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BSIM - SUBSTRATE CURRENT MODELING

APPENDIX C: SPICE IMPLEMENTATION OF THE BSIM SUBSTRATE CURRENT AND DEGRADATION MODELS

by

Peter M. Lee, P. K. Ko and C. Hu

Memorandum No. UCB/ERL M86/64

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Instability and Geometric Transience of the Aloha Protocol[†]

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1. Introduction.

1.1. Instability of the Aloha Protocol.

In this paper we consider the uncontrolled slotted Aloha protocol. This protocol is described in § 2.1.

It is well-known that this protocol is unstable. Mathematically, this means that the embedded Markov chain describing the backlog of the system is transient (e.g., see [2] and [6]). This, in turn, implies that after sufficiently long time the system backlog will drift towards infinity and the throughput of the system will decline towards 0. However, usually the protocol works well for a "long" period of time. During this time the system backlog is restricted to moderate values. It is only after this random "stable" period that the system backlog makes its rare excursion to higher values and then onwards the system is very likely to get further backlogged and hence more sluggish.

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1.2. Quantifying Instability.

In this paper we will quantify instability described above by eigenvalue analysis. We will see that it is meaningful to quantify this instability by a certain first passage time.

The idea of quantifying this instability in this way is due to Schoute [7]. In [7], the author makes some important observations numerically regarding the transition matrix of the embedded Markov chain. In this paper we will prove these observations. It might be possible to use more general work that relies on Spectral theory of Generators (e.g., see Van Doorn [9]) for a part of our proof. However, in the present case, our probabilistic proof is more intuitive and simpler.

1.3. Outline of the Remaining Sections.

In § 2, we describe the Aloha protocol. In § 3, we collect some mathematical facts that will be useful later in the paper. § 4 will be spent describing the main ideas and § 5 will be devoted to proving them. In § 6, we will put the results of § 4 and § 5 in proper perspective to define a new performance measure quantifying instability of the system. In § 6, we will also describe a conjecture that makes a connection between the first passage time considered here and the first passage time considered in the work by Cottrell et al. [1] which can be analyzed using the techniques of Large Deviation theory. Finally, we will summarize the results of this paper in § 7.

2. The Protocol.

2.1. Aloha Model.

Aloha is a multi-access radio channel protocol. Users of the system transmit packets according to a specified rule over the channel as they get requests. The channel time is divided into slots. The duration of a slot equals the transmission time of a packet and this is assumed to be constant. If there are more than one transmission during the same slot then all the transmissions fail and will be retransmitted at some later randomly selected times. All the colliding users will perceive the failure through the channel feedback. Users with messages to be retransmitted are called backlogged users. A user who is not backlogged attempts a transmission during the next time slot after the request. We assume the infinite users model.

The mathematical modelling of this protocol is done in the following widely accepted manner.

Assume that transmissions are attempted only at times $n = o,1,2,\cdots$. Arrivals of new requests to the system for transmissions are assumed to be governed by a global Poisson process with parameter $\lambda > 0$. Hence, for the time slot $[n, n+1)$, there will be k new arrivals with probability $p_k := e^{-\lambda}.\lambda^k / k!$. If there is a collision before during the slot $[n, n+1)$, all the transmitting users will know about the collision before time $n+1$. Then, the backlogged users will attempt to retransmit with probability $p > 0$ at times $n+1, n+2, \cdots$ until they succeed. Let $\{X_n, n = 0,1,2,\cdots\}$ denote the number of backlogged users at time n. For our model $\{X_n\}$ is a time-homogeneous Markov chain with the transition probability matrix $P = \{p_{ij}\}$ given as follows:

$$p_{ij} = 0, \text{ if } j \leqslant i-2 \text{ and } i \geqslant 2,$$

$$p_{i(i-1)} = i.p.(1-p)^{i-1}.e^{-\lambda}, \text{ if } i \geqslant 1,$$

$$p_{ii} = (1-p)^i.\lambda.e^{-\lambda}+(1-i.p.(1-p)^{i-1}).e^{-\lambda}, \text{ if } i \geqslant 1,$$

$$= \lambda.e^{-\lambda}+e^{-\lambda}, \text{ if } i \geqslant 1,$$

$$p_{i(i+1)} = (1-(1-p)^i).\lambda.e^{-\lambda}, \text{ if } i \geqslant 1,$$

$$= 0, \text{ if } i = 0,$$

$$p_{ij} = \frac{e^{-\lambda}.\lambda^{j-i}}{(j-i)!}, \text{ if } j \geqslant i+2 \text{ and } i \geqslant 0. \tag{1}$$

2.2. Some Observations about the Transition Probabilities $\{p_{ij}\}$.

We make the following observations regarding the shapes of the transition probabilities. They will be useful to us in the coming sections.

(1) $p_{i(i-1)}$: There exists $i_0 \geqslant 1$ such that $p_{i(i-1)}$ is increasing for $i < i_0$ and then decreasing to 0 for $i > i_0$. It has its maximum at i_0.

(2) p_{ii}: For $i \geqslant 1$, p_{ii} can be rewritten as

$$p_{ii} = e^{-\lambda} + (1-p)^{i-1}.e^{-\lambda}.((1-p).\lambda - i.p)$$

Then, clearly, there exists $i_1 \geqslant 1$ such that p_{ii} is increasing to $e^{-\lambda}$ for $i \geqslant i_1$.

(3) $p_{i(i+1)}$: For $i \geqslant 1$, $p_{i(i+1)}$ is increasing to $\lambda.e^{-\lambda}$.

3. Some Facts.

In this section we collect some facts about Markov chains. These facts are a well-established part of the theory of Markov chains. We have used Seneta [8] and Kendall ([4] and [5]) as our references.

We consider countable (can be finite) non-negative matrices and denote them generically by T. If T is stochastic, then it can be thought of as the transition probability matrix of a Markov chain evolving in a countable state-space. In throughout this section, we assume that T is irreducible, i.e., for each $i, j = 1,2, \cdots$, there exists $k \equiv k(i, j)$ such that $t_{ij}^{(k)} > 0$.

Fact 1 (Seneta [8], pp. 200) : The power series

$$T_{ij}(s) := \sum_{k=0}^{\infty} t_{ij}^{(k)}.s^k, \quad i, j = 1,2, \cdots,$$

all have common convergence radius R, $0 \leqslant R < \infty$, for each pair (i, j).

Fact 2 (Seneta [8], pp. 201) : If T is finite then $R = 1/r$, where r is the Perron-Frobenius eigenvalue (the largest eigenvalue) of T. (r is a strictly positive real number.)

Fact 3 (Seneta [8], pp. 205) : If T is positive-recurrent or null-recurrent then $R = 1$. There exists stochastic T's which are transient with $R > 1$.

Fact 4 : Let T be aperiodic and stochastic. Then, T^k converges elementwise to T^* as $k \to \infty$.

Definition (Kendall [5]) : Aperiodic and stochastic T is called geometrically ergodic if numbers M_{jk} and δ_{jk} exist such that

$$0 \leqslant M_{jk} < \infty, \; 0 \leqslant \delta_{jk} < 1 \text{ and}$$

$$|t_{jk}^{(k)} - t_{jk}^*| \leqslant M_{jk} . \delta_{jk}^k, \; k = 0,1,2, \cdots .$$

for all pairs (j,k).

Fact 5 (Kendall [5]) : An aperiodic and stochastic T is geometrically ergodic if and only if there exists δ satisfying $0 \leqslant \delta < 1$ such that

$$|t_{jk}^{(k)} - t_{jk}^*| \leqslant M_{jk} . \delta^k .$$

Fact 6 (Seneta [8], pp. 209) : Stochastic T satisfies

$$t_{ij}^{(k)} = O(\delta^k), \tag{2}$$

as $k \to \infty$, for some δ, $0 < \delta < 1$, and for some fixed pair (i,j), if and only if T has convergence parameter $R > 1$. In this case Eqn. (2) holds uniformly in delta for each pair (i,j), for only fixed δ satisfying $R^{-1} \leqslant \delta < 1$; and no $\delta < R^{-1}$ for any pair (i,j).

Fact 7 : For a stochastic transient or null-recurrent T, $T^* \equiv 0$.

Definition : A stochastic transient T which is geometrically ergodic is called geometrically transient.

Let $_{(n)}T$ denote the $(n \times n)$ northwest corner truncation of T. Let $_{(n)}R$ denote the convergence parameter of $_{(n)}T$.

Fact 8 (Seneta [8], pp. 211) : $_{(n+1)}R < {_{(n)}R}$ for all irreducible $_{(n)}T$ and $_{(n)}R \to R$ (decreasing) as $n \to \infty$, where R is the convergence parameter of T.

Fact 9 (Kendall [4]) : Let T be aperiodic and stochastic. Let $f_{ij}^{(k)}$ denote the probability that the system reaches j for the first time with the n^{th} step given that it starts from i. $f_{ij}^{(k)}$'s are called the first passage probabilities. By definition, $f_{ij}^{(0)} = 0$, for all (i,j). Fix any i. Then, T is geometrically ergodic if and only if the series

$$\sum_{k=1}^{\infty} f_{ii}^{(k)} s^k$$

has a radius of convergence greater than one.

Fact 10 (Kendall [5]) : Let T be finite, aperiodic and stochastic. Then, T is geometrically ergodic.

4. Statement of the Main Theorem and the Idea of Domination.

Now we concentrate on the Markov chain $\{X_n\}$ defined in § 2. For simplicity, we denote $p_{i(i-1)}$ by d_i. Recall that we are denoting Poisson probabilities $e^{-\lambda}.\lambda^k / k!$ by p_k.

Theorem (1) : The Markov chain $\{X_n\}$ is geometrically transient.

To prove this theorem we will consider a simpler Markov chain $\{Y_n\}$. We construct $\{Y_n\}$ from $\{X_n\}$ as follows: Choose N large enough so that

$$d_{N+1} < \min\left(\frac{p_1.(1-p)}{(2-p)^2}, \frac{(1-p)}{(2-p)}.e^{-\lambda}, \frac{(1-p)}{(2-p)}.\frac{(1-e^{-\lambda})}{2} \right), \tag{3}$$

$$d_{i+1} < d_i,\ i \geqslant N \text{ and } p_{ii} < e^{-\lambda},\ i \geqslant N. \tag{4}$$

Note that we can always find such N. (See § 2.2.) Let

$$\epsilon = \frac{(2-p)}{(1-p)}.d_{N+1}. \tag{5}$$

Let q_{ij} denote the transition probabilities of $\{Y_n\}$. We define q_{ij}'s as follows:

For $i < N$,

$$q_{ij} = p_{ij},$$

and, for $i \geqslant N$,

$$\begin{aligned} q_{ii} &= e^{-\lambda} \quad (=: f_3), \\ q_{i(i+1)} &= 1-(e^{-\lambda}+\epsilon) = (p_1-\epsilon)+\sum_{k=2}^{\infty} p_k \quad (=: f_1), \\ q_{i(i-1)} &= \epsilon \quad (=: f_2). \end{aligned} \tag{6}$$

The following Lemma will be used to prove the theorem. It also ensures that the expressions in the definition of q_{ij}'s are positive.

Lemma (1) : ϵ defined in Eqn. (5) satisfies

(1) $\epsilon < e^{-\lambda}$,

(2) $\epsilon > d_N$,

(3) $p_1 - \epsilon > 0$,

(4) $(p_1 - \epsilon).\epsilon > p_1.d_{N+1}$,

(5) $f_1 = 1-(e^{-\lambda}+\epsilon) > \epsilon$.

Proof :

(1) Obvious from Eqns. (3) and (5).

(2) From Eqn. (5),

$$\frac{\epsilon}{d_{N+1}} = 1+\frac{1}{(1-p)}$$

and from Eqn. (1)

$$\frac{d_{N+1}}{d_N} = \frac{N+1}{N}.(1-p).$$

Hence,

$$\frac{d_{N+1}}{d_N} > (1-p).$$

Now, multiplying ϵ / d_{N+1} and d_{N+1} / d_N, we get

$$\frac{\epsilon}{d_N} > 1+(1-p) > 1.$$

(3) Note from Eqns. (3) and (5) that

$$\epsilon = \frac{(2-p)}{(1-p}).d_{N+1} < \frac{p_1}{(2-p)} < p_1.$$

(4) From Eqn. (5)

$$\epsilon - d_{N+1} = \frac{d_{N+1}}{(1-p)}.$$

Hence,

$$\frac{\epsilon^2}{\epsilon - d_{N+1}} = \frac{\frac{(2-p)^2}{(1-p)^2} . d_{N+1}^2}{\frac{d_{N+1}}{(1-p)}} = d_{N+1} . \frac{(2-p)^2}{(1-p)}. \tag{7}$$

Now, from Eqn. (3), we have

$$d_{N+1} < \frac{p_1 . (1-p)}{(2-p)^2}. \tag{8}$$

Hence, using Eqns. (7) and (8), we have

$$\frac{\epsilon^2}{(\epsilon - d_{N+1})} < p_1,$$

which is equivalent to

$$(p_1 - \epsilon) . \epsilon > p_1 . d_{N+1}.$$

(5) Again, from Eqns. (3) and (5), we have

$$\epsilon = \frac{(2-p)}{(1-p)} . d_{N+1} < \frac{(1-e^{-\lambda})}{2}.$$

Hence,

$$1-(e^{-\lambda}+\epsilon) > \epsilon.$$

This completes the proof of the lemma.

The following lemma states the idea of domination and hence the purpose of introducing the Markov chain $\{Y_n\}$

Lemma (2) : If $\{Y_n\}$ is geometrically transient then so is $\{X_n\}$.

Proof : Let ${}_W f_{ij}^{(k)}$ denote the first passage probability for a Markov chain $\{W_n\}$. Suppose $\{Y_n\}$ is geometrically transient. Then, the series

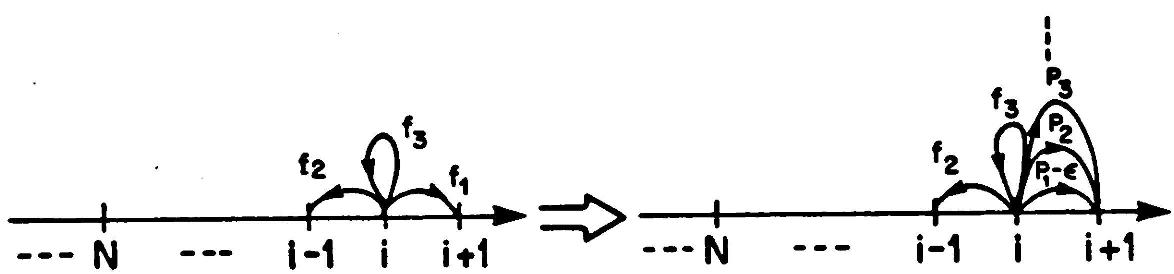
$$\sum_{k=1}^{\infty} {}_Y f_{NN}^{(k)} . s^k$$

has a radius of convergence greater than one (Fact 9, § 3). We will show that ${}_X f_{NN}^{(k)} \leqslant {}_Y f_{NN}^{(k)}$ for $k \geqslant 1$. Then, due to Fact 9, § 3, this will clearly prove the lemma.

We give a pathwise domination argument. Note that, for $i \geq N$, the probability of a forward jump for $\{Y_n\}$ is

$$f_1 = (p_1 - \epsilon) + \sum_{k=2}^{\infty} p_k$$

Envisage such transitions as made up of countable possible steps with weights $(p_1-\epsilon)$ and p_k, $k \geq 2$ respectively. We will refer to them by arcs in the following. The arc with the weight $(p_1-\epsilon)$ will be referred to as arc 1 and the arc with the weight p_k, $k \geq 2$ will be referred to as arc k. This decomposition is shown in Figure (1).



Decomposition of Transition Probabilities of $\{Y_n\}$.

Figure (1)

Consider a realization ω_X of $\{X_n\}$ that contributes to ${}_X f_{NN}^{(k)}$. We now correspond to ω_X a realization ω_Y of $\{Y_n\}$ (with specific arcs specified) by the following rules.

ω_Y imitates the jumps of ω_X except when ω_X takes a forward jump from $i \geq N$. If a forward jump in ω_X from $i \geq N$ is of size one then ω_Y takes a forward jump via arc 1. On the other hand, if a forward jump in ω_X is of size larger than 1 (say Δ) then ω_Y takes a forward jump via arc Δ and to balance the time to return to N it adds $\Delta-1$ self-loop transitions at $N+1$ just before the last transition into N. Following are some illustrations of this correspondence.

$$(1)\ \omega_X = \{N \to N+1 \to N+2 \to N+1 \to N+5 \to N+6 \to N+5 \to N+4 \to N+3 \to N+2 \to N+1 \to N\},$$

$$\omega_Y = \{N \to N+1 \to N+2 \to N+1 \to (arc\ 5) \to N+2 \to N+3 \to N+2 \to N+1 \to N+1 \to N+1 \to N+1 \to N\}.$$

$\to (arc\ 5) \to$ in the above example indicates that the transition from $N+1$ to $N+2$ in ω_Y is via arc 5. If not specified, transitions from $i \geqslant N$ to $i+1$ is via arc 1.

$$(2)\ \omega_X = \{N \to N-1 \to N-1 \to N+3 \to \cdots\},$$

$$\omega_Y = \{N \to N-1 \to N-1 \to N+3 \to \cdots\}.$$

Observe that ω_Y, thus constructed, contributes to ${}_Y f_{NN}^{(k)}$. Also observe that ω_Y's (with arcs specified) constructed above have one-to-one correspondence with ω_X's. We want to show that $P_X(\omega_X) < P_Y(\omega_Y)$ for each ω_X contributing to ${}_X f_{NN}^{(k)}$. Instead of giving a long descriptive proof, we make the following salient observations. They make the gaps in the proof obvious.

(O1) If ω_X starts off by going to $N-1$, then ω_Y has more weight for the first step, since $\epsilon > d_N$, from Lemma (1), (2), § 4. After the first step difference in weights occur only if ω_X visits the set $(N+1, N+2, \cdots)$.

(O2) Since $f_2 = \epsilon < e^{-\lambda}$, $d_N < \epsilon$ (Lemma (1), (1) and (2), § 4) and $d_{i+1} < d_i, i \geqslant N$ (Eqn. (4)), we have $d_i < e^{-\lambda}, i \geqslant N$. Hence the addition of the self-loop transitions instead of having to come back increases the contribution to $P_Y(\omega_Y)$.

(O3) For $\omega_X = \{N \to N+1 \to N\}$ and the corresponding $\omega_Y = \{N \to N+1 \to N\}$ (the transition N to $N+1$ ω_Y is via arc 1), we should compare the weights $p_{N(N+1)}.d_{N+1}$ and $(p_1-\epsilon).\epsilon$. But from Lemma (1), (2) and (4), and Eqns. (1) and (4), we have

$$(p_1-\epsilon).\epsilon > p_1.d_{N+1} > p_{N+1}.d_{N+1}.$$

Now, since $d_{N+1} > d_i, i > N+1$, we also have $(p_1-\epsilon).\epsilon > p_{i(i+1)}.d_{i+1}$, $i > N$. This explain why ω_Y has more weight for a forward jump via arc 1 from $i \geqslant N$ and the required jump to N as compared to that for ω_X

Above observations make the claim that ${}_Y f_{NN}^{(k)} \geqslant {}_X f_{NN}^{(k)}$, $k \geqslant 1$, quite clear. It is well-known that $\{X_n\}$ is transient (e.g., see [2] and [6]). Hence, Fact 9, § 3, shows that if $\{Y_n\}$ is geometrically transient then so is $\{X_n\}$. This completes the proof of the lemma.

5. Proof of the Theorem (1).

Because of Lemma (2), it is sufficient to show that $\{Y_n\}$ is geometrically transient. To show this we further dominate $\{Y_n\}$ by a finite Markov chain $\{Z_n\}$ that evolves in the set $\{0,1,2, \cdots ,N+1\}$. To define $\{Z_n\}$, we modify $\{Y_n\}$ as follows: Replace each jump from $i \leqslant N+1$ to $j > N+1$ by a jump from i to $N+1$ with the same weight as for $\{Y_n\}$. More precisely, r_{ij}, $0 \leqslant i,j < N+1$, the transition probabilities of $\{Z_n\}$, are defined as follows:

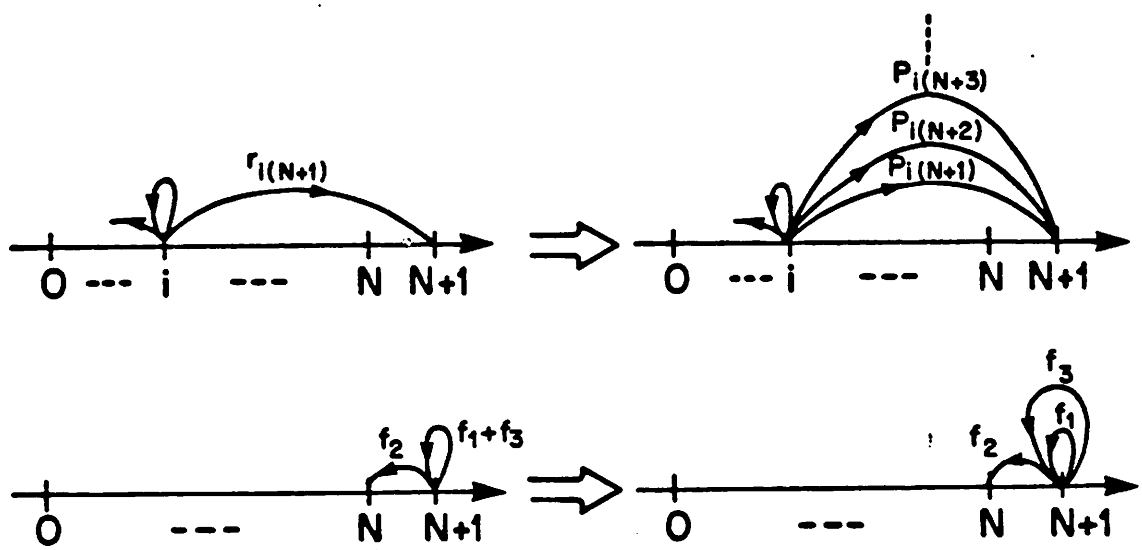
$$\begin{aligned}
&r_{ij} = q_{ij}, \text{ if } i \leqslant N \text{ and } j \leqslant N, \\
&r_{i(N+1)} = \sum_{j=N+1}^{\infty} q_{ij} \ \ (= \sum_{j=N+1}^{\infty} p_{ij}), \text{ if } i \leqslant N-1, \\
&r_{N(N+1)} = q_{N(N+1)} = f_1, \\
&r_{(N+1)N} = q_{(N+1)N} = f_2, \\
&r_{(N+1)(N+1)} = q_{(N+1)(N+1)} + q_{(N+1)(N+2)} = f_1 + f_3.
\end{aligned} \tag{9}$$

From Fact 10, § 3, we know that $\{Z_n\}$ is geometrically ergodic and hence, from Fact 9, § 3, we also know that the series

$$\sum_{k=1}^{\text{if}} {}_Z f_{NN}^{(k)} s^k$$

has the radius of convergence larger than 1. From Lemma (1), (5), § 4, $\{Y_n\}$ is clearly transient. Then, due to Fact 9, § 3, it is sufficient to show that ${}_Z f_{NN}^{(k)} \geqslant {}_Y f_{NN}^{(k)}$, $k \geqslant 1$ (see the definition of geometric transience, § 3).

As in the proof of Lemma (2), § 4, we give a pathwise argument to show that for each $k \geqslant 1$, ${}_Z f_{NN}^{(k)} \geqslant {}_Y f_{NN}^{(k)}$. For the Markov chain $\{Z_n\}$, the transition from $i < N$ to $N+1$, with the probability $\sum_{j=N+1}^{\infty} p_{ij}$, can be decomposed into arcs of weights p_{ij}, $j \geqslant N+1$. (Recall that $\{Y_n\}$ has the same jump distributions as $\{X_n\}$ for jumps from $i < N$.) The transition of $\{Z_n\}$ from $N+1$ to $N+1$, with the probability (f_1+f_3), can be decomposed into arcs of weights f_1 and f_3 respectively. These decompositions of transition probabilities is shown in Figure (2).



Decomposition of Transition Probabilities of $\{Z_n\}$.

Figure (2)

Consider a realization ω_Y of $\{Y_n\}$ that contributes to $_Yf_{NN}^{(k)}$. Now, correspond to ω_Y a realization of $\{Z_n\}$, ω_Z (with arcs specified), as follow: ω_Z imitates ω_Y for all the jumps which are from $i \leqslant N+1$ to $j \leqslant N+1$. For each jump of ω_Y from $i \leqslant N+1$ to $j > N+1$, ω_Z jumps from i to $N+1$ via the arc with the weight p_{ij} and $j-(N+1)$ self-loop transitions at $N+1$ via the arc with the weight f_1. The self-loop transitions compensate the backward jumps to $N+1$ of ω_Y. For each jump of ω_Y from $i > N+1$ to $i+1$ and $i-1$, ω_Z jumps from $N+1$ to $N+1$ via the arc with the weight f_1. Finally, for each jump of ω_Y from $i \geqslant N=1$ to i, ω_Z jumps from $N+1$ to $N+1$ via the arc with the weight f_3. Observe that ω_Z's defined above (with arcs specified) have one-to-one correspondence with ω_Y's.

Since, in the above construction of ω_Z's from ω_Y's, we only substitute some of the jumps of the weight $f_2 = \epsilon$ with the jumps of the weight f_1 and $f_1 > \epsilon$ (Lemma (1), (5), § 4), we have $P_Z(\omega_Z) \geqslant P_Y(\omega_Y)$ for each ω_Y contributing to ${}_Y f_{NN}^{(k)}$. This shows that ${}_Z f_{NN}^{(k)} \geqslant {}_Y f_{NN}^{(k)}, k \geqslant 1$. This completes the proof of the theorem.

6. Consequence of Theorem (1).

6.1. New Performance Measure.

Consider the Markov chain $\{X_n\}$ and P, its transition matrix, which describe the evolution of the number of backlogged users in the uncontrolled slotted Aloha protocol. Let ${}_{(n)}P$ denote the $(n \times n)$ northwest corner truncation of P. Let ${}_{(n)}R$ and β_n denote the convergence parameter (see § 3) and the Perron-Frobenius eigenvalue respectively of ${}_{(n)}P$.

Theorem (2) : $\beta_n \to \beta$ (increasing), where $0 < \beta < 1$.

Proof : Observe that P and ${}_{(n)}P$ are aperiodic and irreducible. From Fact 2, § 3, we know that ${}_{(n)}R = 1/\beta_n$. Also, from Fact 8, § 3, we have ${}_{(n)}R \to R$ (decreasing), where R is the convergence parameter of P. Since, from Theorem (1), we know that P is geometrically transient, Fact 6, § 3, implies that $R > 1$. Define $\beta := 1/R$. Then, clearly $\beta_n \to \beta$ (increasing). Also, from Fact 1, § 3, we have $R < \infty$. Hence, $0 < \beta < 1$. This completes the proof of the theorem.

The above result is counterintuitive because the largest eigenvalue of P is 1.

Let v_n denote a right eigenvector corresponding to β_n. From the Perron-Frobenius theory (e.g., see [8], pp. 4, 21 and 22) it follows that the components of v_n are strictly positive and v_n's are unique up to scalar multiples. Let $\bar{v}_n$ denote the unique scalar multiple of v_n which has the sum of its components equal to 1. Since

$${}_{(n)}P\bar{v}_n = \beta_n \bar{v}_n ,$$

we have

$$_{(n)}P^k \bar{v}_n = \beta_n^k \bar{v}_n \text{, for } k \geq 1.$$

Let $t_{[n]}$ denote the random time before the first visit to the set $(n+1, n+2, \cdots)$. Let $f_{[n]k}(v)$ denote the probability that $t_{[n]} \leq k$ given that the initial distribution is v. Then,

$$f_{[n]k}(\bar{v}_n) = 1-\beta_n^k$$

and

$$E_{\bar{v}_n}\{t_{[n]}\} = \sum_{k=1}^{\infty} k.(\beta_n^{k-1}-\beta_n^k) = \frac{1}{(1-\beta_n)},$$

where $E_\pi\{\ \}$ denotes the expected value given that the initial distribution is π.

From Theorem (2), we readily observe that $E_{\bar{v}_n}\{t_{[n]}\} \to 1/(1-\beta)$ (increasing) as $n \to \infty$ and $1/(1-\beta) < \infty$. We define $1/(1-\beta)$ as a new performance measure of the Aloha protocol. Roughly, $1/(1-\beta)$ "represents" the average time before the system starts "drifting" towards infinity. Closer is the value of β to 1, more stable is the protocol. It is in this sense that $1/(1-\beta)$ quantifies instability of the protocol. Observe that if P were to be recurrent, β would be equal to 1 and this matches the notion of stability.

6.2. Conjecture about the applicability of the Quick Simulation Method of Cottrell et al. [1].

The exit time $t_{[n]}$ starting with the initial distribution $\bar{v}_n$ is termed quasi-stationary exit time in literature (e.g., see Keilson [3], pp. 90).

Let $E_0\{t_{[n]}\}$ denote the expected value of $t_{[n]}$ given that the system starts from 0. It is possible to show by a coupling argument that

$$E_{\bar{v}_n}\{t_{[n]}\} \leq E_0\{t_{[n]}\}.$$

We believe that the above upper bound for $E_{\bar{v}_n}\{t_{[n]}\}$ is tight in the following sense.

It is well-known that the Aloha protocol has two equilibrium points, n_s, a stable one, and n_c, an unstable one (e.g., see [1], pp. 914, and [6]). In the vicinity of n_s the "drift" of the process is towards n_s while in the vicinity of n_c the "drift" of the process is away from n_c. In particular, as the retransmission probability, p, goes to 0, $n_c \to \infty$. We believe that

$$\lim_{p \to 0} \frac{E_{\bar{v}_{n_c}}\{t_{[n_c]}\}}{E_0\{t_{[n_c]}\}} = 1.$$

Now, for small p, $E_0\{t_{[n_c]}\}$ can be estimated using the Quick Simulation Method [1], which relies on the theory of Large Deviation. Since, it may be difficult to calculate v_{n_c}, the above conjecture, makes a useful connection between $1/(1-\beta_{n_c})$ and $E_0\{t_{[n_c]}\}$, where the latter can be estimated efficiently by the the Quick Simulation Method.

7. Conclusions.

In this paper we have established that the embedded Markov chain describing the backlog of the uncontrolled slotted Aloha protocol is geometrically transient. As a consequence of this, we have shown that the largest eigenvalues of the northwest truncations of the original transition matrix P tend towards a constant less than 1. This, in turn, results in the definition of a meaningful performance measure for the protocol. We have also described a conjecture that makes a connection between two different exit times of the protocol. This, in turn, makes a connection to the Quick Simulation Method that is inspired by Large Deviation theory.

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