a = (e^{\theta_0 \xi_0} e^{\theta_1 \xi_1} e^{l_2 \xi_2})^{-1} g O_a \tag{3.2.13}$$

This is again problem 2 of Section (2.4) and θ_3 and θ_4 can be determined. Only θ_5

remains to be determined. From (3.2.8)

$$e^{\theta_5 \xi_5} = (e^{\theta_0 \xi_0} e^{\theta_1 \xi_1} e^{t_2 \xi_2} e^{\theta_3 \xi_3} e^{\theta_4 \xi_4})^{-1} g \tag{3.2.14}$$

Applying both sides of (3.2.14) to $O_n \notin \xi_5$ turns (3.2.14) into problem 1 of Section 2.4 and θ_5 can be determined. As with the elbow manipulator, we have determined all the joint angles of the manipulator using the subproblems of section 2.4.

It is not new that these manipulators have solutions, but it is appealing that the solutions can be described so simply in terms of common subroutines.

Chapter 4

Manipulator Singularities

It is well known that differential motions of a manipulator's gripper may be related through the Jacobian of the forward kinematic map to differential motions of the joints [1]. An interesting subject of study has been those configurations of six degree-of-freedom (d.o.f.) manipulators where the Jacobian is singular (i.e. critical points of the forward kinematic map). Understanding these singular configurations is important for the following reasons.

(1) At singularities, bounded hand velocities may produce unbounded joint velocities.

(2) Points on the boundary[1] of the manipulator's reachable space correspond to singular configurations when the manipulator has all revolute joints.

(3) A technique commonly used to plan bounded error, straight line paths in the reachable workspace generates more knot points near singularities [2]. This is due to the fact that the distance between knot points in ***jointspace*** determines bounds on error in the workspace. The scheme suggested by Taylor [2] reduces distances between knot points in the ***workspace*** to improve tolerances. This is effective everywhere except near singularities where small distances in the workspace do not necessarily correspond to small distances in joint space.

(4) Points in the manipulator's workspace which are reachable only when the manipulator is in a singular configuration may become unreachable under perturbation of link parameters. Since the position and orientation of a manipulator's gripper are given by

[1] Here we restrict "boundary" to mean those points on the boundary of the geometrical model's reachable space where mechanical limits are not an issue.

a smooth function of the joint variables and the link parameters, the implicit function theorem guarantees solutions to the inverse kinematic problem under perturbation of link parameters only where the Jacobian is nonsingular.

Manipulator singularities have received considerable attention. Whitney [1] presents the Jacobian in a cross product form similar to the one that will be used here. Singularities of robot wrists are analyzed and working regions away from these singularities are defined by Paul and Stevenson [3]. Screw calculus has been used to describe singularities [4] and Luh [5] has used screw calculus to analyze redundant manipulators. In particular, Luh presents a method for avoiding singularities by taking advantage of redundancy. Since singularities occur at boundary points, they can be used to define the reachable workspace [6]. Finally, Litvin and Castelli [7] have found singular configurations for a Cincinnati Milacron manipulator and the Unimation Puma manipulator. Additional related work is presented by Paul [8] and Pieper [9].

This chapter focuses on the geometric interpretation of singularities for manipulators with six degrees-of-freedom. Several properties of a manipulator's configuration are independent of the coordinate system used to express them and depend only on the angles and distances between links and joints. The singularity of the Jacobian is such a property and this fact will be exploited to describe singularities geometrically.

The layout of this chapter is as follows: Section 4.1 describes the tangent of the forward kinematic map. Section 4.2 gives some examples of configurations where the the tangent map fails to be surjective. Section 4.3 describes the decoupling of singularities which occurs in manipulators with three consecutive revolute joints with intersecting axes and section 4.4 contains some discussion.

4.1 The Tangent Map, $Tf_{\bar{\xi}}$ (see [10])

From equation (3.1.2), we have that the forward kinematic map associated with $\bar{\xi} = (\xi_0, \xi_1, \cdots, \xi_{n-1})$, $f_{\bar{\xi}}: J \rightarrow G$, is given by

$$f_{\xi}(\theta) = \prod_{i=0}^{n-1} e^{\theta_i \xi_i} \tag{4.1.1}$$

where the ξ_i are the twists corresponding to the joints in the zero configuration of the manipulator. From (4.1.1), $Tf_{\xi}:TJ \to TG$ is given by

$$Tf_{\xi}(\theta, v) = \sum_{k=0}^{n-1} v_k \frac{\partial}{\partial \theta_k} f_{\xi}(\theta) = \sum_{k=0}^{n-1} v_k \frac{\partial}{\partial \theta_k} \left[\prod_{i=0}^{n-1} e^{\theta_i \xi_i} \right] \in T_{f_{\xi}(\theta)}G \tag{4.1.2}$$

where $\theta \in J$ and $v \in T_{\theta}J = \mathbf{R}^n$.

Observe that

$$\begin{aligned} \frac{\partial}{\partial \theta_k} \left[\prod_{i=0}^{n-1} e^{\theta_i \xi_i} \right] &= \left[\prod_{i=0}^{k-1} e^{\theta_i \xi_i} \right] \xi_k \left[\prod_{i=k}^{n-1} e^{\theta_i \xi_i} \right] \\ &= \left[\prod_{i=0}^{k-1} e^{\theta_i \xi_i} \right] \xi_k \left[\prod_{i=0}^{k-1} e^{\theta_i \xi_i} \right]^{-1} \left[\prod_{i=0}^{n-1} e^{\theta_i \xi_i} \right] \end{aligned} \tag{4.1.3}$$

Recall, from proposition 2.3.2, that

$$\xi'_k = \left[\prod_{i=0}^{k-1} e^{\theta_i \xi_i} \right] \xi_k \left[\prod_{i=0}^{k-1} e^{\theta_i \xi_i} \right]^{-1}$$

is a twist with the same pitch and magnitude as ξ_k and the axis of ξ'_k is that of ξ_k translated by $\left[\prod_{i=0}^{k-1} e^{\theta_i \xi_i} \right]$. Thus ξ'_k is the twist representing the kth joint when the manipulator is in the configuration given by $(\theta_0, \theta_1, \cdots, \theta_{n-1})$. Since elements in G are nonsingular, we have that $Tf_{\xi}(\theta)$ is surjective if and only if the ξ'_k span $T_I G$. Since the map tc is a vector space isomorphism we have, in terms of the twist coordinates of ξ'_k, that $Tf_{\xi}(\theta)$ is surjective if and only if the matrix

$$J(\theta) = \left[tc(\xi'_0) \; tc(\xi'_1) \; \cdots \; tc(\xi'_{n-1}) \right] \tag{4.1.4}$$

has full rank (=6). This matrix of twist coordinates is called the manipulator Jacobian. Note that the columns of the Jacobian are determined only by the positions of the joint axes. This means that singularities of the Jacobian are determined only by the geometry of the axes as we might expect.

4.2 Some Common Critical Points of Six d.o.f. Manipulators

Many critical points of the forward kinematic map of 6 d.o.f. manipulators have a simple geometric interpretation. In this section, we describe several of these, and demonstrate them on industrial manipulators. There are, of course, many other critical points which have no simple geometric interpretation other than the linear dependence of twists representing the joints.

For the rest of this chapter, we will violate our own convention and write points and vectors as columns of three real numbers rather than four. This will allow the writing of twist coordinate as a column of two vectors in $\mathbf{R}^3$ and simplify our discussion. If ξ is a zero-pitch unit twist with its axis passing through a point Q and having unit direction vector z, then

$$tc(\xi) = \begin{bmatrix} z \times (O - Q) \\ z \end{bmatrix} \tag{4.2.1}$$

where $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. (We still follow the rules of syntax we had with the 4-vectors; it does not make sense to take the cross product of a point and a vector). If ξ is an ∞-pitch unit twist with unit axis direction vector z, then

$$tc(\xi) = \begin{bmatrix} 0 \\ z \end{bmatrix}. \tag{4.2.2}$$

So for a general six d.o.f. manipulator, the manipulator Jacobian will consist of six columns of the form (4.2.1) or (4.2.2).

Example

Figure 3.3 shows the Stanford manipulator which is a RRPRRR manipulator. Its Jacobian therefore has the form

$$J = \begin{bmatrix} z_0 & z_1 & 0 & z_3 & z_4 & z_5 \\ z_0 \times (O - Q_0) & z_1 \times (O - Q_1) & z_2 & z_3 \times (O - Q_3) & z_4 \times (O - Q_3) & z_4 \times (O - Q_3) \end{bmatrix}. \quad (4.2.3)$$

where the Q_i are points on the revolute axes and the z_i are directions of the axes. Since the final three axes intersect, the points Q_3, Q_4, and Q_5 are chosen equal to Q_3.

Some examples of manipulator configurations where the Jacobian is singular are now given. These examples have particularly simple geometric descriptions and apply to general manipulators, but it is important to remember that, in general, singular configurations have no simple description. The fact that common manipulators are simple geometrically may be the reason for their easily described singularities. To simplify notation we write (Q, z) to represent the directed line through Q having unit direction vector z.

Example 1. Two Collinear Revolute Joints Axes.

Without loss of generality[2], take the joints[3] to be 0 and 1. Then

(a) Their axes, (Q_0, z_0) and (Q_1, z_1), have parallel directions: $z_0 = \pm z_1$

(b) The vector $(Q_0 - Q_1)$ is parallel to z_0 and z_1 : $z_i \times (Q_0 - Q_1)$ for $i \in \{0, 1\}$, and

$$J = \begin{bmatrix} z_0 & z_1 & \cdots \\ z_0 \times (O - Q_0) & z_1 \times (O - Q_1) & \cdots \end{bmatrix} \in \mathbf{R}^{6 \times 6}. \quad (4.2.4)$$

By the elementary row operation[4] row 2 $\leftarrow$ row 2 $+ (O - Q_0) \times$ row 1 we have[5]

$$J \sim \begin{bmatrix} z_0 & z_1 & \cdots \\ 0 & z_1 \times (Q_0 - Q_1) & \cdots \end{bmatrix}. \quad (4.2.5)$$

By (b) it follows that

$$J \sim \begin{bmatrix} z_0 & z_1 & \cdots \\ 0 & 0 & \cdots \end{bmatrix}. \quad (4.2.6)$$

It is now clear by (a) that J is singular. The Stanford manipulator (Figure 3.3) exhibits this singularity when joints 3 and 5 line up. When two revolute joints are collinear there

[2] Elementary column operations allow us to obtain the form of (4.2.4) regardless of the joint numbering.
[3] We number the columns of the Jacobian in the same way that we number the joints; we begin with zero.
[4] By row 1 we mean the first row of vectors and similarly for row 1.
[5] $A \sim B$ means that there exists nonsingular C, D such that $A = CBD$.

are a continuum of solutions since the links between the two revolute joints may be rotated without affecting the position of the gripper.

Example 2. Three Parallel, Coplanar Revolute Joint Axes.

Without loss of generality take the three joints to be 0, 1, and 2, with axes (Q_i, z_i), $i \in \{0,1,2\}$. The condition that the joints are parallel is then

(a) $z_i = \pm z_j \quad i,j \in \{0,1,2\}$

and the condition that the three joints are coplanar is

(b) There exists a plane containing the axes with unit normal n such that $n^T z_i = 0$ and $n^T(Q_i - Q_j) = 0 \quad i,j \in \{0,1,2\}$.

Since joints 0,1, and 2 are revolute the Jacobian has the form

$$J = \begin{bmatrix} z_0 & z_1 & z_2 & \cdots \\ z_0 \times (O - Q_0) & z_1 \times (O - Q_1) & z_2 \times (O - Q_2) & \cdots \end{bmatrix}. \tag{4.2.7}$$

By the elementary row operation row 2 $\leftarrow$ row 2 $+ (O - Q_0) \times$ row 1, we obtain

$$J \sim \begin{bmatrix} z_0 & z_1 & z_2 & \cdots \\ 0 & z_1 \times (Q_0 - Q_1) & z_2 \times (Q_0 - Q_2) & \cdots \end{bmatrix}. \tag{4.2.8}$$

By (a) there exists elementary column operations to yield

$$J \sim \begin{bmatrix} z_0 & 0 & 0 & \cdots \\ 0 & z_1 \times (Q_0 - Q_1) & z_2 \times (Q_0 - Q_2) & \cdots \end{bmatrix}. \tag{4.2.9}$$

By (b) columns 1 and 2 of (4.2.9) are in the range of $[n, 0]^T$ and are therefore linearly dependent. Thus, J is singular. The elbow manipulator in Figure 3.2. has this singularity when the elbow is fully extended as shown. In this configuration the manipulator is at the boundary of its reachable space.

Example 3. Four Intersecting Revolute Joint Axes.

When four axes, say (Q_i, z_i), $i \in \{0,1,2,3\}$, intersect at a point Q, the point Q satisfies

(a) $z_i \times (Q_i - Q) = 0, \quad i \in \{0,1,2,3\}$.

Now

$$J = \begin{bmatrix} z_0 & z_1 & z_2 & z_3 & \cdots \\ z_0 \times (O - Q_0) & z_1 \times (O - Q_1) & z_2 \times (O - Q_2) & z_3 \times (O - Q_3) & \cdots \end{bmatrix}. \tag{4.2.10}$$

By the elementary row operation row 2 $\leftarrow$ row 2 $+ (O - Q) \times$ row 1 and (a) yield

$$J \sim \begin{bmatrix} z_0 & z_1 & z_2 & z_3 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \end{bmatrix} \tag{4.2.11}$$

which is clearly singular since the first four columns are contained in a 3 dimensional subspace of $\mathbf{R}^6$. The Intelledex 605 robot, diagramed in Figure 4.1 has three intersecting axes at its shoulder. This type of singularity occurs when the final joint axis intersects the shoulder adding a fourth axis as shown.

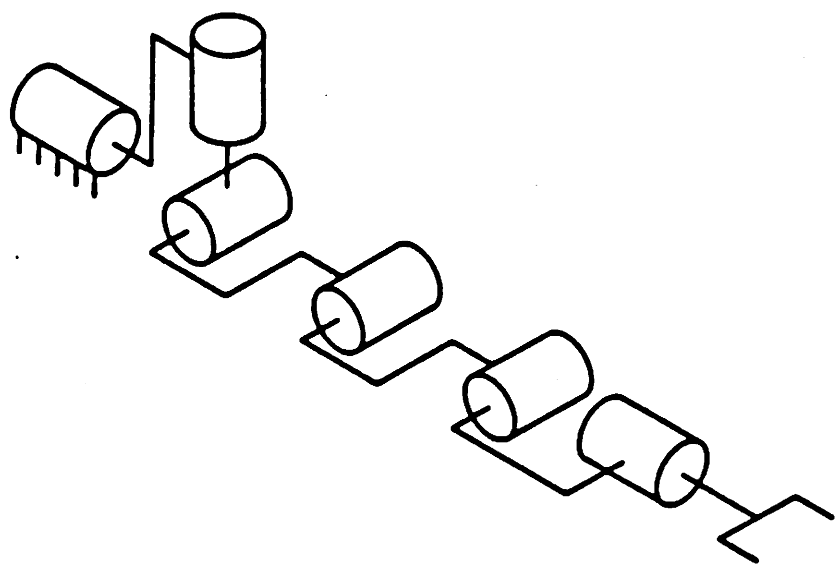

Figure 4.1. Intelledex 605 Robot.

Example 4. Four Parallel Revolute Joint Axes.

If the joint axes (Q_i, z_i), $i \in \{0,1,2,3\}$, are parallel then

(a) $z_i = \pm z_j$, $j \in \{0,1,2,3\}$.

$$J = \begin{bmatrix} z_0 & z_1 & z_2 & z_3 & \\ z_0 \times (O - Q_0) & z_1 \times (O - Q_1) & z_2 \times (O - Q_2) & z_3 \times (O - Q_3) & \cdots \end{bmatrix}. \tag{4.2.12}$$

By the elementary row operation row 2 $\leftarrow$ row 2 $+ (O - Q_0) \times$ row 1, we have

$$J \sim \begin{bmatrix} z_0 & z_1 & z_2 & z_3 & \\ 0 & z_1 \times (Q_0 - Q_1) & z_2 \times (Q_0 - Q_2) & z_3 \times (Q_0 - Q_3) & \cdots \end{bmatrix}. \tag{4.2.13}$$

Using (a) and elementary column operations we obtain

$$J \sim \begin{bmatrix} z_0 & 0 & 0 & 0 & \\ 0 & z_1 \times (Q_0 - Q_1) & z_2 \times (Q_0 - Q_2) & z_3 \times (Q_0 - Q_3) & \cdots \end{bmatrix}. \tag{4.2.14}$$

Now columns[2] 1,2, and 3, in (4.2.14), are in the null space of $\begin{bmatrix} z_0^T & 000 \\ 0 & I \end{bmatrix}$ which has dimension 2. It follows that J is singular.

Example 5. Four Coplanar Revolute Joint Axes.

Let n be the unit normal to the plane containing the four joint axes. These axes (Q_i, z_i),

$i \in \{0,1,2,3\}$, then satisfy

(a) Each axis direction is orthogonal to n : $n^T z_i = 0$, $i \in \{0,1,2,3\}$, and

(b) The vector from Q_i to Q_j is orthogonal to n : $n^T (Q_i - Q_j) = 0$, $i \in \{0,1,2,3\}$.

Now

$$J = \begin{bmatrix} z_0 & z_1 & z_2 & z_3 & \\ z_0 \times (O - Q_0) & z_1 \times (O - Q_1) & z_2 \times (O - Q_2) & z_3 \times (O - Q_3) & \cdots \end{bmatrix}. \tag{4.2.15}$$

By the elementary row operation row 2 $\leftarrow$ row 2 $+ (O - Q_0) \times$ row 1, we obtain

$$J \sim \begin{bmatrix} z_0 & z_1 & z_2 & z_3 & \\ 0 & z_1 \times (Q_0 - Q_1) & z_2 \times (Q_0 - Q_2) & z_3 \times (Q_0 - Q_3) & \cdots \end{bmatrix}. \tag{4.2.16}$$

Then elementary column operations yield

$$J = \begin{bmatrix} z_0 & z_1 - z_0 z_0^T z_1 & z_2 - z_0 z_0^T z_2 & z_3 - z_0 z_0^T z_3 & \\ 0 & z_1 \times (Q_0 - Q_1) & z_2 \times (Q_0 - Q_2) & z_3 \times (Q_0 - Q_3) & \cdots \end{bmatrix}. \tag{4.2.17}$$

Columns 1,2, and 3, are in the range of the rank 2 matrix $\begin{bmatrix} 0 & n \times z_0 \\ n & 0 \end{bmatrix}$ by (a) and (b) above and are therefore linearly dependent.

The Stanford manipulator reaches this configuration when joints 0,1,3, and 4, are coplanar as shown in Figure 3.3.

Example 6. Six revolute Joint Axes Intersecting a Line.

This configuration occurs in a six degree-of-freedom manipulator with all revolute joints when the manipulator is at full reach. For this reason, this configuration is useful for describing the reachable space of a manipulator. Let the line which the six revolute axes intersect be represented by the axis (Q, b). Each axis (Q_i, z_i), $i \in \{0,1,\ldots,5\}$, has a point in common with the axis (Q, b) so there exists γ_i, β_i, $i \in \{0,1,\ldots,5\}$, such that

(a) $Q_i + \gamma_i z_i = Q + \beta_i b$.

For a manipulator with all revolute joints, the columns of J are

$$J_i = \begin{bmatrix} z_i \\ z_i \times (O - Q_i) \end{bmatrix}, \qquad i \in \{0,1,\ldots,5\}. \tag{4.2.18}$$

From (a) we have

$$Q_i = Q + \beta_i b - \gamma_i z_i \tag{4.2.19}$$

Using (4.2.18), (4.2.19), and the fact that the cross product of a vector with itself is zero yields

$$J_i \sim \begin{bmatrix} z_i \\ z_i \times (O - Q - \beta_i b) \end{bmatrix}. \tag{4.2.20}$$

Applying the elementary row operation row 2 ← row 2 + $(O - Q) \times$ row 1, we obtain

$$J_i \sim \begin{bmatrix} z_i \\ -\beta_i z_i \times b \end{bmatrix}. \tag{4.2.21}$$

It follows that J is singular since $[b^T, 0^T]$ is in the left nullspace of the RHS of (4.2.21). The elbow manipulator in Figure 3.2. is at full reach and exhibits this singularity. This singularity occurs at other configurations those of maximum reach.

Example 7. Prismatic Joint Axis Normal to a Plane Containing Two Parallel Revolute Axes.

Label the two revolute joints 0 and 1, and the prismatic joint 2. The revolute axes are therefore (Q_0, z_0) and (Q_1, z_1) , and the prismatic joint axis is (Q_2, z_2) . The condition that (Q_0, z_0) and (Q_1, z_1) are in a plane orthogonal to the prismatic joint axis is

(a) $z_2^T z_i = 0 \quad i \in \{0,1\}$

$z_2^T (Q_0 - Q_1) = 0.$

The condition that the two revolute axes are parallel is

(b) $z_0 = \pm z_1.$

From (4.2.1) and (4.2.2)

$$J = \begin{bmatrix} z_0 & z_1 & 0 & \\ z_0 \times (O - Q_0) & z_1 \times (O - Q_1) & z_2 & \cdots \end{bmatrix}. \qquad (4.2.22)$$

By the elementary operation row 2 $\leftarrow$ row 2 $+ (O - Q_0) \times$ row 1 we have

$$J \sim \begin{bmatrix} z_0 & z_1 & 0 & \\ 0 & z_1 \times (Q_0 - Q_1) & z_2 & \cdots \end{bmatrix}. \qquad (4.2.23)$$

Using the fact that the revolute axes are parallel, (b), together with an elementary column operation yields

$$J \sim \begin{bmatrix} z_0 & 0 & 0 & \\ 0 & z_1 \times (Q_0 - Q_1) & z_2 & \cdots \end{bmatrix}. \qquad (4.2.24)$$

Now by (a), both z_1 and $(Q_0 - Q_1)$ are orthogonal to z_2 so $z_1 \times (Q_0 - Q_1)$ is in the range of z_2 . It follows that columns 1 and 2 of (4.21) are linearly dependent and that J is singular. A schematic diagram of the Rhino robot is shown in Figure 1.1. It reaches this singular configuration when joints 1 and 5 are parallel and in a plane perpendicular to the sliding motion of joint 0.

4.3 Decoupled Singularities.

In this section we demonstrate the decoupling of singularities which occurs in manipulators having three consecutive revolute joints with intersecting axes. For these manipulators, singular configurations are easily recognized. We may choose points on the axes of a manipulator in this class such that three are coincident with the intersection of the three axes. For concreteness we discuss six degree-of-freedom manipulators with revolute joints only. By renumbering the joints, the points on the axis of joints 3,4, and 5 may be chosen to coincide with the intersection of the three axes. The Jacobian is then

$$J = \begin{bmatrix} z_0 & z_1 & z_2 & z_3 & z_4 & z_5 \\ z_0 \times (O - Q_0) & z_1 \times (O - Q_1) & z_2 \times (O - Q_2) & z_3 \times (O - Q_3) & z_4 \times (O - Q_3) & z_5 \times (O - Q_3) \end{bmatrix}. \tag{4.3.1}$$

Note that the last three joints share the same origin labeled Q_3. By the elementary row operation row 2 $\leftarrow$ row 2 + $(O - Q_3) \times$ row 1, we have

$$J \sim \begin{bmatrix} z_0 & z_1 & z_2 & z_3 & z_4 & z_5 \\ z_0 \times (Q_3 - Q_0) & z_1 \times (Q_3 - Q_1) & z_2 \times (Q_3 - Q_2) & 0 & 0 & 0 \end{bmatrix}. \tag{4.3.2}$$

Therefore J is singular if and only if either

(a) z_3, z_4, and z_5 are coplanar.

or

(b) $z_0 \times (Q_3 - Q_0), z_1 \times (Q_3 - Q_1)$, and $z_2 \times (Q_3 - Q_2)$ are coplanar.

For the elbow manipulator in Figure 3.2 we may choose Q_3 to coincide with the intersection of the three wrist axes, and the joint numbering in (4.3.1) is the natural numbering from base to gripper. Looking at the first and last three joints separately we may determine the singularities of this manipulator by inspection using (a) and (b) above. First, z_3, z_4, and z_5 are coplanar if and only if joints 3 and 5 are collinear. This is the only singularity contributed by the wrist. Second, the three vectors in (b) are coplanar in the following two cases.

(i) The elbow is fully extended, or 180 degrees from full extension, so that $z_1 \times (Q_3 - Q_1)$ and $z_2 \times (Q_3 - Q_2)$ are linearly dependent.

(ii) Q_3 is directly above the base on the axis (Q_0, z_0) so that $z_0 \times (Q_3 - Q_0) = 0$.

These singularities may be interpreted in terms of the examples as well as (a) and (b) above.

4.4 Discussion

By using the manipulator Jacobian in cross product form, we have described several singular configurations geometrically. The descriptions are manipulator ***independent*** and therefore apply to any six degree-of-freedom manipulator which can attain the singular configurations. These simple descriptions allow the evaluation of singular configurations without explicitly computing the determinant of the Jacobian. We have also shown that for manipulators with three consecutive intersecting joint axes the evaluation of singularities is particularly simple. Future work in this area should involve the study of branching, or bifurcation, in the solutions of the inverse kinematic solutions. From the inverse function theorem, it follows that branching can only occur at singular configurations. The study of exactly which bifurcation occurs at a singularity requires second and higher order derivatives of the forward kinematic equation.

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Chapter 5

Optimal Design of 6R Manipulators

This chapter develops, for 6R manipulators, the notions of work-volume, maximal work-volume, duality, and well-connected-workspace. With these notions defined, a design theorem is proved which states that a 6R manipulator, M, has maximal work-volume and well-connected-workspace if and only if M or M^* (its dual) is an elbow manipulator.

The ideas of this chapter are motivated by several authors. The notion of work-volume used here is that derived from a translation invariant volume element on the group of rigid motions in 3-dimensional Euclidean space. This definition of volume is natural and corresponds to the integration of Roth's [1] service coefficient over the Reachable Workspace [2]. The application of such a notion of volume to robot manipulators is also suggested by Brockett [3]. The Dextrous and Reachable Workspaces of a manipulator have been used to relate kinematic design to performance in [4] and in [2]. Other work on manipulator workspaces appears in [5] and the references contained therein.

The format of this chapter is the following. Section 1 develops the mathematical framework for proving a basic theorem about 6R manipulators which is contained in section 2, and the Appendix contains a few subproblems encountered in section 2.

5.1 Mathematical Framework

A formal theory of manipulator kinematics requires, first of all, a mathematical representation of manipulators. The amount of information contained in the representation depends on what we are trying to accomplish. For dynamics, a representation must contain information on inertias etc. For the purposes of this chapter the only significant objects on a 6R manipulators are the joint axes. Thus, we will represent a 6R manipulator by an ordered set (ordered from base to gripper) of six zero-pitch unit twists whose axes are coincident with the manipulator axes for some configuration of the manipulator[1]. Figure 5.1 depicts a manipulator and the axes of six zero-pitch unit twists which are coincident with the joint axes for the configuration shown. The ordered set of twists $\bar{\xi} = \{\xi_0,\dots,\xi_5\}$ is called a representative of the manipulator M. Since there are many

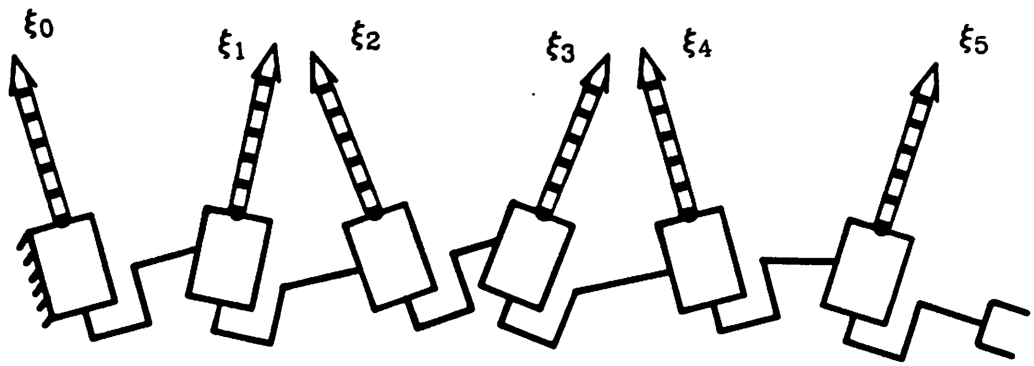

Figure 5.1 Representing a 6R Manipulator by an Ordered Set of Twists

such representatives we say that $\bar{\zeta}$ is equivalent to $\bar{\xi}$ and write $\bar{\zeta} \sim \bar{\xi}$ if $\bar{\zeta}$ and $\bar{\xi}$ are representatives of the same manipulator. Formally, $\bar{\zeta} \sim \bar{\xi}$ if $\exists \Phi \in T^5$ such that

$$\xi_i = \pm(e^{\phi_0\zeta_0} e^{\phi_1\zeta_1} \cdots e^{\phi_{i-1}\zeta_{i-1}})\zeta_i(e^{\phi_0\zeta_0} e^{\phi_1\zeta_1} \cdots e^{\phi_{i-1}\zeta_{i-1}})^{-1} \quad , i \in \{0,\dots,5\}$$

From Proposition 2.3.2 we have that that $g\zeta g^{-1}$,$g \in G$, has the same pitch and magnitude as ζ and that the axis of $g\zeta g^{-1}$ is the axis of ζ translated by the rigid motion g. This gives the interpretation of $\sim$ as $\bar{\zeta} \sim \bar{\xi}$ if the axis of each ζ_i can be rotated successively about the previous axes $\zeta_{i-1},\dots,\zeta_0$ such that its axis is coincident (possibly antiparallel) with that of ξ_i. A manipulator is identified with an equivalence class generated

[1]Since this chapter deals exclusively with 6R manipulators we will drop the 6R and simply write manipulator.

by $\sim$. If $\bar{\zeta}$ is a representative of M, we say $M = [\bar{\zeta}]$. The brackets, $[\cdot]$, are read "the equivalence class containing."

The length of a manipulator is important for obtaining bounds on its work-volume. Presently, we develop a notion of length for 6R manipulators which has operational significance. Define the (non-empty) set of linking curves of $\bar{\xi}$ as follows.

Definition 5.1.1 [2]

$C_{\bar{\xi}} \triangleq \{c : [0,5] \to \mathbf{R}^3 \mid c$ is continuous, linear on $[i, i+1]$, $i \in \{0,1,2,3,4\}$,
and $c_i \cap \xi_i \neq \emptyset\}$ is called the set of ***linking curves*** of $\bar{\xi}$.
(We use the notation c_t to denote c evaluated at t).

(5.1.1)

We refer to the line segments $c_{[i,i+1]}$, $i \in \{0,1,2,3,4\}$ as ***links*** of $\bar{\xi}$.

■

Using the set of linking curves we can define the length of $\bar{\xi}$ as the infimum of the lengths of all linking curves.

Definition 5.1.2 Define the length of $\bar{\xi}$, $l_{\bar{\xi}}$, by

$$l_{\bar{\xi}} \triangleq \inf_{c \in C_{\bar{\xi}}} \int_0^5 \left\| \frac{d}{dt} c_t \right\| dt \qquad (5.1.2)$$

■

The following proposition allows the replacement of the "inf" in (5.1.2) by a "min."

Proposition 5.1.1 There exists $c \in C_{\bar{\xi}}$ such that $\int_0^5 \left\| \frac{d}{dt} c_t \right\| dt = l_{\bar{\xi}}$.

Proof: Let Q_i, $i \in \{0,\dots,5\}$ be arbitrary points on the axes of the ξ_i respectively, and let z_i be unit direction vectors along these axes. Then any point on the axis of ξ_i may be written $Q_i + \lambda^i z_i$ for some $\lambda^i \in \mathbf{R}$.

[2] We allow ourselves the minor abuse of notation by identifying a zero-pitch twists with its axis. $\xi_1 \cap \xi_2 \neq \emptyset$ means the axes of ξ_1 and ξ_2 intersect.

Since a linking curve is linear between its points of intersection with the axes of the ξ_i, the linking curve is described completely by the points of intersection. Let $\lambda \in \mathbf{R}^6$ be such that $Q_i + \lambda^i z_i$ are the points of intersection. Then the length of the linking curve is

$$d(\lambda) \triangleq \sum_{i=1}^{5} \|(Q_i + \lambda^i v_i) - (Q_{i-1} + \lambda^{i-1} v_{i-1})\| \geqslant 0.$$

To show that there is a linking curve having minimal length, it is sufficient to prove the existence of a $\lambda^* \in \mathbf{R}^6$ which minimizes $d(\lambda)$.

Let $\{\lambda_k\}$ be a sequence in $\mathbf{R}^6$ such that $d(\lambda_k) \to l_{\bar{\xi}}$. If $\{\lambda_k\}$ has an accumulation point, λ^*, then by the continuity of d, $d(\lambda^*) = l_{\bar{\xi}}$ and we are done.

If $\{\lambda_k\}$ has no accumulation point, then $\{\lambda_k^i\}$ is unbounded for some $i \in \{0,\ldots,5\}$. Now $d(\lambda_k)$ is bounded so $\|(Q_i + \lambda_k^i v_i) - (Q_j + \lambda_k^j v_j)\|$ is bounded by the triangle inequality. Thus $\{\lambda_k^i\}$ is unbounded for all $i \in \{0,\ldots,5\}$ and the v_i are parallel (i.e. $v_i^T v_j = \pm 1$). Without loss of generality, let $v_i = v_j \; \forall \, i, j \in \{0,\ldots,5\}$. Then $d(\lambda) = d(\lambda + \alpha\tilde{1})$ where $\alpha \in \mathbf{R}$ and $\tilde{1} \triangleq (1,1,1,1,1,1)^T$. Moreover, the sequence $\{\gamma_k\} \triangleq \{\lambda_k - \lambda_k^0 \tilde{1}\}$ is also an infimizing sequence with $\gamma_k^0 \equiv 0$. This implies that $\{\gamma_k\}$ is bounded and has an accumulation point γ^* and $d(\gamma^*) = l_{\bar{\xi}}$. The accumulation point defines the desired linking curve.

■

The next proposition allows the extension of Definition 5.1.2 to a definition of length for manipulators.

Proposition 5.1.2 If $\bar{\xi} \sim \bar{\zeta}$, then $l_{\bar{\xi}} = l_{\bar{\zeta}}$.

Proof: $\bar{\xi} \sim \bar{\zeta} \;\Rightarrow\; \exists \bar{\phi} \in \mathbf{T}^5$ such that $\xi_i = \pm(e^{\phi_0\zeta_0} \cdots e^{\phi_{i-1}\zeta_{i-1}})\zeta_i(e^{\phi_0\zeta_0} \cdots e^{\phi_{i-1}\zeta_{i-1}})^{-1}$ $i \in \{0,\ldots,5\}$. Let $c \in C_{\bar{\zeta}}$ and define the curve d by

$$d_t = e^{\phi_0\zeta_0} \cdots e^{\phi_i\zeta_i} c_t \quad \text{for } t \in [i, i+1], \; i \in \{0,1,2,3,4\} \tag{5.1.3}$$

d_t is linear on $[i, i+1]$ and $e^{\phi_0\zeta_0} \cdots e^{\phi_i\zeta_i}c_i = e^{\phi_0\zeta_0} \cdots e^{\phi_{i-1}\zeta_{i-1}}c_i$ so d is continuous and piecewise linear on [0,5]. Also $c_i \in \zeta_i$ (meaning c_i is on the axis of ξ_i) => $d_i = e^{\phi_0\zeta_0} \cdots e^{\phi_{i-1}\zeta_{i-1}}c_i \in \xi_i$. It follows that $d \in C_{\bar{\xi}}$.

Since rigid motions preserve length it is easy to see that

$$\int_0^5 \|\frac{d}{dt}c_t\|dt = \int_0^5 \|\frac{d}{dt}d_t\|dt \ . \tag{5.1.4}$$

Thus, for every linking curve of $\bar{\zeta}$, $\exists$ a linking curve of $\bar{\xi}$ having the same length. By the symmetry of the equivalence relation, $\sim$, the opposite is also true. Thus, $l_{\bar{\xi}} = l_{\bar{\zeta}}$.

■

Proposition 5.1.2 tells us that each representative of a manipulator has the same length. Thus, the length of a manipulator is well defined as follows.

Definition 5.1.3 Let $M = [\bar{\xi}]$ then its ***length***, l_M, is given by

$$l_M \triangleq l_{\bar{\xi}} \tag{5.1.5}$$

■

It is often the case that a property of a manipulator is invariant under the reversal of the axis order. Reversing the order of the axes is equivalent to using the gripper end of the robot as the base and the base end as a gripper (clearly this is not practical unless we replace the final link with a gripping device and the first link with a mounting flange). This is an important symmetry which is made precise by defining the dual of a manipulator.

Definition 5.1.4 If $M = [\xi_0, \ldots, \xi_5]$ then $M^* \triangleq [\xi_5, \ldots, \xi_0]$ is called the ***dual*** of M. The representative of M^* is denoted $\bar{\xi}^*$.

■

Proposition 5.1.3 Let M be a manipulator, then $l_M = l_{M^*}$.

Proof: If c_t is a linking curve for $\bar{\xi}$, a representative of M, then c_{5-t} is a linking curve for $\bar{\xi}^*$ and vice versa.

■

For manipulation we express the motion of the hand attached after the last joint in terms of the joint angles. If $\bar{\xi}$ represents the configuration of the manipulator axes when the hand is in the identity configuration, then the configuration of the hand as a function of the joint angles is

$$R = e^{\theta_0 \xi_0} e^{\theta_1 \xi_1} \cdots e^{\theta_5 \xi_5} . \tag{5.1.6}$$

The rigid motion of the hand relative to its identity configuration is a rotation about ξ_5 by θ_5 followed by a rotation about ξ_4 by θ_4, etc. As we saw in chapter three, the map $f_{\bar{\xi}}$ defined by $f_{\bar{\xi}} : \bar{\theta} \rightarrow e^{\theta_0 \xi_0} e^{\theta_1 \xi_1} \cdots e^{\theta_5 \xi_5}$ is called the ***forward kinematic map*** for $[\bar{\xi}]$ associated with $\bar{\xi}$. It would be nice to be able to define a unique forward kinematic map, however there does not seem to be a natural way to do this. The following proposition demonstrates the relationship between forward kinematic maps associated with different representatives of the same manipulator.

Proposition 5.1.4 If $\bar{\xi} \sim \bar{\zeta}$ then $\exists \bar{\phi} \in \mathbf{T}^5$ such that

$$f_{\bar{\xi}}(\theta_0, \ldots, \theta_5) = f_{\bar{\zeta}}((\pm\theta_0 + \phi_0), \ldots, (\pm\theta_4 + \phi_4), (\pm\theta_5)) e^{-\phi_5 \zeta_5} \cdots e^{-\phi_0 \zeta_0} \tag{5.1.7}$$

for some choice of signs in the RHS of (5.1.7).

Proof: Since $\bar{\xi} \sim \bar{\zeta}$ $\exists$ $\bar{\phi} \in \mathbf{T}^5$ such that $\xi_i = \pm (e^{\theta_0 \zeta_0} e^{\theta_1 \zeta_1} \cdots e^{\theta_{i-1} \zeta_{i-1}}) \zeta_i (e^{\theta_0 \zeta_0} e^{\theta_1 \zeta_1} \cdots e^{\theta_{i-1} \zeta_{i-1}})^{-1}$. Thus,

$$\begin{aligned}
f_{\bar{\xi}}(\theta_0, \cdots, \theta_5) &= e^{\theta_0 \xi_0} \cdots e^{\theta_5 \xi_5} \\
&= e^{\pm\theta_0 \zeta_0} \exp(\pm\theta_1 e^{(\phi_0 \zeta_0)} \zeta_1 e^{(-\phi_0 \zeta_0)}) \cdots \exp(\pm\theta_5 (e^{\phi_0 \zeta_0} \cdots e^{\phi_4 \zeta_4}) \zeta_5 (e^{\phi_0 \zeta_0} \cdots e^{\phi_4 \zeta_4})^{-1}) \\
&= e^{\pm\theta_0 \zeta_0} [e^{\phi_0 \zeta_0} e^{\pm\theta_1 \zeta_1} e^{-\phi_0 \zeta_0}] \cdots \exp(\pm\theta_5 (e^{\phi_0 \zeta_0} \cdots e^{\phi_4 \zeta_4}) \zeta_5 (e^{\phi_0 \zeta_0} \cdots e^{\phi_4 \zeta_4})^{-1}) \\
&= e^{(\pm\theta_0 + \phi_0) \zeta_0} e^{(\pm\theta_1 + \phi_1) \zeta_1} \cdots e^{(\pm\theta_5) \zeta_5} e^{-\phi_4 \zeta_4} \cdots e^{-\phi_0 \zeta_0} = \text{RHS of (5.1.7)}.
\end{aligned} \tag{5.1.8}$$

■

Thus, $f_{\bar{\xi}} = R_g f_{\xi} \circ h$ where h is an automorphism of T^6 and R_g is a right translation by some $g \in G$. It follows that $f_{\bar{\xi}}(T^6) = R_g f_{\xi}(T^6)$. (i.e., a translated version of $f_{\bar{\xi}}(T^6)$). If we define volume in G such that it is invariant under translations by group elements, then we can associate a unique volume to each manipulator. Let ω be the unique translation invariant volume form on $SO(3)$ [6] such that $\int_{SO(3)} \omega = 8\pi^2$. This normalization gives the units of orientation-volume in radians cubed. Define a volume element on R^3 by $dx \wedge dy \wedge dz$ as usual. Then we construct the volume form $\Omega \triangleq dx \wedge dy \wedge dz \wedge \omega$ on $R^3 \times SO(3)$. From [7] page 399 we have that $dx \wedge dy \wedge dz$ induces a measure μ_1 (Lebesgue measure) on R^3 and ω induces a measure μ_2 on $SO(3)$. The volume form Ω induces a measure on $R^3 \times SO(3)$ which is simply the product measure $\mu_1 \times \mu_2$. This observation will simplify our calculation of volumes in $R^3 \times SO(3)$ since we only calculate volumes of rectangles of the form $A \times SO(3)$, $A \subset R^3$. We make the identification of G with $R^3 \times SO(3)$ and note that Ω is a translation invariant volume form on G.

Definition 5.1.5 The work-volume, V_M, of a manipulator, $M = [\bar{\xi}]$, is given by

$$V_M \triangleq \int_{f_{\bar{\xi}}(T^6)} \Omega \tag{5.1.9}$$

■

Comment: The translation invariance of Ω in 5.1.9 guarantees that V_M is independent of the choice of representative $\bar{\xi}$.

Proposition 5.1.5 For any manipulator, M,

$$V_M \leqslant \frac{4}{3}\pi(l_M)^3 \cdot 8\pi^2$$

Proof: Let $c \in C_{\bar{\xi}}$ with length l_M. Then $\| f_{\bar{\xi}}(\bar{\theta})c_5 - c_0 \| \leqslant l_M \ \forall \bar{\theta} \in T^6$ since we can construct a curve from c_0 to $f_{\bar{\xi}}(\bar{\theta})c_5$ of length $l_M \ \forall \bar{\theta}$ as was done in Proposition

5.1.2. Thus,

$$f_{\bar{\xi}}(\bar{\theta})c_5 - c_0 \in \bar{B}(0, l_M) \qquad (5.1.10)$$

Thus c_5 is always "tethered" to c_0 by a curve of length l_M. (The other c_i are also "tethered" together by shorter curves).

$$\Rightarrow f_{\bar{\xi}}(T^6) \subset \{g_1 g_2 \mid g_1 \in G \text{ is a translation by } v \in \bar{B}(c_0 - c_5, l_M) \text{ and } g_2 \in G_{c_5}\} \cong_V \bar{B}(0, l_m) \times SO(3) \qquad (5.1.11)$$

Thus,

$$V_{[\xi]} \leq \frac{4}{3}\pi(l_M)^3 \cdot 8\pi^2 \qquad (5.1.12)$$

■

Definition 5.1.6 M has *maximal work-volume* (MWV) if $V_M = \frac{4}{3}\pi(l_M)^3 \cdot 8\pi^2$.

■

Let $C_{\bar{\xi}}$ be the set of critical points [7] of $f_{\bar{\xi}}$.

Definition 5.1.7 The manipulator $[\xi]$ has *well-connected-workspace* if $f_{\bar{\xi}}(B) = f_{\bar{\xi}}(T^6) \setminus f_{\bar{\xi}}(C_{\bar{\xi}})$ $\forall$ connected components, B, of $T^6 \setminus C_{\bar{\xi}}$. (Since $f_{\bar{\xi}}$ and $f_{\bar{\zeta}}$ are related, for $\bar{\xi} \sim \bar{\zeta}$, by compositions with diffeomorphisms this property is well-defined for a manipulator).

■

Comment: A manipulator having the well-connected-workspace property has the ability (modulo obstacles and mechanical constraints) to move its gripper from one regular value [7] to another without passing through a critical value (singularity). This is a "nice" property of manipulators since it guarantees that the manipulator need not change configurations (e.g. from elbow up to elbow down) in order to move from one regular value of its forward kinematic map to another

Proposition 5.1.6 Let M be a manipulator. Then $V_M = V_{M^*}$.

Proof: Let $\bar{\xi}$ be a representative of M then

$$
\begin{aligned}
f_{\bar\xi}(T^6) &= \{e^{\theta_0\xi_0}\cdots e^{\theta_5\xi_5} \mid \bar\theta \in T^6\} \\
&= \{(e^{-\theta_5\xi_5}\cdots e^{-\theta_0\xi_0})^{-1} \mid \bar\theta \in T^6\} \\
&= \{(e^{\theta_5\xi_5}\cdots e^{\theta_0\xi_0})^{-1} \mid \bar\theta \in T^6\} \\
&= I f_{\bar\xi^*}(T^6)
\end{aligned}
\qquad (5.1.13)
$$

where I is the inversion map in the group G. I is volume preserving for translation invariant volume elements => $V_M = V_{M^*}$.

■

Corollary M has MWV <=> M^* has MWV.

Using the fact that I is a diffeomorphism we can show.

Proposition 5.1.7 M has well-connected-workspace <=> M^* has a well-connected-workspace.

■

Comment: Proposition 5.1.6, its corollary, and Proposition 5.1.7 tell us that the notion of a dual is a fundamental symmetry in the analysis of manipulators. We will use this symmetry to simplify our proof of Theorem 5.2.1. When we exploit this symmetry we say "by duality ...".

We will see shortly that elbow manipulators are very special 6R manipulators. We first give a formal definition of an elbow manipulator.

Definition 5.1.8 M is an *elbow manipulator* if it has a representative $\bar\xi$ with the following properties (see Fig. 5.2):

$\exists$ a line segment $\overline{c_0c_5}$ with (i) $\xi_0 \cap \xi_1 = c_0$, (ii) $\xi_0 \perp \xi_1$, (iii) $\xi_1 \perp \overline{c_0c_5}$, (iv) $\xi_1 \| \xi_2$ (v) ξ_2 is a perpendicular bisector of $\overline{c_0c_5}$, (vi) $\xi_3 \cap \xi_4 \cap \xi_5 = c_5$, (vii) $\xi_3 \perp \xi_4$ and (viii) $\xi_4 \perp \xi_5$.

■

(We use $\overline{c_0c_5}$ anticipating a linking curve with endpoints c_0 and c_5.)

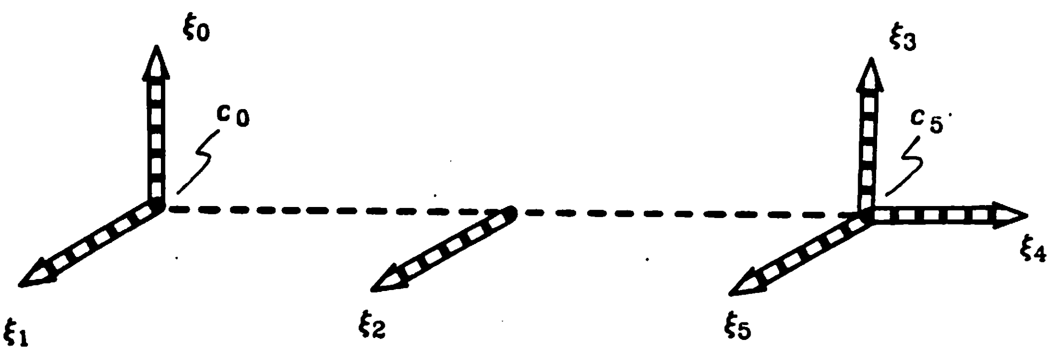

Figure. 5.2 Representative of an Elbow Manipulator

Proposition 5.1.8 Let M be an elbow manipulator and $\overline{c_0c_5}$ be as in the definition of the elbow manipulator. Then $l_M = \|c_5 - c_0\|$.

■

Proof: Let $\bar{\xi}$ be a representative of M which satisfies Definition 5.1.8. Since the last three axes of $\bar{\xi}$ intersect consecutively orthogonally at c_5 we can point joint axis 5 arbitrarily. Thus we can rotate ξ_5 about ξ_4 and then about ξ_3 to a new position ξ'_5 with $\xi'_5 \| \xi_1$. Then $\overline{c_0c_5}$ is a mutual perpendicular of ξ_1 and ξ'_5. There is no curve shorter than $\|c_5 - c_0\|$ connecting ξ_1 and ξ'_5 and therefore $l_M \geqslant \|c_5 - c_0\|$. Now it is easy to construct a linking curve c_t with $c_{[0,5]} = \overline{c_0c_5}$ and whose length is $\|c_5 - c_0\|$. Therefore $l_M = \|c_5 - c_0\|$.

■

This concludes the mathematical setup. In the next section a basic theorem is proved relating the ideas of this section.

5.2 Optimality Theorem

It is generally accepted that "elbow" manipulators have large work-volumes. This is made precise and an interesting converse is proved in the following theorem.

Theorem 5.2.1 A 6R manipulator, M, has well-connected-workspace and maximal work-volume if and only if M or M^* is an elbow manipulator.

Proof: (<=) By duality, it is sufficient to prove this implication for M an elbow manipulator. Let $\overline{\xi}$ be a representative of an elbow manipulator satisfying the properties in the definition and the additional (nonrestrictive) properties $\xi_3 = \xi_5$ and $z_0^T(c_5 - c_0) = \|c_5 - c_0\|$ where z_0 is a unit vector parallel to ξ_0. (These added conditions allow us to write down a closed form inverse kinematic solution for $f_{\overline{\xi}}$.) We prove the well-connectedness and maximal work-volume of M by examining the solution to the kinematic equation:

$$e^{\theta_0 \xi_0} e^{(\theta_1 - \theta_2/2)\xi_1} e^{\theta_2 \xi_2} e^{\theta_3 \xi_3} e^{\theta_4 \xi_4} e^{\theta_5 \xi_5} = R \;^{*} \tag{5.2.1}$$

where $R \in G$ is a desired rigid motion. Let z_i be a unit vector parallel to ξ_i, $i \in \{0,...,5\}$. Also, c_5 and c_0 are as in Definition 5.1.8, and $l_M = \|c_5 - c_0\|$ is the length of the manipulator. A straightforward application of the solution procedures of the subproblems of section 2.4. to 5.2.1 yields that $\overline{\theta}$ is a solution to (5.2.1) if and only if $\overline{\theta}$ satisfies (5.2.2-5.2.5):

$$\theta_2 = \pm \cos^{-1}\left(\frac{2\|Rc_5 - c_0\|^2}{(l_M)^2} - 1\right) \tag{5.2.2}$$

$$u_1 \triangleq Rc_5 - c_0$$

$$\theta_1 = \pm ATAN_2\left(\frac{\sqrt{\|u_1\|^2 - (z_0^T u_1)^2}}{z_0^T u_1}\right) \tag{5.2.3a}$$

$$\theta_0 = ATAN_2\left(\frac{\pm z_1^T u_1}{\pm (z_0 \times z_1)^T u_1}\right) \tag{5.2.3b}$$

(The upper and lower signs are matched in (5.2.3a) and (5.2.3b).)

$$R_1 \triangleq [e^{\theta_0 \xi_0} e^{(\theta_1 - \theta_2/2)\xi_1} e^{\theta_2 \xi_2}]^{-1} R$$

*Note that we make a minor change in joint coordinates $(\theta_1 \to \theta_1 - \theta_2/2)$ in joint coordinates. This is for convenience and has no effect on work-volume or the well-connected-workspace property.

$u_2 \triangleq R_1 z_3$

$$\theta_4 = \pm ATAN_2 \left[\frac{\sqrt{1-(z_3^T u_2)^2}}{z_3^T u_2} \right] \tag{5.2.4a}$$

$$\theta_5 = ATAN_2 \left[\frac{\mp z_4^T u_2}{\pm (z_3 \times z_4)^T u_2} \right] \tag{5.2.4b}$$

$R_2 \triangleq [e^{\theta_3 \xi_3} e^{\theta_4 \xi_4}]^{-1} R_1 \, . \; u_3 \triangleq R_2 z_4$

(The upper and lower signs are matched in (5.2.4a) and (5.2.4b)).

$$\theta_5 = ATAN_2 \left[\frac{(z_5 \times z_4)^T u_3}{z_4^T u_3} \right] \tag{5.2.5}$$

Note that all quantities under the radical in (5.2.3-5.2.4) are nonnegative and that the domain of the $ATAN_2$ function is the entire plane. (The numerator and denominator are 2 distinct arguments of the $ATAN_2$ function. When they are both zero the function is set valued and the θ_i may take any value in $[0, 2\pi\,]$). The only restrictions we have on R are given by (5.2.2). That is, there is a solution for R if and only if

$$R \in \{g \in G \mid (\frac{2\|gc_5-c_0\|^2}{(l_M)^2} - 1) \in [-1,1]\}$$
$$= \{g \in G \mid \|gc_5-c_0\| \leq l_M\}$$
$$\cong_V \bar{B}(0,l_M) \times SO(3)$$

(5.2.6)

Thus the work-volume of the elbow manipulator is $\frac{4}{3}\pi(l_M)^3 \cdot 8\pi^2$ and is therefore maximal.

The singular configurations of the elbow manipulator are those where the elbow is fully extended or retracted, joints 3 and 5 are coincident, or the intersection of the last three joints lies on joint axis zero. A calculation of the Jacobian determinant of f_ξ (see [8] for this type of calculation) yields that the set of critical points for f_ξ in (5.2.1) is

$$C_{\bar\xi} = \{\bar\theta \mid \theta_1 \in \{0,\pi\} \text{ or } \theta_2 \in \{0,\pi\} \text{or } \theta_4 \in \{0,\pi\}\} \tag{5.2.7}$$

It is easy to see that $T^6 \setminus C_{\bar\xi}$ has 8 connected components. We must verify that

$$f_{\bar\xi}(B) = f_{\bar\xi}(T^6) \setminus f_{\bar\xi}(C_{\bar\xi}) \tag{5.2.8}$$

for all connected components, B, of $T^6 \setminus C_{\bar\xi}$. Let $R \in f_{\bar\xi}(T^6) \setminus f_{\bar\xi}(C_{\bar\xi})$, then when we apply the solution procedure (5.2.2-5.2.5) with such an R there must be no choice of signs in (5.2.2-5.2.5) such that $\theta_i \in \{0,\pi\}$, $i \in \{1,2,4\}$ (otherwise R is a critical value). Thus, whenever we choose a solution branch for θ_1, θ_2, and θ_4 (5.2.2, 5.3.3a and 5.3.4a) there is a choice of $\theta_i \in (0,\pi)$ or $\theta_i \in (-\pi,0)$, $i \in \{1,2,4\}$. It follows that we can find a solution in each connected component of $T^6 \setminus C_{\bar\xi}$ => (5.3.8) holds.

(=>) Let $M = [\bar\zeta]$ and $c' \in C_{\bar\xi}$ with length l_M and define $l' \triangleq \max_{\bar\theta \in T^6} \| e^{\theta_i \zeta_0} \cdots e^{\theta_5 \zeta_5} c'_5 - c'_0 \|$. If $l' < l_M$ then $V_M \leqslant \frac{4}{3}\pi(l')^3 \cdot 8\pi^2 < \frac{4}{3}\pi(l_M)^3 \cdot 8\pi^2$ and M does not have MWV. Thus $\exists \bar\theta$ such that $\| e^{\theta_0 \zeta_0} \cdots e^{\theta_5 \zeta_5} c_5 - c_0 \| = l_M$. Let $\bar\xi$ be a representative corresponding to this configuration. As was done in Proposition 5.1.2 we can construct $c \in C_{\bar\xi}$ from c' such that c has length l_M, $c_0 = c_0'$, and $c_5 = e^{\theta_0\zeta_0} \cdots e^{\theta_5\zeta_5} c_5'$. Thus, $\exists c$, a linking curve for $\bar\xi$ with length $l_M = \|c_5 - c_0\|$. So $c_{[0,5]} = \overline{c_0 c_5}$, a line segment. Since c is a linking curve, all ξ_i intersect $\overline{c_0 c_5}$ and the c_i are ordered on $\overline{c_0 c_5}$. (Figure 5.3a shows a representative consistent with our knowledge of M at this point in the proof.) Let $f_{\bar\xi}: T^6 \to G$ be the forward kinematic map for M associated with $\bar\xi$. Now $f_{\bar\xi}(T^6)$ is compact => $x \in f_{\bar\xi}(T^6)^C$ has a neighborhood of nonzero volume. Thus MWV =>

$$f_{\bar\xi}(T^6) = \{g_1 g_2 \mid g_1 \in G \text{ is a translation by } v \in \bar B(c_0 - c_5, l_M),\ g_2 \in G_{c_5}\}$$
(see proof of prop. 5.1.5).

Thus $f_{\bar\xi}(T^6)$ contains

$$\mathbf{O} \triangleq \{g_1 g_2 \mid g_1 \text{ is a translation by } v \in [l_M S^2 + (c_0 - c_5)],\ g_2 \in G_{c_5}\} \cong S^2 \times SO(3) \tag{5.2.9}$$

and $\mathbf{O}$ is a 5-manifold.

Next, define

$$\begin{aligned} \text{min} &= \min\{i \mid \xi_i \cap c_0 = \emptyset\} \\ \text{max} &= \max\{i \mid \xi_i \cap c_5 = \emptyset\} \end{aligned} \tag{5.2.10}$$

By definition of a linking curve min > 0, and max < 5. Also, max $\geqslant$ min; otherwise there would only be one link with nonzero length and MWV would not hold. From (5.2.9), we have that

$$\| e^{\theta_0 \xi_0} \cdots e^{\theta_5 \xi_5} c_5 - c_0 \| = l_M \quad \forall \bar{\theta} \in f_{\bar{\xi}}^{-1}(\mathbf{O}) \ . \tag{5.2.11}$$

Since $c_0 \cap \xi_i \neq \emptyset \ \forall i <$ min and $c_5 \cap \xi_i \neq \emptyset \ \forall i >$ max, (5.2.11) becomes

$$\| e^{\theta_{\min} \xi_{\min}} \cdots e^{\theta_{\max} \xi_{\max}} c_5 - c_0 \| = l_M \quad \forall \bar{\theta} \in f_{\bar{\xi}}^{-1}(\mathbf{O}) \tag{5.2.12}$$

$$\Rightarrow \| (e^{\theta_{\min} \xi_{\min}} \cdots e^{\theta_{\max} \xi_{\max}} c_5 - c_{\min}) + (c_{\min} - c_0) \| = l_M \quad \forall \bar{\theta} \in f_{\bar{\xi}}^{-1}(\mathbf{O}) \tag{5.2.13}$$

Since $\int_0^5 \| \frac{d}{dt} c_t \| dt = \| c_5 - c_0 \|$ we have $\int_a^b \| \frac{d}{dt} c_t \| dt = \| c_b - c_a \|$, $\forall a, b \in [0,5]$. Let $a =$ min, $b = 5$, then by a tethering argument similar to the first part of the proof of Proposition 5.1.5,

$$\| e^{\theta_{\min} \xi_{\min}} \cdots e^{\theta_{\max} \xi_{\max}} c_5 - c_{\min} \| \leqslant \| c_5 - c_{\min} \| \quad \forall \bar{\theta} \in f_{\bar{\xi}}^{-1}(\mathbf{O}) \tag{5.2.14}$$

Since $\| c_{\min} - c_0 \| + \| c_5 - c_{\min} \| = l_M$, the two terms in the norm in (5.2.13) have the same direction and the first term attains the bound in (5.2.14):

$$e^{\theta_{min}\xi_{min}} \cdots e^{\theta_{max}\xi_{max}} c_5 - c_{min} = \frac{c_{min} - c_0}{\| c_{min} - c_0 \|} \| c_5 - c_{min} \|$$

$$= c_5 - c_{min} \qquad (5.2.15)$$

=>

$$e^{\theta_{min}\xi_{min}} \cdots e^{\theta_{max}\xi_{max}} \in G_{c_5} \; \forall \, \overline{\theta} \in f_{\overline{\xi}}^{-1}(O) \; . \qquad (5.2.16)$$

A similar argument with c_{max} (duality) yields

$$(e^{\theta_{min}\xi_{min}} \cdots e^{\theta_{max}\xi_{max}})^{-1} \in G_{c_0} \; \forall \, \overline{\theta} \in f_{\overline{\xi}}^{-1}(O) \; . \qquad (5.2.17)$$

Let η be a zero-pitch unit twist whose axis is coincident with $\overline{c_0 c_5}$, then since $g^{-1} \in G_O$ $\Rightarrow g \in G_O$.

$$\{ e^{\theta_{min}\xi_{min}} \cdots e^{\theta_{max}\xi_{max}} \mid \overline{\theta} \in f_{\overline{\xi}}^{-1}(O) \} \subset G_{c_5} \cap G_{c_0} = \{ e^{\phi\eta} \mid \phi \in S^1 \} \qquad (5.2.18)$$

Now

$$O = f(f^{-1}(O)) = \{ e^{\theta_0\xi_0} \cdots e^{\theta_5\xi_5} \mid \overline{\theta} \in f^{-1}(O) \}$$
$$\subset \{ e^{\theta_0\xi_0} \cdots e^{\theta_{min-1}\xi_{min-1}} e^{\phi\eta} e^{\theta_{max+1}\xi_{max+1}} \cdots e^{\theta_5\xi_5} \mid \theta_i , \phi \in S^1 \} \qquad (5.2.19)$$

The containment in (5.2.19) follows from (5.2.18). Since O is a 5-manifold, the order of $\{\theta_0, \ldots, \theta_{min-1}, \phi, \theta_{max+1}, \ldots, \theta_5\}$ must be 5 (see A4). (Intuitively, we need 5 degrees of freedom to sweep out a 5-manifold.) Thus, max-min $\leqslant 1$. (Fig. 5.3b represents approximately our knowledge of M at this point in the proof -- at most two of the ξ_i do not intersect c_0 or c_5). Suppose max−min = 1 and that $\{ e^{\theta_{min}\xi_{min}} e^{\theta_{max}\xi_{max}} \mid \overline{\theta} \in f_{\overline{\xi}}^{-1}(O) \}$ is a finite set. Then from (5.2.18) we have that $\exists$ a finite set $\{\phi_1, \cdots, \phi_n\}$ such that

$$\{ e^{\theta_{min}\xi_{min}} e^{\theta_{max}\xi_{max}} \mid \overline{\theta} \in f_{\overline{\xi}}^{-1}(O) \} = \{ e^{\phi\eta} \mid \phi \in \{\phi_1, \ldots, \phi_n\} \} \qquad (5.2.20)$$

and

$$O \subset \bigcup_{k=1}^{n} \{ e^{\theta_0\xi_0} \cdots e^{\theta_{min}\xi_{min-1}} e^{\phi_k} e^{\theta_{max+1}\xi_{max+1}} \cdots e^{\theta_5\xi_5} \mid \theta_i \in S^1 \}$$

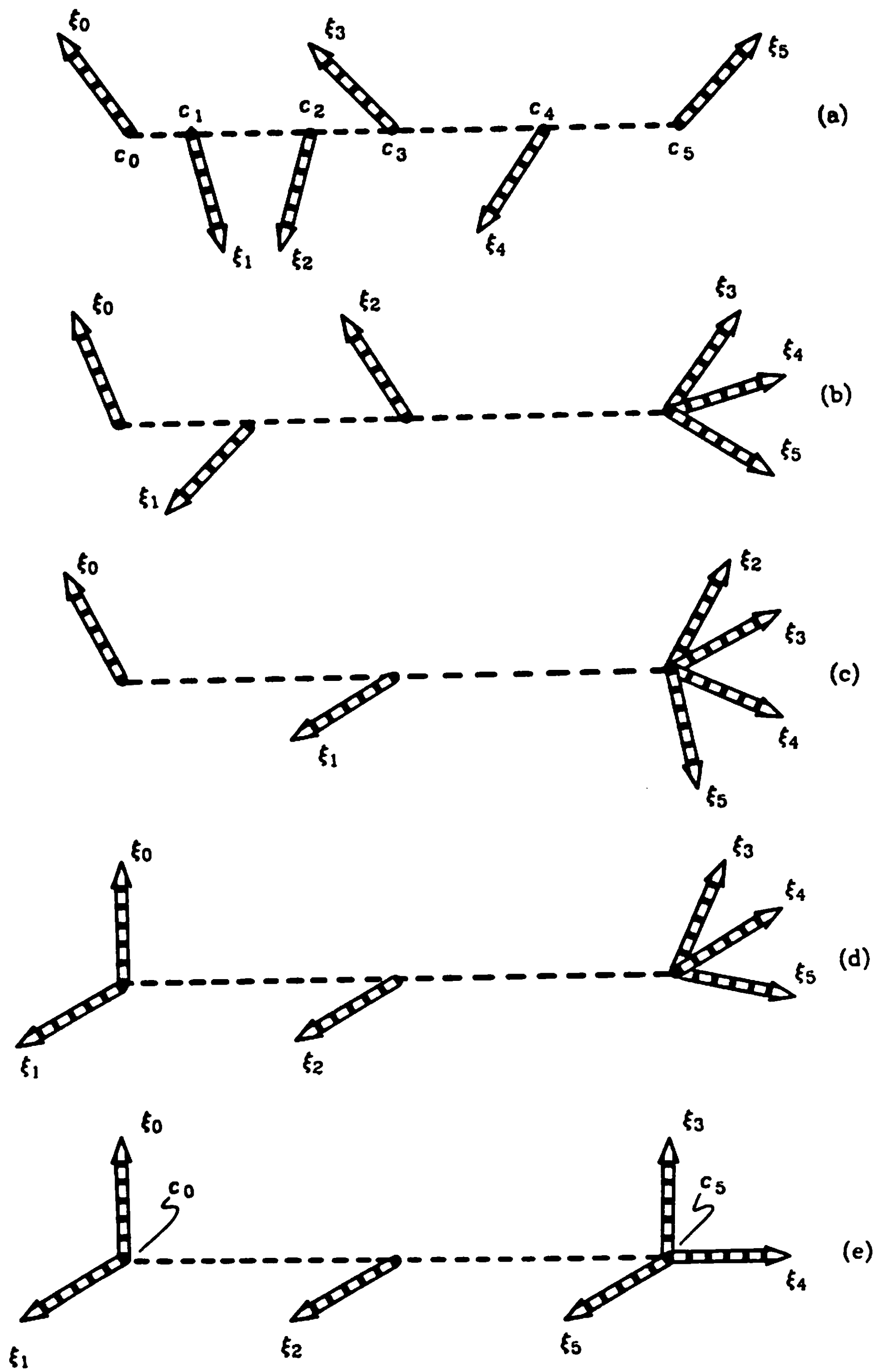

Figure 5.3 The Steps in the Proof of Theorem 5.2.1.

Since the union is finite and each set in the union is compact it follows (Baire Category Theorem [Roy. 1]) that one of the sets in the union has nonempty interior in **O**. Now **O** is a 5-manifold so the order of $\{\boldsymbol{\theta}_0,\ldots,\boldsymbol{\theta}_{\min-1},\boldsymbol{\theta}_{\max+1},\ldots,\boldsymbol{\theta}_5\}$ must be 5 by (A4), but the order is at most 4 when max-min = 1. Thus we must have that $\{e^{\theta_{min}\xi_{min}} e^{\theta_{max}\xi_{max}} \mid \bar{\theta} \in f_{\xi}^{-1}(\theta)\}$ is infinite when max − min = 1. This and (5.2.16) => $e^{\theta_{min}\xi_{min}} e^{\theta_{max}\xi_{max}} c_5 = c_5$ has ∞ solutions. This, in turn, implies (see section 2.4 problem 2) that $\xi_{max} \cap c_5 \neq \emptyset$ or $\xi_{min} \cap c_5 \neq \emptyset$. The definition of max => $\xi_{max} \cap c_5 = \emptyset$. By duality $\xi_{min} \bigcap c_0 = \emptyset$ also. This is a contradiction so max − min = 1 is impossible and max = min. Thus, there is exactly one twist, ξ_{min}, which intersects neither c_0 nor c_5. Next, we show that ξ_{min} is a perpendicular bisector of $\overline{c_0 c_5}$.

M has maximal work-volume => The map $\tilde{f}: \bar{\theta} \to \| e^{\theta_0 \xi_0} \cdots e^{\theta_5 \xi_5} c_5 - c_0 \|$ is onto $[0, l_M]$. Since ξ_{min} is the only axis not intersecting c_5 or c_0 $\tilde{f}(\bar{\theta}) = \| e^{\theta_{min}\xi_{min}} c_5 - c_0 \|$ and by A3 we have that

$$\xi_{min} \text{ is a perpendicular bisector of } \overline{c_0 c_5} \tag{5.2.21}$$

So we have that (v) of definition 5.1.8 is satisfied if min = 2. (We will see this shortly.) (Figure 5.3c is a representative consistent with our knowledge of M at this point in the proof. The drawings are isometric.)

Next, we claim that MWV => $d: \bar{\theta} \to e^{\theta_0 \xi_0} \cdots e^{\theta_{min-1} \xi_{min-1}} c_{min} - c_0$ is onto $\| c_{min} - c_0 \| S^2$. Suppose not, then since $d(\mathrm{T}^6) \subset \| c_{min} - c_0 \| S^2$ $\exists$ a unit vector u such that $u^T d(\bar{\theta}) < \| c_{min} - c_0 \| \; \forall \bar{\theta} \in \mathrm{T}^6$

=> $u^T (f_{\xi}(\bar{\theta}) c_5 - c_0) < \| c_5 - c_0 \| \; \forall \bar{\theta} \in \mathrm{T}^6$ which is a contradiction to MWV since $f_{\xi}(\mathrm{T}^6)$ closed (compact). Thus the claim is true.

Now d is a smooth function onto a 2-manifold so d is a function of 2 or more of the θ_i by (A4). Thus, min $\geq$ 2. By duality, min = max $\leq$ (5−2)=3. Thus, there are 2 axes intersecting one end of $\overline{c_0 c_5}$ and 3 the other. In other words M or M^* has min = 2. Since we are only trying to prove M or M^* is an elbow manipulator there is no loss in generality by assuming M has the property min = 2. It follows from A1 of the Appendix and the fact that d is onto $\| c_{min} - c_0 \| S^2$ that

$$\xi_0 \perp \xi_1 \text{ and } \xi_1 \perp \overline{c_0 c_5} \tag{5.2.22}$$

At this point, (i),(ii),(iii), and (v) of Definition 5.1.8 are satisfied. Next we show that $\xi_1 \| \xi_2$.

Recall that z_0 is the unit vector parallel to ξ_0. Then since M has MWV, $\exists \bar{\theta} \in T^6$ such that $f_{\bar{\xi}}(\bar{\theta})c_5 = c_0 + (\frac{l_M}{2})z_0$ (this point lies in $\bar{B}(c_0, l_M)$). Let $\bar{\xi}'$ be a representative corresponding to this configuration, and $c' \in C_{\bar{\xi}'}$ be the linking curve derived from c (see the proof of Proposition 5.1.2). Figure 5.4 shows the configuration of the manipulator represented by ξ'. Note that c'_0, c'_2, c'_5 are not collinear and consider the plane containing them. Since

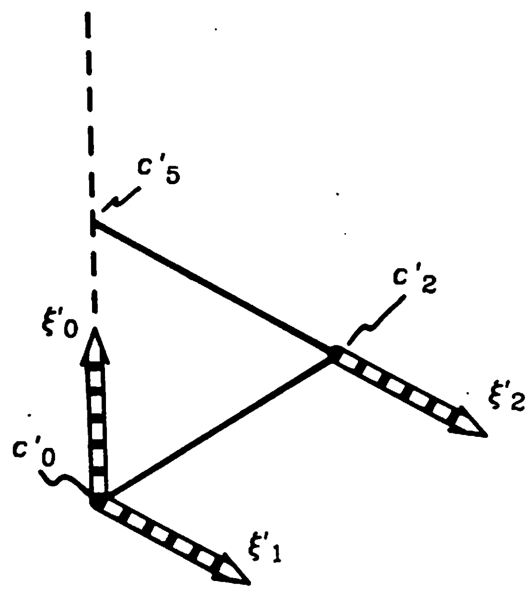

Figure 5.4 Showing that $\xi_1 \| \xi_2$

ξ'_0, and $\overline{c'_0 c'_2}$ are in the plane we have[3] by (5.2.22) that ξ'_1, is perpendicular to the plane. Now $\overline{c'_0 c'_2}$ and $\overline{c'_2 c'_5}$ are in the plane => ξ'_2 is perpendicular to the plane by (5.2.21) and so $\xi'_2 \| \xi'_1$ => $\xi_2 \| \xi_1$ and (iv) is satisfied. (Figure 5.3d represents approximately the infor-

[3]Angles between successive joints, and between joints and adjacent links are independent of choice of representative when the linking curves are related as in the proof of Proposition 5.1.2.

mation we have at this point in the proof.)

Since min = max = 2, the last three twist axes intersect at c_5 satisfying (vi) of the definition of the elbow manipulator. To show that these axes are consecutively orthogonal ((vii) and (viii) of Definition 5.1.8), we write

$$f_{\xi}(\overline{\theta}) = h(\theta_0,\theta_1,\theta_2)\, w(\theta_3,\theta_4,\theta_5) \tag{5.2.23}$$

where $h(\theta_0,\theta_1,\theta_2) \triangleq e^{\theta_0\xi_0}e^{\theta_1\xi_1}e^{\theta_2\xi_2}$ and $w(\theta_3,\theta_4,\theta_5) \triangleq e^{\theta_3\xi_3}e^{\theta_4\xi_4}e^{\theta_5\xi_5}$.

Since the wrist axes of M are intersecting the singularities of the manipulator decouple. That is,

$\overline{\theta}$ is a critical point of f_{ξ}

<=>

$(\theta_0,\theta_1,\theta_2)$ is a critical point of

$h' : (\theta_0,\theta_1,\theta_2) \rightarrow h(\theta_0,\theta_1,\theta_2)c_5$

or

$$(\theta_3,\theta_4,\theta_5) \text{ is a critical point of } w : T^3 \rightarrow G_{c_5}. \tag{5.2.24}$$

More concisely,

$$T^6 \backslash C_{\xi} \;=\; T^3 \backslash C_{h'} \times T^3 \backslash C_w \tag{5.2.24b}$$

Since we know the relationship among ξ_0,ξ_1 and ξ_2 it is straight forward to verify that h' has the property that each connected component, B_1, of $T^3\backslash C_{h'}$ satisfies $h'(B_1) = h'(T^3)\backslash h'(C_{h'})$ and $T^3\backslash C_{h'}$ has 4 connected components. Also h' is 1-1 when restricted to a connected component of $T^3\backslash C_{h'}$.

Suppose that ξ_3, ξ_4, ξ_5 are not consecutively orthogonal. Then w is not onto G_{c_5} by (A2). Then $w(T^3)$ is a compact, proper subset of G_{c_5} and $w(T^3)^C$ is a nonempty open set in G_{c_5}. In addition, let $\overline{\theta}$ be a regular point of f_{ξ} and let $(\theta_{0i},\theta_{1i},\theta_{2i})$, $i \in \{1,2,3,4\}$ be the four solutions to the equation $h'(\cdot,\cdot,\cdot) = h'(\theta_0,\theta_1,\theta_2)$. Since the critical values of w

have measure zero in G_{c_5} (the measure on G_{c_5} is that induced by a translation invariant volume form)

$$D_w \triangleq \bigcup_{i=1}^{4} [h(\theta_0,\theta_1,\theta_2)]^{-1} h(\theta_{0i},\theta_{1i},\theta_{2i}) w(C_w) \tag{5.2.25}$$

has measure zero in G_{c_5}. Therefore $w(T^3)^C \setminus D_w$ is nonempty and

$$\exists R^1 \in w(T^3)^C \setminus D_w . \tag{5.2.26}$$

It follows that $h(\theta_0,\theta_1,\theta_2)R^1$ is a regular value of $f_{\bar{\xi}}$ and by the choice of R^1, $f_{\bar{\xi}}(\bar{\theta}) \neq h(\theta_0,\theta_1,\theta_2)R^1$. The manipulator has maximal work-volume so there exists $\bar{\theta}' \in T^6$ such that

$$f_{\bar{\xi}}(\bar{\theta}') = h(\theta_0,\theta_1,\theta_2)R^1 . \tag{5.2.27}$$

as $h(\theta_0,\theta_1,\theta_2)R^1 \in \{ g \in G \mid \|gc_5 - c_0\| \leq l_M \}$. Also, by (5.2.26), $(\theta_0',\theta_1',\theta_2') \neq (\theta_0,\theta_1,\theta_2)$. Now h' is 1-1 on connected components of $T^3 \setminus C_h \Rightarrow \bar{\theta}'$ and $\bar{\theta}$ are not in the same connected component of $T^6 \setminus C_{\bar{\xi}} = T^3 \setminus C_h \times T^3 \setminus C_w$ thereby contradicting the well-connected-workspace assumption. Thus

$$\xi_3 \perp \xi_4 \text{ . and } \xi_4 \perp \xi_5. \tag{5.2.28}$$

In other words (vii) and (viii) are satisfied and we have that M or M^* is an elbow manipulator.

■

Theorem 5.2.1 reinforces the generally accepted idea that elbow manipulators are good kinematic designs and are optimal with respect to work-volume. It is somewhat surprising that the elbow manipulators and their duals are the *only* designs that meet the criteria of maximal workspace/well-connected workspace. This theorem should encourage special consideration of elbow manipulators and their duals in the study of path planning and collision detection problems.

Admittedly, the class of 6R manipulators is quite limited. The extension of the theorem to 6 dof manipulators with both R and P (prismatic) joints is feasible with an extension of the definition of l_M. The analog of the elbow manipulator when P joints are allowed is the Stanford manipulator with zero shoulder offset. Also note that the definitions of work-volume and length of a manipulator extend directly to redundant nR manipulators. One can conclude from arguments in the proof of theorem 1 that nR redundant manipulators having maximal work-volume have their first two and last two joints intersecting. As redundancy supposedly increases the "connectivity" of the workspace, it will be interesting to see if there is a "very"-well-connectedness measure for redundant manipulators.

5.A Appendix to Chapter Five

Collected here for convenience are several subproblems which appear in the proof of Theorem 5.2.1.

A1) Let ξ_0, ξ_1, be zero-pitch unit twists, $O, P \in \mathbb{R}^3$, $O \neq P$, and $\xi_0 \cap \xi_1 = O$. If $(\theta_0, \theta_1) \xrightarrow{f} e^{\theta_0 \xi_0} e^{\theta_1 \xi_1} P$ is onto $O + \|P-O\| S^2$ (The sphere of radius $\|P-O\|$ centered at O) then

$$\xi_0 \perp \xi_1 \text{ and } \xi_1 \perp \overline{OP} \ . \qquad (5.A.1)$$

Proof: Let z_0 and z_1 be unit vectors parallel to ξ_0 and ξ_1 respectively. Now

$$f(\mathrm{T}^2) = O + \|P-O\| S^2 \qquad (5.A.2)$$

$$\Rightarrow z_0^T (f(\mathrm{T}^2) - O) = [-\|P-O\|, \|P-O\|] \qquad (5.A.3)$$

$\Rightarrow \exists\, \theta_1, \theta_1'$ such that

$$e^{\theta_1 \xi_1}(P-O) = \|P-O\| z_0 \qquad (5.A.4)$$
$$e^{\theta_1' \xi_1}(P-O) = -\|P-O\| z_0$$

$$=> e^{\theta_1\xi_1}(P-O) = -e^{\theta_1'\xi_1}(P-O) \tag{5.A.5}$$

Premultiplying both sides by z_1^T

$$=> z_1^T(P-O) = -z_1^T(P-O) => z_1^T(P-O) = 0 \tag{5.A.6}$$

Thus

$$\xi_1 \perp \overline{OP} \tag{5.A.7}$$

Next from (5.A.4)

$$z_0 = e^{\theta_1\xi_1}\frac{(P-O)}{\|P-O\|} \tag{5.A.8}$$

$$=> z_1^T z_0 = z_1^T e^{\theta_1\xi_1}\frac{(P-O)}{\|P-O\|} = z_1^T\frac{(P-O)}{\|P-O\|} = 0 \tag{5.A.9}$$

$$=> \xi_1 \perp \xi_0 \tag{5.A.10}$$

■

A2) Let $O \in R^3$ and ξ_0, ξ_1, ξ_2 be zero-pitch unit twists with $\xi_0 \cap \xi_1 \cap \xi_2 = O$. If

$$(\theta_0,\theta_1,\theta_2) \xrightarrow{f} \theta^{\theta_0\xi_0}e^{\theta_1\xi_1}e^{\theta_2\xi_2} \text{ is onto } G_O \tag{5.A.11}$$

then $\xi_0 \perp \xi_1, \xi_1 \perp \xi_2$

Proof: f onto G_O =>

$$\forall R \in G_0, \exists(\theta_0,\theta_1\theta_2) \text{ such that } e^{\theta_0\xi_0}e^{\theta_1\xi_1}e^{\theta_2\xi_2} = R \ . \tag{5.A.12}$$

Let $P \neq O$ be a point on ξ_2 and let both sides of (5.A.12) act on P. Then (5.A.12) implies

$$\forall R \in G_O, \exists(\theta_1,\theta_2) \text{ such that } e^{\theta_1\xi_0}e^{\theta_1\xi_1}P = RP$$

but $G_O P = O + \|P-O\|S^2$ so the result follows from A1.

■

A3) Let ξ be a zero-pitch unit twist and $O, P \in R^3$ with $O \neq P$ and $\xi \cap \overline{OP} \neq \emptyset$. If $f : \theta \rightarrow \| e^{\theta \xi} P - O \|$ is onto $[0, \| P - O \|]$ then ξ is a perpendicular bisecter of $\overline{OP}$.

Proof: Let $Q \in \xi \cap \overline{OP}$, then

$$\| e^{\theta \xi} P - O \| = \| e^{\theta \xi}(P - Q) - (O - Q) \| \geqslant \Big| \| P - Q \| - \| O - Q \| \Big| \tag{5.A.13}$$

but $\| e^{\phi \xi} P - O \| = 0$ for some $\phi \Rightarrow \| P - Q \| = \| O - Q \|$

$$\Rightarrow \xi \cap \overline{OP} \text{ contains the midpoint of } \overline{OP} . \tag{5.A.14}$$

Now, let z be a unit vector parallel to ξ then

$$z^T e^{\phi \xi}(P - Q) = z^T (O - Q) \tag{5.A.15}$$

$$\Rightarrow z^T (P - Q) = z^T (O - Q) \tag{5.A.16}$$

$$\Rightarrow z^T (P - O) = 0 \tag{5.A.17}$$

This together with (5.A.14) yields the result.

A4) Let M and N be m and n-manifolds respectively and let $f : M \rightarrow N$ be smooth. If $f(M)$ has non-empty interior then $m \geqslant n$.

Proof: Since the set of regular values of f is dense in N [Mil. 1., pg. 11] there exists a regular value $y \in$ interior of $f(M)$. Thus, there exists $x \in M$ such that $Tf : TM \rightarrow TN$ is surjective $\Rightarrow m \geqslant n$.

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Chapter 6

Variable Structure Control of Robot Manipulators

In this chapter we propose a variable structure controller for robot manipulators. This is an extension and application of the theory of *variable structure systems* (VSS) as described in [1, 2], and [3]. Previous applications of VSS ideas to manipulator control are described by Young [4], Slotine and Sastry [5], and Morgan and Ozguner [6]. These schemes decouple the manipulator dynamics by introducing one hyperplane of control discontinuity for each joint of the manipulator via feedback control. In [4] a hierarchical method (see [2]) is used to move the manipulator state to the hyperplanes of control discontinuity sequentially, whereas in [5] the manipulator state moves to all the hyperplanes simultaneously. Our contribution is twofold: (1)A new analysis technique is developed for variable structure systems by integrating Filippov's solution concept for differential equations with discontinuous RHS and Clarke's generalized gradient. (2) We use this technique to analyze sliding on the intersection of control discontinuities.

The qualitative properties of a VSS are shown in Figure 6.1. Figure 6.1(a) depicts a phase diagram for a hypothetical VSS with control discontinuities at S_1 and S_2. Trajectories for the flow in Figure 6.1(a) move to, and then slide along the switching surface S_1. This motion of the state along the control discontinuity motivates the nomenclatures sliding mode and sliding surface. Although there is a control discontinuity across S_2, no sliding mode exists there. Figure 6.1(b) represents a disturbance flow which is added to the original flow of Figure 6.1(a). The robust nature of the sliding surface is demonstrated in the resulting flow shown in Figure 6.1(c). The flows have changed somewhat but a sliding mode still exists along S_1. The primary reason that sliding modes are introduced into dynamical systems is this robustness to disturbances.

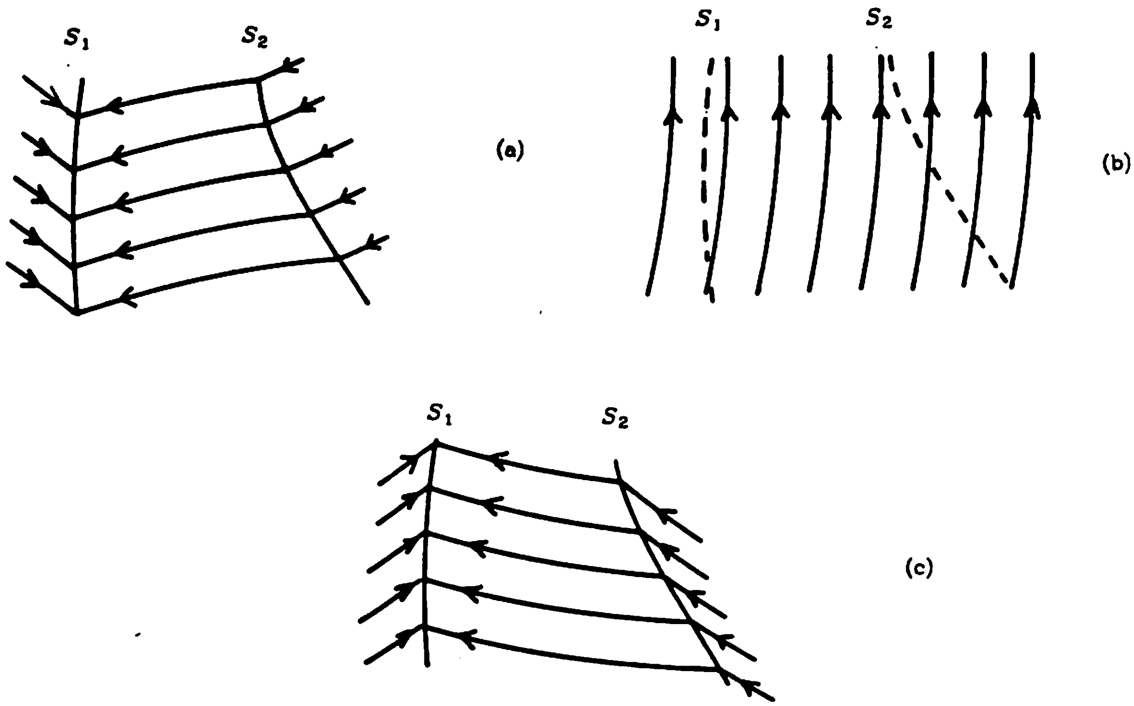

Figure 6.1 Phase Portrait of an Hypothetical VSS.

The VSS control scheme proposed here for robot manipulators is a multivariable design which produces a sliding mode on the *intersection* of several switching surfaces but does not necessarily generate sliding modes on the switching surfaces independently. This type of sliding mode is mentioned in [1] and is analyzed for the first time here. Figure 6.2 is a phase diagram of this type of VSS having a sliding mode at the origin. The techniques used in this chapter to analyze this type of sliding mode are new. Essential to the analysis is the use of Clarke's generalized gradient [7] and Filippov's solution concept for differential equations with discontinuous right-hand side [8]. A simple relationship between these two ideas is proved in Theorem 6.1.1 part (6). As is common in VSS, saturating switching controls are used in our scheme. In addition to providing robust tracking, there is a natural force limiting provided by these saturating controls which allow the manipulator to "give" when a slight misalignment in an assembly operation requires the manipulator to deviate from its nominal trajectory. We show that there is bounded set of forces that the manipulator can apply at its gripper without deviating from the nominal trajectory. The size and shape of this set can be be varied by adjusting the gains in the VSS controller. Thus, the apparent stiffness of the gripper can be varied making the manipulator suitable for compliant assembly.

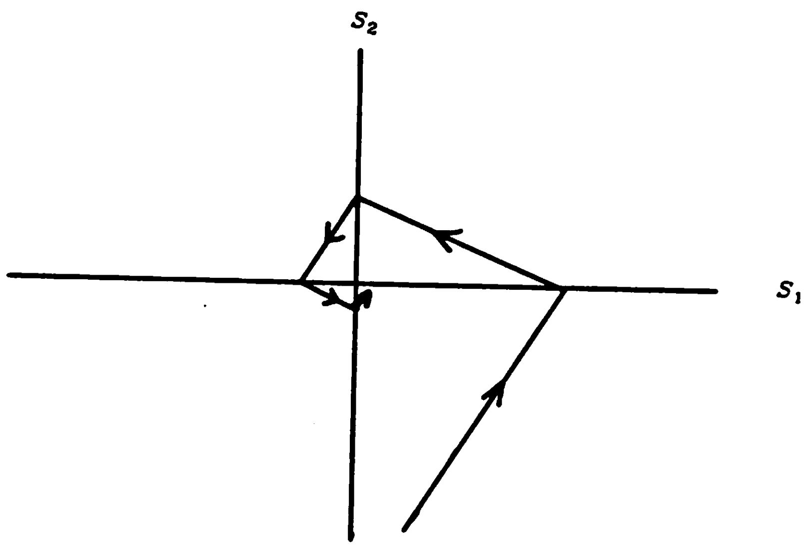

Figure 6.2 Phase Portrait of a Multivariable VSS.

Existing compliance control formulations which are important for comparison are due to Salisbury[9] and Raibert and Craig[10]. Salisbury varies the servo stiffness of a linear controller to control the stiffness of the manipulators gripper. Our approach is similar in that we use the natural stiffness properties of the control scheme to control compliance at the gripper. The resulting compliance forces of the two schemes is however quite different. In [10] Raibert and Craig switch various degrees of freedom of the gripper from position to force control to allow compliant motion. The VSS control scheme presented here switches *implicitly* to force control when the manipulator is perturbed from its nominal trajectory. This is a result of our choice of discontinuous control.

It is important to point out that the direct application of discontinuous control in mechanical systems is almost always impractical since the effects of switching forces on actuators and gear trains can be destructive. Thus, in real systems the control discontinuity is smoothed [5] so that the system trajectory moves to a neighborhood of the approximate discontinuity. The study of the idealized discontinuous control scheme, however, gives a clear picture of the salient properties of the system dynamics. Nonidealities other than smoothed discontinuities such as small delays and hysteresis produce chattering along sliding surfaces rather the ideal sliding described above. Descriptions of the ideal behavior as a limit of these nonideal motions are contained in [8, 2] and [1] and provide additional motivation for studying VSS.

The format of this chapter is as follows. Section 1 contains the non-standard mathematical framework used in the analysis of the control scheme. Section 2 presents the manipulator dynamics and formulates the tracking problem. The control scheme is developed in section 3 and a design example is worked through in section 4. The effects of a linear coordinate transformation of the joint coordinates is discussed in section 5. Compliance properties are analyzed in section 6 and section 7 contains a brief discussion.

6.1 Differential Equations with Discontinuous RHS and Nonsmooth Potential Functions.

Since we will be considering control laws which are discontinuous and potential functions which are not differentiable everywhere, the associated (non-standard) mathematical framework is developed in this section. We begin by defining a solution to differential equations with discontinuous right-hand side. A solution concept for such differential equations has been developed by Filippov and is used here. Other solution concepts are discussed and compared with Filippov's in [11].

Consider the vector differential equation

$$\dot{x} = f(x, t) \qquad (6.1.1)$$

where $f : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^n$ satisfies the following condition [8].

Condition B : f is defined almost everywhere and measurable in an open region $Q \subset \mathbf{R}^{n+1}$. Further, $\forall$ compact $D \subset Q$ $\exists$ integrable $A(t)$ such that $\|f(x,t)\| \leqslant A(t)$ *a.e.* in D.

■

Definition 6.1.1 [Filippov] A vector function $x(\cdot)$ is called a solution of (6.1.1) on $[t_0,t_1]$ if $x(\cdot)$ is absolutely continuous on $[t_0,t_1]$ and for almost all $t \in [t_0,t_1]$

$$\dot{x} \in K[f](x) \tag{6.1.2}$$

where

$$K[f](x) \triangleq \bigcap_{\delta>0} \bigcap_{\mu N=0} \overline{co} f(B(x,\delta)-N,t)$$

and $\bigcap_{\mu N=0}$ denotes the intersection over all sets N of Lebesgue measure zero.

■

The time dependence of $K[f](x)$ is dropped in definition 6.1.1 for economy - all results in this chapter that pertain to $K[\cdot]$ hold with time dependence since t can be viewed as a parameter in the definition. Note that the definition of $K[f]$ makes sense for $f:\mathbf{R}^m \rightarrow \mathbf{R}^n$; this is a minor generalization, but it is useful in theorem 6.1.1. We will *assume* throughout that all functions are defined *a.e.* and Lebesgue measurable.

The definition of K in (6.1.2) is quite cumbersome to use in applications so that the set of properties summarized in Theorem 6.1.1 is useful. Before proceeding with the theorem we need to introduce Clarke's generalized gradient.

Definition 6.1.2: Let $V:\mathbf{R}^n \rightarrow \mathbf{R}$ be locally Lipschitz continuous and define ∂V, the *generalized gradient* of V, by

$$\partial V(x) \triangleq co\{\lim \nabla V(x_i) \mid x_i \rightarrow x, x_i \notin \Omega_V \bigcup N\}$$

where Ω_V is the set of Lebesgue measure zero where ∇V does not exist and N is an arbitrary set of zero measure.

■

Theorem 6.1.1. (Properties of $K[f]$) The map $K:\{f \mid f:\mathbf{R}^m \rightarrow \mathbf{R}^n\} \rightarrow \{g \mid g:\mathbf{R}^m \rightarrow 2^{\mathbf{R}^n}\}$ has the following properties.

(1) Assume that $f:\mathbf{R}^m \to \mathbf{R}^n$ is locally bounded. Then $\exists N_f \subset \mathbf{R}^m, \mu N_f = 0$ such that $\forall N \subset \mathbf{R}^m, \mu N = 0$

$$K[f](x) = co\{\lim f(x_i) \mid x_i \to x, x_i \notin N_f \bigcup N\} \tag{6.1.3}$$

(2) Assume that $f, g:\mathbf{R}^m \to \mathbf{R}^n$ are locally bounded then

$$K[f+g](x) \subset K[f](x) + K[g](x) \tag{6.1.4}$$

(3) Assume that $f_j:\mathbf{R}^m \to \mathbf{R}^{n_j}$, $j \in \{1,2,....,N\}$ are locally bounded, then

$$K[\mathop{\times}_{j=1}^{N} f_j](x) \subset \mathop{\times}_{j=1}^{N} K[f_j](x)^{\ddagger} \tag{6.1.5}$$

(4) Let $f:\mathbf{R}^m \to \mathbf{R}^n$ be C^1, $rank\ Df(x) = n$,and $f:\mathbf{R}^n \to \mathbf{R}^p$ be locally bounded, then

$$K[f \circ g](x) = K[f](g(x)) \tag{6.1.6}$$

(5) (equivalent control [2]) Let $g:\mathbf{R}^m \to \mathbf{R}^{p \times n}$ (i.e. matrix valued) be C^0 and $f:\mathbf{R}^m \to \mathbf{R}^n$ be locally bounded, then

$$K[gf](x) = g(x)K[f](x) \tag{6.1.7}$$

where $gf(x) \triangleq g(x)f(x) \in \mathbf{R}^p$.

(6) Let $V:\mathbf{R}^n \to \mathbf{R}$ be locally Lipschitz continuous, then

$$K[\nabla V](x) = \partial V(x) \tag{6.1.8}$$

(7) Let $f:\mathbf{R}^m \to \mathbf{R}^n$ be continuous, then

$$K[f](x) = \{f(x)\} \tag{6.1.9}$$

Proof: See Appendix 6.A.

■

The manipulator dynamics together with our proposed control law is best described as a nonsmooth gradient system, i.e. a gradient system whose potential function is not differentiable everywhere. The following definition and theorem provide the formalism necessary to calculate certain time derivatives associated with nonsmooth gradient

‡ Cartesian product notation and column vector notation are used interchangeably.

systems.

Definition 6.1.3: $V:\mathbf{R}^m \to \mathbf{R}$ is called a *max function* if $V(x) = \max_{j \in Y} f_j(x)$ where $f_j:\mathbf{R}^m \to \mathbf{R}$ are C^1 and Y is a finite index set.

■

Theorem 6.1.2.

Let $V:\mathbf{R}^m \to \mathbf{R}$ be a max function. If $x:\mathbf{R} \to \mathbf{R}^m$ and $V(x(t))$ are differentiable at t, then

$$\frac{d}{dt}[V(x(t))] = \xi^T \dot{x} \qquad \forall \xi \in \partial V(x)$$

Proof: See Appendix 6.A.

■

6.2 Manipulator Dynamics and Problem Formulation.

The dynamics of an n-joint rigid-link manipulator may be described by the equation

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta}) + G(\theta) + D(\theta,\dot{\theta},t) = F \tag{6.2.1}$$

where

1 1. θ is the n × 1 vector of joint coordinates $M(\theta)$ is the n × n inertia matrix $C(\theta,\dot{\theta})$ is the n × 1 vector of Coriolis and centrifugal forces[1] $G(\theta)$ is the n × 1 vector of gravitational forces $D(\theta,\dot{\theta},t)$ is the n × 1 vector of disturbances F is the n × 1 vector of generalized forces applied by the actuators at the joints of the manipulator.

and (6.2.1) has the following properties.

(P1) $M(\theta)$ is symmetric and positive definite. $M(\cdot)$, $C(\cdot,\cdot)$, and $G(\cdot)$ are C^1 functions of the manipulator state $[\theta,\dot{\theta}]^T$. $D(\cdot,\cdot,\cdot)$ is locally bounded.

■

[1] "Forces" and "generalized forces" will be used interchangeably throughout.

The positive definiteness of $M(\theta)$ is an important property of the manipulator dynamics as it is essential to the stability analysis of the proposed sliding mode control scheme. This property is exploited in [4] and [5] to guarantee the invertibility of $M(\theta)$. Another important feature of the dynamics for earth-bound manipulators is the gravitational force $G(\theta)$ which is usually large. To accommodate this fact the joint forces will include a compensation[2] term for the gravitational forces. For the sake of generality we allow for the compensation of other forces as well. We therefore write the joint forces applied by the actuators in the form

$$F = \hat{C}(\theta,\dot{\theta}) + \hat{G}(\theta) + \hat{D}(\theta,\dot{\theta},t) + u \tag{6.2.2}$$

where the hatted terms are estimates of the corresponding unhatted objects and satisfy the following assumption.

(A1) $\hat{C}$, $\hat{G}$, and $\hat{D}$ are locally bounded. (Note that no continuity assumption is made so that discontinuous models of friction may be used in $\hat{D}$.)

■

The vector u is the additional joint force beyond the compensation forces and will be referred to as the *control*. The expression of the dynamics described by (6.2.1) and (6.2.2) is simplified by defining the "disturbance" vector

$$\widetilde{D}(\theta,\dot{\theta},t) \triangleq \hat{G}(\theta) - G(\theta) + \hat{C}(\theta,\dot{\theta}) - C(\theta,\dot{\theta}) + \hat{D}(\theta,\dot{\theta},t) - D(\theta,\dot{\theta},t). \tag{6.2.3}$$

Using (6.2.1),(6.2.2), and (6.2.3) the manipulator dynamics become

$$M(\theta)\ddot{\theta} = u + \widetilde{D}(\theta,\dot{\theta},t). \tag{6.2.4}$$

Dropping the functional dependencies, the state equation form of (6.2.4) is

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ M^{-1}(u + \widetilde{D}) \end{bmatrix}. \tag{6.2.5}$$

Let $[\theta_d,\dot{\theta}_d]^T$ be the desired state trajectory that we would like the manipulator to follow. Further, let it satisfy

[2] Any or all of the compensation terms may be set to zero.

(A2) $[\theta_d, \dot{\theta}_d]^T$ is C^1 on $[t_0, \infty)$.

Now define the tracking error by

$$\begin{bmatrix} e \\ \dot{e} \end{bmatrix} \triangleq \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} - \begin{bmatrix} \theta_d \\ \dot{\theta}_d \end{bmatrix}. \tag{6.2.6}$$

In terms of (6.2.5) and (6.2.6) the *tracking problem* is the following:

Find a feedback control u such that for any given initial state $[\theta, \dot{\theta}](t_0) = [\theta_0, \dot{\theta}_0]^T$, $[e(t), \dot{e}(t)]^T \to 0$ as $t \to \infty$.

Once u is chosen to achieve accurate tracking, the usefulness of the feedback control scheme for compliant motion is considered. The restoring forces exerted by the manipulator when it is perturbed from a nominal trajectory determine the suitability of the control scheme for tasks that require compliance. These forces are calculated in section 6.6.

6.3 The Control Scheme.

Choose $B \in R^{n \times n}$ such that $\sigma(B) \subset \mathbf{C}_+^0$.

Define the "switching" vector

$$s \triangleq [B \;\; I]\begin{bmatrix} e \\ \dot{e} \end{bmatrix} \tag{6.3.1}$$

and the control u by

$$u = -k(\theta, \dot{\theta}, \theta_d, \dot{\theta}_d, \ddot{\theta}_d)\nabla V(s)\ ^3 \tag{6.3.2}$$

$$V(s) \triangleq \|s\|_1 = \sum_{i=1}^{n} |s_i|. \tag{6.3.3}$$

where the gain k satisfies

(A3) $k : \mathbf{R}^{5n} \to \mathbf{R}$ is C^0.

Clearly, if $s = 0$ then

$$\dot{e} = -Be \tag{6.3.4}$$

and it follows that $[e \; \dot{e}]^T \to 0$ exponentially for arbitrary initial conditions. Our goal,

[3] ∇V is not defined on a set of Lebesgue measure zero. The analysis to follow takes this into account.

then, is to choose k such that s becomes zero in finite time. To find such a k a Lyapunov based design approach is used with the obvious choice of Lyapunov function, $V(s)$.

We begin by computing $\dot{V}$ and then choosing k such that $\dot{V}$ is bounded below zero (i.e. $\dot{V}(t) \leqslant -\epsilon \ \forall t \geqslant t_0$) whenever $s \neq 0$.

This will guarantee that $s \rightarrow 0$ in finite time. Lyapunov theory as developed say in [12] holds for differential equations with continuous right hand side. However, the nondifferentiability of $V(s)$ and the discontinuous nature of the control pose some technical problems. Using the results of section 2 we can compute an upper bound for $\dot{V}$.

Theorem 6.3.1 Let the manipulator dynamics and control be described by (6.2.5) and (6.3.1-6.3.3). Assume that P1, A2 and A3 are satisfied. If $[\theta, \dot{\theta}]$ is a solution to (6.2.5) on $[t_0, \infty)$ in the sense of Fillipov then

(i) $V(s(t))$ is the Lebesgue integral of its derivative

(ii) $\exists \delta \in K[\widetilde{D}]$ such that

$$\dot{V} \leqslant -k\,\xi^T M^{-1} \xi + \xi^T (M^{-1}\delta + B\dot{e} - \ddot{\theta}_d) \qquad a.e.$$

where $\xi = argmin\,\{\|\eta\|_{M^{-1}} \mid \eta \in \partial V(s)\}$.

Proof: From (6.2.5), (6.3.2) and the fact that $[\theta, \dot{\theta}]$ is a solution to (6.2.5) on $[t_0, \infty)$ we have the following *a.e.* in $[t_0, \infty)$.

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} \in K \begin{bmatrix} \dot{\theta} \\ -kM^{-1}\nabla V(s) + M^{-1}\widetilde{D} \end{bmatrix} \tag{6.3.5}$$

Now by Theorem 6.1.1, property (3),

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} \in \begin{bmatrix} K[\dot{\theta}] \\ K[-kM^{-1}\nabla V(s) + M^{-1}\widetilde{D}] \end{bmatrix} \tag{6.3.6}$$

Next, by Properties (2),(5), and (7),

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} \in \begin{bmatrix} \dot{\theta} \\ -kM^{-1}K[\nabla V(s)] + M^{-1}K[\widetilde{D}] \end{bmatrix} \tag{6.3.7}$$

(6.2.6),(6.3.1),(6.3.7), and property (6) yield

$$\dot{s} \in -kM^{-1}\partial V(s) + B\dot{e} + M^{-1}K[\widetilde{D}] - \ddot{\theta}_d \tag{6.3.8}$$

The absolute continuity of the solution $[\theta, \dot{\theta}]^T$ on compact intervals , and the continuous differentiability of $[\theta_d, \dot{\theta}_d]^T$ imply s is absolutely continuous on compact intervals. This, in turn, implies the absolute continuity of V on compact intervals. Thus, $\dot{V}$ exists almost everywhere, V is the Lebesgue integral of its derivative and (i) holds.

From (6.3.3) we have

$$V(s) = \sum_{i=1}^{n} max(-s_i, s_i). \tag{6.3.9}$$

Since the finite sum of max functions is a max function we have by Theorem 6.1.2, and the absolute continuity of V and s that

$$\dot{V} = \xi^T \dot{s} \qquad a.e. \tag{6.3.10}$$
$$\forall \ \xi \in \partial V(s)$$

From (6.3.8) and (6.3.10)

$$\dot{V} = -k\, \xi^T M^{-1}\beta + \xi^T[M^{-1}\delta + B\dot{e} - \ddot{\theta}_d] \qquad a.e. \tag{6.3.11}$$
$$\forall \ \xi \in \partial V(s) \text{, some } \beta \in \partial V(s) \text{, and some } \delta \in K[\widetilde{D}].$$

Choose

$$\xi = argmin\{\|\eta\|_{M^{-1}} \mid \eta \in \partial V(s)\} \tag{6.3.12}$$

Then, from the convexity of the set $\partial V(s)$,

$$\dot{V} \leqslant -k\, \xi^T M^{-1}\xi + \xi^T[M^{-1}\delta + B\dot{e} - \ddot{\theta}_d] \qquad a.e. \tag{6.3.13}$$

Whence we have (ii).

■

Part (i) of Theorem 6.3.1 tells us that we can ignore the set of measure zero where $\dot{V}$ does not exist and obtain an upper bound on V by integrating the bound on $\dot{V}$ in part (ii). The following corollary uses this fact to determine k such that $s \to 0$ in finite time.

Corollary Let (6.2.5) satisfy the conditions of Theorem 6.3.1, and let $[\theta, \dot{\theta}]^T$ be a solution of (6.2.5) on $[t_0, \infty)$. If k satisfies

$$k \geqslant \sigma_{\max} M(\epsilon + \frac{\|\widetilde{D}\|}{\sigma_{\min} M} + \|B\dot{e}\| + \|\ddot{\theta}_d\|) \tag{6.3.14}$$

with $\|\widetilde{D}\| \triangleq \sup\{\|\delta\| \mid \delta \in K[\widetilde{D}]\}$ and $\epsilon > 0$, then $\exists T \in \mathbf{R}$ such that

$$s = 0 \;\forall\; t > T. \tag{6.3.15}$$

Proof: From Theorem 6.3.1 we have

$$\dot{V} \leqslant -k\|\xi\|^2 \sigma_{\min} M^{-1} + \|\xi\|(\sigma_{\max} M^{-1}\|\widetilde{D}\| + \|B\dot{e}\| + \|\ddot{\theta}_d\|) \qquad a.e. \tag{6.3.16}$$

The assumption (6.3.14) on k yields

$$\dot{V} \leqslant (\|\xi\| - \|\xi\|^2)(\sigma_{\max} M^{-1}\|\widetilde{D}\| + \|B\dot{e}\| + \|\ddot{\theta}\|) - \|\xi\|^2\epsilon \quad a.e. \text{ on } [t_0, \infty) \tag{6.3.17}$$

Since $\partial V(0) = [-1,1]^n$, the unit cube in $\mathbf{R}^n$, we have by the convexity of the function V, $\partial V(s) \bigcap (-1,1)^n = \varnothing \;\forall\; s \neq 0$ (see [7] proposition 2.2.9). Thus, by definition of ξ, $\|\xi\| \geqslant 1 \;\forall\; s \neq 0$ and from (6.3.17) we have $\dot{V} \leqslant -\epsilon \;\forall\; s \neq 0$ $a.e.$ on$[t_0, \infty)$. Thus, since $V \geqslant 0$ and $V = 0 \iff s = 0$ we have $s = 0 \;\forall\, t \geqslant T \triangleq t_0 + V(t_0)/\epsilon$.

■

In order to use the corollary we must show that a Filippov solution to (6.2.5) exists on $[t_0, \infty)$. In Appendix 6.B it is proved, under the assumptions of the corollary and (A2), that a solution exists. Thus, under these assumptions, $s \to 0$ in finite time and by the definition of s, $[e, \dot{e}]^T \to 0$ exponentially. These results justify the following design procedure for a manipulator controller. The procedure generates a control law that solves the *tracking problem*.

Design Procedure

Data:

Manipulator dynamics of the form (6.2.1) satisfying (P1) and a class of desired trajectories satisfying (A2).

Step 1:

Choose $\hat{C}, \hat{G}, \hat{D}$ satisfying (A1).

Step 2:

Choose $B \in \mathbf{R}^{n \times n}$ such that $\sigma(B) \subset \mathbf{C}_+^\circ$.

Step 3:

Choose k satisfying (A3) and (6.3.14). (e.g. if $\|\tilde{D}\|$ is C^0 then taking k equal to the RHS of (6.3.14) is a satisfactory choice).

Step 4:

Choose actuator forces according to (6.2.2) and (6.3.1-6.3.3).

Comment: In practice a large value of k may excite unmodeled dynamics (for instance, flexure modes in the manipulator). Thus, in order to minimize the required gain the estimates in step 1 of the procedure should be as close to the true values as possible. These can be determined from the nominal joint inertias and measured values of joint friction. Bounds on the errors of the estimates can be obtained from knowledge of the errors in the link inertias and measurement errors. Also, if the eigenvalues of B are large, the gain k may be large due to (6.3.14); this should be considered in step 2.

Note that the only information necessary to design a controller satisfying (6.3.14) is bounds on $\sigma_{\min}M$, $\sigma_{\max}M$, and $\|\tilde{D}\|$. Thus, variation of M and $\tilde{D}$ within these bounds will not affect the tracking performance of the controller. This robustness to parameter variations and disturbances is common in VSS controllers.

6.4 Design Example

Consider the two-degrees-of-freedom manipulator shown in Figure 6.3. Each link has unit mass concentrated at its endpoint, unit length, and the acceleration of gravity is taken to be one. Each joint actuator has unit inertia also. Given these parameters the dynamics are [13]

$$\begin{bmatrix} 4 + 2\cos(\theta_2) & 1 + \cos(\theta_2) \\ 1 + \cos(\theta_2) & 2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} +$$

$$\begin{bmatrix} 2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2) + \dot{\theta}_2^2\sin(\theta_2) \\ 2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2) + \dot{\theta}_1^2\sin(\theta_2) \end{bmatrix} + \begin{bmatrix} \sin(\theta_1) + \sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}. \qquad (6.4.1)$$

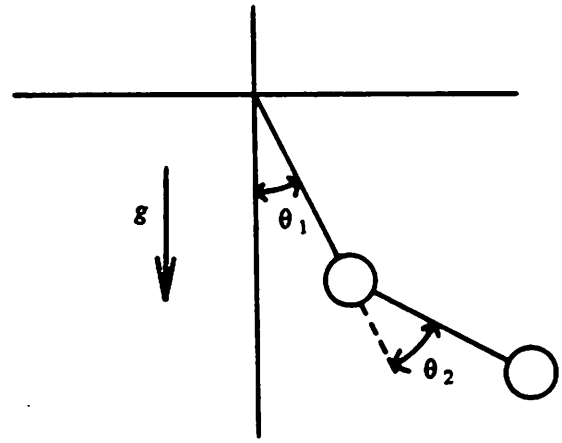

Figure 6.3 Two Degree-of-Freedom Manipulator.

Equation (6.4.1) satisfies (P1) and has the form of (6.2.1) where the disturbance term is equal to zero. The only contribution to $\tilde{D}$ will be from the error in estimating $C(\theta,\dot{\theta})$ and $G(\theta)$. A standard practice, which will be followed here, is to estimate $G(\theta)$ and to approximate the Coriolis and centrifugal terms by zero. With the simplifying assumption that the estimate of the gravitational forces is *exact* it follows that

$$\tilde{D} = C(\theta,\dot{\theta}) = \begin{bmatrix} 2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2) + \dot{\theta}_2^2\sin(\theta_2) \\ 2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2) + \dot{\theta}_1^2\sin(\theta_2) \end{bmatrix} \tag{6.4.2}$$

and (A1) is satisfied. To simplify the form of the gain k the following bound for $\tilde{D}$ will be used.

$$\|\tilde{D}\| \leqslant 2(\dot{\theta}_1 + \dot{\theta}_2)^2. \tag{6.4.3}$$

We begin by choosing the matrix B diagonal;

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6.4.4}$$

Next, for concreteness, set $\epsilon = 1$, and from (6.4.1), and a simple calculation, it follows that

$$\begin{aligned} \sigma_{\max} M &< 7 \\ \sigma_{\min} M &> 1. \end{aligned} \tag{6.4.5}$$

Now verify that

$$k = 7(1 + 2(\dot{\theta}_1 + \dot{\theta}_2)^2 + \|\dot{\theta} - \dot{\theta}_d\| + \|\ddot{\theta}_d\|) \qquad (6.4.6)$$

satisfies (A3) and (6.3.14). Putting together (6.2.2),(6.3.1-6.3.3),(6.4.4), and (6.4.6) yields

$$F = -\hat{g}(\theta) + u \qquad (6.4.7)$$

$$= \begin{bmatrix} \sin(\theta_1) + \sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) \end{bmatrix}$$

$$-(7 + 14(\dot{\theta}_1 + \dot{\theta}_2)^2 + 7\|\dot{\theta} - \dot{\theta}_d\| + 7\|\ddot{\theta}_d\|) \begin{bmatrix} sgn(\theta_{d1} - \theta_1 + \dot{\theta}_{d1} - \dot{\theta}_1) \\ sgn(\theta_{d2} - \theta_2 + \dot{\theta}_{d2} - \dot{\theta}_2) \end{bmatrix} \qquad (6.4.8)$$

where θ_d is any trajectory satisfying (A2). The choice of joint forces in (6.4.8) will move the switching vector s to zero in finite time. Thus, by our choice of B, the tracking error tends to zero exponentially.

There are many possible variations in deriving a gain that satisfies (6.3.14); in practice all bounds used should be made as tight as possible without violating constraints on computation time for the joint forces. The next section discusses a method for reducing the required gain k by scaling.

6.5 Linear Coordinate Transformation.

In the design example of the last section the link masses and lengths were the same so that $\sigma_{\max}M / \sigma_{\min}M$ was not excessively large for any configuration. However, this is not the case for most manipulators as their link masses and lengths vary widely. From equation (6.3.14) it is clear that a large value of $\sigma_{\max}M / \sigma_{\min}M$ will cause the gain k to be large. Also, equation (6.3.2) suggests that all joint forces are approximately the same modulo the gravity compensation. This is not appropriate for a manipulator with differing link sizes. The natural modification to the "normalized" control (6.3.2) is a scaling. This is accomplished by making a linear transformation of the joint coordinates.

Choose nonsingular $A \in \mathbb{R}^{n \times n}$ and define transformed coordinates and forces by

$$q \triangleq A^{-1}\theta \qquad (6.5.1)$$

and

$$f \triangleq A^T F \tag{6.5.2}$$

Multiplying equation (6.2.1) on the left by A^T yields

$$m(q)\ddot{q} + c(q,\dot{q}) + g(q) + d(q,\dot{q},t) = f \tag{6.5.3}$$

where

$$\begin{aligned} m(q) &= A^T M(Aq)A \\ c(\dot{q},q) &= A^T C(Aq,A\dot{q}) \\ g(q) &= A^T G(Aq) \\ d(\dot{q},q,t) &= A^T D(Aq,A\dot{q},t) \end{aligned}$$

This equation in the transformed variable q has the same form as (6.2.1) and satisfies (P1). The design approach, therefore, works on these transformed dynamics as well. The advantage of allowing this transformation is that we may choose A to minimize $\sigma_{\max} m / \sigma_{\min} m$, fit the joint forces to match the actuators more closely, or achieve some compromise between the two.

The force transformation (6.5.2), and equation (6.3.2) suggest that a good choice for A might be a diagonal matrix with A_{ii} equal to the inverse of the ith actuator force rating. A nonlinear transformation may be desirable to achieve a particular dynamic behavior [14] but the discussion here will consider linear transformations only.

6.6 Compliance.

In assembly operations requiring compliance, the forces that are generated when the manipulator moves one workpiece into contact with another must be controlled. For example, consider the peg insertion task depicted in Figure 6.4; in order to execute this task with the proposed VSS control scheme a nominal trajectory must be specified for the manipulator to follow. The manipulator follows this trajectory until some misalignment of the peg or hole causes the manipulator to deviate from the nominal trajectory. If the resulting forces do not cause binding or excessive friction, the manipulator will follow a path close to the nominal path and complete the task.

We use the approach of [9] and describe the compliance of the control scheme by the

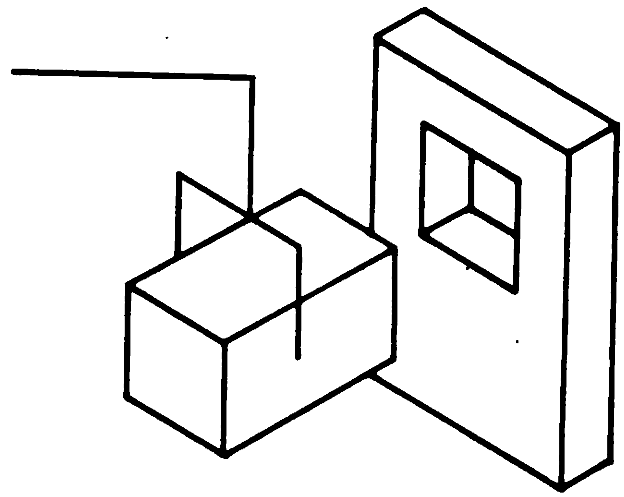

Figure 6.4 Peg-in-Hole Task

restoring forces [1] generated by the control when the manipulator is forced from the nominal trajectory. To study the performance of the proposed control scheme in compliant motion it is assumed that the motion is *quasi−static*. That is, all time derivatives of the manipulator state and the desired trajectory are approximated by zero. This approximation is reasonable for most assembly operations requiring programmed compliance [15].

With this assumption the force exerted by the manipulator on its environment is calculated. Let F_C be the force that the manipulator exerts at its gripper in some set of workspace oriented coordinates. This force is translated into joint forces by the usual Jacobian transformation [13] and is equal to $J^T(\theta)F_C$ where $J(\theta)$ is the Jacobian of the workspace oriented coordinates with respect to the joint coordinates. With this added force, which is not accounted for in the design procedure, equation (6.2.1) becomes

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta}) + G(\theta) + D(\theta,\dot{\theta},t) = F + J^T(\theta)F_C. \qquad (6.6.1)$$

Making the linear transformation described in the last section yields

$$m(q)\ddot{q} + c(q,\dot{q}) + g(q) + d(q,\dot{q},t) = f + A^T J^T(A^{-1}q)F_C \qquad (6.6.2)$$

Applying the design procedure to (6.6.2) with $F_C \equiv 0$, we obtain

$$f = \hat{c}(q,\dot{q}) + \hat{g}(q) + \hat{d}(q,\dot{q},t) + u \qquad (6.6.3)$$

where

$$u = -k \nabla V(s)$$
$$V(s) = \|s\|_1$$
$$k \geq \sigma_{\max} m(\epsilon + \frac{\|\tilde{d}\|}{\sigma_{\min} m} + \|B\dot{e}\| + \|\ddot{\theta}_d\|). \quad (6.6.4)$$

and

$$s = B(q_d - q) + (\dot{q}_d - \dot{q})$$

where q_d is the desired trajectory and $\tilde{d}$ is defined in (6.2.3) with upper-case characters replaced by lower-case. The dynamics are then

$$m(q)\ddot{q} = u + \tilde{d}(q, \dot{q}, t) + A^T J^T (A^{-1} q) F_C. \quad (6.6.5)$$

Choosing B to be the identity matrix and applying the quasi-static assumption we have

$$k \geq \sigma_{\max} m(\epsilon + \frac{\|\tilde{d}\|}{\sigma_{\min} m}). \quad (6.6.6)$$
$$s = (q_d - q) \quad (6.6.7)$$

Given the control specified by (6.6.3) - (6.6.7) the compliance question is: what force does the manipulator apply at its gripper when the manipulator is perturbed slightly to $q_c \neq q_d$? Using the quasi-static assumption again we set $\ddot{q} = 0$ in equation (6.6.5) and from the Filippov definition of solution of section 6.1, it follows that

$$[A^T J^T] F_C \in -K[u + \tilde{d}]. \quad (6.6.8)$$

Equation (6.6.8) defines the compliance of the control scheme. We restrict our attention now to manipulators with six degrees-of-freedom. Let θ_0 be the approximate configuration of the manipulator for an assembly task and assume $J(\theta_0)$ is nonsingular. Choosing $A = J(\theta_0)^{-1}$ for the transformation will simplify the compliance of the control scheme since (6.6.8) becomes

$$F_C \in [k \partial V(s) - K[\tilde{d}]]. \quad (6.6.9)$$

Define

$$\begin{aligned} \Delta q &= q_d - q_c \\ &= A^{-1}(\theta_d - \theta_c) \\ &= J(\theta_d - \theta_c) \\ &\approx \Delta x \end{aligned} \quad (6.6.10)$$

where Δx represents a small change in the gripper coordinates by definition of the Jacobian.

Then we have (approximately)

$$F_C \in k\partial V(s) - K[\widetilde{d}] = k \begin{bmatrix} SGN\,\Delta x_1 \\ SGN\,\Delta x_2 \\ \vdots \\ SGN\,\Delta x_6 \end{bmatrix} - K[\widetilde{d}] \tag{6.6.11}$$

Here the compliance behavior of the quasi-static manipulator is apparent. When $q = q_d$ the manipulator can apply at its gripper any reaction force in $[-k, k]^6$ modulo disturbances and modeling errors. Once the manipulator is forced from the desired position the control applies an approximately constant restoring force. The gain k is a stiffness parameter which may be used to control the compliance behavior of the manipulator. Note that equation (6.6.6) puts a lower bound on the stiffness and that this bound is given primarily by the magnitude of disturbances that must be rejected. In words, the stiffness of the manipulator may be controlled but the manipulator can only be as compliant as modeling errors, joint friction, and other disturbances allow.

Various choices of A and B will give different stiffness behavior to the manipulator. For example, stiffnesses along different axes may be controlled independently by suitable choice of these matrices.

6.7 Discussion

We have shown by using a multivariable approach to VSS that an extremely simple controller can be designed for robot manipulators. The control scheme developed provides for robust tracking and for compliance control and the compliance behavior is such that when the manipulator is forced from a nominal trajectory the control switches implicitly to force control.

The techniques used for proving stability are new for VSS and should be useful for the analysis of a wide variety of VSS described by nonsmooth gradient systems.

6.A Appendix A to Chapter 6

Proof of Theorem 6.1.1:

(1) To prove this property we first need two lemmas.

Lemma (6.1.1)

Let $\{ E_m \}$ be a sequence of compact subsets of $\mathbf{R}^n$ such that $E_{m+1} \subset E_m$. Then

$$\bigcap_{m \in \mathbf{N}} co\, E_m = co \bigcap_{m \in \mathbf{N}} E_m. \tag{6.A.1}$$

Proof: This is a simple application of Caratheodory's theorem for convex sets [16].

■

Lemma (6.1.2)

Let f be defined almost everywhere and measurable on a set E, $\mu E \neq 0$. Then $\exists N_f$ of measure zero such that

$$\bigcap_{\mu N = 0} \overline{co}\, f(E - N) = \overline{co}\, f(E - N_f). \tag{6.A.2}$$

Proof: See [8].

■

Henceforth, N subscripted with a function will be interpreted in terms of this lemma.

Proceeding with the proof of the property.

$$K[f](x) = \bigcap_{\delta > 0} \bigcap_{\mu N = 0} \overline{co} f\,(B(x, \delta) - N) \tag{6.A.3}$$

$$= \bigcap_{m \in \mathbf{N}} \bigcap_{\mu N = 0} \overline{co} f\,(B(x, 1/m) - N) \tag{6.A.4}$$

Now, from lemma (6.1.2) we have

$$K[f](x) = \bigcap_{m \in \mathbf{N}} \overline{co} f\,(B(x, 1/m) - N_{f,m}) \tag{6.A.5}$$

Define

$$N_f = \bigcup_{m \in \mathbf{N}} N_{f,m} \tag{6.A.6}$$

then

$$K[f](x) = \bigcap_{m \in \mathbf{N}} \overline{co} f\,(B(x, 1/m) - N_f) \tag{6.A.7}$$

since $N_{f,m}$ can be enlarged by a set of measure zero in (6.A.2).

Now f is locally bounded $\Rightarrow$

$$K[f(x)] = \bigcap_{m \in \mathbf{N}} co\, \overline{f(B(x,1/m) - N_f)}. \tag{6.A.8}$$

$$= \bigcap_{m \in \mathbf{N}} co\, \{ \lim f(x_i) \mid x_i \in B(x,1/m) - N_f \} \tag{6.A.9}$$

By lemma 6.1.1

$$K[f(x)] = co \bigcap_{m \in \mathbf{N}} \{ \lim f(x_i) \mid x_i \in B(x,1/m) - N_f \} \tag{6.A.10}$$

$$= co\, \{ \lim f(x_i) \mid x_i \to x, x_i \notin N_f \} \tag{6.A.11}$$

Finally, by noting that N_f can be enlarged by any set of measure zero in (6.A.2) the result follows.

(2) By property (1)

$$K[f+g](x) = co\{ \lim (f+g)(x_i) \mid x_i \to x, x_i \notin N_{f+g} \bigcup N_f \bigcup N_g \} \tag{6.A.12}$$

Since f and g are locally bounded, for each sequence $x_i \to x$ such that the limit in (6.A.12) exists, $\exists$ a subsequence (we do not reindex) $x_i \to x$ such that $\lim f(x_i)$ and $\lim g(x_i)$ exist and $\lim f(x_i) + \lim g(x_i) = \lim (f+g)(x_i)$ Thus,

$$\begin{aligned} K[f+g](x) &= co\{ \lim f(x_i) + \lim g(x_i) \mid x_i \to x, x_i \notin N_{f+g} \bigcup N_f \bigcup N_g \} \\ &\subset co\{ \lim g(x_i) \mid x_i \to x, x_i \notin N_{f+g} \bigcup N_f \bigcup N_g \} \\ &\quad + co\{ \lim f(x_i) \mid x_i \to x, x_i \notin N_{f+g} \bigcup N_f \bigcup N_g \} \\ &= K[f](x) + K[g](x). \end{aligned} \tag{6.A.13}$$

(3)

Define

$$g(x) \triangleq \bigtimes_{j=1}^{N} f_j(x) \tag{6.A.14}$$

then by property (1)

$$\begin{aligned} K[\bigtimes_{j=1}^{N} f_j](x) &= co\{\lim \bigtimes_{j=1}^{N} f_j(x_i) \mid x_i \to x, x_i \notin \bigcup_{j=1}^{N} N_{f_j} \bigcup N_g\} \\ &\subset co \bigtimes_{j=1}^{N} \{\lim f_j(x_i) \mid x_i \to x, x_i \notin \bigcup_{j=1}^{N} N_{f_j} \bigcup N_g\} \\ &= \bigtimes_{j=1}^{N} co\{\lim f_j(x_i) \mid x_i \to x, x_i \notin \bigcup_{j=1}^{N} N_{f_j} \bigcup N_g\} \end{aligned}$$

$$= \bigtimes_{j=1}^{N} K[f_j](x) \tag{6.A.15}$$

(4)

We begin the proof of this property with a lemma.

Lemma (6.3.1)

Let $f:\mathbf{R}^{m+n} \to \mathbf{R}^n$ be C^1 and $x \in \mathbf{R}^{m+n}$ be such that $rank(Df(x)) = n$. Then $\exists$ neighborhoods U of x and W of $f(x)$, such that $\forall\ M \subset \mathbf{R}^{m+n}, N \subset \mathbf{R}^n$, with $\mu M = \mu N = 0$, we have

$$\begin{aligned} &\mu\{[f(U \cap M^c)]^c \cap W\} = 0 \\ &\mu\{[f^{-1}(W \cap N^c)]^c \cap U\} = 0 \end{aligned} \tag{6.A.16}$$

Proof: $rank(Df(x)) = n \Rightarrow$ we can choose (without loss of generality) a partition (x_1, x_2) of $\mathbf{R}^{m+n}$ with $x_1 \in \mathbf{R}^m, x_2 \in \mathbf{R}^n$ so that $D_2 f(x_1, x_2)$ is nonsingular. By the implicit function theorem, $\exists$ a C^1 function $g:\mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^n$, and neighborhoods U_1 containing x_1, U_2 containing x_2, and W containing $f(x_1, x_2)$ such that

$$f(x_1, g(x_1, w)) = w \quad \forall\ x_1 \in U_1, w \in W, \tag{6.A.17}$$

and Φ defined by $\Phi(x_1, w) = (x_1, g(x_1, w))$ is a C^1 diffeomorphism of $U_1 \times W$ onto $U \triangleq U_1 \times U_2$. By continuity of Φ it follows (see [17] pg. 551) that $\Phi|_{U_1 \times W}$ maps null (zero-measure) sets to null sets and similarly for $\Phi^{-1}|_U$. It is therefore sufficient to prove the result for $f \circ \Phi|_{U_1 \times W}$ which is simply a projection. This is straight forward and is left to the reader.

■

Now the proof of the property: By lemma (6.3.1), $\exists$ neighborhoods U of x and W of $f(x)$ such that $[g^{-1}(W \cap N_f^c)]^c \cap U$ and $[g(U \cap N_{f \circ g}^c)]^c \cap W$ are null sets. Next, by property (1), and the fact that $K[f](x)$ depends on f only near x we obtain

$$K[f \circ g](x) = co\ L[f \circ g](x) \tag{6.A.18}$$

where

$$L[f \circ g](x) \triangleq \{\lim f \circ g(x_i) \mid x_i \to x, x_i \in U \cap N_{f \circ g}^c \cap g^{-1}(W \cap N_f^c)\}$$

and

$$K[f](g(x)) = co\, L[f](g(x)) \tag{6.A.19}$$

where

$$L[f](g(x)) = \{\lim f(y_i) \mid y_i \to g(x), y_i \in W \cap g(U \cap N_{f \circ g}^c) \cap N_f^c\}$$

For every $z \in L[f \circ g](x)$, $\exists x_i \to x$ such that $x_i \in U \cap N_{f \circ g}^c \cap g^{-1}(W \cap N_f^c)$. and $f(g(x_i)) \to z$.

Now, let $y_i = g(x_i)$, then $y_i \in g(U \cap n_{f \circ g}^c) \cap W \cap N_f^c$ and $y_i \to g(x)$ since g is continuous. Now $f(y_i) \to z \implies$

$$L[f \circ g](x) \subset L[f](g(x)). \tag{6.A.20}$$

For the reverse inclusion, let $z \in L[f]g(x)$ then $\exists y_i \to g(x)$ such that $y_i \in W \cap g(U \cap N_{f \circ g}^c) \cap N_f^c$ and $f(y_i) \to z$. By the rank condition on $Dg(x)$, g is locally surjective (see [17] page 108) so $\exists$ a subsequence of $\{y_i\}$ (we do not index) such that $y_i \in W \cap g(U \cap N_{f \circ g}^c \cap B(x, 2^{-i})) \cap N_f^c$. Thus,

$$\exists x_i \to x, x_i \in U \cap N_{f \circ g}^c \cap g^{-1}(W \cap N_f^c) \text{ such that } y_i = g(x_i) \text{ and } f(g(x_i)) \to z$$
$$\implies z \in L[f \circ g](x)$$

Thus we have $L[f \circ g](x) = L[f](g(x))$ and the result follows by taking the convex hull of both sides.

(5)

By proposition 1, we obtain

$$K[gf](x) = co\, \{\lim g(x_i) f(x_i) \mid x_i \to x, x_i \notin N_{gf} \cup N_f\} \tag{6.A.21}$$

Since g is continuous in its argument and f is locally bounded

$$K[gf](x) = co\, \{g(x) \lim f(x_i) \mid y_i \to x, x \notin N_{gf} \cup N_f\}.$$
$$= g(x) K[f](x). \tag{6.A.22}$$

since co commutes with linear maps.

(6) Since V is locally Lipschitz, ∇V is defined almost everywhere and is locally bounded.

Therefore by (1) we have

$$K[\nabla V](x) = co\{\lim \nabla V(x_i) \mid x_i \to x, x_i \notin N_{\nabla V}\}$$
$$= \partial V(x) \qquad (6.A.23)$$

(7)

This is a corollary of (4) obtained by taking f to be the identity map.

∎

Proof of Theorem 6.1.2: First, we have by definition

$$V(x(t)) = \max_{j \in Y} f_j(x(t)) \qquad (6.A.24)$$

Computing left and right derivatives we obtain

$$\frac{d}{dt}[V(x(t))] \text{ exists} \Leftrightarrow \max_{j \in Y^*(x)} \nabla f_j^T(x)\dot{x}(t) = \min_{j \in Y^*(x)} \nabla f_j^T(x)\dot{x}(t) \qquad (6.A.25)$$

where $Y^*(x) = \{ j \mid f_j(x) = V(x)\}$

Thus, the existence of $\dot{V}$ $\Rightarrow$

$$\frac{d}{dt}[V(x(t))] = \nabla f_j^T(x)\dot{x}(t) \quad \forall \; j \in Y^*(x) \qquad (6.A.26)$$

$\Rightarrow$

$$\frac{d}{dt}[V(x(t))] = [\sum_{j \in Y^*(x)} \lambda_j \nabla f_j^T(x)]\dot{x}(t)$$
$$\forall \{\lambda_j\} \text{ such that } \sum_{j \in Y^*(x)} \lambda_j = 1 \qquad (6.A.27)$$

$\Rightarrow$

$$\frac{d}{dt}[V(x(t))] = \xi^T \dot{x}(t) \quad \forall \xi \in co\{\nabla f_j \mid j \in Y^*(x)\} \qquad (6.A.28)$$

and $co\{\nabla f_j \mid j \in Y^*(x)\} = \partial V(x)$.

∎

6.B Appendix B to Chapter 6

Here we prove the existence and continuation of a Filippov solution to (6.2.5).

Theorem 6.A.1 Let u be defined by (6.3.1-6.3.3) and (6.3.14). If A1,A2,A3, and P1 are satisfied, then, for any initial condition $[\theta, \dot{\theta}]^T(t_0) = [\theta_0, \dot{\theta}_0]^T$, (6.2.5) has a solution continuable on $[t_0, \infty)$.

Proof: Let $Q = \mathbf{R}^{2n} \times \mathbf{R}$ and let D be an arbitrary compact set in Q. By A1,A2,A3, and P1 we have that $\dot{\theta}, M^{-1}, \tilde{D}$, and k are bounded on D. Also, ∇W is defined *a.e.* and bounded. Thus, RHS of (6.2.5) is bounded by, say, L on D. Choose $A(t) = L$ which is integrable on D. The RHS of (6.2.5) is measurable and defined *a.e.* in Q. Thus, the RHS of 3.5 satisfies condition B. Now by theorem 4 of [8] we have the local existence of a solution to (6.2.5).

By theorem 5 of [8] any solution of (6.2.5) is continuable on $[t_0, t_1)$ where $t_1 = \infty$ or $\|[\theta, \dot{\theta}]^T\| \to \infty$. By (6.3.15) we have for any solution of (6.2.5) that s is bounded $\Rightarrow$ $[e, \dot{e}]^T$ is bounded by (6.3.1) $\Rightarrow$ $[\theta, \dot{\theta}]^T$ is bounded on bounded sets by A1. Therefore, there exists a solution continuable on $[t_0, \infty)$ (see[18]. for a discussion of uniqueness).

■

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Chapter Seven

Conclusion

This thesis has dealt with two subfields in robotics: kinematics and control of robot manipulators. Following a tutorial development of manipulator kinematics using twists and their exponentials (chapters 1-4) an optimal design theorem for 6R manipulators was proved using this notation. To address the problem of control for robot manipulators, we developed nonsmooth analysis techniques which made the design of a robust variable structure control scheme a simple exercise.

The tutorial on manipulator kinematics relied on the expression of rigid motions as exponentials of twists. When writing the kinematic equations for manipulators we were able to avoid attaching a coordinate system to each link of the manipulator in contrast to the standard Hartenburg-Denavit approach. The only information needed to write down the forward kinematic map is the positions of the joint axes in the nominal zero position. In addition, the exponential notation makes differentiation and the determination of critical points in the forward kinematic map easy. We claim that the proof of the optimal design theorem is (even more) gruesome in the standard notation.

Future development of this geometric approach to kinematics may include the following. (1) A truly geometric development of twists starting with physical 3-space modeled as a manifold. This would avoid the expression of twists in coordinates which tends to confuse one's intuition. (2) A simplification of manipulator dynamics. Just as the exponential notation simplifies differentiation and the determination of critical points, we expect that the exponential notation will give a clear picture of manipulator dynamics. These twists and their exponentials are not a cure all. They allow us to represent simple ideas simply; difficult problems remain. We do not , for example, expect that twists will help find solutions to general kinematic equations of 6R manipulators. For these, the kinematic equation must be reduced to a set of polynomial equations which will most likely be solved numerically.

The optimality theorem of chapter five answers some fundamental questions on the design of manipulators. It develops the relationships between significant design parameters, and provides a clear statement and proof of a folk theorem that states that elbow manipulators are optimal. On top of this, the theorem is consistent with the design of the human arm. The human arm is like the elbow manipulator in that one degree of freedom is at the elbow (halfway between the shoulder and the wrist) and the remaining degrees of freedom are concentrated at the ends (shoulder and wrist). Also, the definitions of length and work-volume extend to other types of manipulators and set a framework for studying these as well. We expect that this theorem will motivate the special consideration of elbow manipulators and their duals in the study of path-planning, collision detection, and dynamics. The special structure of elbow manipulators may simplify these problems and lead to practical implementation of sophisticated control algorithms.

The calculus developed for analyzing variable structure systems in chapter six allows us to view variable structure control as an extension of standard nonlinear control techniques. We replaced high gain feedback with discontinuous feedback and a smooth Lyapunov function with a nonsmooth Lyapunov function. When applied to the robot control problem, we were able to obtain stability results easily as in standard Lyapunov analysis. We simply exchanged differential equations with differential inclusions and gradients with (set-valued) generalized gradients. Although we developed a simple controller for robot manipulators the impact of this work on VSS controllers is likely to be felt more in the area of VSS. We expect that this (sub-)sub-field will be viewed as an application of nonsmooth analysis.

This thesis has treated only two problems in a field which is incredibly broad. There are many more open problems in robotics which also need careful attention.