GLOBAL LYAPUNOV FUNCTION THEORY

by

Efthimios Kappos and Shankar Sastry

Memorandum No. UCB/ERL M86/81

8 October 1986

GLOBAL LYAPUNOV FUNCTION THEORY

by

Efthimios Kappos and Shankar Sastry

Memorandum No. UCB/ERL M86/81

8 October 1986

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

GLOBAL LYAPUNOV FUNCTION THEORY

by

Efthimios Kappos and Shankar Sastry

Memorandum No. UCB/ERL M86/81

8 October 1986

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

GLOBAL LYAPUNOV FUNCTION THEORY *

Efthimios Kappos, Shankar Sastry,

Electronics Research Laboratory and

Department of Electrical Engineering and Computer Science,

University of California, Berkeley.

October 1986.

ABSTRACT

In this paper, we give a global theory of Lyapunov functions for a class of vector fields. This class is defined in Euclidean space by analogy to Morse functions on a compact manifold. Our class is more general in that we allow a wider class of attractors. The vector fields in that class are dissipative, in a sense that is made precise using a generalization of asymptotic stability. It appears reasonable to state that the dissipative property of a vector field is equivalent to the existence (globally) of Lyapunov function (cf. the work of J.Willems [15]).

Our theory is more complete than previous works in that it works out the classification of all possible global Lyapunov functions, based on the diagram of the types of orbits of the vector field (called here the orbit diagram).

We include a general existence theorem for Lyapunov surfaces (i.e. Lyapunov function level sets), since we believe that this is the more natural concept for an existence theory. The proof is based on purely geometric arguments that exploit the foliation of a region of attraction by the orbits in it.

* Research supported in part by NSF Grant #ECS-8308330.
Submitted for Publication to the IEEE Transactions on Circuits and Systems.

Introduction

A Lyapunov function for a dynamical system is a generalization of the energy function of a classical mechanical system. It is a *scalar function* V on the state space with the property that it is non-increasing on trajectories of the system, i.e. $\frac{dV}{dt} \le 0$. When the system is dissipative, the Lyapunov function is strictly decreasing, $\frac{dV}{dt} < 0$.

In system theory, Lyapunov functions are usually local; in this case the existence of a local Lyapunov function is equivalent to the local stability of an equilibrium. Only if the equilibrium is globally stable is the Lyapunov function also defined globally.

In general, each stable equilibrium is stable in a region of state-space that we call its *region of attraction* . This is the natural domain of definition of a Lyapunov function and in power systems, Lyapunov functions have been used to estimate the region of attraction of an equilibrium.

In the mathematical theory of dynamical systems, on the other hand, global Lyapunov functions are considered and have proved very useful in the study of global dynamics for certain classes of flows (see F.W.Wilson [16],[17], Franks [5] and Pugh and Shub [13]). In particular, just as Morse theory helps to describe the topological structure of manifolds, global Lyapunov functions are used to describe homological properties of dynamical systems, as in Franks.

Our research was developed with a motivation very different from the above. It generalizes global Lyapunov theory in two directions: first, the attracting sets are allowed to be arbitrary compact manifolds. Second, and more importantly, the class of Lyapunov functions we consider is more general: roughly speaking, the level sets of Lyapunov functions considered before tend, asymptotically, to the boundary of the regions of attraction of the critical elements. In our research, these level sets are transverse to the boundary and hence are not contained in a single region of attraction.

It is important to generalize the notion of an isolated, non-degenerate equilibrium point. The natural concept that makes an attractor isolated and non-degenerate is that of *asymptotic stability.* Furthermore, we allow not only equilibrium points, but general compact manifolds which we call *attracting sets* . As in the case of Morse-Smale vector fields, the non-attracting ω- and α-limit sets are required to be *hyperbolic equilibria or closed orbits* .

To get a complete qualitative picture for the class of dynamics to be defined, we impose a global condition of *dissipativeness*. This allows *orbit diagrams* to be drawn for the flow, assuming a no-cycle condition is satisfied.

The object of section 2 is to give a generalization of Morse functions: these are the (strict) *Lyapunov functions* for the flow. In contrast to the Morse function giving a gradient flow, a Lyapunov function is *not unique* for a given flow. It is one of the main contributions of this research that the precise degree of non-uniqueness of Lyapunov functions is found. This leads naturally to a complete classification of all the possible Lyapunov functions of a given vector field.

The motivation for this work comes from *large deviation theory* (see [3],[14]). It turns out that the class of dynamics we define here is a natural one for obtaining global results in large deviations. In particular, in some special cases desribed in Kappos,Sastry [8] the variational problem in the small-noise asymptotics is actually solved by a Lyapunov function.

This paper is structured as follows: in section 1 we define a class of dissipative dynamics for which Lyapunov theory is appropriate. In section 1.1 we define attractors in a way that generalizes naturally the definition of asymptotically stable equilibria and in section 1.2 we use *orbit diagrams* to define globally the class of dynamics we want. Section 2.1 describes Lyapunov surfaces. These are seen to be global analogs of the transverse neighborhoods to the flow obtained from the flow-box theorem (see section 2.1.1). Section 2.2 derives a botany of Lyapunov functions. The most direct kind that follows from the Lyapunov surface concept is a Lyapunov function defined in the region of attraction of a single attracting set. This is the type of Lyapunov function familiar from system theory and power systems.

Using the results of section 2.1.3 on Lyapunov surfaces for saddle critical elements, we are led to global Lyapunov functions whose level sets propagate across saddles in a prescribed order. This allows the classification scheme of section 2.2.2. Section 2.3 draws some conclusions.

1. A Class of Dynamical Systems

1.1. Attracting Sets

Let K be a compact, connected, q-dimensional submanifold of $\mathbf{R}^n$, $q<n$. Define the distance function to K:

$$d(x,K)=\inf_{y\in K}|x-y|$$

Since K is compact, d is well-defined and continuous. Now define, for $\varepsilon>0$, the set:

$$N_\varepsilon^K=\{x\in \mathbf{R}^n | d(x,K)<\varepsilon\}$$

For ε small enough, we can take this to be an ε- *tubular neighborhood* of K in $\mathbf{R}^n$.

More precisely, a *tubular neighborhood* of the submanifold K of $\mathbf{R}^n$ is a pair $(f,\mathbf{B})$, where $\mathbf{B}=(p,E,K)$ is a vector bundle over K and f is an embedding of E in $\mathbf{R}^n$ such that:

(a) f restricted to K is the identity map

(b) $f(E)$ is an open neighborhood of K in $\mathbf{R}^n$.

We shall, by abuse of notation, refer to $f(E)$ as the tubular neighborhood of K (see Hirsch [6] for details).

Note that the fibre over any $x\in K$ can be taken to be the normal space to $T_xK = \mathrm{N}_xK$ which, in $\mathbf{R}^n$, is identified with $(T_xK)^\perp$ (see the proof of Theorem 5.1 in [6]). In this case, $(f,\mathbf{B})$ is called a *normal tubular neighborhood* (n.t.n.) of K and we can take $f(E)$ to be the set N_ε^K, for some small ε.

We use these neighborhoods as the open sets in the definition of an attracting set.

Definition 1:

The manifold K is an **attracting set** *if: given a normal tubular neighborhood* N_δ^K, $\delta>0$ *of K, we can find a* $\varepsilon>0$ *and a n.t.n.* N_ε^K *such that:*

(i) for all $x\in N_\varepsilon^K$, $\phi_t x\in N_\delta^K$ *for all* $t\geq 0$.

(ii) for all $x \in N_\varepsilon^K$, $d(x,K) \to 0$ *as* $t \to 0$.

Call A(b) the set of attracting sets of the vector field b.

Remarks:

1) If $K=\{x\}$, we are back to the definition of an asymptotically stable equilibrium. If K is a limit cycle (the only possible 1-dimensional compact,connected manifold), then we have defined an asymptotically stable limit cycle.

2) In higher dimensions, when $q \geq 0$, the attracting set contains more than one orbit of b (an example is a two-dimensional torus). We make no assumptions on the behavior of b on K. In particular, the flow of b may be conservative when restricted to K. However, we require K to be of dimension strictly less than that of the state space, as should be the case for dissipative systems (see Birkhoff [4]).

3) If $K=\{x\}$ is a hyperbolic attractor, then it is certainly an attracting set, by the Grobman-Hartman theorem.

Definition 2:

The **region of attraction** R_K *of the attracting set K is defined as the set of points* $x \in \mathbf{R}^n$ *satisfying:*

(i) $d(\phi_t x, K) \to 0$ *as* $t \to +\infty$.

(ii) $x \notin K$.

Note that for all $x \in R_K$, $b(x) \neq 0$. We have:

Lemma 1:

If $x \in R_K$, *the orbit of* x : $\{\phi_t x \ , t \in \mathbf{R}\}$ *belongs to* R_K. *Also,* R_K *is an open set and there is a normal tubular neighborhood of K contained in it.*

Proof:

If $x \in R_K$ and $y = \phi_{t'} x$ for some t',

$$d(\phi_t y, K) = d(\phi_{t+t'} x, K)$$

and the first part follows. Now let N_ε^K, N_δ^K be as in definition 1. If $x \in R_K$, there is a time $T>0$ such that $\phi_T x \in N_\varepsilon^K$. Since ϕ_T is a diffeomorphism, we can find a neighborhood U of x such that $\phi_T U \subset N_\varepsilon^K$. It is now clear that for all $y \in U$, $y \in R_K$ and hence R_K is open.

Finally, it is obvious that the normal tubular neighborhood N_ε^K is contained in R_K. ■

1.2. The Class of Flows Considered

First we recall some definitions (see [12] for details) (throughout, b(x) is a complete vector field and $\phi_t x$ is its flow).

Definition 3:

A point $x \in \mathbf{R}^n$ *is said to be* **wandering** *if there is a neighborhood V of x and a time T>0 such that:* $\phi_t x \cap V = \emptyset$ *for* $|t| > T$. *Otherwise, the point is called* **nonwandering** . *Write* $\Omega(b)$ *for the set of nonwandering points of the vector field b, for all points outside the set of attracting sets* $\mathrm{A}(b)$.

Definition 4:

Call a set a **critical element** *if it is a zero of b or a closed orbit (a zero will also be called an equilibrium). A point p is an* α **-limit point** *of the point x if there is a sequence of times* $t_k \rightarrow -\infty$ *such that* $\phi_{t_k} x \rightarrow p$. *It is an* ω **-limit point** *of x if the same holds for a sequence of times* t_k *going to* $+\infty$. *If for the point x,* $|\phi_t x| \rightarrow +\infty$ *as* $t \rightarrow -\infty$ $(+\infty)$ *we say* ∞ *is an* α*-limit point (* ω*-limit point) of x. Call* $L_\alpha(b)$ *and* $L_\omega(b)$ *the set of* α*- and* ω*-limit points of b, for x outside* $\mathrm{A}(b)$ *and belonging to a bounded orbit.*

Note that if $p \in L_\alpha \cup L_\omega$, then p is a non-wandering point. Thus, for a general flow, $\Omega(b) \supset L_\alpha(b) \cup L_\omega(b)$. Also note that $\mathrm{A}(b) \subset L_\omega(b)$.

Definiton 5:

An equilibrium is called **hyperbolic** *if its linearization has no eigenvalues with zero real part. A closed orbit is* **hyperbolic** *if the linearization of its local Poincare map has no eigenvalues of modulus one.*

The stable manifold of a hyperbolic critical element σ is denoted by $W^s(\sigma)$. The unstable manifold of σ by $W^u(\sigma)$. For the existence and properties of the stable and unstable manifolds of critical elements refer to eg. Hirsch et.al.[7].

Definiton 6:

The vector field b on $\mathbf{R}^n$ *is a* **dissipative Morse-Smale** *vector field if:*

(i) there is a finite number of attracting sets and a finite number of critical elements, all hyperbolic.

(ii) (a) the set of ω*-limit points in* $\mathbf{R}^n - \mathrm{A}(b)$ *is equal to* $L_\omega(b)$.

(ii) (b) the set of α*-limit points in* $\mathbf{R}^n - \mathrm{A}(b)$ *is equal to* $L_\alpha(b) \cup \{\infty\}$.

(iii) $\Omega(b) = L_\omega(b) \cup L_\alpha(b) \cup \{\infty\}$.

(iv) if σ_1 *and* σ_2 *are critical elements, then* $W^u(\sigma_1)$ *and* $W^s(\sigma_2)$ *are transversal.*

Remark:

It is a consequence of condition (ii) (a) that every *future* trajectory $\{\phi_t x\}_{t\geq 0}$ of a dissipative Morse-Smale vector field is bounded. Something stronger is actually true: almost all points in $\mathbf{R}^n$ have their ω-limit sets in A(b). This is because the stable manifold of a critical element that is not an attractor has dimension strictly less than n and hence has Lebesgue measure 0 in $\mathbf{R}^n$. The *past* trajectories $\{\phi_t x\}_{t\leq 0}$ of b, however, generically go to ∞ or to a critical element of index n.

Definition 7:

Given critical elements or attracting sets σ_1 , σ_2 *introduce the* **relation** < *by:* $\sigma_1 < \sigma_2$ *if there is an orbit* γ *that has* σ_1 *as its* ω*-limit point and* σ_2 *as its* α*-limit set.*

A set of critical elements satisfies the **no-cycle condition** *if we cannot find distinct critical elements* $\sigma_{i_1}, \sigma_{i_2}, .., \sigma_{i_m}$ *such that:* $\sigma_{i_1} < \sigma_{i_2} < .. < \sigma_{i_m} < \sigma_{i_1}$.

Now construct the **orbit diagram** of a dissipative Morse-Smale vector field that satisfies the no-cycle condition as follows:

Distinguish (n+1) levels, according to the index k of a critical element (i.e. the dimension of its unstable manifold). Thus $0<k<n$. The attracting sets are at the index-0 level. List all the critical elements and attracting sets at each level. Include the repellor at ∞ at the index-n level. Call these the **nodes** of the orbit diagram.

Connect node σ to node τ by an arrow pointing from σ to τ if $\tau<\sigma$.

Lemma:

$\tau<\sigma$ if and only if index τ < index σ.

Proof:

This follows from transversality, by keeping track of the dimensions of the stable and unstable manifolds of the critical elements. ■

Note that the above lemma shows there are no *homoclinic orbits* (i.e. $\sigma<\sigma$ does not happen).

We are now in a position to define the class of dynamical systems for which the global Lyapunov theory of chapter 2 will be developed.

The Class $D(\mathbf{R}^n)$:

A complete vector field b on $\mathbf{R}^n$ belongs to the class $D(\mathbf{R}^n)$ if:

(a) b is a dissipative Morse-Smale vector field and

(b) the no-cycle condition is satisfied.

From now on, the vector fields under consideration will be assumed to be in $D(\mathbf{R}^n)$.

Discussion of the Class $D(\mathbf{R}^n)$:

$D(\mathbf{R}^n)$ is a general enough class of dynamics to include many familiar examples of interest: for example, the Josephson junction models away from bifurcation, multi-machine models in power systems etc. The sense in which a vector field in $D(\mathbf{R}^n)$ is dissipative is consistent with previous attempts to formalize this notion (see G.Birkhoff [4] and, more recently, J.Willems [15]).

As Birkhoff requires ([4],pp.31-32), asymptotically as $t\to\infty$, the motion of a dissipative system takes place close to bounded sets of dimension lower than that of the state space. On such a set, the motion is assumed conservative. In our formulation, the attracting sets are compact manifolds of dimension strictly less than n.

According to Willems, dissipativeness has to do with the existence of a function which decreases with the forward (in time) evolution of the system dynamics. This is captured in our theory by the concept of a global Lyapunov function. Our theory goes further than that in specifying exactly how many such functions we can find and how they are related.

The members of $D(R^n)$ are *structurally stable* In fact, if we compactify the state space using the point at infinity, we see that the class we defined is very similar to the Morse-Smale vector fields, except for the existence of the more general class of asymptotic attractors.

In Figure 1.1, we give some simple examples of orbit diagrams.

2.1. Existence of Lyapunov Surfaces

2.1.1. Basic Definitions and Properties

The region of attraction of an attracting set K is foliated in an obvious way by the 1-dimensional orbits of the vector field b: each x in R_K belongs to a unique orbit and the vector field is non-singular in all of R_K.

The global Lyapunov theory we are developing gives foliations dual to the above: there are foliations of R_K with leaves which are (n-1)-dimensional submanifolds (hypersurfaces), which we call Lyapunov surfaces and which are transverse to the 1-dimensional foliation by the orbits of b.

The construction of Lyapunov surfaces has two steps: first, relying on the flow-box theorem, we get local Lyapunov surfaces. Then, the properties of the attracting set K are used to patch together the local surfaces to obtain a Lyapunov surface intersecting all orbits of b in R_K.

Definition 1:

A **Lyapunov surface** S *for the vector field b is a hypersurface of* $\mathbf{R}^n$*, bounded as a subset of* $\mathbf{R}^n$*, with the property that at all points* $x \in S$*:*

$$<b(x)> + T_x S = T_x \mathbf{R}^n$$

(i.e. S *is transverse to the flow of b) and such that each orbit of b intersects* S *at most once.*

We shall be interested in Lyapunov surfaces that are contained in R_K, for $K \in A(b)$. In this case we have:

Definition 2:

A Lyapunov surface S *is* **complete for the attracting set** *K if* S *is contained in* R_K *and if all orbits of b in* R_K *intersect* S *(exactly once).*

A Lyapunov surface may have more than one connected component. Each component of S has an orientation induced on its normal bundle *N*S by the flow: we say that the vector $n(x) \in N_x S$ points *inwards* if $b(x) \cdot n(x) > 0$.

Every non-singular point x of the vector field b has a Lyapunov surface locally: just find a hypersurface whose tangent space at x is transverse to $\langle b(x) \rangle$; then a neighborhood of x on the hypersurface is transverse to the vector field by openness of transversality.

We shall need the above result in a stronger form, where we simultaneously rectify the vector field around x. This is given by the flow-box theorem (for a proof, see Arnol'd [2],p.227):

Theorem 1 (Flow-Box):

Let b be a C^r vector field. $(r \geq 1)$. Let $x \in \mathbf{R}^n$ be such that $b(x) \neq 0$. Then there is a C^r-diffeomorphism ψ mapping a neighborhood U of x onto an open ball $B_\delta(0) \subset \mathbf{R}^n$, with $\psi(x)=0$ and the vector field b to the constant vector field:

$$\psi_* b(\psi^{-1}(y)) = e_1(y) \quad , \; y \in B_\delta(0)$$

where $\{e_1, \ldots, e_n\}$ is a standard Euclidean basis for $T_0\mathbf{R}^n$.

Corollary 1:

Let $b(x) \neq 0$; then there exists a Lyapunov surface containing x.

Proof:

Let $\bar{V} = \{y \in B_\delta(0), y_1 = 0\}$ where $B_\delta(0)$ is as above. Then $V = \psi^{-1}(\bar{V})$ is the desired Lyapunov surface.

■

A basic property of Lyapunov surfaces is that they can be moved along the flow in an arbitrary manner, while remaining transverse to the flow. We can use the diffeomorphisms ϕ_t of the flow or general maps that move each point on the Lyapunov surface by a variable amount along its orbit.

We summarize the above in the two basic lemmas:

Lemma 1:

Let S *be a Lyapunov surface and* $t \in \mathrm{R}$ *be given. Then the image of* S *under* ϕ_t *is again a Lyapunov surface.*

Lemma 2:

Let a *be a smooth, real-valued function on* S. *Define the map* χ *from* S *to* R_K *by:*

$$\chi(x)=\phi(\mathrm{a}(x),x)$$

Then the image of S *under* χ *is again a Lyapunov surface.*

Proof of Lemma 1:

Let i be the embedding map of S. Then $\phi_t \circ i$ is the embedding map of $\phi_t(\mathrm{S})$ and it is clear that $\phi_t(\mathrm{S})$ is a hypersurface, since ϕ_t is a diffeomorphism.

To prove transversality, choose a basis $\{e_1(x),...,e_{n-1}(x)\}$ for $T_x\mathrm{S}$, viewed as a subspace of $T_x\mathrm{R}^n$, for $x \in \mathrm{S}$. Since $b(x)$ is trasverse to S , $\{e_1(x),...,e_{n-1}(x),b(x)\}$ is a basis for $T_x\mathrm{R}^n$. The vectors $\{(T_x\phi_t)(e_1(x)),...,(T_x\phi_t)(e_{n-1}(x)),(T_x\phi_t)(b(x))\}$ are linearly independent since $T_x\phi_t$ is an isomorphism. Furthermore, we have the identity:

$$(T_x\phi_t)b(x)=b(\phi_t x)$$

It suffices to show that $(T_x\phi_t)(e_i(x)) \in T_{\phi_t(x)}\phi_t(\mathrm{S})$. Since:

$$T_p(\phi_t \circ i)=T_{i(p)}\phi_t \circ T_p i \ , \ p \in \mathrm{R}^{n-1} \ , \ i(p)=x \in \mathrm{S}$$

if $e_i(x)=(T_p i)v$ for some $v \in T_p\mathrm{R}^{n-1}$, we have:

$$T_p(\phi_t \circ i)(v)=T_{i(p)}\phi_t \circ T_p i(v)=$$

$$=T_{i(p)}\phi_t \circ e_i(x)=(T_x\phi_t)(e_i(x))$$

and so $(T_x\phi_t)(e_i(x))$ is in $T_p(\phi_t \circ i)\mathrm{R}^{n-1}=T_{\phi_t(i(p))}\phi_t(\mathrm{S})$. ■

Proof of Lemma 2:

We distinguish two cases:

$$\text{(i) } \chi(x)\neq 0 \text{ and (ii) } \chi(x)=0.$$

Case (i): Pick a neighborhood U of x in S such that χ is non-zero on U. Rescale the vector field b locally:

$$b'(z)=\chi(y)b(z)\ ,\ y\in U\ ,\ z\in \text{orbit of } y$$

It is easily seen that the map defined in the lemma is the map $\phi'_1|_U$ where ϕ' is the flow of b'. By Lemma 2.6, $\phi'_1 U$ is transverse to b' and therefore to b.

Case (ii): Pick a neighborhood U of x and a time τ such that $\tau>\sup_{x\in U}\chi(x)$. $\phi_\tau U$ is tranverse to b at x, by Lemma 2.2. If we define $\chi'(z)=\chi(\phi_{-\tau}z)-\tau$ for $z\in\phi_\tau U$, then $\chi'(z)\neq 0$ and $\phi(\chi(x),x)=\phi(\chi'(\phi_\tau x),\phi_\tau x)$. We are now back to case (i). ■

2.1.2. Complete Lyapunov Surfaces for Attracting Sets

Consider the equivalence relation ~ in $\mathbf{R}^n$:

$$x \sim y \Leftrightarrow \exists\, t \in \mathbf{R} \text{ such that } y = \phi_t\, x$$

In the region of attraction R_K of an attracting set K, this equivalence will yield a quotient space Q that is a compact manifold and is diffeomorphic to any complete Lyapunov surface for K. As a result, R_K is diffeomorphic to $Q\times\mathbf{R}$ and Lyapunov functions are easily obtained from functions on $Q\times\mathbf{R}$.

Theorem: (Existence of Complete Lyapunov Surfaces for Attracting Sets)

Let K be an attracting set for the vector field $b\in\mathbf{D}(\mathbf{R}^n)$. Let an open neighborhood of K be given in R_K.

Then there is a complete Lyapunov surface for K in that neighborhood.

Corollary:

The quotient space Q of R_K under the equivalence relation ~ is a compact manifold. Moreover, R_K is diffeomorphic to the product space $Q\times\mathbf{R}$.

Proof:

There is no loss in generality in assuming that the open neighborhood of K is an ε–normal tubular neighborhood N_ε^K, such that $\overline{N_\varepsilon^K}\subset R_K$. The set $\partial N_\varepsilon^K=\{x{:}d(x,K)=\varepsilon\}$ is closed and bounded and hence compact. It is in fact a manifold, diffeomorphic to $K\times S^{n-q-1}$ (Where q is the dimension of K).

The equivalence relation ~ in R_K yields the quotient space $Q{:}=R_K/\sim$,which has a manifold structure. This is because each element of Q is an orbit γ of b in R_K: $\gamma=\{\phi_t x, t\in\mathbf{R}\}$ and, by the flow-box theorem, we can find a neighborhood U of x in R_K (since R_K is open) mapped onto $B_\delta(0)$. The equivalence relation in $B_\delta(0)$ is simple: it yields the quotient space $\bar{V}$ of Corollary 2.1 and hence a coordinate map β for a neighborhood of γ (see Fig.1) (since $\pi\circ\psi^{-1}$ maps $\bar{V}$ to $\pi(V)=\pi\circ\psi^{-1}(V)$ diffeomorphically and hence the inverse β of $\pi\circ\psi^{-1}$ exists).

The canonical map $\pi:R_K \to Q$, sending each point to its orbit in Q is locally *onto* since, when transformed via ψ to a flow-box, it can be written as:

$$\pi = \psi^{-1} \circ \pi' \circ \psi$$

where π' is the canonical projection in $\mathbf{R}^n$, taking $(y_1, y_2, .., y_n)$ to $(y_2, .., y_n)$ and is clearly onto.

The equivalence relation $\sim$ is called a *regular* equivalence and Q the *quotient manifold* (see Abraham et.al.[1],p173).

Next, we want to show that Q is compact. First, we claim that every orbit of b in R_K intersects ∂N_ε^K. This is because R_K, and therefore $\overline{N_\varepsilon^K}$, contains no α–limit points of b. Suppose, otherwise, that there is an orbit $\gamma = \{\phi_t x, t \in \mathbf{R}\}$ that is contained in N_ε^K. Define $\xi_i := \phi_{-i} x$, $i \geq 0$; the sequence $\{\xi_i\}_{i=1}^{\infty}$ has a limit point in $\overline{N_\varepsilon^K}$ because $\overline{N_\varepsilon^K}$ is compact. This is a contradiction, since $\overline{N_\varepsilon^K}$ contains no α–limit points. All points of R^K tend to K as $t \to +\infty$ and they are eventually in $\overline{N_\varepsilon^K}$. On the other hand, the orbits of all points of $\overline{N_\varepsilon^K}$ intersect ∂N_ε^K. Thus, all orbits in R_K hit ∂N_ε^K.

If we restrict the canonical projection map to ∂N_ε^K we get, by the above, that $\pi(\partial N_\varepsilon^K) = Q$. Since ∂N_ε^K is a smooth manifold and π a smooth map, we get that the image of the continuous map $\pi|_{\partial N_\varepsilon^K}$ is compact, because ∂N_ε^K is. Thus Q is compact (see Munkres [11], p.167).

An open cover of Q is obtained as follows: first, find another neighborhood $N_{\bar{\varepsilon}}^K$ such that $\phi_t(N_{\bar{\varepsilon}}^K) \subset N_\varepsilon^K$ for all $t \geq 0$. For all points $x \in N_{\bar{\varepsilon}}^K$, find a neighborhood U_x in $N_{\bar{\varepsilon}}^K K$ and a local Lyapunov surface V_x as in Cor.2 that is mapped diffeomorphically to $\pi(V_x)$, a neighborhood of $\pi(x)$ in Q. The $\pi(V_x)$ cover Q:

$$Q = \bigcup_{x \in N_{\bar{\varepsilon}}^K} \pi(V_x)$$

Since Q is compact, we can find a finite subcover:

$$Q = \bigcup_{i=1}^{m} \pi(V_i)$$

where the V_i are neighborhoods of the points x_i, $i=1,..,m$.

This finite cover will be used to get a *global section* of the quotient $\pi:R_K \rightarrow Q$, i.e. a smooth map $s:Q \rightarrow R_K$ such that $\pi(s(x))=x$ on Q.

Note that we already have *local* sections (sections defined on open subsets of Q). These are obtained (see Fig.2) by mapping back to R_K, using the rectifying diffeomorphism ψ_i, the neighborhoods of the x_i:

$$s_i=\psi_i^{-1}\circ\beta_i$$

$$s_i:\pi(V_i)\rightarrow V_i \subset R_K$$

To get the global section, we need to patch together the local ones; this is accomplished using a *partition of unity* subordinate to the open sets $\{\pi(V_i), i=1,..,m\}$ and the following *transition maps :*

On a non-empty intersection $\pi(V_i)\cap\pi(V_j)\neq\emptyset$ the map:

$$\gamma_{kj}:s_j(\pi(V_j)\cap\pi(V_k)) \rightarrow \mathbf{R}$$

sending x to:

$$\gamma_{kj}(x)=\inf_{t\in\mathbf{R}}\{|t|:\phi_t x\in V_k\}$$

is smooth and satisfies:

$$s_k(p)=\phi(\gamma_{kj}(s_j(p)),s_j(p)) \ , \ \ p\in\pi(V_j)\cap\pi(V_k)$$

When $j=k$, we have: $\gamma_{jj}(p)=0\ \forall j$. On triple intersections we have the consistency condition: $\gamma_{li}(s_i(p))=\gamma_{lk}(s_k(p))+\gamma_{ki}(s_i(p))$. *

Let $\{\alpha_i, i=1,..,m\}$ be the functions that define the partition of unity (i.e. $supp\,\alpha_i\subset\pi(V_i)$ and $\sum_{i=1}^{m}\alpha_i(p)=1, \forall p\in Q$). The expression for the global section s can now be given on each open set $\pi(V_i)$:

* proved using the group property of the flow:

$$s_l=\phi(\gamma_{lk}(s_k),s_k)=\phi(\gamma_{lk}(s_k),\phi(\gamma_{ki}(s_i),s_i))=\phi(\gamma_{lk}(s_k)+\gamma_{ki}(s_i),s_i)$$

and on the other hand $s_l=\phi(\gamma_{li}(s_i),s_i)$. Comparing, we get the desired result.

$$s(p)=\phi(\sum_{j=1,j\neq i}^{m}\alpha_j(p)\gamma_{ji}(s_i(p)),s_i(p)) \quad \text{for } p\in\pi(V_i)$$

We claim that this gives a consistently defined s on all of Q. To show this, take a non-empty intersection $\pi(V_i)\cap\pi(V_k)\neq\emptyset, i\neq k$. We have two expressions for $s(p)$ for points p in the intersection:

$$s(p)=\phi(\sum\alpha_j(p)\gamma_{ji}(s_i(p)),s_i(p))$$

$$s(p)=\phi(\sum\alpha_l(p)\gamma_{lk}(s_k(p)),s_k(p)) \qquad \text{(b)}$$

We must show that the expressions (a),(b) for $s(p)$ are equal:

$$(b)=\phi(\sum\alpha_l(p)\gamma_{lk}(s_k(p)),s_k(p))=$$

$$=\phi(\sum\alpha_l(p)\gamma_{lk}(s_k(p)),\phi(\gamma_{ki}(s_i(p)),s_i(p)))=$$

$$=\phi(\sum\alpha_l(p)\gamma_{lk}(s_k(p))+\gamma_{ki}(s_i(p)),s_i(p))=$$

substituting for γ_{lk} using the consistency relation:

$$=\phi(\sum\alpha_l(p)[\gamma_{li}(s_i(p))-\gamma_{ki}(s_i(p))]+\gamma_{ki}(s_i(p)),s_i(p))=$$

$$=\phi(\sum\alpha_l(p)\gamma_{li}(s_i(p))+(-\sum\alpha_l(p)+1)\gamma_{ki}(s_i(p)),s_i(p))=$$

and the result follows from the partition of unity:

$$=\phi(\sum\alpha_l(p)\gamma_{li}(s_i(p)),s_i(p))=(a)$$

It is clear that s is smooth and one-to-one. It remains to show that the image of Q under s is *transverse* to the flow. We know the local sections are transverse to b by construction of the local Lyapunov surfaces V_i. On each V_i, s modifies the local section s_i using the flow ϕ and a smooth function f that moves each point of $s_i(\pi(V_i))$ along the orbit. Here:

$$f(s_i(p))=\sum_j\alpha_j(p)\gamma_{ji}(s_i(p))$$

Therefore, Lemma 2 applies and the modified section is also transverse to the flow. □

Proof of Corollary:

It has been proved that Q is a compact manifold. To show R_K is diffeomorphic to $Q \times \mathbf{R}$, assume given a global section s, constructed as in the previous proof. Then, the flow ϕ gives the required diffeomorphism $\bar{\phi}$; we have:

$\bar{\phi}: Q \times \mathbf{R} \to R_K$

$(p,t) \mapsto \phi(t, s(p))$

and $\bar{\phi}$ is obviously smooth.

It is onto, since, for any $y \in R_K$, we can find a $t' \in \mathbf{R}$ such that $\phi_{t'} y = x' \in S$; then $(-t', p')$ goes to y, where $s(p') = x'$.

It is one-to-one since if $\bar{\phi}(t,p) = \bar{\phi}(t',p')$ we must have $p = p'$ since s is one-to-one and orbits do not intersect. Then $t = t'$, by uniqueness of solutions to the flow. □

2.1.3 Lyapunov Surfaces of Critical Elements

Before we can define global Lyapunov functions, we must discuss Lyapunov surfaces for critical elements.

A critical element σ is a **repellor** if its stable manifold is trivial. By reversing the flow direction, a repellor becomes an attracting set. Thus we have:

Lemma 3:

If σ is a repellor, there is a complete Lyapunov surface for σ in any given neighborhood of it.

A critical element σ is called a **saddle** if it has non-trivial stable and unstable manifolds. We distinguish **saddle equilibria** and **saddle orbits** . A saddle's region of attraction is its stable manifold, which is not an open subset of $\mathbf{R}^n$; thus there is no concept of complete Lyapunov surfaces for saddles. However, the 'future' orbits of points close to the stable manifold pass close to the unstable manifold of σ. We are therefore interested in Lyapunov surfaces in neighborhoods of the stable and unstable manifolds. We give separately the cases of σ being an equilibrium and a closed orbit.

Saddle Equilibria:

Let σ be a hyperbolic equilibrium of index $k, 0<k<n$. $W^u(\sigma)$, its unstable manifold, is k-dimensional and $W^s(\sigma)$, the stable manifold, is (n-k)-dimensional. The two manifolds are invariant under b and intersect transversely at σ:

$$T_\sigma W^u(\sigma) \oplus T_\sigma W^s(\sigma) = T_\sigma \mathbf{R}^n$$

As we saw in the previous section, on $W^s(\sigma)$ we can find a complete Lyapunov surface $S^s(\sigma)$ for σ as an attracting set on $W^s(\sigma)$. Similarly, we can find a complete Lyapunov surface $S^u(\sigma)$ on $W^u(\sigma)$, since σ is a repellor on $W^u(\sigma)$.

Consider the normal bundle $NW^s(\sigma)$ and its restriction to S^s, $NW^s(\sigma)|_{S^s}$. We can find a tubular neighborhood of S^s in $NW^s(\sigma)|_{S^s}$ of size ε; this means that all points on the neighborhood are a distance less than ε from S^s. This neighborhood $N_\varepsilon(S^s)$ can be made transverse to the flow, since S^s is transverse and it is also compact (since transversality holds on a neighborhood of any given point, we cover S^s with finitely many such neighborhoods and make ε small enough so that $N_\varepsilon(S^s)$ is inside the union of these neighborhoods). A similar construction gives a transverse neighborhood of S^u, $N_{\varepsilon'}(S^u)$ in $NW^u(\sigma)|_{S^u}$. The two neighborhoods are then Lyapunov surfaces for b. Note, finally, that we can assume that they are disjoint, by making them sufficiently small.

Saddle Orbits:

Let σ be a hyperbolic closed orbit. It has a k-dimensional unstable manifold $W^u(\sigma)$, $(0<k<n-1)$ and a $(n-k-1)$-dimensional stable manifold $W^s(\sigma)$. We have the splitting: $T_x\mathrm{R}^n=T_xW^u(\sigma)+T_xW^s(\sigma)$ where $T_xW^u(\sigma)\cap T_xW^s(\sigma)=\langle b(x)\rangle$.

The construction above for equilibria extends to the present case with minor changes.

2.2 Lyapunov Functions

We are now ready to define Lyapunov functions globally. First, we find Lyapunov functions defined in a region of attraction of an attracting set K. This is a simple application of the existence theory of section 2.1. Then, Lyapunov functions on the whole of $\mathbf{R}^n$ are obtained using the propagation of Lyapunov surfaces past saddle critical elements which is given in section 2.2.2.

2.2.1. Lyapunov Functions from Complete Lyapunov Surfaces

Corollary 2.1 establishes that, for an attracting set K, its region of attraction R_K is diffeomorphic to $Q\times\mathbf{R}$, where Q is the quotient manifold $R_K/\sim$ under the regular equivalence of belonging to the same orbit. The diffeomorphism $\overline{\phi}:Q\times\mathbf{R}\rightarrow R_K$ can then be used to map functions on $Q\times\mathbf{R}$ to Lyapunov functions on R_K.

Definition 3:

A continuous function V defined on an open subset $U\subset\mathbf{R}^n$ is a **Lyapunov function** *for the vector field $b\in\mathbf{D}(\mathbf{R}^n)$ if:*

(i) V is constant on each attracting set and on each critical element,

(ii) V is smooth on $U-\Omega(b)$ and

(iii) $dV(b)<0$ everywhere on $U-\Omega(b)$ (alternatively, if $y=\phi_t x, t>0$, then $V(y)<V(x)$).

Fix an attracting set K. Consider the following class of functions $\mathbf{L}(K)$ on $Q_K\times\mathbf{R}$:

$$\mathrm{a}\in\mathbf{L}(K) \quad \textit{iff}$$

(i) a is smooth and maps $Q_K\times\mathbf{R}\rightarrow\mathbf{R}$: $(p,\tau)\rightarrow \mathrm{a}(p,\tau)$.

(ii) for every $p\in Q_K$, $\mathrm{a}(p,.):\mathbf{R}\rightarrow\mathbf{R}$ is a diffeomorphism and let us assume that $\frac{\partial \mathrm{a}}{\partial\tau}(p,\tau)>0$.

Also consider a smooth strictly monotone increasing function $v:\mathbf{R}\rightarrow\mathbf{R}_+$.

Theorem 3:

To every a$\in$L(K) *and* v *as above, there corresponds a Lyapunov function* V *defined on* R_K. *Conversely, if* V *is a Lyapunov function on* R_K, *we can find functions* a *of class* L(K) *and* v *as above such that* V *is obtained from them.*

Proof:

For each $\tau\in$ R, a(.,τ) is a smooth function from Q_K to R and as τ varies, we foliate $Q_K\times$R with the graphs of the functions a(.,τ). The map:

$$Q_K\times\mathbf{R} \rightarrow Q_K\times\mathbf{R}$$

$$(p,\tau)\mapsto(p,\mathrm{a}(p,\tau))$$

is a diffeomorphism which we call $\overline{\mathrm{a}}$.

We also have the diffeomorphism $\overline{\phi}$ mapping $Q_K\times$R to R_K. Now simply define, for $x\in R_K$:

$$V(x)=v\circ\pi_\tau\circ\overline{\mathrm{a}}^{-1}\circ\overline{\phi}^{-1}(x)$$

where π_τ is the projection $(p,\tau)\rightarrow\tau$. V is obviously a smooth function from R_K to R_+.

To check that it is a Lyapunov function, note that the level sets $\{V=constant\}$ are the images under $\overline{\phi}$ of a(.,τ) for each τ. These, as we can see from Lemma 2.2, are transverse to the flow and hence are complete Lyapunov surfaces. □

2.2.2 Classification of Global Lyapunov Functions

In section 2.1.2 Lyapunov surfaces were obtained for saddle critical elements in neighborhoods of their stable and unstable manifolds. The flow of the vector field can be used to propagate the Lyapunov surfaces near the stable manifold to the Lyapunov surface near the unstable one. With this technique we can then discuss global Lyapunov functions whose level sets intersect more than one region of attraction and are defined globally, on the whole of $\mathbf{R}^n$.

Propagation of Lyapunov Surfaces past a Saddle:

The flow close to the unstable manifold of a saddle σ passes near the saddle and then stays close to the stable manifold of σ (see Fig.2). This allows us to define a diffeomorphism between the "punctured" Lyapunov surfaces $N(S^s)-S^s$ and $N'(S^u)-S^u$. The Lyapunov surfaces S^s and S^u have to be removed since W^s and W^u are invariant under the flow.

Theorem 4:

Let σ be a saddle critical element. We can find punctured Lyapunov surfaces $N(S^s)-S^s$ and $N'(S^u)-S^u$ and a smooth positive real function $\mathrm{a}:N(S^s)-S^s\rightarrow\mathbf{R}_+$ such that $N'(S^u)-S^u$ is the diffeomorphic image of $N(S^s)-S^s$ under the map:

$$x\rightarrow\phi(\mathrm{a}(x),x)$$

Proof:

We do separately the cases of σ an equilibrium and a closed orbit.

Case 1: Saddle Equilibrium:

Let σ have index $k, 0<k<n$. It is known ([12]) that there is a homeomorphism h that takes a neighborhood U of σ to an open set of $\mathbf{R}^n$, sending σ to O and on which the push-forward h^*b of the vector field b is the simple linear vector field:

$$\dot{\xi}=\begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix}\xi$$

The general solution of the above linear equation is:

$$\begin{bmatrix}\xi_1(t)\\ \xi_2(t)\end{bmatrix}=\begin{bmatrix}e^t\xi_1(0)\\ e^t\xi_2(0)\end{bmatrix}, \quad \xi_1(0)\in \mathbf{R}^k , \xi_2(0)\in \mathbf{R}^{n-k}$$

We can assume that the neighborhoods of S^s and S^u found in section 2.2.2, $N_\varepsilon(S^s)$ and $N_{\varepsilon'}(S^u)$ are in U and do not intersect. Also suppose that the image under h of U is a ball around 0, $h(U)=B_\eta(0)$, $\eta>0$.

If we have an initial condition close to S^s, i.e. $\xi_2(0)$ is close to S^s and $|\xi_1(0)|$ is non-zero and small relative to $|\xi_2(0)|$, then we can find a time $T>0$ such that $e^{-T}\xi_2(0)$ is arbitrarily smalll and $e^T\xi_1(0)$ is larger than η.

Thus it is clear that if $\xi_1(0)$ is sufficiently small, the orbit of $\begin{bmatrix}\xi_1(0)\\ \xi_2(0)\end{bmatrix}$ intersects $h(N_{\varepsilon'}(S^u)-S^u)$. It also follows that for the linear vector field, a small punctured neighborhood of $h(S^u)$ is mapped by the flow onto a small neighborhood of $h(S^s)$. Thus $h(N(S^u)-S^u)$ is mapped by the homeomorphism h onto a neighborhood $h(N'(S^s)-S^s)$ of $h(S^u)$.

Having established that there is a continuous 1-1 and onto map between the two neighborhoods, we can map back using h and get a diffeomorphism between them in $\mathbf{R}^n$, by the flow diffeomorphism $\phi(t,x)$.

Case 2: Saddle Closed Orbit:

Let σ have index k , $0<k<n-1$.

The Poincare map P_Σ on a local Lyapunov surface Σ of a point $x\in\sigma$ is locally equivalent to one of the following maps (see Theorem 5.5 of [12],p.72):

$$A_k^1 = \begin{bmatrix}B_+ & 0\\ 0 & C_+\end{bmatrix}, \quad A_k^2 = \begin{bmatrix}B_- & 0\\ 0 & C_+\end{bmatrix}$$

$$A_k^3 = \begin{bmatrix}B_+ & 0\\ 0 & C_-\end{bmatrix}, \quad A_k^4 = \begin{bmatrix}B_- & 0\\ 0 & C_-\end{bmatrix}$$

where $B_+=\frac{1}{2}I_k$ where I_k is the k-dimensional identity matrix, $C_+=2I_{n-1-k}$ and B_- and C_- differ from B_+ and C_- only in that their (1,1) elements have a negative sign. Thus, $A_k^i\in \mathbf{R}^{(n-1)\times(n-1)}$, i=1,2,3,4.

The solution of the discrete system:

$$\xi_{j+1}=A^i_k\xi_j$$

is $\xi_j=(A^i_k)^j\xi_0$. In particular, there are contracting and expanding directions as in the case if the linear flow above. And just as in case 1, a point close to the stable manifold of σ on Σ gets mapped close to the unstable manifold of σ on Σ (see Fig.3). Reasoning as in the previous case, we get a diffeomorphism between the sets $\Sigma\cap(N(S^s)-S^s)$ and $\Sigma\cap(N'(S^u)-S^u)$. □

Remark:

In Morse theory, one is interested in describing how the maps given above for saddle elements contribute to the relative cohomology of the level sets of Morse functions. Although the homological aspects of this transformations of Lyapunov surfaces around saddles is interesting and may lead to a deeeper understanding of optimal control for dynamics of class $\mathbf{D}(\mathbf{R}^n)$ (see chapter 3), we shall leave the topic for future research.

Classification of Global Lyapunov Functions:

First, we motivate the basis for our classification method by an example. Consider a vector field in the plane with the phase portrait of Fig.5(a) (its orbit diagram is shown in Fig.5(b)). There are three attractors and two saddles. In Figs.5(c) and (d) we show two possible global Lyapunov functions by plotting a few of its Lyapunov surfaces. In chapter 3 we shall see how for some optimal control problems (and related large deviation problems) a Lyapunov function is the optimal cost functional starting from a given attractor. It is easy to see that in case (c) optimal exit is from saddle s_2 while in case (d) it is from saddle s_1, since it is less costly to take those paths (one should view the Lyapunov surfaces as isocost surfaces).

We would like a classification scheme that differentiates between these two types of Lyapunov function on the basis of the qualitative feature that exit paths go through different critical elements. The aim is then to generalize this to all vector fields of class $\mathbf{D}(\mathbf{R}^n)$.

In order to get different kinds of Lyapunov functions than those obtained in section 2.2 we need to abandon complete Lyapunov surfaces. In particular, we will use Lyapunov surfaces that are global sections of Q_K but with the unstable manifolds of some critical elements removed (Fig.4).

A global Lyapunov function is always increasing as we move on a stable manifold from the ω-limit set to the α-limit set of the manifold. The converse holds for an unstable manifold. As we move on a chain of manifolds starting from an attractor, we want to define a Lyapunov function consistently. To do this, we have imposed the no-cycle condition in Chapter 1. The best way to book-keep the different possible global Lyapunov functions is to look at the *orbit diagram* of b. There, we can classify all the global Lyapunov functions by specifying an order on the set of critical elements of each index level. We now make precise the above ideas.

Definition 4:

Let σ be a critical element of index k , $k>0$ such that $W^u(\sigma)\cap R_K\neq\varnothing$, where R_K is the region of attraction of some $K\in A(b)$. We call σ a **k-ancestor** *of K.*

Similarly, if σ_1 and σ_2 are critical elements with $k_1<k_2$ and are such that $W^u(\sigma_2)\cap W^s(\sigma_1)\neq\varnothing$, we call σ_2 a **k-ancestor** *of σ_1 and σ_1 a* **k-descendant** *of σ_2.*

Let the orbit diagram of the vector field $b\in D(\mathbf{R}^n)$ have $l_0>0$ attracting sets and $l_k\geq 0$ critical elements of index k ($1\leq k\leq n$) (all numbers are finite). In numbering critical elements and attractors, we shall use superscripts to denote *index* and subscripts for *ordering*. Let $m_i^{k_1k_2}$ be the number of k_2-ancestors of the critical element (or attractor) $\sigma_i^{k_1}$ of index k_1.

Define the sets:

$$N=\{1,2,..,n\}$$

$$c^k=\{\sigma_1^k,\sigma_2^k,..,\sigma_{l_k}^k\}\ ,\ k\in N$$

$$v^{k_2}(\sigma_i^{k_1})=\{\sigma_1^{k_1k_2},\sigma_2^{k_1k_2},..,\sigma_{m_i^{k_1k_2}}^{k_1k_2}\}\ ,\ k_1,k_2\in N\ ,\ k_2>k_1\ ,\ \sigma_i^{k_1}\in c^{k_1}$$

$$\nu(\sigma_i^{k_1})=\bigcup_{k>k_1}\nu^k(\sigma_i^{k_1})$$

c^k is the set of critical elements of b of index k while $\nu_i^{k_i k}$ counts the k-ancestors of σ_{k_i}. Finally, $\nu_i^{k_i}$ is the set of all ancestors of $\sigma_i^{k_i}$. Note that:

$$l_k=\sum_{k_1=0}^{k-1}\sum_{i=1}^{l_{k_1}} m_i^{k_1 k} \ , k>0$$

Since an orbit diagram, considered as a set with the order relation < is not even a partially-ordered set, we shall need to proceed inductively. This will require examining generalized orbit diagrams, where the nodes are not single attracting sets or critical elements but groups of them. The price we pay in added complexity of the relevant statements is offset by the sharpness of the results. To see this we first remark that a given global Lyapunov function orders *all* members of $\Omega(b)=L_\alpha(b)\cup L_\omega(b)$:

Remark:

Suppose a global Lyapunov function V is given for the vector field $b\in D(\mathbf{R}^n)$. *Then the members of* $\Omega(b)$ *(attracting sets and critical elements) and hence the sets* c^k *and* $\nu(\sigma)$ *for all* $\sigma\in\Omega(b)$ *are ordered in a unique way by the Lyapunov function: given any two* σ_i, $\sigma_j\in\Omega(b)$, $\sigma_i<(\leq)\sigma_j$ *if and only if* $V(\sigma_i)<(\leq)V(\sigma_j)$.

We now arrive at the general scheme by giving a number of steps that will lead to the desired classification theorems.

Step 1: Identify *clusters* $\{C_i\}_i$ of nodes. These are subsets such that if $\sigma\in C_i$ of index k, then there is a path down from σ to K, different from itself, passing nodes of the same cluster and each C_i is connected as a graph.

Step 2: For each attractor, look at the set $\nu(K)$, $K\in A(b)$. *Discard* any node σ that is such that there is a chain of nodes (other than itself) joining σ to K. (This is because a Lyapunov function strictly increases as we move up a branch of the orbit diagram; e.g. in Fig.6 we cannot ever have $V(\sigma^1)>V(\sigma^2)$.)

Step 3: Call the nodes in $\nu(K)$ remaining after the discarding process in each cluster a *proper-ancestor set (p.a. set).* Fix orderings on the proper-ancestor sets of all attractors that is *consistent*. This means that if C_1 and C_2 are proper-ancestor sets of two attractors such that $C_1 \cap C_2 \neq \emptyset$, then the intersection the two orderings coincide.

Step 4: After discarding elements of each cluster and ordering *select* the first node for each p.a. set of each attractor. This is the first saddle critical element to be swept by Lyapunov surfaces starting from each R_K. For every attractor look at all attractors that adjoin it by these selected saddle elements. Order the attractors by looking at the order of the selected first elements (there may be more than one first attractors). We have the Lemma:

Lemma:

Consider the set $K_\sigma = (\bigcup K_i) \cup W^u(\sigma) \cup \sigma$ *composed of a saddle critical element that is ordered first according to steps 1-4 above, its unstable manifold and attractors that are connected to* σ *in the orbit diagram. Then* K_σ *is an attracting set (with boundary) and hence has Lyapunov functions defined in its region of attraction.*

The proof follows from the results on the propagation of Lyapunov surfaces past saddles. It consists of patching together the neighborhoods that yield a complete Lyapunov surface (see proof of existence theorem) except the ones that intersect $W^u(\sigma)$ with a local Lyapunov surface $N(W^u(\sigma)) - W^u(\sigma)$.

Step 5: Consider the sets K_σ (by convention, $K_\emptyset$ means that attractor K was not ordered first in step 4. There is a *generalized orbit diagram* associated with this new set of attractors. Its nodes are the sets K_σ and the remaining critical elements and the branches are formed in the obvious way: connect node σ_i to node σ_j if there was at least one arrow connecting them in the original orbit diagram.

Step 6: Repeat steps 1-4 for the new diagram.

Step 7: Apply step 5 again and continue with step 6 until all critical elements are taken care of.

Example: In Fig.7 we give the steps of this procedure until we exhaust the critical elements of b.

The previous discussion has proved the two fundamental theorems on the classification of global Lyapunov functions:

Theorem 5:

Fix an ordering of the proper-ancestor sets.

Then any two global Lyapunov functions with the same ordering yield the same qualitative behavior of exit paths, i.e. exit from each attractror goes through the same sequence of saddle critical elements for both Lyapunov functions.

Conversely, there exists a global Lyapunov function such that the ordering it induces on the proper-ancestor sets according to its value on the elements of $\Omega(b)$ coincides with the given one.

2.3. Conclusions

We have derived a global existence theory of Lyapunov functions and we have been able to classify all Lyapunov functions for a class of dissipative dynamics which is quite general.

The information that is contained in this classification is substantial. It gives not only qualitative results on stability problems, but also leads to quantitative results once the geometric insight contained in this complete description is put to work in solving nonlinear optimal control problems (see Kappos,Sastry [9]).

REFERENCES

[1] R.Abraham, J.Marsden, T.Ratiu: Global Analysis, Manifolds and Tensors, Addison-Wesley, 1984.

[2] V.I.Arnol'd: Ordinary Differential Equations, M.I.T.Press,1973.

[3] R.Azencott: Grandes Deviations et Applications, Ecole d'Ete de Prob. St.Flour, 1980.

[4] G.Birkhoff: Dynamical Systems, AMS Coll.Publ. 9, 1927.

[5] J.Franks: Homology and Dynamical Systems, CBMS #49, 1982.

[6] M.Hirsch: Differential Topology, Springer, 1976.

[7] M.Hirsch, C.Pugh, M.Shub: Invariant Manifolds, Springer, 1977.

[8] E.Kappos, S.Sastry: Large Deviations in Dissipative Dynamics, work in progress.

[9] E.Kappos, S.Sastry: Large Deviation Behavior of the Josephson Junction, paper in preparation.

[10] J.Milnor: Morse Theory, Princeton University Press, 1963.

[11] J.Munkres: Topology, Prentice-Hall, 1975.

[12] J.Palis, DeMelo: Geometric Theory of Dynamical Systems, Springer, 1982.

[13] C.Pugh, M.Shub: The Ω-Stability Theorem for Flows, Invent.Math., 11, pp.150-158, 1970.

[14] A.Wentzell, M.Freidlin: Random Perturbations of Dynamical Systems, Springer, 1984.

[15] J.Willems: Dissipative Dynamical Systems, Arch.Rat.Mech.Appl., 45, 1972.

[16] F.Wilson: Smoothing Derivatives and Applications, Trans.AMS, 139, pp.413-428, 1969.

[17] F.Wilson: The Structure of the Level Surfaces of a Lyapunov Function, J.Diff.Eq., 3, pp.323-329, 1967.

FIGURE CAPTIONS

Fig.1.1: Phase-space flows and orbit diagrams for two examples of dissipative dynamics in the plane.

Fig.2.1: Mapping a neighborhood of x on Q by the projection π and on $\mathbf{R}^n$ by the flow-box diffeomorphism ψ.

Fig.2.2: Flow near a saddle equilibrium: $W^u(\sigma)$ and $W^s(\sigma)$ are the stable and unstable manifolds, S^u and S^s are Lyapunov surfaces on them and the flow maps the punctured neighborhood of S^u to the punctured neighborhood of S^s.

Fig.2.3: Flow near a saddle limit cycle: on the Poincaré surface Σ, a point close to the unstable manifold $W^u(\sigma)$ is mapped to a point close to the stable manifold $W^s(\sigma)$.k

Fig.2.4: Propagation of a complete Lyapunov surface past a saddle: the Lyapunov surface is no longer complete: it misses the unstable manifold of σ.

Fig.2.5: A two-dimensional example of a vector field of class $\mathbf{D}(\mathbf{R}^n)$ that has two different Lyapunov functions: (a) gives the phase-portrait, (b) the orbit diagram and (c) and (d) the two Lyapunov functions (by showing some of their Lyapunov surfaces).

Fig.2.6: Part of an orbit diagram that shows that, if σ_1 is at a higher index level than σ_2, then $V(\sigma_1)$ must be greater than $V(\sigma_2)$.

Fig.2.7: Demonstration of the selection procedure among proper ancestor sets and the iterative collapsing of orbit diagrams: here we ordered σ_2 before σ_1 and we obtained the Lyapunov function of Fig.2.5(c).

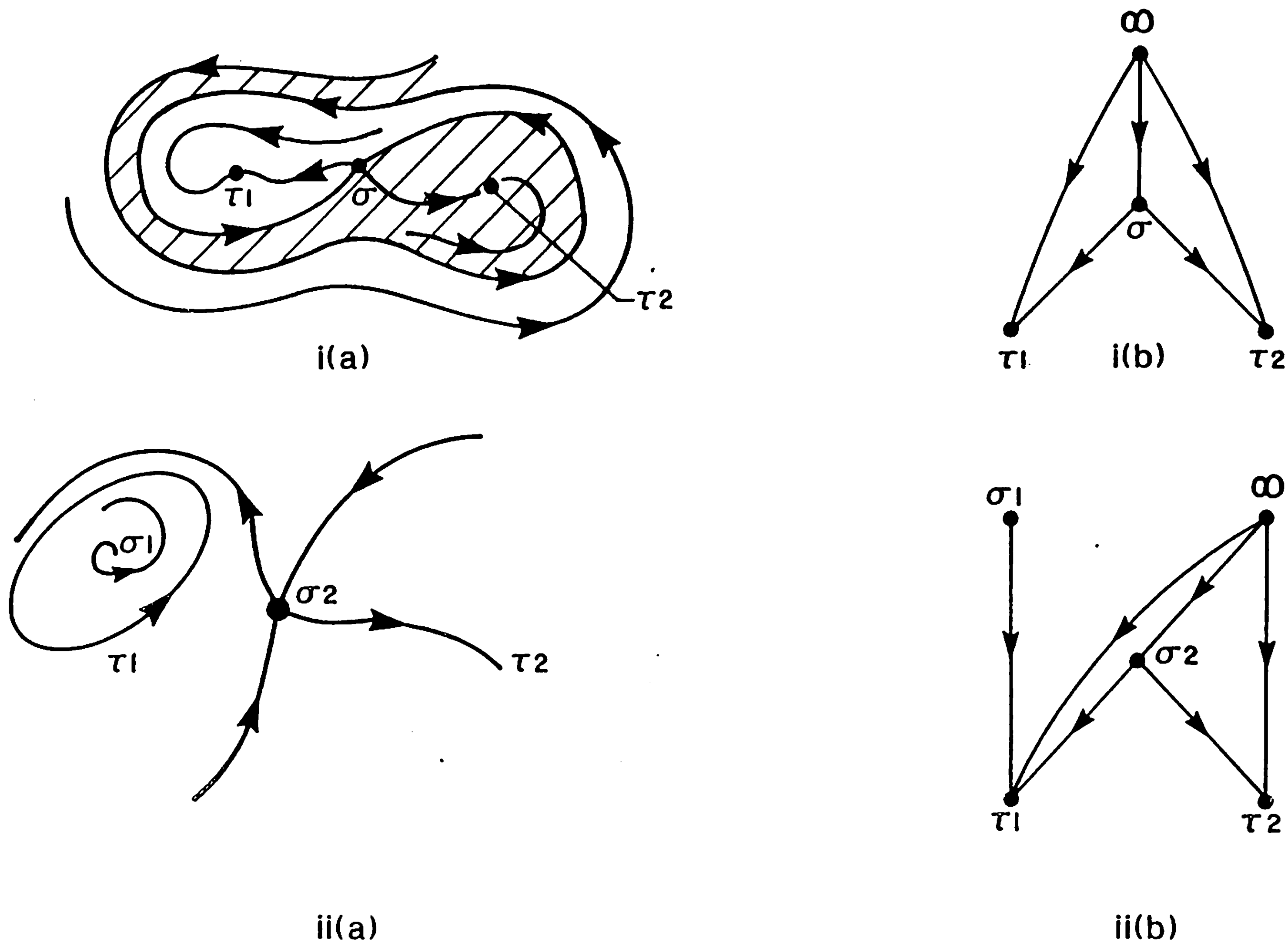

Fig. 1.1

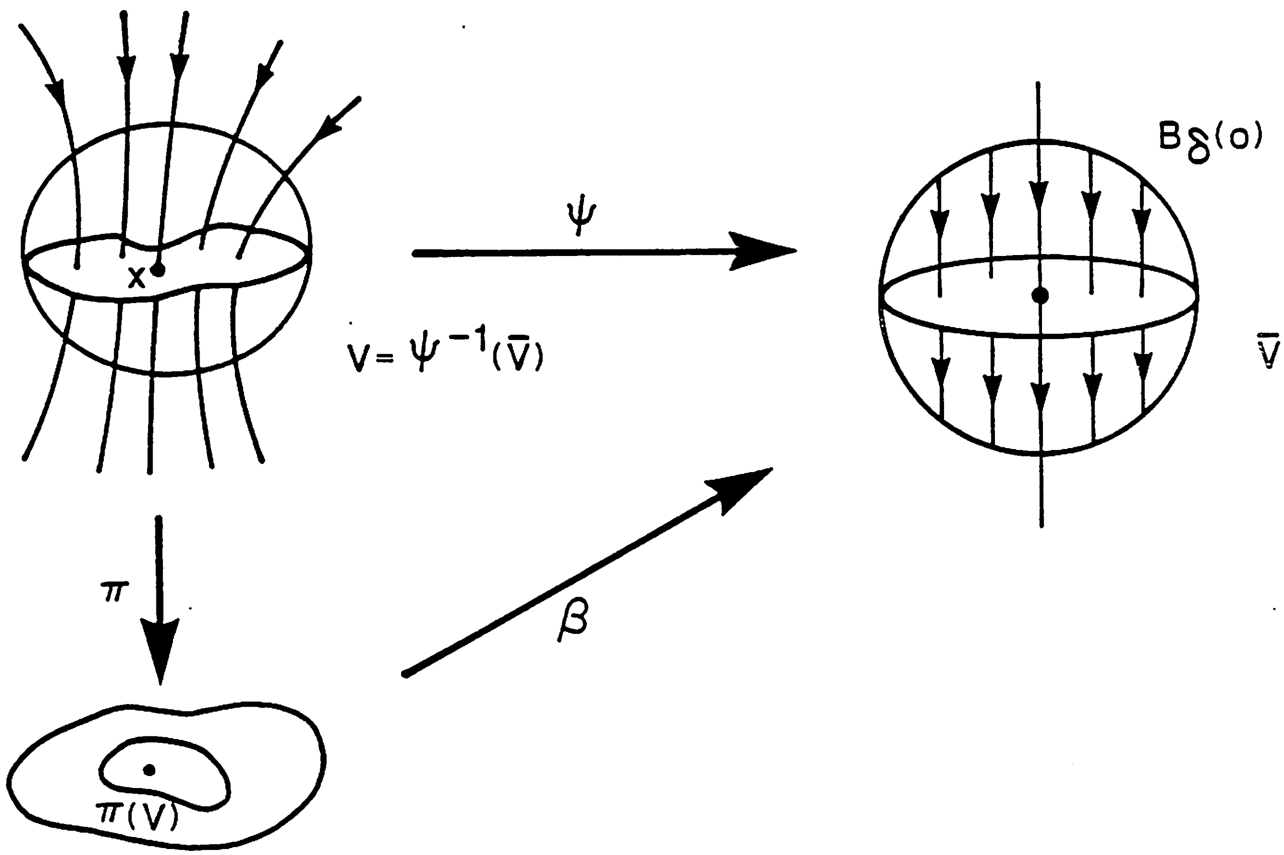

Fig. 2.1

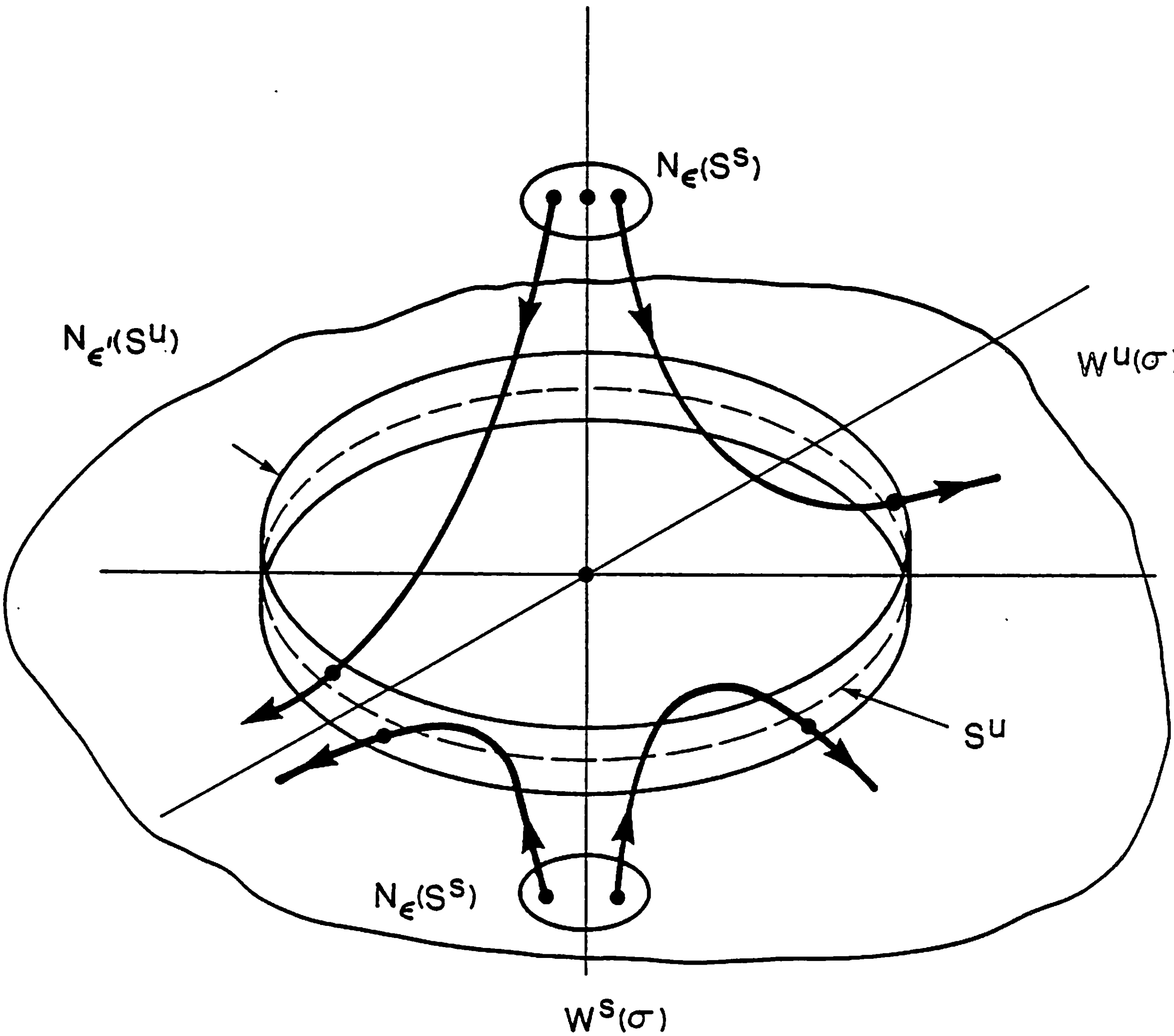

Fig. 2.2

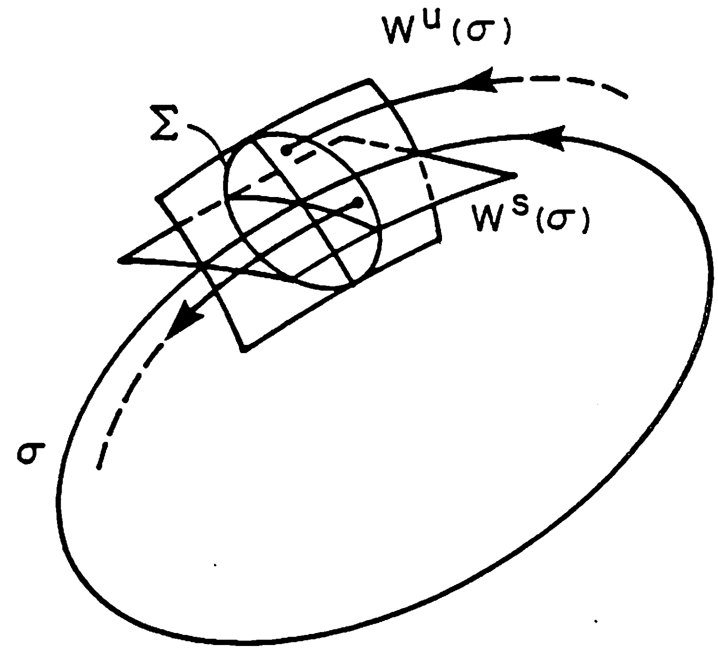

Fig. 2.3

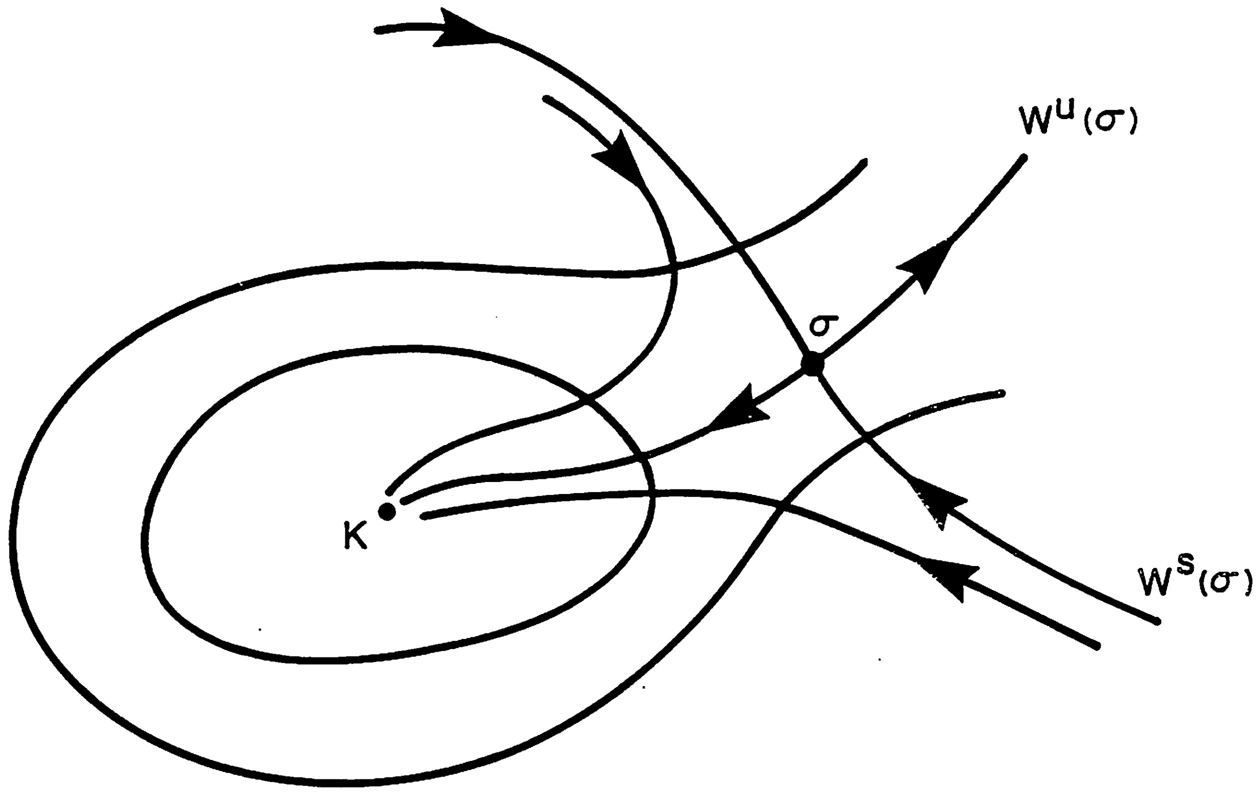

Fig. 2.4

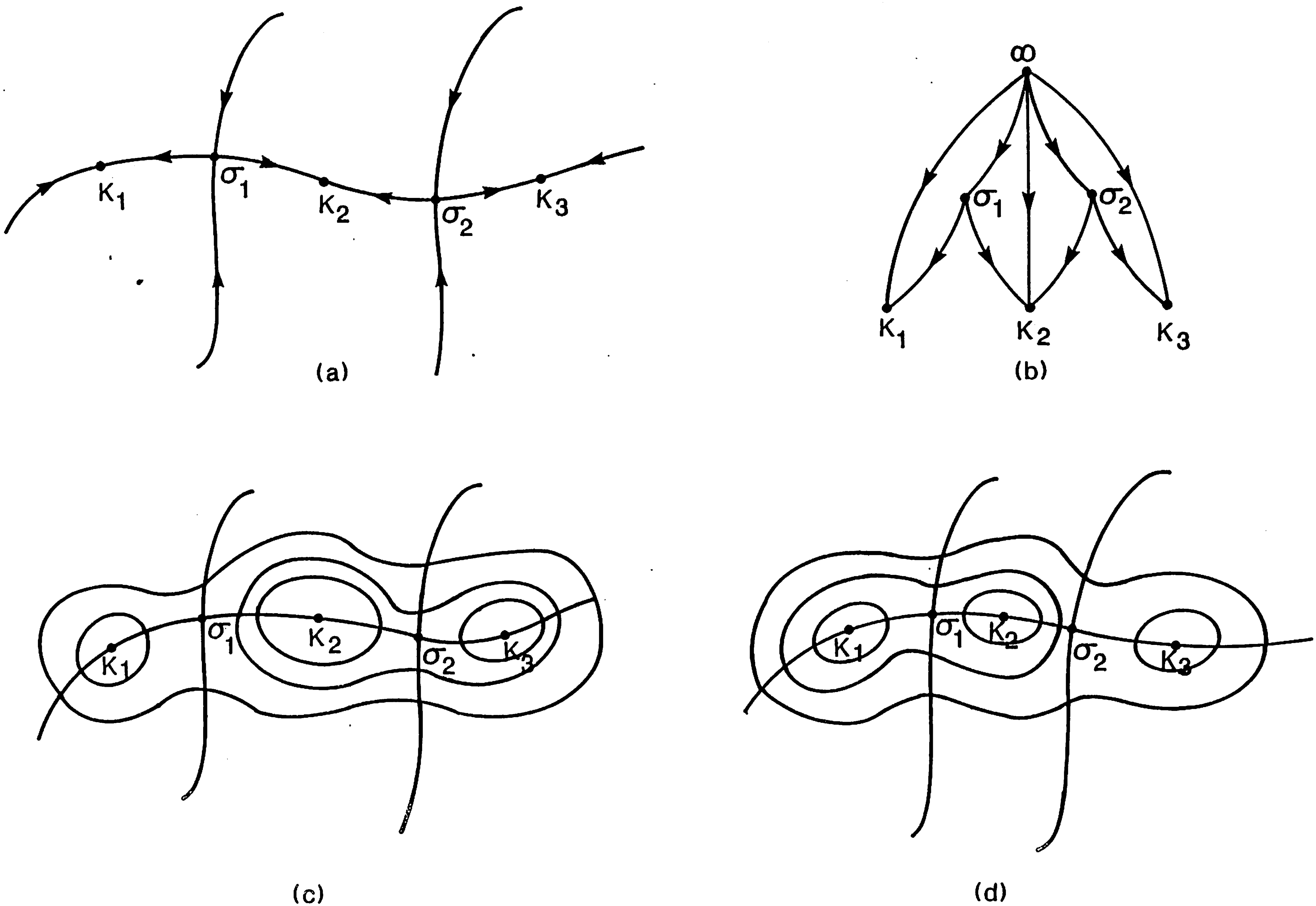

Fig. 2.5

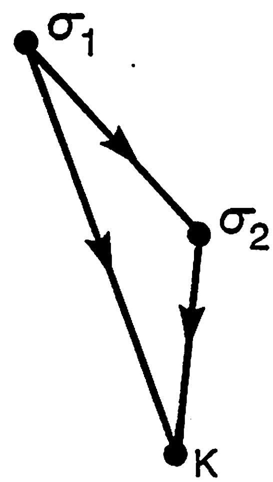

Fig. 2.6

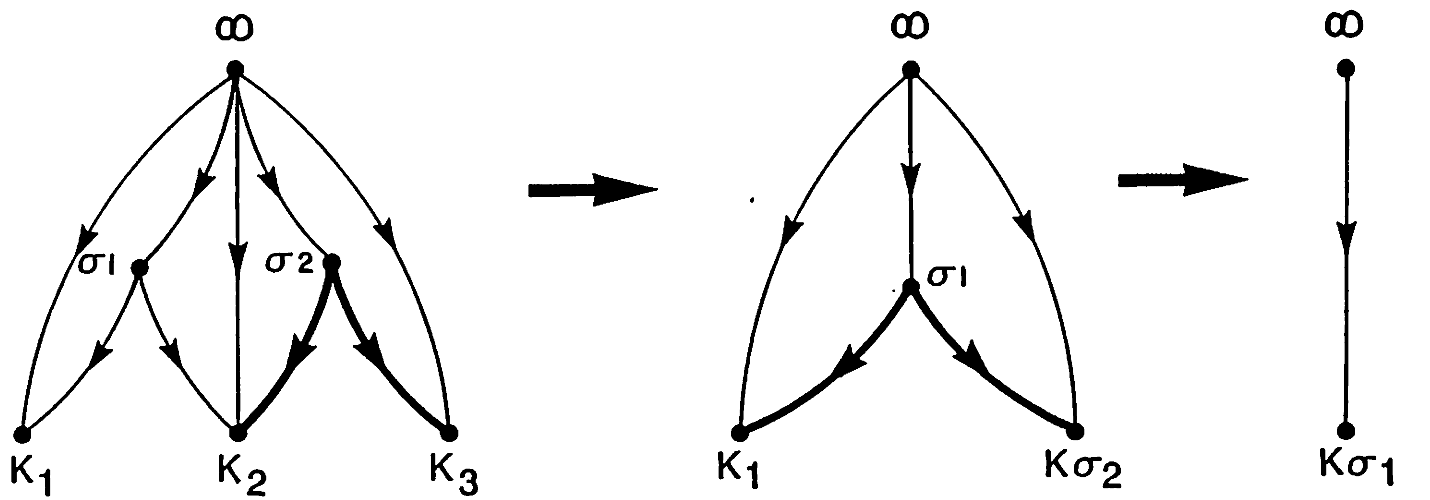

Fig. 2.7