ADAPTIVE STABILIZATION OF SAMPLED SYSTEMS

by

Erwei Bai and S. Shankar Sastry

Memorandum No. UCB/ERL M86/52

11 June 1986

ADAPTIVE STABILIZATION OF SAMPLED SYSTEMS

by

Erwei Bai and S. Shankar Sastry

Memorandum No. UCB/ERL M86/52

11 June 1986

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

TITLE PAGE

ADAPTIVE STABILIZATION OF SAMPLED SYSTEMS

by

Erwei Bai and S. Shankar Sastry

Memorandum No. UCB/ERL M86/52

11 June 1986

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

Adaptive Stabilization of Sampled Systems

Erwei Bai and *S.Shankar Sastry*

Department of Electrical Engineering and
Computer Science
Electronics Research Laboratory
University of California, Berkeley CA 94720

ABSTRACT

In this paper, a simple discrete adaptive control scheme is proposed for stabilizing minimum phase continuous time systems under fast sampling. Even though the sampled system is not necessarily minimum phase, information about the pole-zero locations of the sampled system can be incorporated to complete the proof of stability.

June 11, 1986

Research supported by Army Research Office under grant #DAAG 29-85-K-0072
Keywords: *Adaptive control, Sampled systems, Stability analysis*

Adaptive Stabilization of Sampled Systems

Erwei Bai and *S.Shankar Sastry*
Department of Electrical Engineering and
Computer Science
Electronics Research Laboratory
University of California, Berkeley CA 94720

1 Problem Statement

In this paper, a simple discrete adaptive control scheme is proposed for fast sampling of minimum phase continuous time systems. The subject is directly motivated by a paper of Astrom, Hagander and Sternby [1], where authors have shown that all continuous time systems with relative degree large than 1 will give rise to sampled systems with unstable zeros when the sampling period is sufficient small, i.e. the corresponding discrete time system is no longer minimum phase. In a recent paper [4], Praly, Hung and Rhode have presented a technique for the adaptive control of such rapidly sampled systems. In their approach, however the unstable zeros of the sampled system are assumed to be known exactly. It seems that the stability needs to be further analyed for the control of sampled systems, since the unstable zeros are known only approximately.

Here we propose an indirect adaptive scheme for the control of sampled systems. The principal difficulty associated with the indirect approach arises from the fact that the estimated system may have unstable pole-zero cancellations. Thus in the literature(for example see [3],[6]), more or less stringent conditions on the plant have to be imposed: for instance that the external signal be persistently exciting or the parameters of the system lie in a known convex set in which no unstable pole-zero cancellation may occur. Fortunately, a great deal of prior information is available about pole zero locations of the sampled system with fast sampling and such information can be used in the design of our indirect adaptive controller.

Research supported by Army Research Office under grant #DAAG 29-85-K-0072

We now summarize the problem as follows: Given a minimum phase continuous time system

$$G(s) = k\,\frac{(s-q_1)(s-q_2)...(s-q_m)}{(s-p_1)(s-p_2)...(s-p_n)} \quad m < n \tag{1.1}$$

where m and n are known, k, q_i, p_j unknown but constant with following assumption

A1) There exists some constants ϵ, $M > 0$ such that

$$-M \leqslant \mathrm{Re}q_i \leqslant -\epsilon$$

$$-M \leqslant \mathrm{Im}q_i \leqslant M$$

In particular, the plant (1.1) is minimum phase.

The goal is to design a discrete time controller such that the closed loop consisting of the sampled system (1.1) together with controller is exponentially stable with all signals bounded.

2 Adaptive Scheme and Convergence Analysis

With fast sampling, the sampled system of (1.1) can be approximated (in the sense of closeness of poles and zeros, see [1]) as

$$H(z) = k\frac{h^{n-m}}{(n-m)!}\frac{B(z)\,B_u(z)}{A(z)} \tag{2.1}$$

In (2.1), h is sampling period and

$$B(z) = (z-e^{q_1 h})...(z-e^{q_m h}) \tag{2.2}$$

$$A(z) = (z-e^{p_1 h})...(z-e^{p_n h}) \tag{2.3}$$

Further $B_u(z)$ is a known polynomial of order (n-m-1) which depends only on the relative degree (n-m) of the system (1.1). All unstable zeros of $B_u(z)$ lie in the interval $(-\infty,-1]$. For a few values of n, the polynomials $B_u(z)$ and their unstable zeros are listed below (from [1]).

n-m	$B_u(z)$	*unstable zeros*
1	1	
2	z+1	-1
3	z^2+4z+1	-3.732
4	$z^3+11z^2+11z+1$	−1, −9.899
5	$z^4+26z^3+66z^2+26z+1$	−2.322, −23.20

Thus, the locations of zeros and poles of H(z) are as indicated in Fig.1

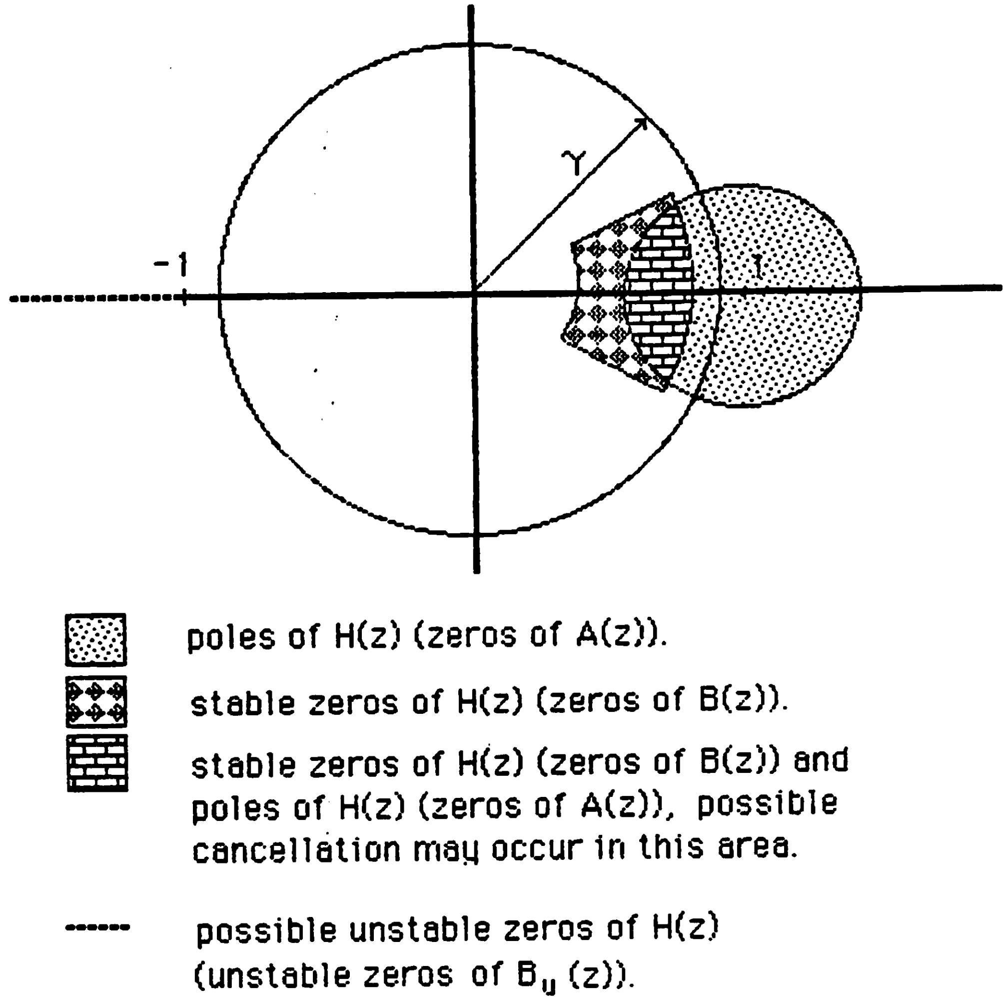

Fig. 1

All poles of H(z) lie inside the small disk centered at (1,0), all zeros of B(z) lie in the fan-like set and all unstable zeros of $B_u(z)$ are in the interval $(-\infty, -1]$. By observation, then there exists $1 > \gamma > e^{-\epsilon h}$, such that all possible pole-zero cancellations are stable and inside a disk with radius γ.

The above discussion leads us to state that the sampled system of (1.1) can be described as

$$H^*(z) = k^* \frac{B^*(z)\, B_u^*(z)}{A^*(z)} = \frac{b_1^* z^{n-1} + \ldots + b_n^*}{z^n + a_1^* z^{n-1} + \ldots + a_n^*} \tag{2.4}$$

where $B^*(z)$, $B_u^*(z)$ and $A^*(z)$ are close to $B(z)$, $B_u(z)$ and A(z) respectively in the same sense as before, namely their zeros are close. Thus we may make following assumption on the sampled system (2.4)

A2) Two convex sets C_a in R^n and C_b in R^n are known such that

(1) the vectors $(b_1^*, \cdots, b_n^*)^T$ and $(a_1^*, \cdots, a_n^*)^T$ belong to C_b and C_a respectively.

(2) for any $(b_1, \ldots, b_n)^T \in C_b$ and $(a_1, \ldots, a_n)^T \in C_a$ all the possible pole-zero cancellations of equation (2.4) are stable and inside the disk with radius $\gamma < 1$.

Remark: The assumption above is somewhat restrictive, since it is generally difficult to get C_a and C_b from prior information about the zeros of the polynomials A(z), B(z) and $B_u(z)$. This is so, because the relation between the zeros and the coefficients of a polynomial is nonlinear. Consequently though zeros of a polynomial lie inside a convex set, it is not necessarily true that the coefficients lie inside some convex set. For low order problems, however, it is easy to obtain C_a and C_b. Examples are the first order case, namely

$$A_1(z) = z + a$$

and the second order case

$$A_2(z) = z^2 + a_1 z + a_2$$

$A_2(z)$ is guaranteed stable provided that a_1 and a_2 are constrained to lie in the following convex region (see [2]).

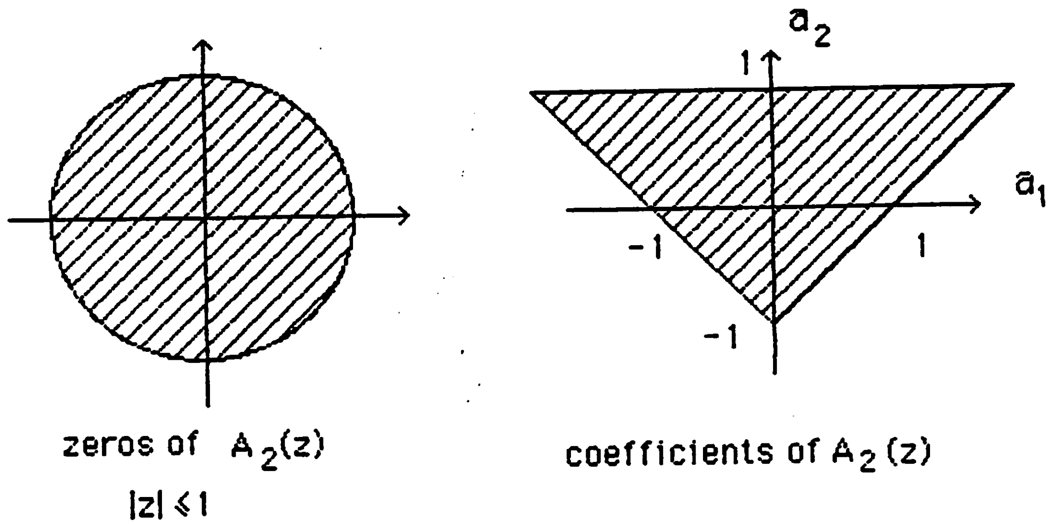

Fig. 2

By direct calculation, the system (2.4) can be realized by the following state space representation. It might be non-minimal, if there are (stable) pole-zero cancellations.

$$x(k+1) = (F + \theta_F{}^* c)\, x(k) + \theta_b{}^* u(k) \tag{2.5a}$$

$$y(k) = c\ x(k) \tag{2.5b}$$

where

$$F = \begin{bmatrix} -f_1 & 1 & 0 & . & 0 \\ . & \ddots & 1 & . & . \\ . & . & . & . & . \\ . & . & . & . & 1 \\ -f_n & 0 & 0 & . & 0 \end{bmatrix}$$

is an arbitrary stable matrix.

$$\theta_F{}^* = \theta_f + \theta_a{}^* = (f_1, \ldots, f_n)^T + (-a_1{}^*, \ldots, -a_n{}^*)^T$$

$$\theta_b{}^* = (b_1{}^*, \ldots, b_n{}^*)^T$$

and

$$c = (1,0,\ldots,0)$$

Also using the techniques of [3], we have, by Fact 1 of the Appendix, another non-minimal realization of (2.4) via equation (2.5) as follows

$$v_1(k+1) = F^T v_1(k) + c^T y(k) \tag{2.6a}$$

$$v_2(k+1) = F^T v_2(k) + c^T u(k) \tag{2.6b}$$

$$y(k) = v_1^T(k)\theta_F^* + v_2^T(k)\theta_b^* \tag{2.6c}$$

A controller is chosen to satisfy the equations

$$x_c(k+1) = k_0(\hat{\theta}(k))x_c(k) + k_1(\hat{\theta}(k))y(k) + k_2(\hat{\theta}(k))r(k) \tag{2.7a}$$

$$u(k) = k_3(\hat{\theta}(k))x_c(k) + k_4(\hat{\theta}(k))y(k) + k_5(\hat{\theta}(k))r(k) \tag{2.7b}$$

where $x_c(k)$ is the state vector of the controller, $r(k)$ is the external input. $\hat{\theta}^T(k) = (\hat{\theta}_a^T(k), \hat{\theta}_b^T(k))$ is the estimate of $(\theta_a^{*T}, \theta_b^{*T})$ at time k. $k_i(\hat{\theta})$ are vectors of appropriate dimension and they depends on the specific control law, choose,for instances, pole placement type, model reference etc. It is assumed that

A3) $k_i(\hat{\theta})$'s are bounded for all $\hat{\theta}_a \in C_a$, $\hat{\theta}_b \in C_b$.

A4) For all $\hat{\theta}_a \in C_a$, $\hat{\theta}_b \in C_b$, we have the following: equation (2.5) (or (2.6)) with θ_a^*, θ_b^* replaced by $\hat{\theta}_a$, $\hat{\theta}_b$, together with controller (2.7), results in a stable closed loop system with all poles inside the disk with radius $\gamma < 1$.

Remark: Assumptions (A3) and (A4) are valid for most control laws, since all uncontrollable modes of equation (2.5) are stable and inside the disk with radius $\gamma < 1$, by assumption A2).

Define the signal vectors w and z by

$$w^T(k) = (v_1^T(k), v_2^T(k))$$

$$z^T(k) = (w^T(k), x_c^T(k)) \tag{2.8}$$

Further the output error e is given by

$$e(k) = y(k) - (\hat{\theta}_F^T(k)v_1(k) + \hat{\theta}_b^T(k)v_2(k))$$

$$= -(v_1^T(k), v_2^T(k)) \begin{bmatrix} \tilde{\theta}_a(k) \\ \tilde{\theta}_b(k) \end{bmatrix} \tag{2.9}$$

where $\tilde{\theta}_a(k) = (\hat{\theta}_a(k) - \theta_a^*)$ and $\tilde{\theta}_b(k) = (\hat{\theta}_b(k) - \theta_b^*)$ denote the parameter errors. Choose the parameter update law as

$$\hat{\theta}(k+1) = P(\hat{\theta}(k) + \frac{w(k)e(k)}{1+z^T(k)z(k)}) \tag{2.10}$$

where P is the projection of $\hat{\theta}_a(k)$ onto C_a and $\hat{\theta}_b(k)$ onto C_b. It is at this point that we need the assumption (A2). This is almost a standard projection type algorithm with the difference that z rather than w is used in the denominator of (2.10). However since $||z(k)||^2 \geqslant ||w(k)||^2$, the following properties of the standard projection type algorithm (see [2]) are readily inherited.

$$||\tilde{\theta}(k)|| \leqslant ||\tilde{\theta}(k-1)|| \tag{2.11}$$

$$\lim_{k \to \infty} \frac{e(k)}{(1+z^T(k)z(k))^{1/2}} = 0 \tag{2.12}$$

$$\lim_{k \to \infty} ||\hat{\theta}(k) - \hat{\theta}(k-1)|| = 0 \tag{2.13}$$

Before analyzing the stability of the overall system, we need the following lemma due to Desoer [5].

Lemma 1. Consider the following discrete time system

$$x(k+1) = A(k)x(k) \tag{2.14}$$

satisfying the following assumptions

$$A(k) \textit{ is bounded.} \tag{2.15}$$

The eigenvalues of $A(k)$ are uniformly stable i.e.

$$|\lambda_i(A(k))| \leqslant \gamma < 1 \quad \text{for } any \; k, i \tag{2.16}$$

$$\lim_{k \to \infty} ||A(k+1) - A(k)|| = 0 \tag{2.17}$$

Then the system (2.14) is exponentially stable.

Proof: Can be found in [5].

We now have the following theorem.

Theorem 1 Consider the system (2.4) with controller of (2.7) and the parameter update law of (2.10). Then if the input r(k) is bounded, the overall system is exponentially stable and all signals are bounded.

Proof: By combining equations (2.6) and (2.7), the overall system can be written as

$$z(k+1) = \begin{bmatrix} F^T + c^T\theta_F^{*T}, & c^T\theta_b^{*T}, & 0 \\ c^T k_4(\hat\theta(k))\theta_F^{*T}, & F^T + c^T k_4(\hat\theta(k))\theta_b^{*T}, & c^T k_3(\hat\theta(k)) \\ k_1(\hat\theta(k))\theta_F^{*T}. & k_1(\hat\theta(k))\theta_b^{*T}. & k_0(\hat\theta(k)) \end{bmatrix} z(k) + \begin{bmatrix} 0 \\ c^T k_5(\hat\theta(k)) \\ k_2(\hat\theta(k)) \end{bmatrix} r(k)$$

$$= \begin{bmatrix} F^T + c^T\hat\theta_F^T(k), & c^T\hat\theta_b^T(k), & 0 \\ c^T k_4(\hat\theta(k))\hat\theta_F^T, & F^T + c^T k_4(\hat\theta(k))\hat\theta_b^T(k), & c^T k_3(\hat\theta(k)) \\ k_1(\hat\theta(k))\hat\theta_F^T(k)), & k_1(\hat\theta(k))\hat\theta_b^T(k), & k_0(\hat\theta(k)) \end{bmatrix} z(k)$$

$$+ \begin{bmatrix} c^T \\ c^T k_4(\hat\theta(k)) \\ k_1(\hat\theta(k)) \end{bmatrix} e(k) + \begin{bmatrix} 0 \\ c^T k_5(\hat\theta(k)) \\ k_2(\hat\theta(k)) \end{bmatrix} r(k)$$

$$= A(\hat\theta(k))\, z(k) + Q(k)e(k) + b(\hat\theta(k))r(k) \tag{2.18}$$

where

$$A(\hat\theta(k)) = \begin{bmatrix} F^T + c^T\hat\theta_F^T(k), & c^T\hat\theta_b^T(k), & 0 \\ c^T k_4(\hat\theta(k))\hat\theta_F^T, & F^T + c^T k_4(\hat\theta(k))\hat\theta_b^T(k), & c^T k_3(\hat\theta(k)) \\ k_1(\hat\theta(k))\hat\theta_F^T(k)), & k_1(\hat\theta(k))\hat\theta_b^T(k), & k_0(\hat\theta(k)) \end{bmatrix}$$

$$Q(k) = \begin{bmatrix} c^T \\ c^T k_4(\hat\theta(k)) \\ k_1(\hat\theta(k)) \end{bmatrix}$$

$$b(\hat\theta(k)) = \begin{bmatrix} 0 \\ c^T k_5(\hat\theta(k)) \\ k_2(\hat\theta(k)) \end{bmatrix} \tag{2.19}$$

Now write Q(k)e(k) as

$$Q(k)e(k) = \frac{e(k)Q(k)}{1+z^T(k)z(k)} + \frac{e(k)Q(k)z^T(k)}{1+z^T(k)z(k)}\, z(k) \tag{2.20}$$

Hence, the equation (2.18) can be rewritten as

$$z(k+1) = \tilde A(k)z(k) + \frac{e(k)Q(k)}{1+z^T(k)z(k)} + b(\hat\theta(k))r(k) \tag{2.21}$$

with

$$\tilde{A}(k) = A(\hat{\theta}(k)) + \frac{e(k)}{(1+z^T(k)z(k))^{1/2}} \frac{Q(k)z^T(k)}{(1+z^T(k)z(k))^{1/2}} \tag{2.22}$$

From the properties of the identifier (2.12), we have that the second term in (2.22) tends to zero as $k \to \infty$. By the assumption (A4), we have that

$$|\lambda_i(A(\hat{\theta}(k)))| \leq \gamma \quad \text{for } any\ k, i$$

Thus the exponential stability of $\tilde{A}(k)$ follows from (2.13) and lemma 1. Furthermore, the boundedness of r(k) and Q(k) implies that all signals are bounded. This completes the proof.

3 Concluding Remarks

In this paper, we have presented a simple discrete adaptive scheme for stabilizing a minimum phase continuous system with fast sampling. The chief assumption was that the parameters needed to lie in a convex set, this is only for the sake of the projection.

4 References

(1) Astrom.K.J., Hagander.P. and Sternby.J. 'Zeros of Sampled Systems' *Automatica Vol.* 20 *No.* 1 *pp* 31—38 1984

(2) Goodwin.G.C. and Sin.K.S. 'Adaptive Filtering, Prediction and Control' Prentice-Hall, Englewood Cliffs.NJ 1984

(3) Kreisselmeier, G. 'An Approach to Stable Indirect Adaptive Control' *Automatica Vol.* 21 *No.* 4 *pp* 425—431 1985

(4) Praly.L., Hung.S.T. and Rhode.D.S. 'Towards A Direct Adaptive Scheme for a Discrete-time Control of a Minimum Phase Continuous-time System' *Proc. of 24th Conference on Decision* and *Control pp* 1188—1191 *Ft.Lauderdale, FL* Dec.1985

(5) Desoer.C.A. 'Slowly Varying Discrete System $X_{i+1} = A_i X_i$' *Electronics Letters* 6,11,pp.339-340, 1970

(6) Bai.E.W. and Sastry.S.S. 'Global Stability Proofs For Continuous Time Indirect Adaptive Control Schemes' Mem. UCB/ERL M86/20 University of California, Berkeley

5 Appendix

Fact 1. Consider the transfer function of the system of equation (2.5). It has a non-minimal realization given by

$$v_1(k+1)=F^T v_1(k)+c^T y(k) \tag{A1}$$

$$v_2(k+1)=F^T v_2(k)+c^T u(k) \tag{A2}$$

$$y(k)=v_1^T(k)\theta_F^* + v_2^T(k)\theta_b^* \tag{A3}$$

Proof: Notice that

$$x(k+1)=(F+\theta_F^* c)x(k)+\theta_b^* u(k)$$

$$=Fx(k)+\theta_F^* y(k)+\theta_b^* u(k) \tag{A4}$$

Then

$$y(k)=cF^k x(0)+\left(\sum_{i=0}^{k-1}(F^T)^{k-i-1}c^T y(i)\right)^T \theta_F^*$$

$$+\left(\sum_{i=0}^{k-1}(F^T)^{k-i-1}c^T u(i)\right)^T \theta_b^* \tag{A5}$$

Define

$$v_1(k+1)=F^T v_1(k)+c^T y(k) \tag{A6}$$

$$v_2(k+1)=F^T v_2(k)+c^T u(k) \tag{A7}$$

Now it follows that

$$y(k)=cF^k x(0)+(v_1(k)-(F^T)^k v_1(0))^T \theta_F^* +(v_2(k)-(F^T)^k v_2(0))^T \theta_b^*$$

$$=v_1^T(k)\theta_F^* + v_2^T(k)\theta_b^* + \textit{exponentially decaying terms} \tag{A8}$$

This completes the proof.