FREQUENCY DOMAIN ANALYSIS AND SYNTHESIS TECHNIQUES FOR ADAPTIVE SYSTEMS

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Li-Chen Fu

Memorandum No. UCB/ERL M87/58

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Frequency Domain Analysis and Synthesis Techniques for Adaptive Systems

Li-Chen Fu

Signature: ______________________
Prof. S. Shankar Sastry
Committee Chairman

Abstract

In this thesis we use a mathematical technique, referred to as the method of averaging, to thoroughly analyze both adaptive identification and adaptive control schemes. In principle the results hold when the rate of parameter update in the adaptive loop is slow compared with the dynamics of the other state variables, but in practice they work for normal rates of parameter adaptation. Our analysis is not confined to the ideal case which consists of knowing the order of the unknown plant exactly and assuming there exist no external disturbances, but it also allows for unmodelled dynamics and additive output disturbances. We also make use of the method of averaging to solve the optimal input problem, i.e. the problem of choosing the input which produces the fastest rate of parameter convergence.

The results of this thesis are many. The first is a set of stability theorems which determine when a dynamical system possesses exponential stability, partial exponential stability or ultimate boundedness. Instability theorems for one- and two-time-scale systems are also given. Under the assumptions of a stationary reference input and slow adaptation these results are applied to adaptive systems. The next result is a calculable estimate of the rate of parameter convergence when various adaptation algorithms are used. When the plant contains unmodelled dynamics, we use the method of averaging to formally define the notion of a set of

"tuned parameters". Under the assumptions of slow adaptation and persistency of excitation, we show that for the adaptive identifier, the actual identifier parameters converge to a ball which is centered at the tuned parameters and whose radius goes to zero as the adaptation gain goes to zero. Similar results, though slightly more complicated, are also obtained for the adaptive control case. To illustrate the importance of the choice of input signals, the phenomenon of slow-drift instability is analyzed. Finally a frequency domain technique, for the synthesis of reference inputs which solve the optimal input problem, is given. An expression for what we call the *average information matrix* is derived and its properties are studied. The objective of the input synthesis technique is to specify the frequency content of a power constrained input signal, which maximizes the smallest eigenvalue of the average information matrix, and hence maximize the parameter convergence rate. A convergent numerical algorithm is given which obtains globally optimal solutions to the above problem. When the plant contains unmodelled dynamics, practical considerations of the range of support of the frequency content of the reference input is given.

To my parents

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Finally, I would like to dedicate this thesis, at heart, to my family and my dear friend Shu-Yu in Taiwan. Despite the distance, their unending love has supported and encouraged me to stand firmly over the years.

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Chapter 1 Introduction

1.1 Motivation

Frequency domain techniques constitute a powerful method for the analysis and design of linear systems. The analysis and design of linear systems is particularly easy since there is a simple relationship between their input-output behavior and their internal dynamics. Further it is easy to characterize the input-output behavior in the *frequency domain* as the Fourier transform of the impulse response. For linear time-varying or general nonlinear systems, "frequency response" is not a well defined notion. One, instead, analyzes the time domain trajectories in the state space.

However, if there is a certain separation between the time rate of change of state variables, a powerful technique, namely averaging, can be used to approximate the time-varying dynamics by time-invariant dynamics. Consequently, averaging serves as a useful method by which one can replace a complicated nonautonomous system by a simpler autonomous (averaged) system. Further since the dynamics of adaptive identification and control systems (for linear models) are "asymptotically linear", the averaged version of their asymptotic dynamics is linear time-invariant, for example, of the form:

$$\dot{x} = R(0)\,x \tag{1.1}$$

where $R(0)$ is an autocovariance matrix (defined in Chapter 3) which has a frequency domain interpretation. This fact enables averaging to become a technique that bridges state space and frequency domain techniques.

Motivated by this, we present a complete set of averaging results which are sufficient for the analysis of general adaptive systems. Using these averaging results, a detailed analysis of adaptive systems with various update algorithms is presented, and a synthesis of the exogenous input signals subject to a certain optimality criterion (defined in Chapter 5) for the system is also provided.

1.2 Review of Previous Work

Adaptive identification is a technique for the estimation of the parameters of an unknown system from input-output data. The algorithms are usually designed on the assumption that the system is fixed, but in practice they work even when the parameters of the system are slowly varying.. An adaptive identification scheme was first devised at least as early as 1967 by Lion, and was extended later by Anderson (1974), Luders and Narendra (1973), and Kreisselmeier (1977). Their original objective was an adaptive observer, i.e. one which provides estimates of state variables of an unknown time-invariant linear system as well as an estimate of the system parameters from measurements of its input and output. Their work, together with that of Sondhi and Mitra (1976), Anderson (1977), Morgan and Narendra (1977a), provides proofs of global exponential convergence of the system parameters and state variables whenever the system has a sufficiently rich input.

Model reference adaptive control schemes were implicitly suggested in the work of Astrom and Wittenmark (1973) and were formalized by Monopoli (1974). Model reference schemes were further extended by Narendra and Valavani (1978) and Landau (1979), and rigorous proofs of stability of these schemes in the general case appeared in Narendra, Lin and Valavani (1980), Morse (1980), and in Goodwin, Ramadge and Caines (1980) for the discrete-time case. While the adaptation algorithms of the schemes mentioned earlier were based on the sensitivity of the output error to the adjustable parameters, Bodson and Sastry (1987) developed a scheme which used the input error for the sensitivity vector. Boyd and Sastry (1986) extended these results further to the case of adaptive control (not just identification) using generalized harmonic analysis. They translated the persistency of excitation condition on the regressor vector to a condition on the spectral content of the reference input.

A great deal of interest in questions of robustness arose from the paper of Rohrs et al (1981)(1985) indicating the extreme sensitivity of the model reference schemes to unmodelled dynamics and output disturbances. Further investigations followed by Astrom (1983)(1984), Krause et al (1983), Chen and Cook (194), Kosut and Johnson (1984), Riedle et al (1984), Riedle and Kokotovic (1985a,b), and Fu and Sastry (1987). Lately, several attempts have been made to make model reference schemes robust by modifying the adaptation law, such as, in Peterson and Narendra (1982), Kreisselmeier and Narendra (1982), Sastry (1984), where a dead zone (fixed size) is used. In Kreisselmeier (1986), Kreisselmeier and Anderson (1986) robustness is achieved using a *relative* dead zone and a projection in the adaptation law, and in Ioannou and Kokotovic (1984), Ioannou and Tsakalis (1986), and Narendra and Annaswamy (1986), the robustness is established by means of an additional, linear feedback term in the adaptation law.

Astrom introduced averaging as a technique for the study of instability mechanisms in adaptive systems (1983)(1984). This tool was more fully exploited by Riedle and Kokotovic (1985a)(1986), Kokotovic et al (1986), Kosut et al (1986), Anderson et al (1986), to obtain sharp conditions on the boundary between stability and instability; and by Fu et al (1985), Bodson et al (1986), and Kosut (1986) to obtain estimates on the rate of parameter convergence. In Mason et al (1987) we applied the averaging method to study the robustness of the adaptive identifier to unmodelled dynamics.

The problem of input design for estimating parameters in a linear stochastic dynamical system has been extensively studied for over two decades. The first systematic attempt to obtain an "optimal" input for identifiers seems to have been that of Levin (1960) who showed that the optimal energy or amplitude constrained input that minimizes the trace or the determinant of the error covariance matrix is a white noise sequence. This work was further developed, for example, in Levadi (1966), Aoki and Staley (1970), Arimoto and Kimura (1971), Goodwin et al (1973), Lopez-Toledo (1974), and Mehra (1974). In the statistical literature, the same problem has also been addressed implicitly in Elfving (1952), Kiefer and

Wolfowitz (1959), and Fedorov (1972), and more rigorously in Box and Jenkins (1970), Minnich (1972), and Viort (1972). An excellent survey was given by Mehra (1974), and at that time an important conclusion reached was that input design problem could be reduced to a finite dimensional optimization problem. These results were extended in recent work that appeared in Goodwin and Payne (1977), Zarrop (1979), Mehra (1981), Goodwin (1982), Ljung and Soderstrom (1983), and Yuan and Ljung (1985). In the deterministic literature, Mareels et al (1986) studied the problem of "optimal" input design for identification through a heuristic discussion and simulations. This further led to the work of Fu and Sastry (1987).

1.3 Contributions of the Thesis

In this dissertation, we develop frequency domain techniques to analyze:

(1) parameter convergence rates in an adaptive system,

(2) robustness of adaptation in the presence of unmodelled dynamics or measurement noise,

(3) synthesis of optimal inputs for adaptive systems.

The outline of the thesis is as follows:

In Chapter 2, we present a complete package of the averaging results that have been developed for adaptive systems (see also Fu et al (1986) and Bodson et al (1986)). These are used to obtain estimates of the rate of parameter convergence. In addition, we develop new results on partial exponential stability and bounded stability which allow for the bounded input bounded state (BIBS) stability analysis of general adaptive systems.

In Chapter 3, the results of Chapter 2 are applied to analyze the performance of the adaptive identifier of Kreisselmeier (1977) in the presence of unmodelled dynamics. Results obtained with coworkers and published in Mason et al (1987) are also reviewed.

Another significant contribution is an averaging based analysis of direct model reference direct adaptive control schemes, including both input and output error schemes with various parameter update algorithms. In Chapter 4, the notion of a tuned model, in the presence of unmodelled dynamics and bounded output disturbances, is established and serves as a basis for developing the reduced order controller. The importance of the spectral content of the reference input to the robustness of the controller is emphasized. The insufficiency of the conventional persistency of excitation (PE) condition is pointed out and is remedied by the so called positive definite PE condition. One type of instability observed due to the non-satisfaction of this condition is discussed in detail. From this discussion, a distinction is drawn between frequency ranges that improve and those that impair the robustness of these schemes.

In Chapter 5, we propose a frequency domain technique of synthesizing reference inputs for adaptive systems. The optimum choice of inputs to maximize the smallest eigenvalue of an average information matrix is formulated and characterized in the frequency domain. A numerical algorithm is provided to obtain the globally optimal inputs. In the presence of unmodelled dynamics, input design guidelines are then presented to assure robustness in addition to the original objective.

Chapter 2 Frequency Domain Approach ---- Averaging

2.1 Introduction

The method of *averaging* typically deals with a system with different time scales, such as the so called *one-time-scale systems*

$$\dot{x} = \varepsilon f(t,x) \ . \tag{2.1.1}$$

In (2.1.1) above, a small $\varepsilon > 0$ models the fact that x varies slowly in comparison with $f(t,x)$. This method, for sufficiently small $\varepsilon > 0$, relates the properties of solutions of the above systems with solutions of the autonomous "*averaged system*"

$$\dot{x}_{av} = \varepsilon f_{av}(x_{av}) \tag{2.1.2}$$

where

$$f_{av}(x) := \lim_{T \to \infty} \frac{1}{T} \int_{s}^{s+T} f(t,x)\, dt \tag{2.1.3}$$

(the limit is assumed to exist uniformly in s and x). This method was proposed originally by Bogoliuboff and Mitropolskii (1961), developed subsequently by Volosov (1962), Sethna (1973), Balachandra and Sethna (1975), Hale (1980), and stated in a geometrical form in Arnold (1982), and Guckenheimer and Holmes (1983).

Averaging methods were first introduced to the study of adaptive systems at first heuristically in the work of Krause et al (1983), and increasingly rigorously in the work of Astrom (1983), (1984), Riedle and Kokotovic (1985), (1986), Fu et al (1985), Bodson et al (1986), Mareels et al (1986), and Anderson et al (1986). Early informal use of averaging was also made in Astrom and Wittenmark (1977), and Ljung and Soderstrom (1983).

In general, an adaptive system is a nonlinear, nonautonomous system. Under the assumption that the parameter is slow, the dynamics fall into the class of systems (2.1.1) or (2.1.2)-

(2.1.3). As we shall see, averaging techniques combined with *generalized harmonic analysis* turn out to be an extremely useful tool for analyzing the stability/instability properties of this class of systems in the *frequency domain.*

In this chapter, we review some averaging results from Fu et al (1985) and Bodson et al (1986). Moreover, we develop theorems for stability including *bounded stability* (cf. Yoshizawa (1975), definition 12.1, p. 126) and *partial exponential stability*, as well as instability of systems (2.1.1) and the so called two-time-scale systems. These results will then be applied to analyze adaptive systems.

2.2 Averaging Theory for Stability of a Dynamical System

In this section, we present averaging results concerning exponential stability, and bounded stability of one- and two-time-scale systems respectively.

2.2.1 One-Time Scale Systems

We consider differential equations of the form:

$$\dot{x} = \varepsilon f(t,x,\varepsilon) \qquad x(0) = x_0 \tag{2.2.1.1}$$

where $x, x_0 \in R^n$, $t \geq 0$, $0 < \varepsilon \leq \varepsilon_0$, and f is piecewise continuous with respect to time t. We will concentrate our attention on the behavior of the solutions in a closed ball, B_r, of radius r, centered at the origin.

For small ε, the variation of x with time is slow compared to the rate of time variation of f. To apply the method of averaging to the system (2.2.1.1), the mean value of $f(t,x,0)$, $f_{av}(x)$, defined by the limit

$$f_{av}(x) = \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} f(\tau, x, 0) d\tau \tag{2.2.1.2}$$

must exist uniformly in t and x. This is formulated more precisely in the following definition.

Definition 2.2.1.1 (Mean Value of a Function, Convergence Function)

The function $f(t,x,0)$ is said to have mean value $f_{av}(x)$ if there exists a continuous function γ: $R_+ \to R_+$, strictly decreasing, such that $\gamma(T) \to 0$ as $T \to \infty$, and

$$\left\| \frac{1}{T} \int_t^{t+T} f(\tau,x,0)d\tau - f_{av}(x) \right\| \leq \gamma(T) \tag{2.2.1.3}$$

for all $t,T \geq 0$, $x \in B_r$. The function γ will be called the *convergence function.* □

Note that the function $f(t,x,0)$ has a mean value $f_{av}(x)$ if and only if the function

$$d(t,x) = f(t,x,0) - f_{av}(x) \tag{2.2.1.4}$$

has a zero mean value. The following definition (Hahn (1967), p. 7) will also be useful.

Definition 2.2.1.2 (Class K Function)

A function $\alpha : R_+ \to R_+$ belongs to class K ($\alpha \in K$), if it is continuous, strictly increasing, and $\alpha(0) = 0$. □

In the literature on averaging, it is common to assume that the function $f(t,x,\varepsilon)$ is periodic in t, or almost periodic in t. Then, the existence of the mean value is guaranteed, without further assumption (Hale (1980), theorem 6, p. 344). We do not make the assumption of (almost) periodicity, but consider instead the assumption of the existence of the mean value as the starting point of our analysis.

Remark: Note that if the function $d(t,x)$ is periodic in t and is bounded, then the integral of the function $d(t,x)$ is also a bounded function of time. This is equivalent to saying that there exists a convergence function $\gamma(T)=a/T$ (i.e. of the order of $1/T$) such that (2.2.1.3) is satisfied. On the other hand, if the function $d(t,x)$ is bounded, but is only required to be almost periodic, then the integral of the function $d(t,x)$ need not be a bounded function of time, even if its mean value is zero ([3], p 346). The function $\gamma(T)$ is bounded (by the same bound as $d(t,x)$), and converges to zero as $T \to \infty$, but the convergence function need not be bounded by a/T as $T \to \infty$ (it may be of order $1/\sqrt{T}$ for example). In general, a zero mean function need not have a bounded integral, although the converse is true. In this paper, we do not make the distinction between the periodic, and the almost periodic case, but we do distinguish the bounded integral case from the general case, and indicate the importance of the function $\gamma(T)$ in the subsequent development.

Assuming the existence of the mean value for the original system (2.2.1.1), the *averaged system* is defined to be:

$$\dot{x}_{av} = \varepsilon f_{av}(x_{av}) \qquad\qquad x_{av}(0) = x_0 \tag{2.2.1.5}$$

Note that the averaged system is autonomous and, for T fixed and ε varying, the solutions over intervals $[0, T/\varepsilon]$ are identical, modulo a simple time scaling by ε. We address the following two questions:

(i) the closeness of the response of the original and averaged systems,

(ii) the relationships between the stability properties of the two systems.

To compare the solutions of the original and of the averaged system, it is convenient to transform the original system in such a way that it becomes a *perturbed* version of the averaged system. An important lemma that leads to this result is attributed to Bogoliuboff and Mitropolskii ((1961), p 450, and Hale (1980), lemma 4, p 346). We state a generalized version of this lemma.

Lemma 2.2.1.1: (Approximate Integral of a Zero Mean Function)

If $d: R_+ \times B_r \to R^n$ is a bounded function, $d(\cdot, x)$ is piecewise continuous, and d has a zero mean value with convergence function γ,

then there exists $\xi_1 \in K$, and a function $w_\varepsilon : R_+ \times B_r \to R^n$ such that: $w_\varepsilon(0, x) = 0$ for all $x \in B_r$ and

$$\| \varepsilon w_\varepsilon(t,x) \| \leq \xi_1(\varepsilon) \tag{2.2.1.6}$$

$$\| \frac{\partial w_\varepsilon(t,x)}{\partial t} - d(t,x) \| \leq \xi_1(\varepsilon) \tag{2.2.1.7}$$

for all $t \geq 0$, $x \in B_r$.

Moreover, if $\gamma(T) = a/T^b$ for some $a \geq 0$, $b \in (0,1]$, then $\xi_1(\varepsilon)$ can be chosen to be $2a\varepsilon^b$. □

Proof: See Appendix A.

Remarks:

(1) The construction of the function $w_\varepsilon(t,x)$ is identical to that in Bogoliuboff and Mitropolskii (1961), but the proof of (2.2.1.6), (2.2.1.7) is different, and leads to the relationship between the convergence function $\gamma(T)$ and the function $\xi_1(\varepsilon)$.

(2) The main point of Lemma 2.2.1 is that, although the exact integral of $d(t,x)$ may be an unbounded function of time, there exists a bounded function $w_\varepsilon(t,x)$, whose first partial derivative with respect to t is arbitrarily close to $d(t,x)$. Although the bound on $w_\varepsilon(t,x)$ may increase as $\varepsilon \to 0$, it increases slower than $1/\varepsilon$, as indicated by (2.2.1.6).

It is necessary to obtain a function $w_\varepsilon(t,x)$, as in Lemma 2.2.1, that has some additional smoothness properties. A useful lemma is given by Hale ((1980) Lemma 5, p. 349). For the price of additional assumptions on the function $d(t,x)$, the following lemma leads to stronger conclusions that are useful in the sequel.

Lemma 2.2.1.2: (Smooth Approximate Integral of a Zero Mean Lipschitz Function)

If $d: R_+ \times B_r \to R^n$ satisfies the following assumptions:

(i) $d(\cdot,x)$ is piecewise continuous, $d(t,\cdot)$ has bounded and continuous first derivatives, and $d(t,0) = 0$ for all $t \geq 0$,

(ii) $d(t,x)$ has a zero mean value, with convergence function $\gamma \|x\|$, and $\frac{\partial d(t,x)}{\partial x}$ has a zero mean value, with convergence function γ,

then there exist $\xi_1 \in K$ and a function $w_\varepsilon : R_+ \times B_r \to R^n$ such that $w_\varepsilon(0,x)=0$, and

$$\| \varepsilon w_\varepsilon(t,x) \| \le \xi_1(\varepsilon) \|x\| \tag{2.2.1.8}$$

$$\| \frac{\partial w_\varepsilon(t,x)}{\partial t} - d(t,x) \| \le \xi_1(\varepsilon) \|x\| \tag{2.2.1.9}$$

$$\| \varepsilon \frac{\partial w_\varepsilon(t,x)}{\partial x} \| \le \xi_1(\varepsilon) \tag{2.2.1.10}$$

for all $t \ge 0$, and for all $x \in B_r$.

Moreover, if $\gamma(T) = a/T^b$ for some $a \ge 0$, $b \in (0,1]$, then $\xi_1(\varepsilon)$ can be chosen to be $2a\varepsilon^b$. □

Proof: See Appendix A.

Remarks:

(1) The difference from Lemma 2.2.1.1 is in the condition on the partial derivative of $w_\varepsilon(t,x)$ with respect to x in (2.2.1.10), and the dependence on $\|x\|$ in (2.2.1.8), (2.2.1.9).

(2) Note that if the original system is linear, i.e.

$$\dot{x} = A(t)\, x \qquad x(0) = x_0 \tag{2.2.1.11}$$

for some $A : R_+ \to R^{n \times n}$, then the main assumption of Lemma 2.2.1.2 is that there exists A_{av} such that $A(t) - A_{av}$ has a zero mean value.

Given some ε_0, $r > 0$, the following assumptions will hereafter be in effect.

Assumptions:

(A1) $x=0$ is an equilibrium point of system (2.2.1.1), and $f(t,x,\varepsilon)$ is Lipschitz in x, i.e. $f(t,0,\varepsilon)=0$, and for some $l_1 \ge 0$

$$\| f(t,x_1,\varepsilon) - f(t,x_2,\varepsilon) \| \le l_1 \| x_1 - x_2 \| \tag{2.2.1.12}$$

for all $t \geq 0$, $x_1, x_2 \in B_r$, $\varepsilon \leq \varepsilon_0$.

(A2) $f(t,x,\varepsilon)$ is Lipschitz in ε, linearly in x, i.e. for some $l_2 \geq 0$

$$\| f(t,x,\varepsilon_1) - f(t,x,\varepsilon_2) \| \leq l_2 \|x\| \, |\varepsilon_1 - \varepsilon_2| \tag{2.2.1.13}$$

for all $t \geq 0$, $x \in B_r$, $\varepsilon_1, \varepsilon_2 \leq \varepsilon_0$.

(A3) $f_{av}(0)=0$, and $f_{av}(x)$ has continuous and bounded first partial derivative with respect to x, for all $x \in B_r$, so that for some $l_{av} \geq 0$

$$\| f_{av}(x_1) - f_{av}(x_2) \| \leq l_{av} \| x_1 - x_2 \| \tag{2.2.1.14}$$

for all x_1, $x_2 \in B_r$.

(A4) The function $d(t,x)=f(t,x,0)-f_{av}(x)$ satisfies the conditions of Lemma 2.2.1.2.

Remark: Note particularly that the equilibrium points of the original and the averaged systems are coincident at $x=0$.

Lemma 2.2.1.3: (Perturbation Formulation of Averaging with Coincident Equilibrium Point)

If the systems (2.2.1.1) and (2.2.1.5) satisfy assumptions (A1)-(A4),

then there exist functions w_ε and ξ_1, as in Lemma 2.2.1.2, and a transformation of the form,

$$x = z + \varepsilon w_\varepsilon(t,z) \tag{2.2.1.15}$$

under which system (2.2.1.1) becomes

$$\dot{z} = \varepsilon f_{av}(z) + \varepsilon p(t,z,\varepsilon) \qquad z(0)=x_0 \tag{2.2.1.16}$$

where $p(t,z,\varepsilon)$ satisfies

$$\| p(t,z,\varepsilon) \| \leq \psi_1(\varepsilon) \| z \| \tag{2.2.1.17}$$

for some $\psi_1 \in K$, $\varepsilon_1 > 0$, and for all $\varepsilon \leq \varepsilon_1$. Further, $\psi_1(\varepsilon)$ is of the order of $\varepsilon + \xi_1(\varepsilon)$. □

Proof: See Appendix A.

Remarks:

(1) A similar lemma can be found in Hale (1980) (Lemma 3.2, p 192). Inequality (2.2.1.17) is a Lipschitz type of condition on $p(t,z,\varepsilon)$, which is not found in Hale (1980), and results from the stronger conclusions of Lemma 2.2.1.2.

(2) Lemma 2.2.1.3 is fundamental to the theory of averaging presented in the following. It separates the error in the approximation of the original system by the averaged system $(x-x_{av})$ into two components: $(x-z)$ and $(z-x_{av})$. The first component results from a pointwise (in time) transformation of variable and is guaranteed to be small by inequality (2.2.1.8). For ε sufficiently small ($\varepsilon \le \varepsilon_1$), the transformation $z \to x$ is invertible, and as $\varepsilon \to 0$, it tends to the identity transformation. The second component is due to the perturbation term $p(t,z,\varepsilon)$. Inequality (2.2.1.17) guarantees that this perturbation is small as $\varepsilon \to 0$.

(3) At this point, we can relate the convergence of the function $\gamma(T)$ to the order of the two components of the error $(x-x_{av})$ in the approximation of the original system by the averaged system. The relationship between the functions $\gamma(T)$ and $\xi_1(\varepsilon)$ was indicated in Lemma 2.2.1.1. Lemma 2.2.1.3 relates the function $\xi_1(\varepsilon)$ to the error due to the averaging. If $d(t,x)$ has a bounded integral (i.e. $\gamma(T) \sim 1/T$), then both $(x-z)$ and $p(t,z,\varepsilon)$ are of the order of ε with respect to the main term $f_{av}(z)$. In general, these terms go to zero as $\varepsilon \to 0$, but possibly more slowly than linearly (as $\sqrt{\varepsilon}$ for example). The proof of Lemma 2.2.1.1 provides a direct relationship between the order of the convergence to the mean value, and the order of the error terms.

We now focus attention on the approximation of the original system by the averaged system. We will need the following assumption:

Assumption:

(A5) $\| x_0 \|$ is small enough that, for fixed T and some $r' < r$, $x_{av}(t) \in B_{r'}$ for all $t \in [0, T/\varepsilon]$ (this is possible, from (A3)).

Theorem 2.2.1.4: (Basic Averaging Theorem)

If the original system (2.2.1.1) and the averaged system (2.2.1.5) satisfy assumptions (A1)-(A5),

then there exists ψ_1 as in Lemma 2.2.1.3 such that, given $T \geq 0$,

$$\| x(t) - x_{av}(t) \| \leq \psi_1(\varepsilon)\, b_T \tag{2.2.1.18}$$

for some b_T, $\varepsilon_T > 0$, and for all $t \in [0, T/\varepsilon]$, $\varepsilon \leq \varepsilon_T$. □

Proof: From Lemma 2.2.1.2 and Lemma 2.2.1.3, we have that

$$\| x - z \| \leq \xi(\varepsilon) \| z \| \leq \psi_1(\varepsilon) \| z \| \tag{2.2.1.19}$$

for $\varepsilon \leq \varepsilon_1$. On the other hand, we have that

$$\frac{d}{dt}(z - x_{av}) = \varepsilon\,(f_{av}(z) - f_{av}(x_{av})) + \varepsilon\, p(t, z, \varepsilon) \qquad z(0) - x_{av}(0) = 0 \tag{2.2.1.20}$$

for all $t \in [0, T/\varepsilon]$, $x_{av} \in B_{r'}$, $r' < r$. We will now show that, on this time interval, and for as long as $x, z \in B_r$, the errors $(z - x_{av})$ and $(x - x_{av})$ can be made arbitrarily small by reducing ε. Integrating (2.2.1.20),

$$\| z(t) - x_{av}(t) \| \leq \varepsilon l_{av} \int_0^t \| z(\tau) - x_{av}(\tau) \|\, d\tau + \varepsilon\, \psi_1(\varepsilon) \int_0^t \| z(\tau) \|\, d\tau\,. \tag{2.2.1.21}$$

Using the generalized Bellman-Gronwall Lemma,

$$\| z(t)-x_{av}(t) \| \leq \varepsilon\psi_1(\varepsilon)\int_0^t \| z(\tau) \| e^{\varepsilon l_{av}(t-\tau)}d\tau \leq \psi_1(\varepsilon)r\left[\frac{e^{\varepsilon l_{av}T}-1}{l_{av}}\right]$$

$$=: \psi_1(\varepsilon)\,a_T\,. \tag{2.2.1.22}$$

Combining these results,

$$\begin{aligned}\| x(t)-x_{av}(t) \| &\leq \| x(t)-z(t) \| + \| z(t)-x_{av}(t) \| \\ &\leq \psi_1(\varepsilon) \| x_{av}(t) \| + (1+\psi_1(\varepsilon)) \| z(t)-x_{av}(t) \| \\ &\leq \psi_1(\varepsilon)\,(r+(1+\psi_1(\varepsilon_1))\,a_T) \\ &=: \psi_1(\varepsilon)b_T\,.\end{aligned} \tag{2.2.1.23}$$

By assumption, $\| x_{av}(t) \| \leq r' < r$. Let ε_T (with $0<\varepsilon_T\leq\varepsilon_1$) be such that $\psi_1(\varepsilon_T)\,b_T<r-r'$. It follows, from a simple contradiction argument, that the estimate in (2.2.1.23) is valid for all $t\in[0,T/\varepsilon]$, whenever $\varepsilon\leq\varepsilon_T$.

■

Remarks:

(1) Theorem 2.2.1.4 establishes that the trajectories of the original and the averaged system are arbitrarily close on intervals $[0,T/\varepsilon]$, as ε is reduced. The error is of the order of $\psi_1(\varepsilon)$, and the order is related to the order of convergence of $\gamma(T)$. If $d(t,x)$ has a bounded integral (i.e. $\gamma(T)\sim 1/T$), then the error is of the order of ε.

(2) It is important to remember that, although the intervals $[0,T/\varepsilon]$ are unbounded, Theorem 2.2.1.4 does not state that

$$\| x(t)-x_{av}(t) \| \leq \psi_1(\varepsilon)\,b_\infty \tag{2.2.1.24}$$

for all $t\geq 0$, and some finite $b_\infty>0$. Consequently, Theorem 2.2.1.4 does not allow us to relate the stability of the original and of the averaged system. This relationship is investigated in Theorem 2.2.1.5, after a preliminary definition.

Definition 2.2.1.3: (Exponential Stability, Rate of Convergence)

The equilibrium point $x=0$ of a differential equation is said to be *exponentially stable*, with *rate of convergence* $\alpha(\alpha>0)$, if

$$\| x(t) \| \leq m \| x(t_0) \| e^{-\alpha(t-t_0)} \tag{2.2.1.25}$$

for all $t \geq t_0 \geq 0$, $x(t_0) \in B_{r_0}$ for some $r_0 > 0$, and some $m \geq 1$. □

In the following, we assume that $r_0 \leq r/m$, so that all trajectories are guaranteed to remain in B_r.

Theorem 2.2.1.5: (Exponential Stability Theorem)

If the original and averaged systems satisfy assumptions (A1)-(A5), and $x_{av}=0$ is an exponentially stable equilibrium point of the averaged system,

then there exists $\varepsilon_2 > 0$ such that the equilibrium point $x=0$ of the original system is exponentially stable for all $\varepsilon \leq \varepsilon_2$. □

Proof: The proof relies on a converse theorem of Lyapunov for exponentially stable systems (see, for example, Hahn (1967) p. 273). Under the hypotheses, there exists a function $v: R^n \to R_+$, and non-zero positive constants α_1, α_2, α_3, α_4, such that, for all $x_{av} \in B_r$,

$$\alpha_1 \| x_{av} \|^2 \leq v(x_{av}) \leq \alpha_2 \| x_{av} \|^2 \tag{2.2.1.26}$$

$$\dot{v}(x_{av}) \leq -\varepsilon \alpha_3 \| x_{av} \|^2 \tag{2.2.1.27}$$

$$\left\| \frac{\partial v}{\partial x_{av}} \right\| \leq \alpha_4 \| x_{av} \| \tag{2.2.1.28}$$

The derivative in (2.2.1.27) is to be taken along the trajectories of the averaged system (2.2.1.5), i.e., by the chain rule,

$$\dot{v}(x_{av}) = \frac{d\,v(x_{av})}{d\,x_{av}}\, \dot{x}_{av}$$

where $\dot{x}_{av}$ is given in (2.2.1.5). The function v is now used to study the stability of the perturbed system (2.2.1.16). Considering $v(z)$, inequalities (2.2.1.26) and (2.2.1.28) are still verified, with z replacing x_{av}. The derivative of $v(z)$ along the trajectories of (2.2.1.16) is given by

$$\dot{v}(z) = \dot{v}(x_{av}) \Big|_{x_{av}=z} + (\frac{\partial v}{\partial z})(\varepsilon p(t,z,\varepsilon)) \tag{2.2.1.29}$$

and, using previous inequalities (including those from Lemma 2.2.1.3),

$$\begin{aligned} \dot{v}(z) &\leq -\varepsilon\alpha_3 \| z \|^2 + \varepsilon\alpha_4 \psi_1(\varepsilon) \| z \|^2 \\ &\leq -\varepsilon \left[\frac{\alpha_3 - \psi_1(\varepsilon)\,\alpha_4}{\alpha_2} \right] v(z) \end{aligned} \tag{2.2.1.30}$$

for all $\varepsilon \leq \varepsilon_1$. Let ε'_2 be such that $\alpha_3 - \psi_1(\varepsilon'_2)\,\alpha_4 > 0$, and define $\varepsilon_2 = \min(\varepsilon_1, \varepsilon'_2)$. Denote

$$\alpha(\varepsilon) := \frac{\alpha_3 - \psi_1(\varepsilon)\alpha_4}{2\alpha_2} . \tag{2.2.1.31}$$

Consequently, (2.2.1.30) implies that

$$v(z) \leq v(z(t_0))\, e^{-2\varepsilon\alpha(\varepsilon)(t-t_0)} \tag{2.2.1.32}$$

and

$$\| z(t) \| \leq \sqrt{\alpha_2 / \alpha_1}\, \| z(t_0) \|\, e^{-\varepsilon\alpha(\varepsilon)(t-t_0)} . \tag{2.2.1.33}$$

Since $\alpha(\varepsilon) > 0$ for all $\varepsilon \leq \varepsilon_2$, system (2.2.1.16) is exponentially stable. Using (2.2.1.8) and (2.2.1.15), it follows that

$$\| x(t) \| \leq \frac{1+\xi_1(\varepsilon)}{1-\xi_1(\varepsilon)} \sqrt{\alpha_2 / \alpha_1}\, \| x(t_0) \|\, e^{-\varepsilon\alpha(\varepsilon)(t-t_0)} \tag{2.2.1.34}$$

for all $t \geq t_0 \geq 0$, $\varepsilon \leq \varepsilon_2$, and $x(t_0)$ sufficiently small that all signals remain in B_r. In conclusion, the original system is exponentially stable, with rate of convergence (at least) $\varepsilon\alpha(\varepsilon)$.

■

Remarks:

(1) Theorem 2.2.1.5 is a *local* exponential stability result. The original system will be *globally* exponentially stable, if the averaged system is globally exponentially stable, and provided that all assumptions are valid globally.

(2) The proof of Theorem 2.2.1.5 gives a useful bound on the rate of convergence of the original system. As ε tends to zero, $\varepsilon\alpha(\varepsilon)$ tends to $\frac{\varepsilon}{2}\frac{\alpha_3}{\alpha_2}$, which is the bound on the rate of convergence of the averaged system that one would obtain using (2.2.1.26)-(2.2.1.27). In other words, the proof provides a bound on the rate of convergence, and this bound gets arbitrarily close to the corresponding bound for the averaged system, provided that ε is sufficiently small. This is a useful conclusion because it is in general very difficult to obtain a guaranteed rate of convergence for the original, nonautonomous system. The proof assumes the existence of a Lyapunov function satisfying (2.2.1.26)-(2.2.1.28), but does not depend on the specific function chosen. Since the averaged system is autonomous, such a function is usually easier to find than for the original system, and any such function will provide a bound on the rate of convergence of the original system for ε sufficiently small.

(3) The conclusion of Theorem 2.2.1.5 is quite different from the conclusion of Theorem 2.2.1.4. Since both x and x_{av} go to zero exponentially with t, the error $x-x_{av}$ also goes to zero exponentially with t. Yet, Theorem 2.2.1.5 does not relate the bound on the error to ε. It is possible, however, to combine Theorem 2.2.1.4 and Theorem 2.2.1.5 to obtain a uniform approximation result, with an estimate similar to (2.2.1.24).

Now we consider the same system (2.2.1.1), but $x=0$ may not be an equilibrium point of that system. Conceivably, only boundedness of $x(t)$ instead of its exponential stability should be expected. This will be established in Theorem 2.2.1.9 after some preliminary lemmas, of which the first two lemmas are variations of Lemma 2.2.1.2-2.2.1.3. To start with, we modify some of the assumptions on f to indicate the change of system property as just mentioned.

Modified Assumptions:

(MA1) $f(t,x,\varepsilon)$ satisfies (A1) except that $x=0$ is no longer an equilibrium point of the system (2.2.1.1).

(MA2) $f(t,x,\varepsilon)$ is Lipschitz in ε, i.e. for some $l_2 \geq 0$,

$$\| f(t,x,\varepsilon_1) - f(t,x,\varepsilon_2) \| \leq l_2 \, | \varepsilon_1 - \varepsilon_2 | \tag{2.2.1.35}$$

for all $t \geq 0$, $x \in B_r$, and ε_1, $\varepsilon_2 \leq \varepsilon_0$.

(MA4) The function $d(t,x)=f(t,x,0)-f_{av}(x)$ satisfies conditions of the following Lemma 2.2.1.6.

Lemma 2.2.1.6: (Smooth Approximate Integral of a Zero Mean Non-Lipschitz Function)

If $d: R_+ \times B_r \rightarrow R^n$ satisfies the following assumptions:

(i) $d(\cdot,x)$ is piecewise continuous, and has a zero mean value, with convergence function γ.

(ii) $\dfrac{\partial d(t,x)}{\partial x}$ has a zero mean value, with convergence function γ,

then there exist $\xi_1 \in K$, $h>0$, and a function $w_\varepsilon : R_+ \times B_r \rightarrow R^n$ such that $w_\varepsilon(0,x)=0$, and

$$\| \varepsilon w_\varepsilon(t,x) \| \leq h\, r \xi_1(\varepsilon) \tag{2.2.1.36}$$

$$\| \frac{\partial w_\varepsilon(t,x)}{\partial t} - d(t,x) \| \leq h\, r \xi_1(\varepsilon) \tag{2.2.1.37}$$

$$\| \varepsilon \frac{\partial w_\varepsilon(t,x)}{\partial x} \| \leq \xi_1(\varepsilon) \tag{2.2.1.38}$$

for all $t \geq 0$, and for all $x \in B_r$.

Moreover, if $\gamma(T)=a/T^b$ for some $a \geq 0$, $b \in (0,1]$, then $\xi_1(\varepsilon)$ can be chosen to be $2a\varepsilon^b$. □

Proof: See Appendix A.

Remark: The difference from Lemma 2.2.1.3 is mainly in the condition that (2.2.1.35), (2.2.1.36) are no longer Lipschitz in x. In other words, the transformation (2.2.1.15) given in Lemma 2.2.1.3 may not be invariant at the origin.

Lemma 2.2.1.7: (Perturbation Formulation of Averaging with Non-Coincident Equilibrium Point)

If the systems (2.2.1.1) and (2.2.1.5) satisfy assumptions (MA1)-(MA2), (A3), and (MA4),

then there exist functions w_ε and ξ_1, as in Lemma 2.2.1.6, and a transformation of the form,

$$x = z + \varepsilon w_\varepsilon(t,z) \tag{2.2.1.39}$$

under which system (2.2.1.1) becomes

$$\dot{z} = \varepsilon f_{av}(z) + \varepsilon p(t,z,\varepsilon) \qquad z(0)=x_0 \tag{2.2.1.40}$$

where $p(t,z,\varepsilon)$ satisfies

$$\| p(t,z,\varepsilon) \| \leq r\psi_2(\varepsilon) \tag{2.2.1.41}$$

for some $\psi_2 \in K$, $\varepsilon_1 > 0$, and for all $\varepsilon \leq \varepsilon_1$. Further, $\psi_2(\varepsilon)$ is of the order of $\varepsilon + \xi_1(\varepsilon)$. □

Proof: See Appendix A.

Definition 2.2.1.3: (Uniformly Ultimately Bounded (UUB) for a Bound β)

The solutions of system (2.2.1.1) are *uniformly ultimately bounded* for a bound β if for any x_0 there exists $0 < T < \infty$ such that

$$\| x(t) \| \leq \beta \qquad t \geq t_0 + T \tag{2.2.1.42}$$

□

Lemma 2.2.1.8: (UUB lemma)

Consider a set of differential equations:

$$\dot{x} = \varepsilon f(x) \qquad x(t_0) = x_0 \in R^n \tag{2.2.1.43}$$

$$\dot{\hat{x}} = \varepsilon f(\hat{x}) + \varepsilon g(t,\hat{x}) \qquad \hat{x}(t_0) = x_0 \tag{2.2.1.44}$$

satisfying

(i) $x=0$ is an exponentially stable equilibrium point of (2.2.1.43),

(ii) $g(t,\hat{x}) \le l_g$ for some $l_g > 0$, for all $t \ge 0$ and $\hat{x} \in R^n$.

Then the solution $\hat{x}(t)$ of (2.2.1.44) is UUB for a bound $k_1 l_g$ for some $k_1 > 0$. Moreover, there exists a $k_2 > 0$ such that for all $x_0 \in R^n$ and for all $t \ge t_0$

$$\| \hat{x}(t) \| \le \max(k_1 l_g , k_2 \| x_0 \|) . \tag{2.2.1.45}$$

□

Proof: cf. Bodson and Sastry (1985).

Remarks:

(1) In the lemma, k_1 and k_2 can be quantified if one uses a converse theorem of Lyapunov for the exponentially stable system, i.e. there exists a function $v : R^n \to R_+$ which satisfies (2.2.1.26), (2.2.1.28) and

$$\dot{v}(x) \le - \alpha_3 \| x \|^2 . \tag{2.2.1.46}$$

Then k_1 and k_2 can be chosen as

$$k_1 := \sqrt{\frac{\alpha_2}{\alpha_1}} \frac{\alpha_4}{\alpha_3} + \kappa \qquad \text{and} \qquad k_2 := \sqrt{\frac{\alpha_2}{\alpha_1}} \tag{2.2.1.47}$$

where κ can be any non-zero positive number.

(2) Although this lemma is stated in a global form, its local version also exists. The differences are: conditions (i)-(ii) are satisfied locally in, for example, B_r so that for some

$o<r_0<r$, $x(t)$, $\hat{x}(t)\in B_r$ for all $t\geq t_0$ whenever $x_0\in B_{r_0}$, which is possible because of (2.2.1.45) and if $k_1 l_g<r$.

Theorem 2.2.1.9: (Bounded Stability Theorem)

If the original system (2.2.1.1) and the averaged system (2.2.1.5) satisfies assumptions (MA1)-(MA2), (A3), (MA4), and $x_{av}=0$ is an exponentially stable equilibrium point of the averaged system,

then there exists $\beta_1\in K$, $T>0$, and $\varepsilon_2>0$ such that the solution $x(t)$ of the original system is UUB for all $\varepsilon\leq\varepsilon_2$, and

$$\| x(t) \| \leq \beta_1(\varepsilon) \qquad t\geq t_0+T \tag{2.2.1.48}$$

for sufficiently small $x(t_0)$. □

Proof: Since, by hypothesis, $x_{av}=0$ is an exponentially stable equilibrium point, Lemmas 2.2.1.7-2.2.1.8 directly imply that $z(t)$ of the perturbed system (2.2.1.40) is UUB for a bound $kr\psi_2(\varepsilon)$ for some $k>0$. It then follows from (2.2.1.36), (2.2.1.39), and (2.2.1.45) that

$$\begin{aligned} \| x(t) \| &\leq \| z(t) \| + hr\xi_1(\varepsilon) \\ &\leq kr\psi_2(\varepsilon) + hr\xi_1(\varepsilon) \\ &=: \beta_1(\varepsilon) \qquad t\geq t_0+T \end{aligned} \tag{2.2.1.49}$$

for some $T>0$, sufficiently small $x(t_0)$, and for all $\varepsilon\leq\varepsilon_2$, for some $\varepsilon_2>0$.

■

Remark: If system (2.2.1.1) is of the form,

$$\dot{x} = \varepsilon A(t)x + \varepsilon g(t) \qquad x(t_0)=x_0 \tag{2.2.1.50}$$

for some $A:R_+\to R^{n\times n}$ and $g:R_+\to R^n$, with its averaged system (2.2.1.5) of the form,

$$\dot{x}_{av} = \varepsilon A_{av}x_{av} + \varepsilon g_{av} \tag{2.2.1.51}$$

then main assumptions in the theorem require (i) $A(t)$ and $g(t)$ be bounded time functions, (ii) $g_{av}\equiv 0$, and (iii) A_{av} be Hurwitz.

After discussions on system (2.2.1.1) with different properties in Theorem 2.2.1.5 and 2.2.1.9, we now focus our attention on the same system but with its property being a mixture of both. Consider systems of the form,

$$\dot{x} = \varepsilon f_1(t,x,y,\varepsilon) \qquad x(0)=x_0 \in R^n \tag{2.2.1.52}$$

$$\dot{y} = \varepsilon f_2(t,y,\varepsilon) \qquad y(0)=y_0 \in R^m \tag{2.2.1.53}$$

with the corresponding averaged systems,

$$\dot{x}_{av} = \varepsilon f_{1av}(x_{av}, y_{av}) \qquad x(0)=x_0 \tag{2.2.1.54}$$

$$\dot{y}_{av} = \varepsilon f_{2av}(y_{av}) \qquad y(0)=y_0 \tag{2.2.1.55}$$

The following are assumptions about f_i and f_{iav}, i = 1,2, for some given $\varepsilon_0>0$, $r>0$, and $y^o \in R^m$.

Assumptions:

(A6) $f_1(t,x,y,\varepsilon)$ is Lipschitz in y, linearly in x, i.e. for some $l_3 \ge 0$

$$\| f_1(t,x,y_1,\varepsilon) - f_1(t,x,y_2,\varepsilon) \| \le l_3 \; \| x \| \; \| y_1 - y_2 \| \tag{2.2.1.56}$$

for all $t\ge 0$, $x \in B_r$, $y_1, y_2 \in B_r(y^o)$, and $\varepsilon \le \varepsilon_0$.

(A7) If y is treated as a time function $y(t)$, then $f_1(t,x,y(t),\varepsilon)$ satisfies (A1)-(A2).

(A8) f_2 satisfies (MA1)-(MA2).

(A9) $(0,y^o)$ is an equilibrium point of the averaged system, i.e. $f_{1av}(0,y^o)=0$ and $f_{2av}(y^o)=0$, f_{2av} satisfies (A3), and for some $\hat{l}_{av} \ge 0$

$$\| f_{1av}(x_1,y) - f_{1av}(x_2,y) \| \le \hat{l}_{av} \; \| x_1 - x_2 \| \tag{2.2.1.57}$$

for all $x_1, x_2 \in B_r$, and $y \in B_r(y^o)$.

(A10) The function $d_y(t,x)=f_1(t,x,y,0)-f_{1av}(x,y)$ satisfies conditions of Lemma 2.2.1.2 for any $y\in B_r(y^o)$, and $d(t,y)=f_2(t,y,0)-f_{2av}(y)$ satisfies conditions of Lemma 2.2.1.6.

Theorem 2.2.1.10: (Partial Exponential Stability Theorem)

If the original system (2.2.1.52)-(2.2.1.53) and its averaged system (2.2.1.54)-(2.2.1.55) satisfy assumptions (A6)-(A10), $y_{av}=y^o$ is an exponentially stable equilibrium point of (2.2.1.55), and there exists a function $v: R^n\times R^m\to R_+$, and non-zero positive constants α_1, α_2, α_3, α_4, α_5, such that, for all $x_{av}\in B_r$, $y_{av}\in B_r(y^o)$,

$$\alpha_1 \| x_{av} \|^2 \le v(x_{av}, y_{av}) \le \alpha_2 \| x_{av} \|^2 \tag{2.2.1.58}$$

$$\left\| \frac{\partial v}{\partial x_{av}} \right\| \le \alpha_3 \| x_{av} \| \tag{2.2.1.59}$$

$$\left\| \frac{\partial v}{\partial y_{av}} \right\| \le \alpha_4 \| x_{av} \|^2 \tag{2.2.1.60}$$

$$\dot v(x_{av}, y_{av}) \le -\varepsilon\alpha_5 \| x_{av} \|^2 \tag{2.2.1.61}$$

where the derivative is taken along the trajectories (2.2.1.54)-(2.2.1.55),

then there exists $\varepsilon_3>0$ such that $x(t)$ of the original system converges to zero exponentially whenever $\varepsilon\le\varepsilon_3$. □

Proof: First of all, by a Lyapunov theorem on exponential stability, the above assumptions readily imply the exponential stability of the averaged system (2.2.1.54)-(2.2.1.55). Moreover, by a proof similar to that of Lemmas 2.2.1.2-2.2.1.3, and Lemmas 2.2.1.6-2.2.1.7, using in particular (A10), it follows that there exists a transformation of the form,

$$x = z_1 + \varepsilon\, w_{1\varepsilon}(t, z_1, z_2) \tag{2.2.1.62}$$

$$y = z_2 + \varepsilon\, w_{2\varepsilon}(t, z_2) \tag{2.2.1.63}$$

under which system (2.2.1.52)-(2.2.1.53) becomes

$$\dot{z}_1 = \varepsilon f_{1av}(z_1, z_2) + \varepsilon p_{11}(t, z_1, z_2, \varepsilon) \qquad z_1(0) = x_0 \tag{2.2.1.64}$$

$$\dot{z}_2 = \varepsilon f_{2av}(z_2) + \varepsilon p_{12}(t, z_2, \varepsilon) \qquad z_2(0) = y_0 \tag{2.2.1.65}$$

where $\varepsilon w_{1\varepsilon}$, $\varepsilon w_{2\varepsilon}$, and p_{11}, p_{12} satisfy

$$\| \varepsilon w_{1\varepsilon}(t, z_1, z_2) \| \leq \xi_1(\varepsilon) \| z_1 \| \qquad \text{and} \qquad \| \varepsilon w_{2\varepsilon}(t, z_2) \| \leq h r \xi_1(\varepsilon) \tag{2.2.1.66}$$

$$\| p_{11}(t, z_1, z_2, \varepsilon) \| \leq \psi_1(\varepsilon) \| z_1 \| \qquad \text{and} \qquad \| p_{12}(t, z_2, \varepsilon) \| \leq r \psi_2(\varepsilon) \tag{2.2.1.67}$$

for all $z_1 \in B_r$, $z_2 \in B_r(y_o)$, $\varepsilon \leq \varepsilon_1$. We now take an approach similar to that in the proof of Theorem 2.2.1.5. The function v is used to study the stability of the perturbed system (2.2.1.64)-(2.2.1.65). Considering $v(z_1, z_2)$, we have inequalities (2.2.1.58)-(2.2.1.60) still in effect, with z_1 and z_2 replacing x_{av} and y_{av} respectively. The derivative of v along trajectories of (2.2.1.64)-(2.2.1.65) is given by

$$\dot{v}(z_1, z_2) = \dot{v}(x, y) \Big|_{(x,y)=(z_1,z_2)} + \left(\frac{\partial v}{\partial z_1}\right) \varepsilon p_{11}(t, z_1, z_2, \varepsilon)$$

$$+ \left(\frac{\partial v}{\partial z_2}\right) \varepsilon p_{12}(t, z_1, z_2, \varepsilon) \tag{2.2.1.68}$$

where $\dot{v}(x, y)$ is taken along the trajectories (2.2.1.52)-(2.2.1.53), and can be simplified by (2.2.1.61), (2.2.1.66)-(2.2.1.67) as

$$\dot{v}(z_1, z_2) \leq -\varepsilon \alpha_5 \| z_1 \|^2 + \varepsilon \alpha_3 \psi_1(\varepsilon) \| z_1 \|^2 + \varepsilon \alpha_4 \psi_2(\varepsilon) \| z_1 \|^2$$

$$= -\varepsilon \left[\alpha_5 - \alpha_3 \psi_1(\varepsilon) - \alpha_4 r \psi_2(\varepsilon)) \right] \| z_1 \|^2 . \tag{2.2.1.69}$$

Denote

$$\alpha(\varepsilon) := \left[\frac{\alpha_5 - \alpha_3 \psi_1(\varepsilon) - \alpha_4 r \psi_2(\varepsilon)}{2 \alpha_2} \right], \tag{2.2.1.70}$$

then $\alpha(\varepsilon) \to \dfrac{\alpha_5}{2\alpha_2}$ as $\varepsilon \to 0$. Let ε_3 be such that $\alpha(\varepsilon_3) > 0$, and z_2 (and hence y) remains in $B_r(y^o)$ for sufficiently small $(y(t_0) - y^o)$ whenever $\varepsilon \leq \varepsilon_3$, as has been guaranteed by Theorem 2.2.1.9. Thus it follows that

$$\dot{v}(z_1,z_2) \leq -2\,\alpha(\varepsilon)\; v(z_1,z_2) \qquad\qquad t \geq t_0 \tag{2.2.1.71}$$

which, by a proof similar to that of Theorem 2.2.1.5, implies that $x(t)$ converges to zero exponentially for all $\varepsilon \leq \varepsilon_3$ and for sufficiently small $x(t_0)$, with rate of convergence (at least) $\varepsilon\,\alpha(\varepsilon)$.

■

2.2.2 Two-Time Scale Systems

Systems of the form (2.2.1.5) studied in section 2.2.1 are to be thought of as one time scale systems in that the entire state variable x is varying slowly in comparison with the rate of time variation of the right hand side of the differential equation. In this section, we will study averaging for the case when only some of the state variables are slowly varying.

2.2.2.1 Decoupled Time-Scales

We consider a class of systems of the differential equation

$$\dot{x} = \varepsilon f(t,x,y,\varepsilon) \qquad\qquad x(0)=x_0 \tag{2.2.2.1}$$

$$\dot{y} = A(x)y + \varepsilon\, g(t,x,y,\varepsilon) \qquad\qquad y(0)=y_0 \tag{2.2.2.2}$$

where $x \in R^n$ is called the *slow* state, $y \in R^m$ is called the *fast* state, and f, g are piecewise continuous functions of time. It can be seen that the system (2.2.2.1)-(2.2.2.2) are decoupled when $\varepsilon=0$.

As previously, we define the limit

$$f_{av}(x) = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} f(\tau,x,0,0)\,d\tau \tag{2.2.2.3}$$

and assume that the limit exists uniformly in t and x. Then the averaged system of the system (2.2.2.1) and (2.2.2.2) is given by

$$\dot{x}_{av} = \varepsilon f_{av}(x_{av}) \qquad\qquad x_{av}(0)=x_0\,. \tag{2.2.2.4}$$

Additionally, for some $r>0$, $\varepsilon_0>0$ we make the following assumptions:

Assumptions

(B1) $(x,y)=(0,0)$ is an equilibrium point of the system (2.2.2.1) and (2.2.2.2), i.e. $f(t,0,0,\varepsilon)=0$ and $g(t,0,0,\varepsilon)=0$ for all $t\geq 0$, and both f and g are Lipschitz in x and y, i.e.

$$\| f(t,x_1,y_1,\varepsilon) - f(t,x_2,y_2,\varepsilon) \| \leq l_1 \| x_1-x_2 \| + l_2 \| y_1-y_2 \| \tag{2.2.2.5}$$

$$\| g(t,x_1,y_1,\varepsilon) - g(t,x_2,y_2,\varepsilon) \| \leq l_3 \| x_1-x_2 \| + l_4 \| y_1-y_2 \| \tag{2.2.2.6}$$

for all $t\geq 0$, x_1, $x_2\in B_r$, y_1, $y_2\in B_r$, and $\varepsilon\leq\varepsilon_0$.

(B2) $f_{av}(0)=0$, and f_{av} has continuous and bounded first partial derivatives with respect to x for all $x\in B_r$, so that for some $l_{av}\geq 0$

$$\| f_{av}(x_1) - f_{av}(x_2)\| \leq l_{av} \| x_1-x_2 \| \tag{2.2.2.7}$$

for all x_1, $x_2\in B_r$.

(B3) $f(t,x,y,\varepsilon)$ and $g(t,x,y,\varepsilon)$ are Lipschitz in ε, linearly in x and y, i.e.

$$\| f(t,x,y,\varepsilon_1) - f(t,x,y,\varepsilon_2) \| \leq l_5 (\| x \| + \| y \|) \,|\varepsilon_1-\varepsilon_2| \tag{2.2.2.8}$$

$$\| g(t,x,y,\varepsilon_1) - g(t,x,y,\varepsilon_2) \| \leq l_6 (\| x \| + \| y \|) \,|\varepsilon_1-\varepsilon_2| \tag{2.2.2.9}$$

for all $t\geq 0$, $x\in B_r$, $y\in B_r$, and ε_1, $\varepsilon_2\leq\varepsilon_0$.

(B4) The function $d(t,x)=f(t,x,0,0)-f_{av}(x)$ satisfies conditions of Lemma 2.2.1.2.

(B5) $A(x)\in R^{m\times m}$ is *uniformly stable* (Hurwitz) for all $x\in B_r$, i.e. there exist λ_1, $\lambda_2<0$ such that

$$\lambda_1 \leq \mathrm{Re}\,\lambda(A(x)) \leq \lambda_2 \tag{2.2.2.10}$$

for all $x\in B_r$, where $\mathrm{Re}\,\lambda(A(x))$ is the real part of an eigenvalue of the matrix $A(x)$. Moreover, for some $k_a>0$,

$$\left\| \frac{\partial A(x)}{\partial x_i} \right\| \leq k_a \qquad i=1,2,\cdots,n \tag{2.2.2.11}$$

for all $x \in B_r$.

(B6) For some $0 < r_1 < r$, $x_{av}(t) \in B_{r_1}$ on the time intervals considered, and for some $r_2 > 0$, $y_0 \in B_{r_2}$ (where r_1 and r_2 are some constants to be specified later).

Remark: The assumption (B5) implies that there exists $Q, P(x) \in R^{m \times m}$ such that, for some non-zero positive constants p_1, p_2, q_1, and q_2,

$$p_1 I \le P(x) \le p_2 I \tag{2.2.2.12}$$

$$-q_2 I \le -Q = A(x)^T P(x) + P(x) A(x) \le -q_1 I \tag{2.2.2.13}$$

for all $x \in B_r$.

As in the One-Time-Scale case, we will first give the following preliminary lemma analogous to Lemma 2.2.1.3. This lemma allows one to perform a similar transformation so that the original system (2.2.2.1) is reformulated as a perturbed version of the averaged system (2.2.2.4).

Lemma 2.2.2.1 (Perturbation Formulation of Averaging with Coincident Equilibrium Point)

If the original system (2.2.2.1) and (2.2.2.2), and the averaged system (2.2.2.4) satisfy assumptions (B1)-(B4),

then there exist functions w_ε and ξ_2 as in Lemma 2.2.1.2, and a transformation of the form,

$$x = z + \varepsilon w_\varepsilon(t,z) \tag{2.2.2.14}$$

under which system (2.2.2.1) becomes

$$\dot{z} = \varepsilon f_{av}(z) + \varepsilon p_1(t,z,\varepsilon) + \varepsilon p_2(t,z,y,\varepsilon) \qquad z(0) = x_0 \tag{2.2.2.15}$$

where

$$\| p_1(t,z,\varepsilon) \| \le \xi_2(\varepsilon) k_1 \| z \| \quad \text{and} \quad \| p_2(t,z,y,\varepsilon) \| \le k_2 \| y \| \tag{2.2.2.16}$$

for some k_1, $k_2 > 0$. □

Proof: See Appendix A.

We are now ready to state the averaging theorems concerning system (2.2.2.1)-(2.2.2.2). The first Theorem 2.2.2.2 is an approximation theorem similar to Theorem 2.2.1.4. It guarantees that the trajectories of the slow variable x of the original system and those of the averaged system are arbitrarily close on compact intervals when ε tends to zero. We prove the following theorems for the case where $A(x) \equiv A$ is constant (see assumption (B5)). The proof in the general case can be found in Bodson et al (1986).

Theorem 2.2.2.2: (**Basic Averaging Theorem**)

If the original system (2.2.2.1) and (2.2.2.2), and the averaged system (2.2.2.3) satisfy assumptions (B1)-(B6),

then there exists $\psi_3 \in K$ such that, given $T \ge 0$,

$$\| x(t) - x_{av}(t) \| \le \psi_3(\varepsilon)\, b_T \tag{2.2.2.17}$$

for some b_T, $\varepsilon_T > 0$, and for all $t \in [0, T/\varepsilon]$, $\varepsilon \le \varepsilon_T$. Further, $\psi_3(\varepsilon)$ is of the order of $\varepsilon + \xi_2(\varepsilon)$ (as defined in Lemma 2.2.1.2). □

Proof: We estimate the error $(x - x_{av})$, following a proof similar to the that of Theorem 2.2.1.4. First, we have that

$$\| x - z \| \le \xi_2(\varepsilon) \| z \| \ . \tag{2.2.2.18}$$

Then, the error $z - x_{av}$ can be estimated using from (2.2.2.15):

$$\frac{d}{dt}(z - x_{av}) = \varepsilon(f_{av}(z) - f_{av}(x_{av})) + \varepsilon p_1(t,z,\varepsilon) + \varepsilon p_2(t,z,y,\varepsilon) \quad z(0) - x_{av}(0) = 0 \tag{2.2.2.19}$$

for all $t\in[0,T/\varepsilon]$, $x_{av}(t)\in B_{r'}$, $r'<r$. As in the proof of Theorem 2.2.1.4, we will show that, on this interval, and for as long as x, $z\in B_r$, the errors $z-x_{av}$, and $x-x_{av}$ can be made arbitrarily small by reducing ε.

Using Lemma 2.2.2.1, we integrate (2.2.2.19):

$$\| z(t)-x_{av}(t)\| \leq \varepsilon l_{av}\int_0^t \| z(\tau)-x_{av}(\tau)\| d\tau + \varepsilon\xi_2(\varepsilon)k_1\int_0^t \| z(\tau)\| d\tau$$

$$+ \varepsilon k_2\int_0^t \| y(\tau)\| d\tau . \tag{2.2.2.20}$$

Further, $y(t)$ can be calculated from (2.2.2.2):

$$y(t) = e^{At}y_0 + \varepsilon\int_0^t e^{A(t-\tau)}g(\tau,x,y)d\tau . \tag{2.2.2.21}$$

Since A is Hurwitz, we have that

$$\| e^{At}\| \leq m e^{-\lambda t} \tag{2.2.2.22}$$

for some m, $\lambda>0$, and

$$\| y(t)\| \leq m \| y_0\| e^{-\lambda t} + \varepsilon m\int_0^t e^{-\lambda(t-\tau)}(l_3\| x(\tau)\| + l_4\| y(\tau)\|)d\tau \tag{2.2.2.23}$$

or

$$\| e^{\lambda t}y(t)\| \leq m \| y_0\| + \varepsilon m l_3\int_0^t e^{\lambda\tau} \| x(\tau)\| d\tau + \varepsilon m l_4\int_0^t \| e^{\lambda\tau}y(\tau)\| d\tau . \tag{2.2.2.24}$$

Applying the generalized Bellman-Gronwall Lemma,

$$\| e^{\lambda t}y(t)\| \leq m \| y_0\| e^{\varepsilon m l_4 t}+\int_0^t \varepsilon m l_3 e^{\lambda\tau}\| x(\tau)\| e^{\varepsilon m l_4(t-\tau)}d\tau . \tag{2.2.2.25}$$

Define $\lambda(\varepsilon)=\lambda-\varepsilon m l_4$, and ε'_1 $(0<\varepsilon'_1\leq\varepsilon_1)$ so that $\lambda(\varepsilon)>0$ for $\varepsilon\leq\varepsilon'_1$. It follows that

$$\| y(t)\| \leq m r e^{-\lambda(\varepsilon)t} + \varepsilon m l_3 r/\lambda(\varepsilon) . \tag{2.2.2.26}$$

Using this estimate in (2.2.2.20), and using the generalized Bellman-Gronwall Lemma again

$$\| z(t)-x_{av}(t) \| \le \int_0^t \left[\xi_2(\varepsilon)k_1 h + mk_2 r e^{-\lambda(\varepsilon)\tau} + \frac{\varepsilon m k_2 l_3 r}{\lambda(\varepsilon)} \right] \varepsilon e^{\varepsilon l_{av}(t-\tau)} d\tau$$

$$\le (\varepsilon+\xi_2(\varepsilon)) \left[k_1 r + \frac{mk_2 r\, l_{av}}{\lambda(\varepsilon)+\varepsilon l_{av}} + \frac{mk_2 l_3 r}{\lambda(\varepsilon)} \right] \left[\frac{e^{l_{av}T}}{l_{av}} \right]$$

$$=: \psi_3(\varepsilon)\, a_T \,. \tag{2.2.2.27}$$

As in Theorem 2.2.1.4, it follows that, for some b_T

$$\| x(t)-x_{av}(t) \| \le \psi_3(\varepsilon)\, b_T \,. \tag{2.2.2.28}$$

By assumption, $\| x_{av}(t) \| \le r' < r$. Let ε_T $(0<\varepsilon_T \le \varepsilon'_1)$ such that $\psi_3(\varepsilon_T) b_T < r - r'$. Further, let y_0, and ε_T sufficiently small that, by (2.2.2.26), $y(t) \in B_r$, for all $t \in [0, T/\varepsilon]$. It follows, from a simple contradiction argument, that the estimate in (2.2.2.28) is valid for all $t \in [0, T/\varepsilon]$, whenever $\varepsilon \le \varepsilon_T$.

■

Theorem 2.2.2.3: (Exponential Stability Theorem)

If the original system (2.2.2.1) and (2.2.2.2), and the averaged system (2.2.2.4) satisfy assumptions (B1)-(B6), and $x_{av}=0$ is an exponentially stable equilibrium point of the averaged system,

then there exists $\varepsilon_4>0$ such that the equilibrium point $(x,y)=(0,0)$ of the original system is exponentially stable for all $\varepsilon \le \varepsilon_4$. □

Proof: Since $x_{av}=0$ is an exponentially stable equilibrium point of the averaged system, there exists a function $v(x_{av})$ satisfying (2.2.1.26)-(2.2.1.28). We now study the stability of the system (2.2.2.15), (2.2.2.2), and consider the following Lyapunov function,

$$\hat{v}(z,y) = v(z) + \frac{\alpha_2}{p_2} y^T P y \tag{2.2.2.29}$$

$$\alpha'_1(\|z\|^2+\|y\|^2) \le \hat{v}(z,y) \le \alpha_2(\|z\|^2+\|y\|^2) \tag{2.2.2.30}$$

with $\alpha'_1=\min(\alpha_1,\frac{\alpha_2}{p_2}p_1)$. The derivative of $\hat{v}$ along the trajectories of (2.2.2.15), (2.2.2.2) can be estimated, using the previous results,

$$\begin{aligned}\dot{v}_1(z,y) \le &-\varepsilon\alpha_3\|z\|^2+\varepsilon k_1\xi_2(\varepsilon)\alpha_4\|z\|^2\\ &+\varepsilon k_2\alpha_4\|z\|\ \|y\| -\frac{\alpha_2}{p_2}q_1\|y\|^2\\ &+4\varepsilon l_3\alpha_2\|z\|\ \|y\| +2\varepsilon l_4\alpha_2\|y\|^2\end{aligned} \tag{2.2.2.31}$$

for $\varepsilon\le\varepsilon_1$ (so that, in particular, $\|x\| \le 2\|z\|$). Note that since $ab\le(a^2+b^2)/2$ for all $a,b\in R$, we have

$$\varepsilon\|z\|\ \|y\| \le \frac{1}{2}\left[\varepsilon^{4/3}\|z\|^2+\varepsilon^{2/3}\|y\|^2\right] \tag{2.2.2.32}$$

so that

$$\begin{aligned}\dot{v}_1(z,y) \le &-\varepsilon\left[\alpha_3-\xi_2(\varepsilon)k_1\alpha_4-\varepsilon^{1/3}\frac{k_2\alpha_4}{2}-2\varepsilon^{1/3}l_3\alpha_2\right]\|z\|^2\\ &-\left[\frac{\alpha_2}{p_2}q_1-2\varepsilon l_4\alpha_2-\varepsilon^{2/3}\frac{k_2\alpha_4}{2}-2\varepsilon^{2/3}l_3\alpha_2\right]\|y\|^2\\ &=:-2\varepsilon\alpha_2\alpha(\varepsilon)\|z\|^2-q(\varepsilon)\|y\|^2 .\end{aligned} \tag{2.2.2.33}$$

Note that, with this definition, $\alpha(\varepsilon)\to\frac{1}{2}\frac{\alpha_3}{\alpha_2}$ as $\varepsilon\to0$. Let $\varepsilon_4(0<\varepsilon_4\le\varepsilon_1)$ be sufficiently small that $\alpha(\varepsilon)>0$, $q(\varepsilon)>0$, and $2\varepsilon\alpha_2\alpha(\varepsilon)\le q(\varepsilon)$ whenever $\varepsilon\le\varepsilon_4$. Consequently,

$$\dot{\hat{v}}(z,y) \le -2\varepsilon\alpha(\varepsilon)\hat{v}(z,y) \tag{2.2.2.34}$$

and

$$\hat{v}(z,y) \le \hat{v}(z(t_0),y(t_0))\,e^{-2\varepsilon\alpha(\varepsilon)(t-t_0)} . \tag{2.2.2.35}$$

As in Theorem 2.2.1.5, this implies the exponential convergence of the original system, with rate of convergence $\varepsilon\alpha(\varepsilon)$. Also, for $x(t_0)$ and $y(t_0)$ sufficiently small, all signals are guaranteed to remain in B_r, so that all assumptions are applicable as ε goes to zero.

■

Remark: The proof of Theorem 2.2.2.3, as of Theorem 2.2.1.5, gives a useful bound on the rate of convergence of the original system, and this bound again tends to the bound on the rate of convergence of the averaged system.

Now we change our focus of attention to a two-time-scale system which does not have $(x,y)=(0,0)$ as an equilibrium point, similar to the case of one-timer-scale systems under assumption (MA1). However, here we will consider a more general case where $x_{av}=0$ also may not be an equilibrium point of the averaged system. These are indicated in the following modified assumptions:

Modified Assumptions:

(MB1) f and g satisfy assumption (B1) except that $(x,y)=(0,0)$ is not an equilibrium point of the system (2.2.2.1)-(2.2.2.2), and for all $x\in B_r$, $y\in B_r$, $\varepsilon\leq\varepsilon_0$ there exist l_f, $l_g\geq 0$ such that

$$\| f(t,x,y,\varepsilon) \| \leq l_f \qquad \text{and} \qquad \| g(t,x,y,\varepsilon) \| \leq l_g\,. \tag{2.2.2.36}$$

(MB2) f_{av} satisfies (B2) except that $f_{av}(0)\neq 0$.

(MB3) f and g are Lipschitz in ε, but not linearly in x and y, i.e.

$$\| f(t,x,y,\varepsilon_1)-f(t,x,y,\varepsilon_2) \| \leq l_5\,|\,\varepsilon_1-\varepsilon_2\,| \tag{2.2.2.37}$$

$$\| g(t,x,y,\varepsilon_1)-g(t,x,y,\varepsilon_2) \| \leq l_6\,|\,\varepsilon_1-\varepsilon_2\,|\,. \tag{2.2.2.38}$$

(MB4) The function $d(t,x)$ satisfies conditions of Lemma 2.2.1.6.

Lemma 2.2.2.4: (Perturbation Formulation of Averaging with Non-Coincident Equilibrium Point)

If the original system (2.2.2.1)-(2.2.2.2), and the averaged system (2.2.2.4) satisfy assumptions (MB1)-(MB4),

then there exist functions w_ε and ξ_2, as in Lemma 2.2.1.6, and a transformation of the form,

$$x = z + \varepsilon w_\varepsilon(t,z) \tag{2.2.2.39}$$

under which system (2.2.2.1) becomes

$$\dot{z} = \varepsilon f_{av}(z) + \varepsilon p_1(t,z,\varepsilon) + \varepsilon p_2(t,z,y,\varepsilon) \qquad z(0) = x_0 \tag{2.2.2.40}$$

where for some k_3, $k_4 > 0$

$$\| p_1(t,z,\varepsilon) \| \le k_3 r \xi_2(\varepsilon) \qquad \text{and} \qquad \| p_2(t,z,y,\varepsilon) \| \le k_4 \| y \| \ . \tag{2.2.2.41}$$

□

Proof: See Appendix A.

Theorem 2.2.2.5: (Bounded Stability Theorem)

If the original system (2.2.2.1) and (2.2.2.2), and the averaged system (2.2.2.4) satisfy assumptions (MB1)-(MB4), (B5)-(B6), and there exist a function $v: R^n \to R_+$, and non-zero positive constants α_1, α_2, α_3, α_4, and δ such that for all $x_{av} \in B_r$

$$\alpha_1 \| x_{av} \|^2 \le v(x_{av}) \le \alpha_2 \| x_{av} \|^2 \tag{2.2.2.42}$$

$$\left\| \frac{\partial v}{\partial x_{av}} \right\| \le \alpha_3 \| x_{av} \| \tag{2.2.2.43}$$

$$\dot{v}(x_{av}) \le -\varepsilon \| x_{av} \| \ (\alpha_4 \| x_{av} \| - \delta) \tag{2.2.2.44}$$

$$\sqrt{\frac{\alpha_2}{\alpha_1}} \left[\frac{1}{\alpha_4} + \frac{p_2}{\alpha_2 q_1} \right] \delta < r \tag{2.2.2.45}$$

where the derivative in (2.2.2.44) is taken along the trajectories (2.2.2.4), then there exist $\varepsilon_5>0$, $\rho>0$, and $\beta_2 \in K$ such that the solutions of the original system are UUB whenever $\varepsilon \le \varepsilon_5$, and

$$\| x(t) \| \le \beta_2(\varepsilon) + \rho\, \delta \qquad t \ge t_0 + T \tag{2.2.2.46}$$

for some $T>0$, and for sufficiently small $x(t_0)$ and $y(t_0)$. □

Proof: The proof of Lemma 2.2.1.8 and the current hypothesis imply that $x_{av}(t)$ of the averaged system (2.2.2.4) is UUB for a bound $\rho'\delta$ for some $\rho'>0$. For sufficiently small $x(t_0)$, eq. (2.2.2.45) directly implies that $x_{av}(t) \in B_r$ for all $t \ge t_0$. Now an approach similar to that used in the proof of Theorem 2.2.2.3 is taken here. Consider the following constructed Lyapunov function (i.e. $P(x)$ replaces P in (2.2.2.29)),

$$\hat{v}(z,y) := v(z) + \frac{\alpha_2}{p_2}\, y^T P(x) y \tag{2.2.2.47}$$

so that by (2.2.2.12) and (2.2.2.42)

$$\alpha'_1 \left(\| z \|^2 + \| y \|^2 \right) \le \hat{v}(z,y) \le \alpha_2 \left(\| z \|^2 + \| y \|^2 \right) \tag{2.2.2.48}$$

where $\alpha'_1 = \min(\alpha_1, \frac{\alpha_2}{p_2} p_1)$. The derivative of $\hat{v}$ along the trajectories of (2.2.2.40), (2.2.2.2) can be estimated, using Lemma 2.2.2.4 and (2.2.2.43)-(2.2.2.44),

$$\begin{aligned}
\dot{\hat{v}}(z,y) &= \dot{v}(x_{av}) \Big|_{x_{av}=z} + \left(\frac{\partial v}{\partial z}\right) \Big[\varepsilon p_1(t,z,\varepsilon) + \varepsilon p_2(t,z,y,\varepsilon) \Big] \\
&\quad - \frac{\alpha_2}{p_2} y^T Q y + \frac{2\alpha_2}{p_2} y^T P(x) g + \frac{\alpha_2}{p_2} y^T \left[\sum_{i=1}^{n} \frac{\partial P(x)}{\partial x_i} \dot{x}_i \right] y \\
&\le -\varepsilon \| z \| \left(\alpha_4 \| z \| - \delta \right) + \varepsilon r \xi_2(\varepsilon) k_3 \alpha_3 \| z \| + \varepsilon \alpha_3 k_4 \| z \| \, \| y \| \\
&\quad - \frac{\alpha_2 q_1}{p_2} \| y \|^2 + \varepsilon \frac{\alpha_2 l_f}{p_2} \left\| \frac{\partial P(x)}{\partial x} \right\| \| y \|^2 + 2 \varepsilon \alpha_2 l_g \| y \| \, .
\end{aligned} \tag{2.2.2.49}$$

Since $P(x)$ satisfies the linear Lyapunov equation (2.2.2.13), *Cramer's rule* and assumption

(B5) implies the continuous differentiability of $P(x)$ for all $x \in B_r$. Hence, there exists some $k_p > 0$ such that

$$\left\| \frac{\partial P(x)}{\partial x} \right\| \leq k_p \qquad x \in B_r \tag{2.2.2.50}$$

Using this result, and (2.2.2.32), we can reduce (2.2.2.49) to give

$$\dot{v} \leq -\varepsilon \, \|z\| \left[(\alpha_4 - \varepsilon^{1/3} \frac{\alpha_3 k_4}{2}) \|z\| - \delta - \alpha_3 k_3 r \xi_2(\varepsilon) \right]$$

$$- \|y\| \left[\frac{\alpha_2 q_1}{p_2} (1 - \varepsilon \frac{l_f k_p}{q_1} - \varepsilon^{1/3} \frac{\alpha_3 k_4 p_2}{2 \alpha_2 q_1}) \|y\| - 2 \varepsilon \alpha_2 l_g \right]$$

$$:= -\varepsilon \, \|z\| \, (\alpha(\varepsilon) \, \|z\| - \delta - \alpha_3 k_3 r \xi_2(\varepsilon)) - \|y\| \, (q(\varepsilon) \, \|y\| - 2 \varepsilon \alpha_2 l_g) \, . \tag{2.2.2.51}$$

Note that $\alpha(\varepsilon) \to \alpha_4$ and $q(\varepsilon) \to \alpha_2 q_1 / p_2$ as $\varepsilon \to 0$, and then let $\varepsilon'_5 > 0$ be such that $\alpha(\varepsilon'_5) > 0$ and $q(\varepsilon'_5) > 0$. Moreover, since $(a_1^2 + b_1^2) \geq (a_1 a_2 + b_1 b_2)^2 / (a_2^2 + b_2^2)$ for all $a_1, b_1, a_2, b_2 \in R$, we have

$$(\varepsilon \alpha(\varepsilon) \, \|z\|^2 + q(\varepsilon) \, \|y\|^2) \geq (\frac{1}{\alpha(\varepsilon)} + \frac{1}{q(\varepsilon)})^{-1} \, (\varepsilon^1/2 \, \|z\| + \|y\| \,)^2$$

$$:= \beta_3(\varepsilon) \, (\varepsilon^{1/2} \, \|z\| + \|y\| \,)^2 \tag{2.2.2.52}$$

for all $\varepsilon \leq \varepsilon'_5$ so that (2.2.2.51) can further be simplified by

$$\dot{v} \leq -(\varepsilon^{1/2} \, \|z\| + \|y\| \,) \left[\beta_3(\varepsilon) (\varepsilon^{1/2} \, \|z\| + \|y\| \,) - \varepsilon^{1/2} (\psi_4(\varepsilon) + \delta) \right] \tag{2.2.2.53}$$

where $\psi_4(\varepsilon)$ defined by

$$\psi_4(\varepsilon) := \alpha_3 \, k_3 \, r \xi_2(\varepsilon) + 2 \, \varepsilon^{1/2} \alpha_2 \, l_g \tag{2.2.2.54}$$

and is a class K function of ε. Again, by a proof similar to that of Lemma 2.2.1.8, it follows that $z(t)$ and $y(t)$ are UUB, and for some $T > 0$ and $\kappa > 0$,

$$\|z\|^2 + \|y\|^2 \leq (1 + \varepsilon + \kappa) \frac{\alpha_2}{\alpha'_1} (\frac{\delta}{\beta_3(\varepsilon)} + \frac{\psi_4(\varepsilon)}{\beta_3(\varepsilon)})^2 \qquad t \geq t_0 + T \, . \tag{2.2.2.55}$$

This implies that

$$\|z\| \text{ and } \|y\| \le \sqrt{1+\varepsilon+\kappa}\sqrt{\frac{\alpha_2}{\alpha'_1}}\left(\frac{\delta}{\beta_3(\varepsilon)}+\frac{\psi_4(\varepsilon)}{\beta_3(\varepsilon)}\right) \qquad t\ge t_0+T\ . \quad (2.2.2.56)$$

Using (2.2.2.45), it then follows that there exists $\varepsilon''_5 \le \varepsilon'_5$ such that, for sufficiently small $x(t_0)$ and $y(t_0)$, $z(t)\in B_r$ and $y(t)\in B_r$ for all $t\ge t_0$. Finally, following the proof of Lemma 2.2.2.4, we have

$$\|x(t)\| \le \|z(t)\| + h\,r\xi_2(\varepsilon)\ . \qquad (2.2.2.57)$$

Denote

$$\beta_2(\varepsilon) := \sqrt{\frac{\alpha_2}{\alpha'_1}}\frac{\sqrt{1+\varepsilon+\kappa}}{\beta_3(\varepsilon)}\psi_4(\varepsilon) + h\,r\xi_2(\varepsilon) \qquad (2.2.2.58)$$

and

$$\rho_\varepsilon := \sqrt{\frac{\alpha_2}{\alpha'_1}}\frac{\sqrt{1+\varepsilon+\kappa}}{\beta_3(\varepsilon)}\ . \qquad (2.2.2.59)$$

Then there exists $\varepsilon_5 \le \varepsilon''_5$ such that

$$\beta_2(\varepsilon_5) + \rho_{\varepsilon_5}\delta < r \qquad (2.2.2.60)$$

so that $x(t)$ remains in B_r for all $t\ge t_0$ whenever $\varepsilon\le\varepsilon_5$. In conclusion, $x(t)$ is UUB for a bound $\beta_2(\varepsilon)+\rho\,\delta$, where $\rho:=\rho_{\varepsilon_5}$, whenever $\varepsilon\le\varepsilon_5$.

∎

Remark: Note that, when $\delta\equiv 0$, $x(t)$ is UUB for a bound $\beta_2\in K$. This result will then be similar to that of Theorem 2.2.1.9.

As before, we consider a system of the following form:

$$\dot{x} = \varepsilon f_1(t,x,\zeta,y,\varepsilon) \qquad x(t_0)=x_0\in R^{n_1} \qquad (2.2.2.61)$$

$$\dot{\zeta} = \varepsilon f_2(t,x,\zeta,y,\varepsilon) \qquad \zeta(t_0)=\zeta_0\in R^{n_2} \qquad (2.2.2.62)$$

$$\dot{y} = A(x)y + \varepsilon g(t,x,\zeta,y,\varepsilon) \qquad y(t_0)=y_0 \in R^m \tag{2.2.2.63}$$

of which the averaged system exists as follows:

$$\dot{x}_{av} = \varepsilon f_{1av}(x_{av}, \zeta_{av}) \qquad x_{av}(t_0)=x_0 \tag{2.2.2.64}$$

$$\dot{\zeta}_{av} = \varepsilon f_{2av}(x_{av}, \zeta_{av}) \qquad \zeta_{av}(t_0)=\zeta_0 \tag{2.2.2.65}$$

For some given $\zeta^o \in R^{n_2}$, the following are assumptions about f_i , f_{iav} , i = 1,2, and g.

Assumptions:

(B7) Treat ζ as a time function $\zeta(t)$, then $f_1(t,x,\zeta(t),y,\varepsilon)$ and $g(t,x,\zeta(t),y,\varepsilon)$ satisfies (B1), (B3), whenever $\zeta(t) \in B_r(\zeta^o)$.

(B8) f_1 is Lipschitz in ζ, linearly in x and y, i.e. for some $l_7 \geq 0$

$$\| f_1(t,x,\zeta_1,y,\varepsilon) - f_1(t,x,\zeta_2,y,\varepsilon) \| \leq l_7(\|x\| + \|y\|) \|\zeta_1 - \zeta_2\| \tag{2.2.2.66}$$

for all $t \geq 0$, and for all $x \in B_r$, $y \in B_r$, $\zeta_1 , \zeta_2 \in B_r(\zeta^o)$.

(B9) $(x,\zeta,y)=(0,\zeta^o,0)$ is not an equilibrium point of f_2, and for some $l_{f_2} \geq 0$

$$\| f_2(t,x,\zeta,y,\varepsilon) \| \leq l_{f_2} \tag{2.2.2.67}$$

for all $t \geq 0$, and for all $x \in B_r$, $\zeta \in B_r(\zeta^o)$, $y \in B_r$, and $\varepsilon \leq \varepsilon_0$.

(B10) $(0,\zeta^o)$ is an equilibrium point of the averaged system (2.2.2.64)-(2.2.2.65), and for some $l_{1av}, l_{2av} \geq 0$

$$\| f_{1av}(x_1,\zeta) - f_{1av}(x_1,\zeta) \| \leq l_{1av} \| x_1 - x_2 \| \tag{2.2.2.68}$$

$$\| f_{2av}(x,\zeta_1) - f_{2av}(x,\zeta_2) \| \leq l_{2av} \| \zeta_1 - \zeta_2 \| \tag{2.2.2.69}$$

for all $t \geq 0$, and for all $x_1 , x_2 \in B_r$, $\zeta_1 , \zeta_2 \in B_r(\zeta^o)$.

(B11) f_2 is Lipschitz in ε, i.e. for some $l_8 \geq 0$

$$\| f_2(t,x,\zeta,y,\varepsilon_1) - f_2(t,x,\zeta,y,\varepsilon_2) \| \leq l_8 | \varepsilon_1 - \varepsilon_2 | \tag{2.2.2.70}$$

for all $t \geq 0$, and for all $x \in B_r$, $\zeta \in B_r(\zeta^o)$, $y \in B_r$, $\varepsilon_1 , \varepsilon_2 \leq \varepsilon_0$.

(B12) The function $d_\zeta(t,x) := f_1(t,x,\zeta,0,0) - f_{1av}(x,\zeta)$ satisfies conditions of Lemma 2.2.1.2 for all $\zeta \in B_r(\zeta^o)$, and the function $d_x(t,\zeta) := f_2(t,x,\zeta,0,0) - f_{2av}(x,\zeta)$ satisfies conditions of Lemma 2.2.1.6.

Theorem 2.2.2.6: (Partial Exponential Stability Theorem)

If the original system (2.2.2.1)-(2.2.2.2), and the averaged system (2.2.2.4) satisfy assumptions (B5)-(B12), $(0,\zeta^o)$ is an exponentially stable equilibrium point of the averaged system (2.2.2.64)-(2.2.2.65), and there exists a function $v: R^{n_1} \times R^{n_2} \rightarrow R_+$, and non-zero positive constants α_1, α_2, α_3, α_4, α_5 such that for all $x_{av} \in B_r$, $\zeta_{av} \in B_r(\zeta^o)$,

$$\alpha_1 \| x_{av} \|^2 \leq v(x_{av}, \zeta_{av}) \leq \alpha_2 \| x_{av} \|^2 \tag{2.2.2.71}$$

$$\left\| \frac{\partial v}{\partial x_{av}} \right\| \leq \alpha_3 \| x_{av} \| \tag{2.2.2.72}$$

$$\left\| \frac{\partial v}{\partial \zeta_{av}} \right\| \leq \alpha_4 \| x_{av} \|^2 \tag{2.2.2.73}$$

$$\dot{v}(x_{av}, \zeta_{av}) \leq -\varepsilon \alpha_5 \| x_{av} \|^2 \tag{2.2.2.74}$$

where the derivative in (2.2.2.74) is taken along the trajectories (2.2.1.64)-(2.2.1.65),

then there exists $\varepsilon_6 > 0$ such that $x(t)$ of the original system (2.2.2.61) converges to zero exponentially for all $\varepsilon \leq \varepsilon_6$. □

Proof: The proof relies on both a theorem and a converse theorem of Lyapunov for exponentially stable systems. Under the above assumptions, in particular (B12), Lemmas 2.2.1.2, 2.2.2.1, 2.2.1.6, and 2.2.2.4 imply that there exists a transformation of the form

$$x = z_1 + \varepsilon w_{1\varepsilon}(t, z_1, z_2) \tag{2.2.2.75}$$

$$\zeta = z_2 + \varepsilon w_{2\varepsilon}(t, z_1, z_2) \tag{2.2.2.76}$$

under which the system (2.2.2.61)-(2.2.2.62) becomes

$$\dot{z}_1 = \varepsilon f_{1av}(z_1,z_2) + \varepsilon p_{11}(t,z_1,z_2,\varepsilon) + \varepsilon p_{12}(t,z_1,z_2,y,\varepsilon) \qquad z_1(t_0)=x_0 \tag{2.2.2.77}$$

$$\dot{z}_2 = \varepsilon f_{2av}(z_1,z_2) + \varepsilon p_{21}(t,z_1,z_2,\varepsilon) + \varepsilon p_{22}(t,z_1,z_2,y,\varepsilon) \qquad z_2(t_0)=\zeta_0 \tag{2.2.2.78}$$

where

$$\|p_{11}\| \le \xi_2(\varepsilon) k_1 \|z_1\| \qquad \text{and} \qquad \|p_{12}\| \le k_2 \|y\| \tag{2.2.2.79}$$

$$\|p_{21}\| \le r\xi_2(\varepsilon) k_3 \qquad \text{and} \qquad \|p_{22}\| \le k_4 \|y\| \,. \tag{2.2.2.80}$$

Since $(0,\zeta^o)$ is an exponentially stable equilibrium point of the averaged system, by a converse Lyapunov theorem, there exists a function $v_1 : R^{n_1} \times R^{n_2} \to R_+$ that satisfies (2.2.2.42), (2.2.2.43), and

$$\dot{v}_1(x_{av},\zeta_{av}) \le -\varepsilon\alpha_4(\|x_{av}\|^2 + \|\zeta_{av}\|^2) \tag{2.2.2.81}$$

where the derivative is taken along the trajectories (2.2.2.64)-(2.2.2.65) (i.e. $\delta \equiv 0$). Hence, by the remark after Theorem 2.2.2.5, there exists $\varepsilon'_6 > 0$ such that, for sufficiently small x_0, ζ_0, and y_0, the solutions of the original system (2.2.2.61)-(2.2.2.63) are UUB for a bound which is a class K function of ε, and all signals remain in B_r or $B_r(\zeta^o)$ respectively for all $t \ge t_0$.

We now study the variable $x(t)$ of the system (2.2.2.61) by considering the following function $\hat{v}$, similar to that in the proof of Theorem 2.2.2.5, defined by

$$\hat{v}(z_1,z_2,y) := v(z_1,z_2) + \frac{\alpha_2}{p_2} y^T P(x) y \tag{2.2.2.82}$$

which then satisfies

$$\alpha'_1(\|z_1\|^2 + \|y\|^2) \le \hat{v}(z_1,z_2,y) \le \alpha_2(\|z_1\|^2 + \|y\|^2) \tag{2.2.2.83}$$

where $\alpha'_1 := \min(\alpha_1, \frac{\alpha_2}{p_2} p_1)$. The derivative of $\hat{v}$ taken along the trajectories of the perturbed system (2.2.2.77)-(2.2.2.78), (2.2.2.66) becomes

$$\dot{\hat{v}} = \dot{v}(x_{av},\zeta_{av})\Big|_{(x_{av},\zeta_{av})=(z_1,z_2)} + \varepsilon\left(\frac{\partial v}{\partial z_1}\right)(p_{11}+p_{12}) + \varepsilon\left(\frac{\partial v}{\partial z_2}\right)(p_{21}+p_{22})$$

$$-\frac{\alpha_2}{p_2} y^T Q y + 2\frac{\alpha_2}{p_2} y^T P(x) g + \frac{\alpha_2}{p_2} y^T \left[\sum_{i=1}^{n_1} \frac{\partial P(x)}{\partial x_i} \dot{x}_i\right] y \tag{2.2.2.84}$$

where $\dot{v}(x_{av},\zeta_{av})$ is taken along the trajectories (2.2.2.64)-(2.2.2.65). By carrying through the estimation of the R.H.S. of above equation, similar to that used in the proof of Theorem 2.2.2.5, we have

$$\dot{v} \le -2\varepsilon\alpha_2\alpha(\varepsilon)\,\|z_1\|^2 - q(\varepsilon)\,\|y\|^2 \tag{2.2.2.85}$$

where $\alpha(\varepsilon) \to \frac{1}{2}\frac{\alpha_5}{\alpha_2}$ and $q(\varepsilon) \to \frac{\alpha_2}{p_2}q_1$ as $\varepsilon \to 0$. Let $\varepsilon_6 \le \varepsilon'_6$ be such that $\alpha(\varepsilon) > 0$, $q(\varepsilon) > 0$, and $2\varepsilon\alpha_2\alpha(\varepsilon) \le q(\varepsilon)$ for all $\varepsilon \le \varepsilon_6$, then the rest of the proof will just follow that of Theorem 2.2.2.3.

■

The following, which is a corollary to Theorem 2.2.2.5 and 2.2.2.6, deals with a case, similar to that in Theorem 2.2.2.6, where only the bounded stability instead of the exponential stability can be expected.

Corollary 2.2.2.7:

If the original system and the averaged system satisfy the conditions in Theorem 2.2.2.6 except that

$$\dot{v}(x_{av},\zeta_{av}) \le -\varepsilon\alpha_5\,\|x_{av}\|\,(\,\|x_{av}\| - \delta) \tag{2.2.2.86}$$

where

$$\sqrt{\frac{\alpha_2}{\alpha_1}}\left[\frac{1}{\alpha_4} + \frac{p_2}{\alpha_2 q_1}\right]\delta < r, \tag{2.2.2.87}$$

then there exists $\varepsilon'_6 > 0$, $\rho > 0$, and $\beta_4 \in K$ such that the solutions of the original system are UUB whenever $\varepsilon \le \varepsilon'_6$, and

$$\|x(t)\| \le \beta_4(\varepsilon) + \rho\,\delta \qquad t \ge t_0 + T \tag{2.2.2.88}$$

for some $T > 0$, and for sufficiently small $x(t_0)$, $y(t_0)$, and $\zeta(t_0) - \zeta^o$. □

Proof: Using the proof of Theorem 2.2.2.6, the derivative of the constructed Lyapunov function $\hat{v}$ in (2.2.2.82) along the trajectories (2.2.2.77)-(2.2.2.78), (2.2.2.66) satisfies a condition similar to that in (2.2.2.53). Hence, the conclusion readily follows from the proof of Theorem 2.2.2.5.

■

2.2.2.2 Mixed Time-Scales

We now discuss a more general class of two-time-scale systems, arising in adaptive control:

$$\dot{x} = \varepsilon f'(t,x,y',\varepsilon) \tag{2.2.2.89}$$

$$\dot{y}' = A(x)y' + h(t,x) + \varepsilon g'(t,x,y',\varepsilon) \ . \tag{2.2.2.90}$$

We will show that system (2.2.2.89), (2.2.2.90) can be transformed into the system (2.2.2.1), (2.2.2.2). In this case, x is a slow variable, but y' has both a fast, and a slow component.

The averaged system corresponding to (2.2.2.89), (2.2.2.90) is obtained as follows. Define the function

$$w(t,x) := \int_0^t e^{A(x)(t-\tau)} h(\tau,x)\, d\tau \tag{2.2.2.91}$$

and assume that the following limit exists uniformly in t and x.

$$f_{av}(x) = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} f'(\tau,x,w(\tau,x),0)\, d\tau \ . \tag{2.2.2.92}$$

Intuitively, $w(t,x)$ represents the steady-state value of the variable y with x frozen and $\varepsilon=0$ in (2.2.2.90). Consider the transformation,

$$y = y' - w(t,x) \ . \tag{2.2.2.93}$$

From (2.2.2.91), $w(t,x)$ satisfies

$$\frac{\partial}{\partial t} w(t,x) = A(x)w(t,x) + h(t,x) \qquad w(t,0)=0 \tag{2.2.2.94}$$

Differentiating (2.2.2.93), we have that

$$\dot{y}=A(x)y+\varepsilon\left[-\frac{\partial w(t,x)}{\partial x}f'(t,x,y+w(t,x),\varepsilon)+g'(t,x,y+w(t,x),0)\right] \tag{2.2.2.95}$$

so that system (2.2.2.89), (2.2.2.90), is of the form (2.2.2.1), (2.2.2.2), with

$$f(t,x,y,\varepsilon) = f'(t,x,y+w(t,x),\varepsilon) \tag{2.2.2.96}$$

$$g(t,x,y,\varepsilon) = -\frac{\partial w(t,x)}{\partial x} f'(t,x,y+w(t,x),\varepsilon)+g'(t,x,y+w(t,x),\varepsilon) \tag{2.2.2.97}$$

The averaged system is obtained by averaging the R.H.S. of (2.2.2.96) with $y=0$, so that the definitions (2.2.2.3), and (2.2.2.92) agree.

To apply Theorem 2.2.2.2-2.2.2.3, we require that assumptions (B1)-(B6) be satisfied. In particular, we assume similar Lipschitz conditions on f', g', and the following assumption on $h(t,x)$:

Assumption:

(B13) For all $t\geq 0$ and $x\in B_r$, $h(t,0)=0$ and

$$\left\| \frac{\partial h(t,x)}{\partial x_i} \right\| \leq k \qquad i=1,\cdots,n\,. \tag{2.2.2.98}$$

This new assumption implies that $w(t,0)=0$, and

$$\left\| \frac{\partial w(t,x)}{\partial x_i} \right\| \leq k' \qquad i=1,\cdots,n \tag{2.2.2.99}$$

for all $t\geq 0$, $x\in B_r$ since

$$\frac{\partial w(t,x)}{\partial x_i} = \int_0^t \left[e^{A(x)(t-\tau)}\frac{\partial h(\tau,x)}{\partial x_i} + \left[\frac{\partial}{\partial x_i} e^{A(x)(t-\tau)}\right] h(\tau,x)\right] d\tau \tag{2.2.2.100}$$

and entries of $e^{A(x)t}$ and $\frac{\partial}{\partial x_i} e^{A(x)t}$ are of the form $\alpha(x)\,t^{\gamma} e^{\beta(x)t}$, where $\alpha(x)$, $\beta(x)$ are continuous functions of x, $\gamma \in Z_+$, and $\beta(x) \leq \lambda_2 < 0$ by assumption (B5). This condition is sufficient to guarantee Lipschitz conditions for the system (2.2.2.1), (2.2.2.2), given Lipschitz conditions for the system (2.2.2.89), (2.2.2.90).

However, if Theorem 2.2.2.5 is to be applied here, the following modified assumption is imposed instead:

Modified Assumption:

(MB13) For all $t \geq 0$ and $x \in B_r$ $h(t,x)$ is a bounded function, and

$$\left\| \frac{\partial h(t,x)}{\partial x_i} \right\| \leq k \qquad i=1,\cdots,n \tag{2.2.2.101}$$

This assumption implies, in general, $w(t,0) \neq 0$, which then leads to the condition (MB1), i.e. $f(t,0,0,\varepsilon) \neq 0$ and $g(t,0,0,\varepsilon) \neq 0$.

Consequently, the theory developed earlier can be directly applied to systems of the form (2.2.2.89), (2.2.2.90). The key to the preceding transformation is the fact that the new state variable y is truly a *fast* variable, so that the two time scales have been decoupled.

2.3 Averaging Theory for Instability of a Dynamical System

In this section, we will develop averaging theory for *instability* of one-time and two-time-scale dynamical systems respectively. To start with, we give the following preliminary definitions.

Definition 2.3.1.1: (Instability of a Dynamical System)

The equilibrium point $x=0$ of a differential equation is said to be unstable if there is *some* ball B_r of radius r such that for *every* $\delta>0$, no matter how small, there is a non-zero initial state $x(0)=x_0 \in B_\delta$ such that the trajectory starting from x_0 eventually leaves B_r. □

Definition 2.3.1.2: (Decrescent Function)

A continuous function $v: R_+ \times R^n \rightarrow R$ is said to be *decrescent* if there exists a class K function β such that, for all $t \geq 0$ and $x \in B_r$,

$$v(t,x) \leq \beta(\|x\|) . \tag{2.3.1.1}$$

□

2.3.1 One-Time Scale Systems

Again, we consider system (2.2.1.1) and its averaged system (2.2.1.5) with the same setup as given in subsection 2.2.1, where the stability and boundedness properties of both systems are closedly related. In this subsection, such a relation will be shown to hold for the instability property as well. The following theorem provides sufficient conditions under which the instability property of the averaged system will imply that of the original system.

Theorem 2.3.1.1: (Instability Theorem)

If the original system (2.2.1.1) and the averaged system (2.2.1.5) satisfy assumptions (A1)-(A5), and there exists a decrescent function $v: R^n \rightarrow R$ and non-zero positive constants α_1, α_2 such that for some x_{av} arbitrarily close to the origin $v(x_{av}) \geq 0$, and

$$\left\| \frac{\partial v(x_{av})}{\partial x_{av}} \right\| \leq \alpha_1 \|x\| \tag{2.3.1.2}$$

$$\dot{v}(x_{av}) \geq \varepsilon \alpha_2 \|x_{av}\|^2 \tag{2.3.1.3}$$

where the derivative in (2.3.1.3) is taken along the trajectories (2.2.1.5),

then there exists $\varepsilon_7 > 0$ such that the equilibrium point $x=0$ of the original system is unstable for all $\varepsilon \leq \varepsilon_7$. □

Proof: The proof is based on a theorem of Lyapunov for unstable systems (see, for example, Hale (1980), p. 314). Under the hypotheses, the equilibrium point $x_{av}=0$ of the averaged system (2.2.1.5) is unstable. From Lemmas 2.2.1.2 and 2.2.1.3, the original system can be transformed into the perturbed system (2.2.1.16). Now, we will study the instability property of that system, using the same function v. Consider $v(z)$ where inequalities in the hypothesis are still valid, with z replacing x. The derivative of $v(z)$ taken along the trajectories of (2.2.1.16) is given by

$$\dot{v}(z) = \dot{v}(x_{av})\Big|_{x_{av}=z} + (\frac{\partial v}{\partial z})(\varepsilon p(t,z,\varepsilon)) \tag{2.3.1.4}$$

and, using previous inequalities (including those from Lemma 2.2.1.3)

$$\begin{aligned}\dot{v}(z) &\geq \varepsilon\alpha_2 \|z\|^2 - \varepsilon\psi_1(\varepsilon)\alpha_1 \|z\|^2 \\ &= \varepsilon(\alpha_2 - \psi_1(\varepsilon)\alpha_1)\|z\|^2 := \varepsilon\alpha(\varepsilon)\|z\|^2\end{aligned} \tag{2.3.1.5}$$

for all $\varepsilon \leq \varepsilon_1$. Let $\varepsilon_7 \leq \varepsilon_1$ be such that $\alpha(\varepsilon) > 0$ for all $\varepsilon \leq \varepsilon_7$, then again by use of a Lyapunov instability theorem $z=0$ is an unstable equilibrium point. Since Lemma 2.2.1.2 implies

$$\|z\| \leq \frac{\|x\|}{1 - \xi_1(\varepsilon)}, \tag{2.3.1.6}$$

the instability of the equilibrium point $x=0$ of the original system is obvious.

■

Remark: Note that if the original system is linear as in (2.2.1.11), and its averaged system exists and has the form

$$\dot{x}_{av} = \varepsilon A_{av} x_{av}, \tag{2.3.1.7}$$

then conditions in the theorem can easily be verified if, for example, one of the eigenvalues of A_{av} has positive real part and none of the rest of the eigenvalues have zero real parts. Thus, the function $v(x_{av})$ can be chosen as

$$v(x_{av}) = x_{av}^T P x_{av} \tag{2.3.1.8}$$

where P satisfies the Lyapunov equation

$$Q = A_{av}^T P + P A_{av} \tag{2.3.1.9}$$

for some positive definite matrix Q (see Ostrowski and Schneider (1962)).

Sometimes, a function v may not be found to satisfy (2.3.1.3). However, by imposing more conservative conditions, we can obtain stronger results which will be useful in the sequel. This is stated in the following as a corollary.

Corollary 2.3.1.2: (Regional Instability)

If the original system (2.2.1.1) and the averaged system (2.2.1.5) satisfy assumptions (A1)-(A5), and there exists a decrescent function $v: R^n \rightarrow R$ and non-zero positive constants α_1, α_2, α_3, δ, such that $\delta \ll r$ and

$$\alpha_1 \|x_{av}\|^2 \leq v(x_{av}) \leq \alpha_2 \|x_{av}\|^2 \tag{2.3.1.10}$$

$$\left\| \frac{\partial v}{\partial x_{av}} \right\| \leq \alpha_3 \|x_{av}\| \tag{2.3.1.11}$$

$$\dot{v}(x_{av}) \geq \varepsilon \alpha_4 \|x_{av}\| \; (\|x_{av}\| - \delta) \tag{2.3.1.12}$$

then there exists $\varepsilon_8 > 0$ and $r_{in} < r$ such that, for any $x_0 \in B_r | B_{r_{in}}$, $x(t)$ will eventually leaves B_r whenever $\varepsilon \leq \varepsilon_8$ $\square$.

Proof: Again by a theorem of Lyapunov for unstable system, (2.3.1.10)-(2.3.1.12) imply that, if x_0 satisfies

$$\|x_0\| \geq \sqrt{\frac{\alpha_2}{\alpha_1}}\, \delta := r' \tag{2.3.1.13}$$

where $r' < r$ (by assumption $\delta \ll r$), then $x_{av}(t)$ will always stay outside the ball B_δ and eventually leaves B_r. To study the original system for the same property, we consider the function $v(z)$ as in the proof of Theorem 2.3.1.1 and take its derivative along the trajectories of the perturbed system (2.2.1.16):

$$\dot{v}(z) = \dot{v}(x_{av})\Big|_{x_{av}=z} + (\frac{\partial v}{\partial z})(\varepsilon p(t,z,\varepsilon))$$

$$\geq \varepsilon \alpha_4 \|z\| \left[(1-\frac{\alpha_3}{\alpha_4}\xi_1(\varepsilon))\|z\| - \delta \right]$$

$$=: \varepsilon \alpha_4 \|z\| \;(\alpha(\varepsilon)\|z\| - \delta)\,. \tag{2.3.1.14}$$

Note that with this definition, $\alpha(\varepsilon) \to 1$ as $\varepsilon \to 0$. Let $\varepsilon_8 \leq \varepsilon_1$ be such that $\alpha(\varepsilon_8) > 0$ and

$$r_{in} := \frac{r'}{\alpha(\varepsilon_8)} < r \tag{2.3.1.15}$$

so that $z(t)$ will stay outside a ball $B_{\frac{\delta}{\alpha(\varepsilon)}}$ for all $t \geq t_0$ whenever $\|x_0\| \geq r_{in}$ and $\varepsilon \leq \varepsilon_8$, and $z(t)$ will eventually leaves B_r. The conclusion follows from (2.3.1.6).

■

<u>2.3.2 Two-Time Scale Systems</u>

In this subsection, we will only be concerned with case of decoupled time-scales. Obviously, as has been discussed in subsection 2.2.2, such results can be easily extended to the case of mixed time-scales.

Here, we consider the system (2.2.2.1)-(2.2.2.2) and its averaged system (2.2.2.4). The following theorem will provide conditions under which the instability property of the equilibrium point $x_{av}=0$ of the autonomous averaged system will indicate the same property of the equilibrium point $(x,y)=(0,0)$ of the original nonautonomous system.

Theorem 2.3.2.1: (Instability Theorem)

If the original system (2.2.2.1), (2.2.2.2) and the averaged system (2.2.2.4) satisfy assumptions (B1)-(B6), and there exists a decrescent function $v: R^n \to R$ that satisfies conditions given in Theorem 2.3.1.1, then there exists $\varepsilon_9>0$ such that the equilibrium point $(x,y)=(0,0)$ of the original system is unstable for all $\varepsilon\leq\varepsilon_9$. □

Proof: The proof will be similar to that of Theorem 2.2.2.3. By hypothesis, the equilibrium point $x_{av}=0$ of the averaged system is unstable. Now, we construct another decrescent function $\hat{v}$, using the given v:

$$\hat{v}(z,y) = v(z) - \frac{\alpha_3}{q_1} y^T P(x) y \qquad \alpha_3>0 \tag{2.3.2.1}$$

where $P(x)$ and q_1 are defined in (2.2.2.13). It is clear that $\hat{v}$ is a decrescent function, $\hat{v}(z,y)>0$ for some (z,y) arbitrarily close to the origin in $R^n\times R^m$, and it satisfies (2.3.1.2), x being replaced by (z,y). This new function $\hat{v}$ is then used to study the instability of the equilibrium point at origin of the original system (2.2.2.1)-(2.2.2.2) through the perturbed system (2.2.2.15) and (2.2.2.2). The derivative of $\hat{v}(z,y)$ along the trajectories (2.2.2.15), (2.2.2.2) can be shown to be bounded below. Using the previous inequalities:

$$\dot{\hat{v}}(z,y) \geq \dot{v}(x_{av})\Big|_{x_{av}=z} + \alpha_3 \|y\|^2 - 2\varepsilon\frac{\alpha_3}{q_1}p_2 \|y\| \,(l_3\|z\| + l_4\|y\|)$$

$$- \varepsilon\alpha_3 \left\| \frac{\partial P(x)}{\partial x} \right\| (l_1+l_2) r \,\|y\|^2 \tag{2.3.2.2}$$

for $\varepsilon\leq\varepsilon_1$, where $\dot{v}(x_{av})$ is taken along the trajectories (2.2.2.4). Then, using the proof of Theorem 2.2.2.3, we can express the bound more concisely as:

$$\dot{\hat{v}}(z,y) \geq \varepsilon \left[\alpha_2 - \xi_2(\varepsilon)\alpha_1 k_1 - \varepsilon^{1/3}\left(\frac{\alpha_1 k_2}{2} + \frac{l_3\alpha_3 p_2}{q_1}\right)\right] \|z\|^2$$

$$+ \left[\alpha_3 - \varepsilon\frac{\alpha_1 k_2}{2} - \varepsilon 2\frac{l_4\alpha_3 p_2}{q_1} - \varepsilon\frac{2r\,k_p}{q_1}\right.$$

$$\left. \cdot\,(l_{av} + \xi_2(\varepsilon)k_1 + k_2) - \varepsilon^{2/3}\frac{l_3\alpha_3 p_2}{q_1}\right] \|y\|^2$$

$$=: \varepsilon\alpha(\varepsilon)\,\|z\|^2 + q(\varepsilon)\,\|y\|^2 \tag{2.3.2.3}$$

Note that, with this definition, $\alpha(\varepsilon) \to \alpha_2$ and $q(\varepsilon) \to \alpha_3$ as $\varepsilon \to 0$. Let $\varepsilon_9 \le \varepsilon_1$ be such that $\alpha(\varepsilon)$, $q(\varepsilon) > 0$ for all $\varepsilon \le \varepsilon_9$. Then, using a Lyapunov instability theorem as before, we can readily conclude that the equilibrium point $(z,y)=(0,0)$ of the perturbed system (2.2.2.15), (2.2.2.2) is unstable for all $\varepsilon \le \varepsilon_9$. Consequently, from Lemma 2.2.1.2 and 2.2.2.1, the same conclusion will hold for the original system (2.2.2.1)-(2.2.2.2).

■

As in the case of one-time-scale, we also consider a two-time-scale system which fails to have a function v satisfying (2.3.1.3). A useful result for this type of system is given in the following corollary.

Corollary 2.3.2.2: (Regional Instability)

If the original system (2.2.2.1)-(2.2.2.2) and the averaged system (2.2.2.4) satisfy assumptions (MB1)-(MB2), (B3)-(B6), and there exists a decrescent function $v: R^n \to R$ and non-zero positive constants α_1, α_2, α_3, δ, such that $\delta \ll r$ and

$$\alpha_1 \| x_{av} \| \le v(x_{av}) \le \alpha_2 \| x_{av} \|^2 \tag{2.3.2.4}$$

$$\left\| \frac{\partial v}{\partial x_{av}} \right\| \le \alpha_3 \| x_{av} \| \tag{2.3.2.5}$$

$$\dot{v}(x_{av}) \ge \varepsilon \alpha_3 \| x_{av} \| \, (\| x_{av} \| - \delta) \tag{2.3.2.6}$$

where the derivative in (2.3.2.6) is taken along the trajectories (2.2.2.4),

then there exists $\varepsilon_{10} > 0$ and $r_{in} < r$ such that, for any $x_0 \in B_r \,|\, B_{r_{in}}$, $x(t)$ will eventually leaves B_r whenever $\varepsilon \le \varepsilon_{10}$. □

The proof is similar to that of Corollary 2.3.1.2 and Theorem 2.3.2.1, and therefore is omitted.

The following is an analog of Corollary 2.3.2.2 and also a corollary of Theorem 2.3.2.1. It deals with a system for which assumptions (B5)-(B12) hold.

Corollary 2.3.2.3:

If the original system (2.2.2.1)-(2.2.2.2), and the averaged system (2.2.2.4) satisfy assumptions (B5)-(B12), and there exists a function $v : R^{n_1} \times R^{n_2} \to R_+$, and non-zero positive constants α_1, α_2, α_3, α_4, α_5, and $\delta \ll r$ such that

$$\alpha_1 \| x_{av} \|^2 \leq v(x_{av}, \zeta_{av}) \leq \alpha_2 \| x_{av} \|^2 \tag{2.3.2.7}$$

$$\left\| \frac{\partial v}{\partial x_{av}} \right\| \leq \alpha_3 \| x_{av} \| \tag{2.3.2.8}$$

$$\left\| \frac{\partial v}{\partial \zeta_{av}} \right\| \leq \alpha_4 \| x_{av} \|^2 \tag{2.3.2.9}$$

$$\dot{v}(x_{av}, \zeta_{av}) \geq -\varepsilon \alpha_5 \| x_{av} \| \left(\| x_{av} \| - \delta \right) \tag{2.3.2.10}$$

where the derivative in (2.3.2.10) is taken along the trajectories (2.2.2.64)-(2.2.2.65),

then there exists $\varepsilon_{11} > 0$ and $r_{in} < r$ such that, for any $x_0 \in B_r \setminus B_{r_{in}}$, $x(t)$ will eventually leave B_r whenever $\varepsilon \leq \varepsilon_{11}$. □

Proof: The proof is similar to that of Theorem 2.2.2.6. A Lyapunov function $\hat{v}$, similar to that in (2.2.2.82), is constructed. The derivative of $\hat{v}$ along the trajectories of (2.2.2.77)-(2.2.2.78), (2.2.2.66) satisfies a condition similar to that in (2.3.2.10). The conclusion then follows from the proof of Corollary 2.3.2.2.

■

2.4 Concluding Remarks

In this chapter, averaging theory both for stability and instability of one-time and two-time-scale systems has been presented. The elegance of this theory lies in that a complicated analysis of a nonlinear *nonautonomous* dynamical system can be replaced by a simpler analysis of its approximate *automonous* averaged system. Theorems developed under this theory provide sufficient conditions which allow one to extract properties, such as exponential stability, bounded stability, and instability, of the original system from its averaged system. Averaging also serves as a good approximation method that provides useful information such as the rate of convergence of an exponentially stable system.

The novelty of the averaging theory presented here includes a relaxation of the traditional almost periodic condition and a more concise proof of the construction of the coordinate transformation as a fundamental tool. Though more conditions are required, stronger results have been obtained. These conclusions will be especially useful in later analysis of adaptive systems.

Chapter 3 Frequency Domain Analysis of Adaptive Identifiers

3.1 Introduction

The problem of *parameter identification* (or *estimation*) is to identify the parameters of a given model of a plant using input-output data. Algorithms for identifying transfer functions can generally be distinguished into two classes: *off-line* and *on-line*. In the off-line case, it is presumed that all the data are available prior to the analysis. C.K. Sanathanan and J. Koemer (1963), P.A. Payne (1970), and H. Stall (1984) use frequency response data to estimate the parameters and, in turn, to synthesize the transfer function off line. In contrast to the off-line approach, the on-line approach requires that the parameter estimate be updated in *real time*. On-line parameter identification is referred to as adaptive identification. Parameter convergence proofs for adaptive identifier were given by Sondhi and Mitra (1976), Anderson (1977), Kreisselmeier (1977), Morgan and Narendra (1977a & b), and Goodwin and Sin (1984).

In this chapter, we use averaging to simplify the dynamics of adaptive identifiers. In order to bring the tools of Chapter 2 to bear on this topic, we choose a slow rate of parameter adjustment. An additional pay-off of this assumption is a frequency domain interpretation of the convergence analysis.

3.2 General Identifier Structure

In this section, we consider the identification of a transfer function

$$\hat{p}(s) = k_p \frac{\hat{n}_p(s)}{\hat{d}_p(s)} \tag{3.2.1}$$

describing a single-input single-output (SISO) LTI plant, satisfying the following assumptions:

(A1) $\hat{p}(s)$ is a proper, exponentially stable transfer function.

(A2) $\hat{n}_p(s)$ and $\hat{d}_p(s)$ are coprime monic polynomials, and $\hat{d}_p(s)$ is of known degree n.

The adaptive identifier considered here has a structure similar to that of Kreisselmeier (1977) and is shown in the Figure 3.2.1. The filter blocks F_1 and F_2 generate signals $v_1(t)$ and $v_2(t)$, which are smoothed derivatives of the input $u(t)$ and of the output $y_p(t)$ of the plant respectively. These blocks have identical transfer functions:

$$\hat{F}_1(s) = \hat{F}_2(s) = \frac{1}{\hat{\Lambda}(s)} \begin{bmatrix} 1 \\ s \\ \cdot \\ \cdot \\ s^{n-1} \end{bmatrix} \in R^n(s) \tag{3.2.2}$$

where $\hat{\Lambda}(s)$ is, by choice, an nth order Hurwitz polynomial. The output of the identifier $y_o(t)$ is obtained through the adaptive gains $C(t)$, $D(t) \in R^n$, and $c_{n+1}(t) \in R$:

$$y_o(t) = C^T(t)v_1(t) + D^T(t)v_2(t) + c_{n+1}(t)u(t) \,. \tag{3.2.3}$$

From Lemma B1 (in Appendix B), there exists a unique choice of adaptive gains, denoted C^*, D^*, and c^*_{n+1}, such that the transfer function from the input $u(t)$ to the output $y_o(t)$ is identical to the plant transfer function $\hat{p}(s)$. Before we start the analysis of the identifier, we make an assumption on the input $u(t)$:

Assumption:

(A3) The input $u(t)$ satisfies

$$u \,,\, \dot{u} \;\in L_\infty \,. \tag{3.2.4}$$

We define the parameter vector $\theta : R_+ \to R^{2n+1}$:

$$\theta(t) := [\; C^T(t) \,,\, D^T(t) \,,\, c_{n+1}(t) \;]^T \tag{3.2.5}$$

and the signal vector $w : R_+ \rightarrow R^{2n+1}$:

$$w(t) := [\, v^{(1)T}(t)\,,\, v^{(2)T}(t)\,,\, u(t)\,]^T \tag{3.2.6}$$

so that (3.2.3) implies

$$y_o(t) = \theta(t)^T w(t)\,. \tag{3.2.7}$$

Again by Lemma B1, the output of the plant $y_p(t)$ is then given by an equation similar to that of the identifier, i.e.

$$y_p(t) = \theta^{*T} w(t) + \eta(t) \tag{3.2.8}$$

where $\theta^* \in R^{2n+1}$ and

$$\theta^* = [C^{*T}, D^{*T}, c^*_{n+1}]^T \tag{3.2.9}$$

is the vector of "true" parameters corresponding to $\hat{p}(s)$, and $\eta(t)$ is an exponentially decaying function that accounts for effects due to the initial conditions of the stable plant and filters. Define the parameter error $\phi(t)$ as

$$\phi(t) := \theta(t) - \theta^* \tag{3.2.10}$$

and relate the output error $e_o := y_o - y_p$ to the parameter error ϕ by

$$e_o = \phi^T w - \eta\,. \tag{3.2.11}$$

The objective then is to design a parameter update law, using the information of the output error e_o, such that the parameter vector $\theta(t)$ will asymptotically converge to the true parameter θ^* regardless of any initial error. For our interest here, we will only consider two types of update laws, namely, (i) the Gradient Algorithm, and (ii) the Least-Squares with Forgetting Factor Algorithm, which are respectively defined as follows:

(i) **Gradient Algorithm:**

$$\dot{\theta} = -\Gamma e_o w \qquad \theta(0) = \theta_0 \tag{3.2.12}$$

where $\Gamma \in R^{(2n+1)\times(2n+1)}$ is a positive definite (>0) matrix, usually called the adaptation gain matrix.

(ii) **Least-Squares with Forgetting Factor Algorithm:**

$$\dot{\theta} = -g\,P\,e_o\,w \tag{3.2.13}$$

for some $g>0$, where P is called covariance matrix and is updated by the so called covariance propagation equation:

$$\dot{P} = \lambda P(t) - g\,P\,w\,w^T P \qquad P(0)=I \tag{3.2.14}$$

for some $\lambda>0$.

Remark: Whenever the covariance matrix P is invertible, it follows that $\dot{P}^{-1} = -P^{-1}\dot{P}\,P^{-1}$ so that the covariance propagation equation (3.2.14) can also be expressed as

$$\dot{P}^{-1} = -\lambda P^{-1} + g\,w\,w^T \,. \tag{3.2.15}$$

The following theorem guarantees the stability of the identifier and the convergence of the output error e_o.

Theorem 3.2.1: (Stability and Output Convergence)

Consider the identification of an LTI plant described by a transfer function $\hat{p}(s)$ using the identifier described above. Let assumptions (A1)-(A3) be satisfied, and the parameter vector $\theta(t)$ be updated by either (3.2.12) or (3.2.13).

Then the identifier remains stable, i.e. $\theta(t) \in L_\infty^{2n+1}$, and

$$\lim_{t\to\infty} e_o(t) = 0 \,. \tag{3.2.16}$$

□

Proof: See Kreiselmeier (1977).

3.3 Parameter Convergence Analysis

Theorem 3.2.1 assures the stability of the identifier and the convergence of output error e_o, but not the convergence of parameter errors ϕ. In this section, we will first focus our attention on conditions under which $\theta(t)$ will converge to the "true" parameter vector θ^*. Subsequently, we analyze the parameter convergence through the use of averaging theory. To start with, we give a definition which will be frequently used in the sequel.

Definition 3.3.1: (Persistently Exciting (PE))

A vector signal $w : R_+ \to R^m$ is said to be persistently exciting if there exist α_1, α_2, $\delta > 0$, such that

$$\alpha_1 I \le \int_s^{s+\delta} w(t) w^T(t) \mathrm{d}t \le \alpha_2 I \tag{3.3.1}$$

uniformly in $s \ge 0$. □

Remark: If w is replaced by a scalar signal u, the PE condition suggests that the average power of u over a time window with length Such a signal will usually have a frequency representation.

Theorem 3.3.1: (Parameter Convergence)

Consider the same identification problem as given in section 3.2. Let assumptions (A1)-(A3) be satisfied, and the parameters $\theta(t)$ be updated by either (3.2.12) or (3.2.13).

If the signal vector w defined in (3.2.6) is PE,

then the parameter errors $\phi=\theta-\theta^*$ satisfies

$$\lim_{t \to \infty} \phi(t) = 0 \tag{3.3.2}$$

exponentially. □

Proof: See Kreisselmeier (1977).

Definition 3.3.2: (Stationarity, Autocovariance)

A signal vector $w : R_+ \to R^m$ is said to be *stationary* if the following limit exists uniformly in $s \geq 0$:

$$R_w(\tau) := \lim_{T \to \infty} \frac{1}{T} \int_s^{s+T} w(t+\tau) w^T(t)\, dt \tag{3.3.3}$$

In this case, the limit $R_w(\tau) \in R^{m \times m}$ is called the *autocovariance* matrix of w. □

Lemma 3.3.2: (PE Condition on Stationary Signals)

Consider a stationary signal vector w with autocovariance matrix $R_w(\tau)$. w is PE if and only if $R_w(0) > 0$. □

Proof: See Boyd and Sastry (1986).

Lemma 3.3.3: (Positive Semidefinite Function)

Consider the autocovariance matrix $R_w(\tau) \in R^{m \times m}$ of a stationary signal vector w. $R_w(\tau)$ is a positive semidefinite function, i.e. for some $\tau_1, \cdot, \cdot, \cdot, \tau_k \in R$, and $C_1, \cdot, \cdot, \cdot, C_k \in C^m$,

$$\sum_{i,j} C_i^* R_w(\tau_j - \tau_i) C_j \geq 0 \,. \tag{3.3.4}$$

□

Proof: See Boyd and Sastry (1986).

Remark: By a matrix version of Bochner's Theorem (See Wiener (1930)), Lemma 3.3.3 implies that the autocovariance $R_w(\tau)$ can be represented as the inverse Fourier transform of a positive semidefinite, bounded *power spectral measure* $S_w(d\omega)$, i.e.

$$R_w(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega\tau} S_w(d\omega) \ . \tag{3.3.5}$$

We now consider the same identification problem as given in section 3.2, assuming that the input $u(t)$ of the plant is stationary. Denote by $\hat{Q}(s)$ the transfer function from the input u to the output w, as defined in (3.2.6). Then $\hat{Q}(s)$ has the form

$$\hat{Q}(s) = \begin{bmatrix} \hat{F}_1(s) \\ \hat{F}_2(s)\hat{p}(s) \\ 1 \end{bmatrix} \in R^{2n+1}(s) \tag{3.3.6}$$

From Lemma B2, $S_w(d\omega)$ can be computed in terms of $S_u(d\omega)$ as

$$S_w(d\omega) = \hat{Q}(j\omega)\, S_u(d\omega)\, \hat{Q}^*(j\omega) \tag{3.3.7}$$

so that, by Lemma 3.3.2, w is PE if and only if

$$R_w(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{Q}(j\omega)\hat{Q}^*(j\omega) S_u(d\omega) > 0 \ . \tag{3.3.8}$$

This will then allow us to relate the persistency of excitation of w to the frequency content of the input u. The result will be stated in Theorem 3.3.4 after a preliminary definition.

Definition 3.3.3: (Spectral Support)

The *spectral support* of a scalar stationary signal $u(t)$ with power spectral measure $S_u(d\omega)$ is defined as

$$\mathrm{Supp}(u) := \left\{ \omega \;\middle|\; \omega \in R \;\; , \text{ and for all } \delta > 0 \; , \int_{\omega-\delta}^{\omega+\delta} S_u(d\omega) > 0 \right\} . \tag{3.3.9}$$

Remark: Define $\mathrm{F}_u(\omega)$ by:

$$\mathrm{F}_u(\omega) = \int_{-\infty}^{\omega} S_u(d\omega') \; . \tag{3.3.10}$$

Then $\mathrm{F}_u(\omega)$ is a spectral distribution function which is monotonically increasing and continuous from the right. If $\mathrm{F}_u(d\omega)$ is absolutely continuous, then the spectral support Supp(u) defines a continuous spectrum, which denotes the smallest closed set outside which the power spectral measure $S_u(d\omega)$ vanishes. On the other hand, if $\mathrm{F}_u(d\omega)$ is a stair-case function with n jumps, then Supp(u), which has exactly n points in the frequency support, defines the discrete spectrum.

Theorem 3.3.4: (PE Condition on Supp(u))

Consider the signal vector w defined in (3.2.6). w is PE if and only if the spectral support of the input u, Supp(u), contains at least $2n+1$ points. □

Proof: See Boyd and Sastry (1986).

Remark: The input u that results in a PE w will be called *sufficiently rich* (SR). From the theorem, u is SR if and only if Supp(u) contains at least $2n+1$ points.

Using all these definitions and results, we are now ready to analyze the identification system with a parameter update law of either (i) Gradient Algorithm or (ii) Least-Squares with Forgetting Factor Algorithm. However, throughout the sequel, we will drop the term η (the effects of initial conditions) from the output error e_o in (3.2.11) by assuming that the dynamics of the plant and filters are much faster than that of the parameter θ in the context of our later analysis using averaging.

3.3.1 Gradient Algorithm

Application of averaging to the dynamics of $\theta(t)$ in (3.2.12) is focused on the case where $\Gamma = \varepsilon I$ and ε is a non-zero small positive number, i.e.

$$\dot{\phi} = -\varepsilon e_o w \ . \tag{3.3.1.1}$$

Substituting (3.2.11) for e_o in (3.3.1.1) (with η being neglected), we have

$$\dot{\phi} = -\varepsilon w w^T \phi \tag{3.3.1.2}$$

Assuming that u and w are stationary, the averaged system of (3.3.1.2), using definition (2.1.5), is well defined and is given by

$$\dot{\phi}_{av} = -\varepsilon R_w(0) \phi_{av} \tag{3.3.1.3}$$

where $R_w(0)$ is the autocovariance of w. This system is particularly easy to study since it is linear. Now if the input u is SR, then, by Theorem 3.3.4 and Lemma 3.3.2, $R_w(0)$ is positive definite. This further implies the exponential stability of the averaged system (3.3.1.3). A natural Lyapunov function for this system is

$$v(\phi_{av}) = \frac{1}{2} \phi_{av}^T \phi_{av} = \frac{1}{2} \| \phi_{av} \|^2 \tag{3.3.1.4}$$

so that

$$\dot{v}(\phi_{av}) \le -\varepsilon \lambda_{\min}(R_w(0)) \| \phi_{av} \|^2 \tag{3.3.1.5}$$

where $\lambda_{\min}$ stands for the minimum eigenvalue. The rate of exponential convergence of the averaged system is therefore at least $\varepsilon\lambda_{\min}(R_w(0))$. By the remark after Theorem 2.2.1.5, we can readily conclude that the bound on the rate of convergence of the original system (3.3.1.2) for ε sufficiently small is $\varepsilon\lambda_{\min}(R_w(0)) + o(\varepsilon)$.

Remarks:

(1) In fact, the rate of convergence of the averaged system is at most $\varepsilon\lambda_{\max}(R_w(0))$ so that, for ε small, the rate of convergence of (3.3.1.2) will actually be close to the interval:

$$[\ \varepsilon\lambda_{\min}(R_w(0))\ ,\ \varepsilon\lambda_{\max}(R_w(0))\ ] \tag{3.3.1.6}$$

(2) Eq. (3.3.8) gives an interpretation of $R_w(0)$ in the frequency domain and a means of computing an estimate of the rate of convergence of the adaptive algorithm, given the spectral content of the reference input u. If the input u has only point spectrum, the integral in (3.3.8) may be replaced by a summation. Since the transfer function $\hat{Q}(s)$ depends on the unknown plant to be identified, the use of the averaged system to determine the rate of convergence is limited. If, however, prior estimates of the plant, similar to those used in a Bayesian context in stochastic parameter estimation, are available, then some bounds on $R_w(0)$ and on the rates of convergence can be deduced. These in turn can be used to determine the spectral content of the reference input u that will optimize the rate of convergence of the identifier, given physical constraints on u. Such a procedure is very reminiscent of that indicated in Goodwin and Payne (1977) (Chapter 6) for the design of the reference input. The autocovariance matrix defined here can be characterized as an *average information matrix* (that will be defined in Chapter 5), for example, in Goodwin and Payne (1977). Our interpretation, however, is in terms of rates of parameter convergence of the averaged system rather than in terms of parameter error covariance.

To illustrate the result obtained through this analysis, we present the following example.

Example 3.3.1:

Consider the adaptive identification of a first order plant:

$$\hat{p}(s)=2\frac{s+1}{s+3} . \tag{3.3.1.7}$$

The filter of the compensator block is chosen to be $\frac{5}{s+5}$. Denote the parameter error vector $\phi=\theta-\theta^*$, where $\theta^*=[c_1^*, d_1^*, c_2^*]^T$ is computed to be $[-1.6, 0.4, 2.0]^T$. Since the number of unknown parameters is 3, parameter convergence will occur when Supp(u) contains at least 3 points. For the simulations, we considered an input of the form $a_0+a_1\sin(\omega t)$. By virtue of (3.3.8), (3.3.1.3) now becomes

$$\begin{bmatrix} \dot{\phi}_{av1} \\ \dot{\phi}_{av2} \\ \dot{\phi}_{av3} \end{bmatrix} = -\varepsilon \begin{bmatrix} a_0^2+\frac{25a_1^2}{2(25+\omega^2)} & \frac{2}{3}a_0^2+\frac{25(3+\omega^2)a_1^2}{(9+\omega^2)(25+\omega^2)} & a_0^2+\frac{25a_1^2}{2(25+\omega^2)} \\ \frac{2}{3}a_0^2+\frac{25(3+\omega^2)a_1^2}{(9+\omega^2)(25+\omega^2)} & \frac{4}{9}a_0^2+\frac{50(1+\omega^2)a_1^2}{(9+\omega^2)(25+\omega^2)} & \frac{2}{3}a_0^2+\frac{5(15+7\omega^2)a_1^2}{(9+\omega^2)(25+\omega^2)} \\ a_0^2+\frac{25a_1^2}{2(25+\omega^2)} & \frac{2}{3}a_0^2+\frac{5(15+7\omega^2)a_1^2}{(9+\omega^2)(25+\omega^2)} & a_0^2+\frac{a_1^2}{2} \end{bmatrix} \cdot \begin{bmatrix} \phi_{av1} \\ \phi_{av2} \\ \phi_{av3} \end{bmatrix} \tag{3.3.1.8}$$

With $a_0=2$, $a_1=2$ and $\omega=4$, the three positive eigenvalues of $R_w(0)$ are computed to be 0.28ε, 0.64ε and 15.39ε. Figures 3.3.1 and 3.3.2 show the plots of parameter errors ϕ_1 and ϕ_2 for both the original and averaged systems with two different adaptation gains ε=0.1 , 1. Figure 3.3.3 is a plot of the Lyapunov function of (3.3.1.4) for both systems using a log scale. Note the closeness in the rate of convergence of the two systems.

3.3.2 Least-Squares with Forgetting Factor Algorithm

In order to study parameter update law (3.2.13) and the covariance propagation equation (3.2.14) using averaging, we set $g = \varepsilon > 0$ and λ is replaced by $\varepsilon\lambda$, i.e.

$$\dot{\phi} = -\varepsilon P\, e_o\, w \qquad \phi(0) = \phi_0 \tag{3.3.2.1}$$

$$\dot{P} = \varepsilon\lambda P - \varepsilon P\, w\, w^T P \qquad P(0) = I\ . \tag{3.3.2.2}$$

Again, substitution of (3.2.11) for $e_o(t)$ in (3.3.2.1) leads to the following form:

$$\dot{\phi} = -\varepsilon P\, w\, w^T \phi\ . \tag{3.3.2.3}$$

Note that, for small ε, ϕ and P now characterize the slow variables of the identification system, in contrast with the previous case where ϕ is the only slow variable. Consequently, when averaging is applied to this system, both differential equations (3.3.2.2) and (3.3.2.3) should be averaged. As indicated previously, if u is stationary, the averaged equations of (3.3.2.2)-(3.3.2.3) are well defined:

$$\dot{\phi}_{av} = -\varepsilon P_{av}\, R_w(0)\, \phi_{av} \qquad \phi_{av}(0) = \phi_0 \tag{3.3.2.4}$$

$$\dot{P}_{av} = \varepsilon\lambda P_{av} - \varepsilon P_{av}\, R_w(0)\, P_{av} \qquad P_{av}(0) = I\ . \tag{3.3.2.5}$$

Again by Theorem 3.3.4 and Lemma 3.3.2, the reference input u being SR implies $R_w(0) > 0$ so that, by referring to (3.2.15), we have

$$P_{av}(t)^{-1} = I\, e^{-\varepsilon\lambda t} + \frac{1}{\lambda} R_w(0)\,(1 - e^{-\varepsilon\lambda t}) \tag{3.3.2.6}$$

so that

$$\min\left(1, \frac{1}{\lambda}\lambda_{\min}(R_w(0))\right) I \ \le\ P_{av}(t)^{-1} \ \le\ \max\left(1, \frac{1}{\lambda}\lambda_{\max}(R_w(0))\right) I \tag{3.3.2.7}$$

for all $t \ge 0$. Thus it can be easily seen that $(\phi_{av}, P_{av}) = (0, R_w(0)^{-1})$ is an unique equilibrium point of (3.3.2.4)-(3.3.2.5); in particular, $P_{av} = R_w(0)^{-1}$ is an exponentially stable equilibrium point of (3.3.2.5). Consider the following Lyapunov function:

$$v(\phi_{av}, P_{av}) = \frac{1}{2}\, \phi_{av}^T P_{av}^{-1} \phi_{av}\ . \tag{3.3.2.8}$$

Denote

$$\alpha_1 := \frac{1}{2} \min\,(1, \frac{1}{\lambda} \lambda_{\min}(R_w(0))) \tag{3.3.2.9}$$

and

$$\alpha_2 := \frac{1}{2} \max\,(1, \frac{1}{\lambda} \lambda_{\max}(R_w(0)))\,. \tag{3.3.2.10}$$

Then, from (3.3.2.6), v satisfies

$$\alpha_1 \| \phi_{av} \|^2 \le v(\phi_{av}, P_{av}) \le \alpha_2 \| \phi_{av} \|^2 \tag{3.3.2.11}$$

$$\left\| \frac{\partial v}{\partial \phi_{av}} \right\| \le 2\alpha_2 \| \phi_{av} \| := \alpha_3 \| \phi_{av} \| \,, \tag{3.3.2.12}$$

and, using $\dfrac{\partial P_{av}^{-1}}{\partial P_{av(i,j)}} = - P_{av}^{-1} \dfrac{\partial P_{av}}{\partial P_{av(i,j)}} P_{av}^{-1}$,

$$\left\| \frac{\partial v}{\partial P_{av}} \right\| \le 4n^2 \alpha_2^2 \| \phi_{av} \|^2 := \alpha_4 \| \phi_{av} \|^2 \,. \tag{3.3.2.13}$$

The derivative of v taken along the trajectories of (3.3.2.4)-(3.3.2.5) is such that

$$\begin{aligned} \dot{v} + \varepsilon \lambda v &= - \frac{\varepsilon}{2} \phi_{av}^T R_w(0) \phi_{av} \\ &\le - \frac{\varepsilon}{2} \lambda_{\min}(R_w(0)) \,) \| \phi_{av} \|^2 := - \varepsilon \alpha_5 \| \phi_{av} \|^2 \\ &\le - \varepsilon \frac{\alpha_5}{\alpha_2} v \,. \end{aligned} \tag{3.3.2.14}$$

This and inequality (3.3.2.11) readily imply the exponential stability of the averaged system with the rate of convergence at least $\frac{\varepsilon}{2}(\lambda + \frac{\alpha_5}{\alpha_2})$. It can be easily checked that this setup satisfies the assumptions in Theorem 2.2.1.10. Consequently, it follows that, for sufficiently small ε, the bound on the rate of convergence of $\phi(t)$ of the original system (3.3.2.3) is $\frac{\varepsilon}{2}(\lambda + \frac{\alpha_5}{\alpha_2}) + o(\varepsilon)$.

Remarks:

(1) Defining

$$\alpha_6 := \frac{1}{2}\lambda_{\max}(R_w(0)) \tag{3.3.2.15}$$

it can be seen that the rate of convergence of the averaged system is at most $\frac{\varepsilon}{2}(\lambda+\frac{\alpha_6}{\alpha_1})$. Hence for small enough ε the rate of convergence of $\phi(t)$ is actually close to the interval:

$$[\,\frac{\varepsilon}{2}(\lambda+\frac{\alpha_5}{\alpha_2}),\ \frac{\varepsilon}{2}(\lambda+\frac{\alpha_6}{\alpha_1})\,] \tag{3.3.2.16}$$

It is interesting to note that, if $\lambda_{\max}(R_w(0))\geq\lambda$ and $\lambda_{\min}(R_w(0))\leq\lambda$, then the above interval can be replaced by

$$[\,\frac{\varepsilon}{2}\left[\lambda+\frac{1}{\mathrm{cond}(R_w(0))}\right],\ \frac{\varepsilon}{2}\left[\lambda+\mathrm{cond}(R_w(0))\right]\,] \tag{3.3.2.17}$$

where $\mathrm{cond}(R_w(0))$ is the condition number of $R_w(0)$, in contrast with the interval given in (3.3.1.6) in the previous case.

(2) Note that

$$\begin{aligned}
\dot{v} + \varepsilon\lambda v &= -\frac{\varepsilon}{2}\phi_{av}^T P_{av}^{-1/2}(P_{av}^{1/2}R_w(0)P_{av}^{1/2})P_{av}^{-1/2}\phi_{av} \\
&\leq -\frac{\varepsilon}{2}\lambda_{\min}(P_{av}^{1/2}R_w(0)P_{av}^{1/2})\,\|\,P_{av}^{-1/2}\phi_{av}\,\|^2 \\
&= -\,\varepsilon\lambda_{\min}(P_{av}^{1/2}R_w(0)P_{av}^{1/2})\,v\ .
\end{aligned} \tag{3.3.2.18}$$

When ϕ_{av} is small, $P_{av}(t)^{-1}$ (by (3.3.2.6)) is close to $\frac{1}{\lambda}R_w(0)$ so that (3.3.2.18) becomes

$$\dot{v} + \varepsilon\lambda v \leq -\varepsilon\lambda v\ . \tag{3.3.2.19}$$

This implies that the bound on the rate of convergence ϕ_{av} is close to $\varepsilon\lambda$. By an argument as before, it can be checked that, in fact

$$\dot{v} + \varepsilon\lambda v \approx -\varepsilon\lambda v \tag{3.3.2.20}$$

which then implies that rate of parameter convergence is close to $\varepsilon\lambda + o(\varepsilon)$ when ϕ is small. This leads to the conclusion that the rate of "tail" parameter convergence will not be affected much by different choices of reference inputs.

3.4 Robustness to Unmodelled Dynamics

In previous sections, we discussed the identification of an ideal plant. However, unmodelled dynamics will inevitably exist in practice becauses less significant dynamics are often too hard to model or purposely neglected to permit reasonable computation. The operation of an adaptive identifier must be reexamined to assure stability and performance in the presence of unmodelled dynamics.

In this section, we will consider finite dimension (FD) and linear time-invariant (LT-I) unmodelled dynamics so that the overall plant can be represented by

$$\hat{p}_u(s) := \hat{p}(s) + \Delta\hat{p}(s) = k_p \frac{\hat{n}_p(s)}{\hat{d}_p(s)} \frac{\hat{n}_u(s)}{\hat{d}_u(s)} \tag{3.4.1}$$

where $\hat{p}(s)$ is the nominal plant transfer function with order n as described in section 3.2. To study the robustness of the identification scheme, we make two assumptions additional to (A1)-(A3):

Assumptions:

(A4) $\Delta\hat{p}(s)$ is a stable, proper transfer function and $\hat{p}_u(s)$ is of the order N where $N > n$.

(A5) Input $u(t)$ is stationary.

3.4.1 Matching Criterion

Refer to the definition of $\hat{Q}(s)$ in (3.2.6). In the case of no unmodelled dynamics, Lemma B1 guarantees that there exist unique "true" parameters $\theta^* \in R^{2n+1}$ such that

$$\hat{p}(s) = \theta^{*T}\hat{Q}(s) \qquad \text{for all } s \in C \tag{3.4.1.1}$$

Denote

$$\hat{C}(s) = C^T \begin{bmatrix} 1 \\ s \\ \cdot \\ \cdot \\ s^{n-1} \end{bmatrix} \quad \text{and} \quad \hat{D}(s) = D^T \begin{bmatrix} 1 \\ s \\ \cdot \\ \cdot \\ s^{n-1} \end{bmatrix}. \tag{3.4.1.2}$$

Then eq. (3.4.1.1) implies

$$\begin{aligned} \hat{p}(s) &= \frac{\hat{C}^*(s)}{\hat{\Lambda}(s)} + \frac{\hat{D}^*(s)}{\hat{\Lambda}(s)}\hat{p}(s) + c_{n+1}^* \\ &= \frac{c_{n+1}^*\hat{\Lambda}(s) + \hat{C}^*(s)}{\hat{\Lambda}(s) - \hat{D}^*(s)} \end{aligned} \tag{3.4.1.3}$$

where $\hat{\Lambda}(s)$ is defined in section 3.2. Thus, when the parameter convergence takes place under PE condition, the plant transfer function is identified through the relationship (3.4.1.3).

In case the plant has unmodelled dynamics, the order of the identifier is less than that of the true plant by assumption. Then there may not exist a $\theta \in R^{2n+1}$ such that

$$\hat{p}_u(s) = \theta^T \hat{Q}_u(s) \qquad \text{for all } s \in C \tag{3.4.1.4}$$

where $\hat{Q}_u(s)$ is defined by

$$\hat{Q}_u(s) = \begin{bmatrix} \hat{F}_1(s) \\ \hat{F}_2(s)\hat{p}_u(s) \\ 1 \end{bmatrix}. \tag{3.4.1.5}$$

This, in turn, implies the lack of a $\theta \in R^{2n+1}$ to satisfy

$$\hat{p}_u(j\omega) = \theta_0^T \hat{Q}_u(j\omega) \qquad \text{for all } \omega \in R\ . \tag{3.4.1.6}$$

However, under the assumption (A5), there may exist a $\theta_0 \in R^{2n+1}$, corresponding to a specific reference input u, such that

$$\hat{p}_u(j\omega) = \theta_0^T \hat{Q}_u(j\omega) \qquad \text{for all } \omega \in \text{Supp}(u)\,. \tag{3.4.1.7}$$

In other words, due to unmodelled dynamics, perfect matching of parameters for all stationary inputs will no longer be possible; only conditional matching should be expected instead. Now consider a matching and define a pseudo signal e_* by

$$e_* := \theta_0^T w - y_p\,. \tag{3.4.1.8}$$

It follows that

$$\lim_{t \to \infty} e_*(t) = 0 \tag{3.4.1.9}$$

with exponential convergence (Callier and Desoer (1982) p. 127). The convergence of this pseudo signal becomes extremely important in establishing a result similar to Theorem 3.2.1. This will be shown in Theorem 3.4.1.3. First, however, we will present some results which provide conditions for a possible matching (3.4.1.7).

Lemma 3.4.1.1:

Consider the transfer function $\hat{Q}_u(s)$ given in (3.4.1.5). For any $k \in Z_+$, $k \le n$ there exists a set of $2k+1$ frequencies, $(\omega_1, \ldots, \omega_{2k+1})$, such that

$$\left[\hat{Q}_u(j\omega_1), \cdot, \cdot, \cdot, \hat{Q}_u(j\omega_{2k+1})\right]$$

forms a linearly independent set of vectors. □

Proof: See Mason et al (1987).

Theorem 3.4.1.2: (Almost Always Matching Condition)

Consider the above identification problem. Let assumptions (A1)-(A5) be satisfied. For any $k \in Z_+$, $k \le n$, there exists a subset $U_k \subset R^{2k+1}$ which is nowhere dense and measure zero such that the matching (3.4.1.7) is possible if Supp(u) contains $2k+1$ points which form a $(2k+1)$ tuple not contained in U_k. □

Proof: cf. Mason et al (1987).

Remark: Theorem 3.4.1.2 does not guarantee that any stationary input u whose Supp(u) contains at most $2n+1$ points will result in a matching. However, the result indicates that *almost every* input u that satisfies this condition will yield a matching.

Theorem 3.4.1.3: (Stability and Output Convergence under Matching)

Consider the identification problem given above. Let assumptions (A1)-(A5) be satisfied with the parameter vector $\theta(t)$ updated either by (3.2.12) or by (3.2.13).

If the input u satisfies conditions of Theorem 3.4.1.2,

then the identifier will remain stable, i.e. $\theta(t) \in L_\infty^{2n+1}$, and

$$\lim_{t \to \infty} e_o(t) = 0 \, . \tag{3.4.1.10}$$

□

Proof: Replace the output error e_o by

$$\begin{aligned} e_o &= (y_o - \theta_0^T w) + (\theta_0 - y_p) = (\theta - \theta_0)^T w + e_* \\ &=: \phi^T w + e_* \end{aligned} \tag{3.4.1.11}$$

Then the conclusion follows from (3.4.1.9), Theorem 3.4.1.2, Bodson (1986) p. 30, and Kreiselmeier (1977).

■

Remark: The theorem implies that, for almost every input u such that Supp(u) contains at most $2n+1$ points, the true plant behaves like an nth order (nominal order) plant so that the properties obtained in Theorem 3.2.1 also hold here. However, one should note that, due to lack of perfect matching (3.4.1.6), output convergence to zero may fail to hold when Supp(u) contains more than $2n+1$ points. Despite this, in the following subsections, we will show that the stability of the system will be preserved.

3.4.2 Tuned Model

As indicated in the previous remark, the true plant behaves differently from an nth order plant when Supp(u) contains more points than necessary, namely, $2n+1$. As a result, the parameter vector $\theta(t)$ may fail to converge to a fixed value in the parameter space but rather wander around in that space. Yet, this failure of convergence does not suggest the disadvantage of an input u whose Supp(u) contains too many points. On the contrary, in the case where output disturbances, such as measurement noise, deteriorate the plant output, or where rates of parameter convergence are to be optimized, the aforementioned input may be useful.

In this subsection, we aim at seeking a good model of the plant when Supp(u) contains more than $2n+1$ points. Such a model will be arrived at through the use of a frequency-domain interpretation. Conceivably, this model will play a major role in identifying "a" transfer function, and, at the same time, will be quite input dependent. So the choice of reference input will become relatively important considering the future control task of this plant. These points will be made clear in a later subsection, where we relate the so obtained model with signals in time domain by use of averaging, and in Chapter 5. The following theorem, complementary to Theorem 3.4.1.2, will be a useful tool in later analysis.

Theorem 3.4.2.1: (Almost Always PE Condition)

Consider the same identification problem as given before. Let assumptions (A1)-(A5) be satisfied. There exists a nowhere dense, measure zero subset U_n in R^{2n+1} such that the signal vector w is PE if and only if Supp(u) contains at least $2n+1$ points which form a $(2n+1)$ tuple not contained in U_n. □

Proof: See Mason et al (1987).

Remark: This theorem, unlike Theorem 3.3.4, provides only *almost always* PE condition due to unmodelled dynamics. As we know, persistency of excitation of w will make the homogeneous part of the identification system exponentially stable. The essence of this theorem lies in the fact that, though the order (here n) of identifier may be much smaller than that of the true plant, the number of frequencies needed to excite the system persistently could "almost always" still be $2n+1$. This allows one to have confidence in the identification schemes even in the face of unmodelled dynamics.

The following is a preliminary definition, similar to Definition 3.3.2, that will be frequently used in the sequel.

Definition 3.4.2.1: (Crosscovariance, Cross-Power Spectral Measure)

The crosscovariance of two stationary signals $w_1 : R_+ \to R^{m_1}$ and $w_2 : R_+ \to R^{m_2}$ is defined by

$$R_{w_1 w_2}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_s^{s+T} w_1(t+\tau) w_2^T(\tau) \, d\tau \tag{3.4.2.1}$$

or

$$R_{w_2 w_1}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_s^{s+T} w_2(t+\tau) w_1^T(\tau) \, d\tau \, . \tag{3.4.2.2}$$

The Fourier transform of the crosscovariance $R_{w_1w_2}(\tau)$ ($R_{w_2w_1}(\tau)$) gives the cross-power spectral measure $S_{w_1w_2}(d\omega)$ ($S_{w_2w_1}(d\omega)$). □

Remark: As indicated by the definition, the crosscovariance matrix $R_{w_1w_2}(0)$ can be represented by

$$R_{w_1w_2}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{w_1w_2}(d\omega) \,. \tag{3.4.2.3}$$

If $\hat{H}_{uw_1}$ and $\hat{H}_{uw_2}$ are transfer functions from u to w_1 and w_2 respectively, then by a proof similar to Lemma B2 we have

$$R_{w_1w_2}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{H}_{uw_1}(j\omega)\hat{H}^*_{uw_2}(j\omega)S_u(d\omega) \,. \tag{3.4.2.4}$$

As has been pointed out in the beginning of this section, when Supp(u) contains more than $2n+1$ points, there may not exist fixed parameters θ_0 such that the matching (3.4.1.7) will occur. In the off-line identification scheme, a *least-mean-squares* method is used to solve for the plant parameters. Here, a similar approach will be used. Define a cost function $J(\theta)$ by

$$J(\theta) := \lim_{T\to\infty}\frac{1}{T}\int_0^T e_*^2(t)\,dt \tag{3.4.2.5}$$

where e_* is defined in (3.4.1.8). By definition 3.3.2, $J(\theta)$ is the autocovariance of e_* and can be written as

$$J(\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\theta^T\hat{Q}_u(j\omega) - \hat{p}_u(j\omega)\right|^2 S_u(d\omega) \tag{3.4.2.6}$$

as a result of (3.3.8). Clearly, $J(\theta)$ is a quadratic function of θ which has a global minimum but the minimizer may not be unique. Let the optimum θ_T be defined as

$$\theta_T := \operatorname{argmin}\left\{ J(\theta) \mid \theta \in R^{2n+1} \right\}. \tag{3.4.2.7}$$

A necessary condition for optimality is

$$\frac{\partial J(\theta)}{\partial \theta}\bigg|_{\theta=\theta_T} = 0 \tag{3.4.2.8}$$

which leads to

$$\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{Q}_u(j\omega)\hat{Q}_u^*(j\omega)\, S_u(d\omega)\right]\theta_T = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{Q}_u(j\omega)\hat{p}_u^*(j\omega)\, S_u(d\omega)\right] \tag{3.4.2.9}$$

or, by (3.3.8) and (3.4.2.4),

$$R_w(0)\,\theta_T = R_{wy_p}(0)\,. \tag{3.4.2.10}$$

Consequently, by Theorem 3.4.2.1 and Lemma 3.3.2, for almost every input u whose Supp(u) contains at least $2n+1$ points, $R_w(0)>0$ so that θ_T is uniquely defined by

$$\theta_T = R_w(0)^{-1} R_{wy_p}(0)\,. \tag{3.4.2.11}$$

Denote

$$\theta_T = [\, C_T^T\,,\, D_T^T\,,\, c_{(n+1)T}\,]^T\,. \tag{3.4.2.12}$$

Now we define the *tuned model* of the true plant, depending on the specific input u, to be the nth order transfer function $\hat{p}_T(s)$ which is obtained by setting $\theta(t)$ to θ_T. From (3.4.1.2) and (3.4.1.3) we get

$$\hat{p}_T(s) = \frac{c_{(n+1)T}\,\hat{\Lambda}(s) + \hat{C}_T(s)}{\hat{\Lambda}(s) - \hat{D}_T(s)}\,. \tag{3.4.2.13}$$

θ_T is defined to be the *tuned parameter*. Since the tuned model (or tuned parameter) depends on the unknown plant, the above transfer function seems to be more conceptual than practical at this point. However, in subsection 3.4.3, we will apply averaging to show that, when the input u is SR, the parameters $\theta(t)$ will stay within a ball centered at θ_T with a radius $\psi(\varepsilon)$ of class K for ε small enough.

The tuned model, however, may not be stable even when the real plant is in fact stable. This is possible if the input u has spectral energy concentrated in high frequency spectrum where the unmodelled dynamics become too significant to be neglected. Hence, if unmodelled dynamics are present, the choice of input frequencies may become important when the identified transfer function is to be used for control purposes. More discussions on this aspect will be given later in Chapter 5.

3.4.3 Stability Analysis

The fact that $\hat{p}_T$ could be unstable, however, does not imply the instability of the overall identification system. After all, the tuned model is simply a fictitious plant. In this subsection, we will first formally analyze the stability of the identification system in the presence of unmodelled dynamics, and then study its behavior using averaging. Though the case with bounded output disturbance is not discussed here, it can be easily seen that the following results can be applied there equally well.

In subsection 3.4.1, Theorem 3.4.1.3 has guaranteed stability for almost every input u whose Supp(u) contains at most $2n+1$ points. There, the existence of a matching (3.4.1.7) (so that e_* converges to zero exponentially) is the key to the proof. However, such a matching may be lost when Supp(u) contains more than $2n+1$ points. Theorem 3.4.2.1 becomes essential to the proof of stability of the overall system when such is the case. The following theorem will present a result similar to Theorem 3.3.1.

Theorem 3.4.3.1: (BIBO Stability Theorem)

Consider the identification problem given above. Let assumptions (A1)-(A5) be satisfied and the parameter vector $\theta(t)$ be updated either by (3.2.12) or by (3.2.13).

If the input u satisfies conditions of Theorem 3.4.2.1, then the identifier will remain stable, i.e. $\theta \in L_{\infty}^{2n+1}$.

Moreover, if, additionally, Supp(u) contains exactly $2n+1$ points and θ_T is the tuned parameter corresponding to the particular u, then

$$\lim_{t \to \infty} \theta(t) = \theta_T \tag{3.4.3.1}$$

with exponential convergence. □

Proof: Substitute $\theta^T w - y_p$ for e_o in (3.2.12) and (3.2.13) respectively, and we have

$$\dot{\theta} = -\Gamma w\, w^T \theta + \Gamma y_p\, w \tag{3.4.3.2}$$

and

$$\dot{\theta} = -g\, P\, w\, w^T \theta + g\, P\, y_p\, w \tag{3.4.3.3}$$

By hypothesis, Theorem 3.4.2.1 implies the persistency of excitation of w so that, from Theorem 3.3.1, the homogeneous systems of (3.4.3.2) and (3.4.3.3) are exponentially stable. We then conclude the result by using Lemma 2.2.1.8. Furthermore, by Theorem 3.4.1.3, it then follows that $\theta(t)$ converges to θ_T exponentially.

■

Remark: The importance of this theorem and Theorem 3.2.1.3 lies in the fact that, when performing an identification task, we are assured of stability by choosing almost any stationary input u. Furthermore, the convergence of the parameter vector $\theta(t)$ can almost always be achieved when Supp(u) contains exactly 2n+1 points, where n could be much smaller than N.

We now study the relationship between the parameter vector $\theta(t)$, obtained in the time domain, and the tuned parameter vector θ_T, derived in the frequency domain, through averaging. The following theorem will summarize the result.

Theorem 3.4.3.2: (**Tuned Parameter Approximation Theorem**)

Consider the identification problem given above. Let assumptions (A1)-(A5) be satisfied.

If the signal vector w is PE,

then there exists a class K function $\psi(\varepsilon)$, $\varepsilon_1 > 0$, and $0 \le T < \infty$ such that, for all $\varepsilon \le \varepsilon_1$,

$$\| \theta(t) - \theta_T \| \le \psi(\varepsilon) \qquad t \ge t_0 + T \; . \tag{3.4.3.4}$$

□

Proof: We proceed in two parts (a) and (b).

(a) Gradient Algorithm:

The averaged differential equation of (3.4.3.2) with $\Gamma = \varepsilon I$ can be easily found to be

$$\dot{\theta}_{av}(t) = -\varepsilon R_w(0)\theta_{av}(t) + \varepsilon R_{wy_p}(0) \; . \tag{3.4.3.5}$$

By hypothesis and Lemma 3.3.2, $R_w(0) > 0$, and, using the definition (3.4.2.11), it follows that θ_T is the unique, exponentially stable equilibrium point of (3.4.3.5). The conclusion then follows from Theorem 2.2.1.9 and its remark.

(b) Least-Squares with Forgetting Factor Algorithm:

The averaged differential equations of (3.4.3.3) with $g = \varepsilon$ and (3.3.2.2) can be found to be

$$\dot{\theta}_{av}(t) = -\varepsilon P_{av} R_w(0)\theta_{av}(t) + \varepsilon P_{av} R_{wy_p}(0) \tag{3.4.3.6}$$

and

$$\dot{P}_{av} = \varepsilon \lambda P_{av} - \varepsilon P_{av} R_w(0) P_{av} \; . \tag{3.4.3.7}$$

By the analysis in subsection 3.3.2, it follows that $(\theta_{av}, P_{av}) = (\theta_T, R_w(0)^{-1})$ is the unique, exponentially stable equilibrium point of (3.4.3.6)-(3.4.3.7). From (a), the same conclusion then follows.

■

Remark: The conclusions of Theorems 3.4.3.1 and 3.4.3.2 together form similar results as in Theorem 3.3.1. Theorem 3.4.3.1 per se only provides a stability proof for the parameters $\theta(t)$ when Supp(u) contains more than $2n+1$ points, while Theorem 3.4.3.2 provides a good approximation of the steady state of $\theta(t)$ when ε is small. In other words, under the PE condition, the *point-convergence* of the parameter vector $\theta(t)$ in the ideal case has to be replaced by a *set-convergence* in the case of unmodelled dynamics. However, performance of the identifier can be enhanced by making ε small so that the tuned model transfer function can be actually identified.

3.5 Concluding Remarks

In this chapter, we first reviewed the adaptive identifier of Kreisselmeier (1977) under the assumption of no unmodelled dynamics. Both gradient and least-squares algorithms have been considered. When the reference input signals are stationary and the rate of adaptation is slow, the governing differential equations are similar to those of the one-time-scale systems discussed in Chapter 2. We applied the averaging results developed in Chapter 2 as an approximation method to obtain the estimates of rates of parameter convergence. An example using the gradient algorithm was given to illustrate the performance of these estimates. Study of these estimates suggests a means of optimizing parameter convergence rates, namely, maximizing either the smallest eigenvalue or the condition number of the autocovariance matrix of w, $R_w(0)$, under some physical constraints on u. These become basic topics of Chapter 5 and will be explored there in great detail.

Later, assuming the existence of FDLT-I unmodelled dynamics, we re-examined the same adaptive identifier for two fundamentally important objects: stability and performance. A sufficient condition for *almost always* matching, and a sufficient and necessary condition for *almost always* persistently exciting were derived. These are fundamental to establishing the stability of the identification schemes. A physical interpretation of the notion of tuned parameters was obtained: they are the values of identifier parameters which minimize the mean squared power in the output error between the identifier and the unknown plant. An interesting result shows that these tuned parameters turn out to be equilibrium points of the averaged dynamical equations governing the identifier parameters. Under PE and slow adaptation assumptions, it was shown the identifier parameters converge to a ball centered at the tuned parameters with a radius which is a class K function of the adaptation gain.

While the tuned model can be identified when the adaptation gain is small, clearly it depends on the choice of input. The validity of this model in the ultimate control task is unclear and thus requires further study. A treatment on this issue will be given as an input design guideline in Chapter 5.

Chapter 4 Frequency Domain Analysis of Adaptive Controllers

4.1 Introduction

Adaptive control needs to be used when plant parameters are either unknown or are varying sufficiently rapidly with time so that conventional robust control schemes do not provide satisfactory performance. Two adaptive control schemes have attracted a lot of attentions: (i) Indirect Adaptive Control (Self-Tuning Regulator (STR)), and (ii) Model Reference Adaptive Control (MRAC). In the indirect adaptive control, the identification of the unknown plant through recursive parameter estimation and the design of the controller are separated. In the model reference adaptive control scheme, the system is driven to behave like a reference model.

In this chapter, only the MRAC schemes will be considered. The MRAC schemes of Narendra and Valavani (1978), Narendra, Lin, and Valavani (1980), Bodson and Sastry (1987) will be analyzed by the use of averaging, introduced in Chapter 2. Such an analysis will allow us to relate the frequency content of the signals, including inputs and noise, to the parameter convergence rates for the *nominal* system, as well as to the stability and instability of the adaptive system with *unmodelled dynamics*.

4.2 General Structure of a Model Reference Adaptive Controller

In this section, we consider the model reference adaptive control of an SISO LT-I plant described by a transfer function:

$$\hat{p}(s) = k_p \frac{\hat{n}_p(s)}{\hat{d}_p(s)} \tag{4.2.1}$$

where $\hat{n}_p(s)$ and $\hat{d}_p(s)$ are coprime monic polynomials of degrees m and n respectively, and k_p is a scalar. The following assumptions will be made about the plant transfer function.

Assumptions:

(A1) The degrees of the polynomials $\hat{n}_p(s)$ and $\hat{d}_p(s)$ are known and assumed to be m and n respectively.

(A2) The sign of k_p is assumed known, and we assume it positive without loss of generality.

(A3) The plant transfer function $\hat{p}(s)$ is minimum phase, i.e. $\hat{n}_p(s)$ is a Hurwitz polynomial.

The reference model is described by a transfer function:

$$\hat{m}(s) = k_m \frac{\hat{n}_m(s)}{\hat{d}_m(s)} \tag{4.2.2}$$

where $\hat{n}_m(s)$ and $\hat{d}_m(s)$ are monic but not necessarily coprime polynomials of degree m and n respectively. The model transfer function satisfies the following.

Assumptions:

(A4) The model transfer function $\hat{m}(s)$ is stable and minimum phase.

(A5) The sign of k_m is the same as that of k_p, i.e. $k_m > 0$.

The controller structure of the direct model reference adaptive control scheme is shown in Figure 4.2.1. The dynamical compensator blocks F_1 and F_2 (reminiscent of those in the adaptive identifiers) are identical single input, n-1 output systems described by transfer functions:

$$\hat{F}_1(s) = \hat{F}_2(s) = \frac{1}{\hat{\Lambda}(s)} \begin{bmatrix} 1 \\ s \\ \cdot \\ \cdot \\ \cdot \\ s^{n-2} \end{bmatrix} \in R^{n-1}(s) \tag{4.2.3}$$

where $\hat{\Lambda}(s)$ is a multiple of $\hat{n}_m(s)$, i.e.

$$\hat{\Lambda}(s) = \hat{n}_m(s) \cdot \hat{\Lambda}_0(s) \tag{4.2.4}$$

and $\hat{\Lambda}_0(s)$ is a $(n-m-1)$th order Hurwitz polynomial. There are a total of $2n$ parameters to be tuned for the controller. The parameter $C \in R^{n-1}$ in the precompensator block serves to locate the closed loop plant zeros, while $D \in R^{n-1}$ and $d_0 \in R$ assign the closed loop plant poles. The parameter $c_0 \in R$ then adjusts the overall gain of the closed loop plant.

Now define the parameter vector $\theta \in R^{2n}$ by

$$\theta = [\, c_0, C^T, d_0, D^T \,]^T \; . \tag{4.2.5}$$

It is shown in Lemma C1 (see Appendix C) that there exists a unique $\theta^* \in R^{2n}$ such that when $\theta = \theta^*$ the transfer function of the plant plus the controller equals that of the model, $\hat{m}(s)$. The problem left to be addressed here is how one should adjust the parameters θ so that the plant output $y_p(t)$ *converges* to the model output $y_m(t)$ and the *stability*, i.e. all variables remain bounded, of the overall system is maintained for arbitrary inputs satisfying:

Assumption:

(A6) $u : R_+ \rightarrow R$ and $u \in L_\infty$.

We shall consider two of several schemes which solve problem: the first is the *output error direct adaptive control scheme* given by Narendra and Valavani (1978), Narendra, Lin and Valavani (1980), Sastry (1984); the second is the *input error direct adaptive control scheme* by Bodson and Sastry (1987). These two schemes are briefly reviewed below.

4.2.1 Output Error Direct Adaptive Control

This scheme is based on the output error,

$$e_o(t) := y_p(t) - y_m(t) \; . \tag{4.2.1.1}$$

Referring to the controller structure in Figure 4.2.1, we define the signal vector $w : R_+ \rightarrow R^{2n}$ by

$$w(t) = [u(t), v_p^{(1)T}(t), y_p(t), v_p^{(2)T}]^T \tag{4.2.1.2}$$

so that the input to the plant, u_p, can be implemented as

$$u_p(t) = \theta(t)^T w(t) . \tag{4.2.1.3}$$

Moreover, for simplicity we realize the compensator blocks F_1 and F_2 by a controllable canonical pair (Λ, b) where $\Lambda \in R^{(n-1)\times(n-1)}$, $b \in R^{n-1}$ so that $\hat{F}_1(s)=\hat{F}_2(s)=(sI-\Lambda)^{-1}b$. Thus the state space representation of the plant loop is given by

$$\begin{bmatrix} \dot{x}_p \\ \dot{v}_p^{(1)} \\ \dot{v}_p^{(2)} \end{bmatrix} = \begin{bmatrix} A_p & 0 & 0 \\ 0 & \Lambda & 0 \\ b\,c_p^T & 0 & \Lambda \end{bmatrix} \begin{bmatrix} x_p \\ v_p^{(1)} \\ v_p^{(2)} \end{bmatrix} + \begin{bmatrix} b_p \\ b \\ 0 \end{bmatrix} \theta^T w \tag{4.2.1.4}$$

$$y_p = [c_p^T, 0, 0] \begin{bmatrix} x_p \\ v_p^{(1)} \\ v_p^{(2)} \end{bmatrix} \tag{4.2.1.5}$$

where (A_p, b_p, c_p^T) is a minimal realization of the plant and x_p is the plant state corresponding to this realization.

By the assertion in Lemma C1, we can represent the model (in non-minimal form) as the plant loop with θ set equal to θ^*. Thus the resulting state space representation of the model loop is given by

$$\begin{bmatrix} \dot{x}_m \\ \dot{v}_m^{(1)} \\ \dot{v}_m^{(2)} \end{bmatrix} = \begin{bmatrix} A_p + b_p\, d_0^*\, c_p^T & b_p\, C^{*T} & b_p\, D^{*T} \\ b\, d_0^*\, c_p^T & \Lambda + b\, C^{*T} & b\, D^{*T} \\ b\, c_p^T & 0 & \Lambda \end{bmatrix} \begin{bmatrix} x_m \\ v_m^{(1)} \\ v_m^{(2)} \end{bmatrix} + \begin{bmatrix} b_p \\ b \\ 0 \end{bmatrix} c_0^*\, u \tag{4.2.1.6}$$

$$y_m = [c_p^T, 0, 0] \begin{bmatrix} x_m \\ v_m^{(1)} \\ v_m^{(2)} \end{bmatrix} . \tag{4.2.1.7}$$

In (4.2.1.6) and (4.2.1.7), the $(3n-2)\times(3n-2)$ matrix is hereafter referred to as A_m, the

$(3n-2)\times 1$ vector is as b_m, and the $1\times(3n-2)$ vector as c_m^T. Then subtracting (4.2.1.6) from (4.2.1.4) with

$$e^T := [x_p^T, v_p^{(1)T}, v_p^{(2)T}] - [x_m^T, v_m^{(1)T}, v_m^{(2)T}] \tag{4.2.1.8}$$

we have that

$$\dot{e} = A_m e + b_m \phi^T w \tag{4.2.1.9}$$

and

$$e_o = y_p - y_m = c_m^T e \tag{4.2.1.10}$$

where $\phi := \theta - \theta^*$ is the parameter error. Note from (4.2.1.6) that

$$\hat{m}(s) = c_0^* c_m^T (sI - A_m)^{-1} b_m \tag{4.2.1.11}$$

and hence

$$e_o = \frac{1}{c_0^*} \hat{m}(s)(\phi^T w) \, . \tag{4.2.1.12}$$

Comparing (4.2.1.12) with (3.2.11) (neglecting the term η) in section 3.2, we see that the output error e_o is no longer the correlated signal $\phi^T w$ but rather is a filtered version of it. In the case where the relative degree of the plant is one, if the model transfer function $\hat{m}(s)$ is chosen to be strictly positively real (SPR), then, roughly speaking, (4.2.1.12) and (3.2.11) will be "equivalent" as far as the sign of $\phi^T w$ is concerned. As for the case with higher relative degree, a stable filter $\hat{l}(s)$ is sought to make $\hat{m}(s)\hat{l}(s)$ SPR and a similar treatment is carried out. Thus, intuitively, the results obtained for the adaptive identifiers should also work for adaptive controllers, possibly with some modifications. We start with a definition of strictly positive real.

Definition 4.2.1.1 **(Strictly Positive Real)**

A transfer function $\hat{m}(s)$ is said to be *strictly positive real* (SPR) if, $\hat{m}(s)$ is real for real s, all the poles of $\hat{m}(s)$ lie in $\mathring{\mathbb{C}}_-$ and $\mathrm{Re}\,\hat{m}(j\omega) > 0$ for all real ω. □

Case I: Relative Degree n − m = 1

Two types of parameter update laws which are particularly suitable for *averaging analysis* are considered here:

(i) **Gradient Algorithm:**

$$\dot{\phi} = -\Gamma\, e_0 w \qquad \phi(0) = \phi_0 \tag{4.2.1.13}$$

where $\Gamma \in R^{2n \times 2n}$ is a positive definite adaptation gain matrix.

(ii) **Least-Squares with Forgetting Factor Algorithm:**

To apply this algorithm, we would require the model transfer function $\frac{1}{c_0^*}\hat{m}(s) - 1/2$ to be SPR in addition to assumption (A4). This is, however, not possible since $\hat{m}(s)$ needs to be proper but not strictly proper to satisfy this condition. Hence we consider the modified scheme shown in the Figure 4.2.2 with $\hat{l}(s) = (s+\delta)$ and $\delta > 0$. Instead of the original signal vector w, its filtered version ζ,

$$\zeta^T = \hat{l}^{-1}(s)(w)^T = [\,\hat{l}^{-1}(s)(u)\,,\hat{l}^{-1}(s)(v_p^{(1)T})\,,\hat{l}^{-1}(s)(y_p)\,,\hat{l}^{-1}(s)(v_p^{(2)T})\,] \tag{4.2.1.14}$$

is used in the parameter update law,

$$\dot{\phi} = -g\, P e_o \zeta \qquad \phi(0) = \phi_0 \tag{4.2.1.15}$$

for some $g > 0$, where e_o now reads

$$e_o = \frac{1}{c_0^*}\hat{m}(s)\hat{l}(s)\,(\phi^T \zeta) \tag{4.2.1.16}$$

and the covariance matrix P is updated by the covariance propagation equation,

$$\dot{P} = \lambda\, P - g\, P\zeta\zeta^T P \qquad P(0) = I \tag{4.2.1.17}$$

for some $\lambda > 0$.

We will briefly review some results concerning stability of the closed loop system and convergence of the output error e_o using these parameter update laws. These are summarized in the following theorems.

Theorem 4.2.1.1: (Stability and Output Convergence Using Gradient Algorithm)

Consider the above adaptive control problem with plant relative degree one. Let assumptions (A1)-(A6) be satisfied.

If the parameter θ is updated as in (4.2.1.13), and the model transfer function $\hat{m}(s)$ is chosen to be SPR,

then the closed loop system remains stable, i.e.

$$\theta \in L_\infty^{2n} \quad x_p \in L_\infty^{n} \quad v_p^{(1)}, v_p^{(2)} \in L_\infty^{n-1}, \tag{4.2.1.18}$$

$e \in L_2^{3n-2}$, and the plant output y_p converges to the model output y_m, i.e.

$$\lim_{t \to \infty} e_o(t) = 0. \tag{4.2.1.19}$$

□

Proof: See Narendra and Valavani (1978).

To apply the least-squares algorithm, we need the following assumption.

Assumption:

(A7) The lower bound on k_p is known, i.e. $k_p \geq k_{\min}$ for some $k_{\min} > 0$.

Theorem 4.2.1.2: (Stability and Output Convergence Using Least-Squares with Forgetting Factor Algorithm)

Consider the given adaptive control problem satisfying the same conditions as before and assumption (A7).

If the parameter θ is updated by (4.2.1.15) and $\frac{1}{c_0^*}\hat{m}(s)\,\hat{l}(s) - 1/2$ is SPR,

then the closed loop system remains stable as before, $e \in L_2^{3n-2}$, and

$$\lim_{t\to\infty} e_o(t) = 0\,. \tag{4.2.1.20}$$

□

Proof: cf. Narendra and Valavani (1978).

Remarks:

(1) From (A7) and the fact $c_0^*=k_m/k_p$, $\hat{l}(s)$ can always be chosen such that $\frac{k_p}{k_m}(\hat{m}(s)\hat{l}(s))-\frac{1}{2}$ is SPR.

(2) Note that, in the modified scheme shown in Figure 4.2.2, the parameter vector θ is replaced by $\hat{l}(s)\,\theta\,\hat{l}(s)^{-1}$ which in turn is given by

$$\hat{l}(s)\,\theta\,\hat{l}(s)^{-1} = \theta + \dot{\theta}\,\hat{l}(s)^{-1} \tag{4.2.1.21}$$

As a result of (4.2.1.20),

$$\dot{\theta} \to 0 \quad \text{as} \quad t \to \infty \tag{4.2.1.22}$$

which implies that the plant loop converges asymptotically to the one shown in Figure 4.2.1.

Case II: Relative Degree n − m ≥ 2

Since the model transfer function $\hat{m}(s)$ fails to be SPR, and replacing θ by $\hat{l}(s)\,\theta\,\hat{l}(s)^{-1}$, where $\hat{l}(s)$ is a second order polynomial, involves $\ddot{\theta}$, the approach proposed above is not applicable. The scheme can however be modified as in Narendra, Lin and Valavani (1980). For simplicity, we will only consider their scheme for the case when the high frequency gain k_p is known, that is, $c_0^* = k_m / k_p$ is known. However, the analysis for the case where k_p is unknown is more involved.

One should, however, note that the modified scheme shown in Figure 4.2.2 can actually work for gradient algorithm in this particular case. Yet, we will take only a general modified approach so that no further distinction between two algorithms need to be made except that $\hat{l}(s)$ is always sought such that $\hat{m}(s)\,\hat{l}(s)$ is SPR using gradient algorithm, whereas $(\frac{1}{c_0^*}\hat{m}(s)\,\hat{l}(s) - \frac{1}{2})$ is SPR using the least-squares algorithm.

Consider the modified scheme shown in Figure 4.2.3. Since $c_0 = c_0^*$ by assumption, we can define the shortened parameter and signal vectors as follows:

$$\overline{\theta} = [\, C^T \,, d_0 \,, D^T \,]^T \tag{4.2.1.23}$$

so that $\overline{\phi} = \overline{\theta} - \overline{\theta}^*$, and

$$\overline{w} = [\, v_p^{(1)T} , y_p , v_p^{(2)T} \,]^T \,. \tag{4.2.1.24}$$

Then (4.2.1.12) can be represented using this notation as

$$e_o = \frac{1}{c_0^*}\, \hat{m}(s)(\overline{\phi}^T \overline{w}) \,. \tag{4.2.1.25}$$

Let $\hat{l}(s)$ be a Hurwitz polynomial of degree $n-m-1$ (resp. $n-m$) such that $\hat{m}(s)\hat{l}(s)$ (resp. $\hat{m}(s)\hat{l}(s) - 1/2$) is SPR, and rewrite (4.2.1.25) as

$$\begin{aligned} e_o &= \frac{1}{c_0^*}\, \hat{m}\hat{l}\,(\, \hat{l}^{-1}(\overline{\phi}^T \overline{w}) - \overline{\phi}^T \hat{l}^{-1}(\overline{w})\,) + \frac{1}{c_0^*}\, \hat{m}\hat{l}\,(\, \overline{\phi}^T \hat{l}^{-1}(\overline{w})\,) \\ &= \frac{1}{c_0^*}\, \hat{m}\hat{l}\,(\xi) + \frac{1}{c_0^*}\, \hat{m}\hat{l}\,(\, \overline{\phi}^T \zeta\,) \end{aligned} \tag{4.2.1.26}$$

where $\zeta = \hat{f}^{-1}(\overline{w})$, and

$$\xi = \hat{f}^{-1}(\overline{\theta}^T \overline{w}) - \overline{\theta}^T \zeta \tag{4.2.1.27}$$

Difficulties arise when the output error e_o is used for the parameter update because of the extra term $\frac{1}{c_0^*}\hat{m}\hat{l}(\xi)$ in (4.2.1.26) in contrast with (4.2.1.12). A remedy for this is to construct another error signal e_1,

$$e_1 = \frac{1}{c_0^*}\hat{m}\hat{l}(\overline{\phi}^T\zeta - \gamma e_1 \zeta^T \zeta) \tag{4.2.1.28}$$

which for some $\gamma > 0$ is actually implemented by augmenting the output error e_o by y_a,

$$y_a = \frac{1}{c_0^*}\hat{m}\hat{l}(\xi + \gamma e_1 \zeta^T \zeta) \tag{4.2.1.29}$$

so that

$$e_1 = e_o - y_a \,. \tag{4.2.1.30}$$

This signal will usually be referred to as the *augmented error*. Using such an error signal, the parameter update laws discussed in the previous case are modified as follows:

$$\dot{\overline{\phi}} = -\overline{\Gamma}\, e_1 \zeta \qquad \overline{\phi}(0) = \overline{\phi}_0 \tag{4.2.1.31}$$

for gradient algorithm, where $\overline{\Gamma} \in R^{(2n-1)\times(2n-1)}$, and

$$\dot{\overline{\phi}} = -g\,\overline{P} e_1 \zeta \qquad \overline{\phi}(0) = \overline{\phi}_0 \tag{4.2.1.32}$$

$$\dot{\overline{P}} = \lambda \overline{P} - g\,\overline{P\zeta\zeta^T P} \qquad \overline{P}(0) = I \tag{4.2.1.33}$$

for the least-squares with forgetting factor algorithm.

The following theorem summarizes the conditions which guarantee stability of the system, and the convergence of the output error e_o and the augmented error e_1.

Theorem 4.2.1.3: (Stability and Error Convergence)

Consider the above adaptive control problem with the modified scheme of Figure 4.2.3. Let assumptions (A1)-(A6) be satisfied, the high frequency gain k_p be known, and the parameter $\overline{\theta}$ be updated by either (4.2.1.31) or (4.2.1.32).

If the filter $\hat{l}(s)^{-1}$ is chosen such that $\hat{m}(s)\hat{l}(s)$ or $\frac{1}{c_0^*}\hat{m}(s)\hat{l}(s)-\frac{1}{2}$ is SPR for the corresponding algorithm,

then the closed loop system remains stable, i.e.

$$\overline{\theta} \in L_\infty^{2n-1} \quad , \quad x_p \in L_\infty^n \quad , \quad v_p^{(1)} \, , v_p^{(2)} \in L_\infty^{n-1} \, , \tag{4.2.1.34}$$

$e \in L_2^{3n-2}$, and

$$\lim_{t\to\infty} e_o(t) = 0 \qquad \lim_{t\to\infty} e_1(t) = 0 \, . \tag{4.2.1.35}$$

□

Proof: See Sastry (1984).

4.2.2 Input Error Direct Adaptive Control

The input error of an adaptive system is defined as:

$$e_i := \hat{m}^{-1} \left(y_p - y_m \right) = \hat{m}^{-1} \left(e_o \right) \tag{4.2.2.1}$$

The input error scheme by definition is based on this input error e_i or an approximation of it. By rewriting (4.2.2.1) as:

$$e_i = \hat{m}^{-1} \hat{p} \left(u_p \right) - u \tag{4.2.2.2}$$

it may seem that e_i is well defined since $\hat{m}$ and $\hat{p}$ have the same relative degrees. However, if the model transfer function has relative degree at least one, its inverse is not proper. Due to measurement noise in the plant output, we will not implement the input error e_i defined in

(4.2.2.1)

but rather we will construct an approximation of e_i as follows.

Since $\hat{m}(s)$ is minimum phase with relative degree $n-m$, for any stable, minimum phase transfer function $\hat{l}^{-1}(s)$ of relative degree $n-m$, the transfer function $\hat{m}\,\hat{l}$ has a proper and stable inverse. A simple example is to let $\hat{l}$ be a Hurwitz polynomial of degree $n-m$. Recall that

$$\begin{aligned} e_o &= \frac{1}{c_0^*}\,\hat{m}\,(\phi^T w) \\ &= \frac{1}{c_0^*}\,\hat{m}\,\hat{l}\,(\,\hat{l}^{-1}\,(\theta^T w) - \theta^{*T}\hat{l}^{-1}\,(w)\,) \\ &= \frac{1}{c_0^*}\,\hat{m}\,\hat{l}\,(\,\hat{l}^{-1}\,(u_p) - \theta^{*T}\zeta\,) \end{aligned} \qquad (4.2.2.3)$$

where $\zeta = \hat{l}^{-1}\,(w)$ and $u_p = \theta^T w$. By the choice of $\hat{l}$, $\hat{m}\,\hat{l}$ is invertible so that

$$\begin{aligned} \hat{l}^{-1}\,(u_p) &= c_0^*\,(\hat{m}\,\hat{l})^{-1}\,(e_o) + \theta^{*T}\zeta \\ &= c_0^*\,(\hat{m}\,\hat{l})^{-1}\,(y_p) + \overline{\theta}^{*T}\zeta \\ &=: \theta^{*T} v \end{aligned} \qquad (4.2.2.4)$$

where

$$v = [\,(\hat{m}\hat{l})^{-1}\,(y_p)\,,\,\hat{l}^{-1}\,(v_p^{(1)T})\,,\,\hat{l}^{-1}\,(y_p)\,,\,\hat{l}^{-1}\,(v_p^{(2)T})\,]^T\,. \qquad (4.2.2.5)$$

Now since $\hat{l}^{-1}\,(u_p)$ and the signal vector v are all available, we can define an error signal e_2 by

$$e_2 := \theta^T v - \hat{l}^{-1}\,(u_p) = \phi^T v \qquad (4.2.2.6)$$

which is a measurable quantity linearly dependent upon the parameter ϕ. Such an error signal turns out to be an approximation of the input error e_i in the sense that, when $\theta(t)$ is fixed at θ,

$$
\begin{aligned}
e_2 &= c_0\,(\hat{m}\,\hat{l})^{-1}\,(y_p) + \bar{\theta}^T\zeta - c_0\,\hat{l}^{-1}\,(u) - \bar{\theta}^T\zeta \\
&= c_0\,(\hat{m}\,\hat{l})^{-1}\,(y_p - y_m) \\
&= c_0\,\hat{l}^{-1}\,(e_i)
\end{aligned}
\tag{4.2.2.7}
$$

Note that the expression (4.2.2.6) is the same as (3.2.11) in the adaptive identification case. Their difference however exists in the fact that the signal vector v defined in (4.2.2.5) is not automatically bounded whereas the signal vector w defined in (3.2.6) is. Hence the parameter update laws discussed in the adaptive identification case can not directly be applied in this instance. Instead, their normalized versions are considered here, which require the following additional assumption:

Assumption:

(DA7)The upper bound on k_p is known, i.e. $k_p \leq k_{\max}$ for some $k_{\max} > 0$.

(i) Normalized Gradient Algorithm Plus Projection

$$
\dot{\phi} = -\Gamma\,\frac{e_2 v}{1 + \gamma\, v^T v} \qquad \text{if } c_0 = c_{\min} \text{ and } \dot{c}_0 < 0\text{ , then let } \dot{c}_0 = 0 \tag{4.2.2.8}
$$

(ii) Normalized Least-Squares with Forgetting Factor Algorithm Plus Projection

$$
\dot{\phi} = -g\,P\,\frac{e_2 v}{1 + \gamma\, v^T v} \qquad \text{if } c_0 = c_{\min} \text{ and } \dot{c}_0 < 0\text{ , then let } \dot{c}_0 = 0 \tag{4.2.2.9}
$$

where

$$
\dot{P} = \lambda\,P - g\,P\,\frac{v\,v^T}{1 + \gamma\, v^T v}\,P\,. \tag{4.2.2.10}
$$

A result concerning stability of the closed loop system and output error convergence is reviewed here in the following theorem.

Theorem 4.2.2.1: (Stability and Output Error Convergence)

Consider the above adaptive control problem, using the same setup as given above. Let assumptions (A1)-(A7) be satisfied, and the parameter θ be updated by either (4.2.2.8) or (4.2.2.9).

Then the closed loop system remains stable, i.e.

$$\theta \in L_\infty^{2n} \quad , \quad x_p \in L_\infty^n \quad , \quad v_p^{(1)} , v_p^{(2)} \in L_\infty^{n-1} , \tag{4.2.2.11}$$

$e \in L_2^{3n-2}$, and

$$\lim_{t \to \infty} e_o(t) = 0 . \tag{4.2.2.12}$$

□

Proof: See Bodson and Sastry (1987).

4.3 Parameter Convergence Analysis

In this section, we will examine conditions under which the parameter vector $\theta(t)$ converges to the true parameter value θ^* in both the output and input error schemes. Later, we analyze parameter convergence by using averaging to obtain an estimate of the rate of parameter convergence.

4.3.1 Output Error Direct Adaptive Control

In subsection 4.2.1, Theorems 4.2.1.1 - 4.2.1.3 guarantee the stability of all signals inside the closed loop systems, and hence the boundedness of the signal vectors w or $\zeta = \hat{\Gamma}^{-1}(\overline{w})$. Intuitively, if w or ζ is PE, exponential parameter convergence can be achieved. But the PE conditions on w or ζ are not practical since these signals are not exogenously specified. This can be seen as follows:

$$w = w_m + Q e \tag{4.3.1.1}$$

and

$$\zeta = \hat{\Gamma}^{-1}(\overline{w}_m + \overline{Q} e) = \zeta_m + \overline{Q}\hat{\Gamma}^{-1}(e) \tag{4.3.1.2}$$

where Q and $\overline{Q}$ are constant matrices defined by

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ c_p & 0 & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} 0\;0\;0 \\ \overline{Q} \end{bmatrix} \tag{4.3.1.3}$$

and w_m, $\overline{w}_m$ are exogenous signals defined by

$$w_m = [\, u \,,\, v_m^{(1)T} \,,\, y_m \,,\, v_m^{(2)T} \,]^T = [\, u \,,\, \overline{w}_m^T \,]^T . \tag{4.3.1.4}$$

It is not surprising that such a PE condition on w or ζ can be translated to the same condition on the signal vector w_m or $\overline{w}_m$ due to the fact that

$$w - w_m = Q e \in L_2^{2n} \qquad \text{or} \qquad \overline{w} - \overline{w}_m = \overline{Q} e \in L_\infty^{2n-1} \tag{4.3.1.5}$$

and the following result.

Lemma 4.3.1.1: (Filtered PE Lemma)

Let $\overline{w}: R_+ \rightarrow R^{2n-1}$.

If $\overline{w}$ is PE, $\overline{w}$ and $\dot{\overline{w}} \in L_\infty^{2n-1}$, and $\hat{H}(s)$ is a stable minimum phase rational transfer function, then $\hat{H}(s)(\overline{w})$ is PE. □

Proof: See Bodson (1987) (Ph.D Dissertation, p. 46)

Using the above results, the following theorem provides conditions under which the *exponential stability* of the system can be guaranteed. Note that this, in particular, implies the exponential convergence of the parameter errors ϕ.

Theorem 4.3.1.2: (Exponential Stability Under PE Condition)

Consider the output error direct adaptive control scheme in subsection 4.2.1. Let assumptions (A1)-(A7) be satisfied.

If the signal vector w_m (or: $\overline{w}_m$) is PE,

then the adaptive system with relative degree one (or: greater than one) is exponentially stable.

In particular,

$$\lim_{t \to \infty} \phi(t) = 0 \qquad (\text{ or: } \lim_{t \to \infty} \overline{\phi}(t) = 0 \text{ }) \tag{4.3.1.6}$$

with exponential convergence. □

Proof: See Narendra and Valavani (1978), and Narendra, Lin and Valavani (1980).

Remark: Notice that (A7) is needed only for the least-squares algorithm, but not for gradient algorithm.

Now if we are only concerned with *stationary* input signals, it is possible to relate the PE condition on w_m (or $\overline{w}_m$) with the spectral condition on the input u. Denote by $\hat{Q}_m(s)$ the transfer function from the input u to w_m so that

$$\hat{Q}_m = \begin{bmatrix} 1 \\ \hat{F}_1 \hat{m} \hat{p}^{-1} \\ \hat{m} \\ \hat{F}_2 \hat{m} \end{bmatrix} \tag{4.3.1.7}$$

and the autocovariance matrix $R_{w_m}(0)$ can be represented as

$$R_{w_m}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{Q}_m(j\omega)\, \hat{Q}_m^*(j\omega)\; S_u(d\omega)\ . \tag{4.3.1.8}$$

From Lemma 3.3.1, w_m is PE if and only if $R_{w_m}(0) > 0$, thereby leading to the following theorem.

Theorem 4.3.1.3: (PE Condition on Supp(u))

Consider the signal vector w_m defined in (4.3.1.4).

Then w_m (resp. $\overline{w}_m$) is PE if and only if the spectral support of input u, Supp(u), contains at least $2n$ (resp. $2n-1$) points. □

Proof: See Boyd and Sastry (1986).

Remarks:

(1) The theorem presents a result similar to that of Theorem 3.3.4. Here, u is called SR when Supp(u) contains at least $2n$ (or: $2n-1$) points. This theorem and Theorem 4.3.1.2 allow one to achieve exponential stability of the adaptive system by providing sufficiently many points in the spectral support of the reference input.

(2) Since $w - w_m \in L_2^{2n}$ from (4.3.1.5), the theorem also implies that w is PE if and only if u is SR. Similarly, the same conclusion will hold for $\overline{w}$.

With these results, we are now ready to analyze the adaptive system using averaging. The objective here is to obtain estimates of the rates of parameter convergence under the PE assumption.

Case I: Relative Degree $n - m = 1$

Recall from (4.2.1.9) and (4.3.1.1) that the dynamics of the state error e are given by

$$\dot{e} = (A_m + b_m \phi^T Q)\, e + b_m \phi^T w_m$$

$$:= A(\phi)\, e + b_m \phi^T w_m \tag{4.3.1.9}$$

and

$$e_o = c_m^T e = \frac{1}{c_0^*}\, \hat{m}(s)(\phi^T w)\,. \tag{4.3.1.10}$$

Let the parameter $\theta(t)$ be frozen at θ, then it should be clear that by symmetry e_o can also be represented as

$$e_o = \frac{1}{c_0}\, \hat{H}_{y_p u}(\theta, s)(\phi^T w_m) \tag{4.3.1.11}$$

where $\hat{H}_{y_p u}(\theta, s)$ denotes the closed loop plant transfer function from the input u to the output y_p with $\theta(t)$ fixed at θ. Denote

$$\hat{D}_m(s) := \det\,(sI - A_m) \tag{4.3.1.12}$$

then from (4.2.1.11) it follows that there exists a Hurwitz monic polynomial $\hat{L}(s)$ of order $2n$–2, whose roots correspond to all the unobservable modes of the model loop shown in Figure 4.2.1, such that

$$\hat{m}(s) = k_m \frac{\hat{n}_m(s)\hat{L}(s)}{\hat{D}_m(s)}\,. \tag{4.3.1.13}$$

Now denote

$$\hat{D}_\phi(s) := \det\,(sI - A(\phi)) \tag{4.3.1.14}$$

then, by C.T. Chen ((1984) p. 339), the expression of $\hat{H}_{y_p u}(\theta, s)$ can be obtained as follows,

$$\hat{H}_{y_p u}(\theta, s) = k_m \frac{\hat{n}_m(s)\,\hat{L}(s)}{\hat{D}_\phi(s)} . \tag{4.3.1.15}$$

To apply averaging, we consider

$$\dot{\phi} = -\,\varepsilon\,\Gamma(t)\,e_o w \qquad\qquad \phi(0) = \phi_0 \tag{4.3.1.16}$$

for $\varepsilon > 0$ small, where (i) $\Gamma(t) = I$ when gradient algorithm is used and (ii) $\Gamma(t) = P(t)$ satisfying

$$\dot{P} = \varepsilon\lambda P - \varepsilon\, P w w^T P \qquad\qquad P(0) = I \tag{4.3.1.17}$$

when the least-squares with forgetting factor algorithm is used. Let Φ denote a compact subset in R^{2n}, containing the origin, where, for all $\phi \in \Phi$, there exist $\lambda_1, \lambda_2 < 0$ such that

$$\lambda_1 \leq \mathrm{Re}\,\lambda(A(\phi)) \leq \lambda_2 \tag{4.3.1.18}$$

where $\lambda(A(\phi))$ stands for an eigenvalue of $A(\phi)$. This set then induces a compact subset $\Theta := \theta^* + \Phi$ (a vector addition) in the parameter space such that all the poles of $\hat{H}_{y_p u}(\theta, s)$ (equivalently the eigenvalues of $A(\phi)$) satisfy (4.3.1.18) for all $\theta \in \Theta$. Consequently, when ε is small, the adaptive system (4.3.1.9), (4.3.1.16) can be classified as a mixed-time scale system, as defined in subsection 2.2.2.2, where ϕ characterizes slow variables, and e contains both fast and slow components. Hence, with the assumption that input signals are stationary, the averaging results developed in subsection 2.2.2 can be readily applied here.

Recall that, in the context of averaging, e is expressed in terms of ϕ through (4.3.1.9), assuming ϕ is a constant, and then the dynamics of ϕ is averaged. However, when ϕ is a constant (the same as: θ is a constant), w is related to u through a transfer function $\hat{H}_{wu}(\theta, s)$, depending on θ. Such a notion will be fundamental to the following analysis using averaging.

(i) Gradient Algorithm:

Following the above argument, the averaged differential equation of the slow variables ϕ in (4.3.1.16) can be found, using (4.3.1.10), as

$$\dot{\phi}_{av} = -\,\varepsilon\, R_{ww_f}(\phi_{av})\,\phi_{av} \qquad \phi_{av}(0) = \phi_0 \tag{4.3.1.19}$$

where $R_{ww_f}(\phi)$ is defined by

$$R_{ww_f}(\phi) := \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{1}{c_0^*}\, w(t)\,\hat{m}(s)\,(w(t)^T)\; dt$$

$$= \frac{1}{2\pi c_0^*} \int_{-\infty}^{\infty} \hat{H}_{wu}(\theta, j\omega)\; \hat{m}^*(j\omega)\; \hat{H}^*_{wu}(\theta, j\omega)\; S_u(d\omega) \tag{4.3.1.20}$$

which is at least positive semi-definite (but not necessarily symmetric). Now when the input u is SR, it follows from Theorem 4.3.1.3, Lemma 3.3.2, and the fact that $\hat{m}(s)$ is SPR that the matrix $R_{ww_f}(\phi)$ is positive definite for all $\phi \in \Phi$. To study the stability of the nonlinear averaged differential equation (4.3.1.19), we consider the Lyapunov function $v : R^{2n} \to R_+$:

$$v(\phi_{av}) = \frac{1}{2}\,\|\,\phi_{av}\,\|^2 . \tag{4.3.1.21}$$

Taking derivative of v along trajectories of (4.3.1.19), we have

$$\dot{v}(\phi_{av}) = -\,\varepsilon\; \phi_{av}^T\, R_{ww_f}(\phi_{av})\; \phi_{av}$$

$$= -\,\varepsilon\; \phi_{av}^T\, SM(R_{ww_f}(\phi_{av}))\; \phi_{av} \tag{4.3.1.22}$$

where $SM(R_{ww_f}(\phi))$ denotes the symmetric part of $R_{ww_f}(\phi)$, i.e.

$$SM(R_{ww_f}(\phi)) = \frac{1}{2\pi c_0^*} \int_{-\infty}^{\infty} \hat{H}_{wu}(\theta\,, j\omega)\; Re\,\hat{m}(j\omega)\; \hat{H}^*_{wu}(\theta\,, j\omega)\; S_u(d\omega)\ . \tag{4.3.1.23}$$

Since Φ is a compact set, there exists $\alpha_1 > 0$ such that:

$$\lambda_{\min}(\, SM(R_{ww_f}(\phi))\,) \geq \alpha_1 \tag{4.3.1.24}$$

so that:

$$\dot{v}(\phi_{av}) \le -\varepsilon\alpha_1 \, \| \, \phi_{av} \, \|^2 \tag{4.3.1.25}$$

which implies the exponential stability of ϕ_{av} with the rate of convergence at least $\varepsilon\alpha_1$. By the remark after Theorem 2.2.2.3, we can readily conclude that the bound on the rate of convergence of the original differential equation (4.3.1.16) for ε small enough is $\varepsilon\alpha_1 + o(\varepsilon)$.

Remarks:

(1) In fact, there exists $\alpha_2 > 0$ such that

$$\lambda_{max}(\, SM(R_{ww_f}(\phi))\,) \le \alpha_2$$

for all $\phi \in \Phi$ so that the rate of convergence of the averaged differential equation (4.3.1.19) is at most $\varepsilon\alpha_2$. Consequently, for ε sufficiently small, we shall say that the rate of convergence of (4.3.1.16) is "close" to the interval:

$$[\, \varepsilon\alpha_1 \,,\, \varepsilon\alpha_2 \,]\,. \tag{4.3.1.26}$$

This fact is similar to that in the remark of subsection 3.3.1.

(2) The symmetric matrix $SM(R_{ww_f}(\phi))$ can also be expressed in terms of $\hat{Q}_m$ instead of $\hat{H}_{wu}(\theta, s)$, using the following fact deduced from (4.3.1.10)-(4.3.1.11) and (4.3.1.15),

$$\begin{aligned} \phi^T \hat{H}_{wu}(\theta, s) &= \frac{c_0^*}{c_0} \left[\hat{m}(s)^{-1} \hat{H}_{y_p u}(\theta, s) \right] \phi^T \hat{Q}_m(s) \\ &= \frac{\hat{D}_m(s)}{\hat{D}_\phi(s)} \, \phi^T \hat{Q}_m(s) \end{aligned} \tag{4.3.1.27}$$

so that

$$SM(R_{ww_f}(\phi)) = \frac{1}{2\pi c_0^*} \int_{-\infty}^{\infty} \left| \frac{\hat{D}_m(j\omega)}{\hat{D}_\phi(j\omega)} \right|^2 \hat{Q}_m(j\omega) \, Re\hat{m}(j\omega) \, \hat{Q}_m^*(j\omega) \, S_u(d\omega)\,. \tag{4.3.1.28}$$

As an illustration of the preceding results, we present an example in which a linearized adaptive system is considered.

Example 4.3.1:

Consider the model reference adaptive control of a first order plant with an unknown pole and an unknown gain,

$$\hat{p}(s) = \frac{k_p}{s+a_p} . \tag{4.3.1.29}$$

The adaptive process is to adjust the feedforward gain c_0 and the feedback gain d_0 so as to make the closed loop transfer function match the model transfer function

$$\hat{m}(s) = \frac{k_m}{s+a_m} . \tag{4.3.1.30}$$

To guarantee persistency of excitation, we use a sinusoidal input signal of the form,

$$u(t) = a\,sin(\omega t) \tag{4.3.1.31}$$

Thus, equations (4.3.1.9) and (4.3.1.16) become

$$\dot{e} = -a_m e + k_p(\phi_1 u + \phi_2 y_m) \tag{4.3.1.32}$$

$$\dot{\phi}_1 = -\varepsilon e r \tag{4.3.1.33}$$

$$\dot{\phi}_2 = -\varepsilon e y_m \tag{4.3.1.34}$$

where

$$\phi_1 = c_0 - c_0^* \quad , \quad \phi_2 = d_0 - d_0^* \tag{4.3.1.35}$$

With a_m=3, k_m=3, a_p=1, k_p=2 , a=3, the true parameter value $\theta^* = [c_0^*, d_0^*]^T$ is computed as $[1.5, -1]$. Let ω=2. By (4.3.1.20) with $\hat{H}_{wu}(\theta, s)$ being replaced by $\hat{Q}_m(s)$, the linearized version of (4.3.1.19) now becomes

$$\begin{bmatrix} \dot{\phi}_{av1} \\ \dot{\phi}_{av2} \end{bmatrix} = -\frac{\varepsilon a^2}{6} \begin{bmatrix} \dfrac{18}{(9+\omega^2)} & \dfrac{18(9-\omega^2)}{(9+\omega^2)^2} \\ \dfrac{18}{(9+\omega^2)} & \dfrac{162}{(9+\omega^2)^2} \end{bmatrix} \begin{bmatrix} \phi_{av1} \\ \phi_{av2} \end{bmatrix} . \tag{4.3.1.36}$$

The two eigenvalues of the averaged system are computed to be -3.10ε and -0.43ε, both real negative. Figures 4.3.1 and 4.3.2 show the plots of the parameter errors ϕ_1 and ϕ_2 for the original and averaged system, with two different adaptation gains. Figure 4.3.3 illustrates the case of a higher frequency input signal $\omega=4$. Here the eigenvalues of the matrix $R_{w_m w_{mf}}(0)$ are complex $(-0.49\pm0.30i)\varepsilon$, and hence the oscillatory behavior of the original and averaged systems.

(ii) Least-Squares with Forgetting Factor Algorithm:

Now ϕ and P are slow variables governed by the following dynamics,

$$\dot{\phi} = -\varepsilon P e_o \zeta \qquad \phi(0)=\phi_0 \tag{4.3.1.37}$$

$$\dot{P} = \varepsilon\lambda P - \varepsilon P\zeta\zeta^T P \qquad P(0) = I \tag{4.3.1.38}$$

which can be averaged, using the same technique as above, to yield

$$\dot{\phi}_{av} = -\varepsilon P_{av} R_{\zeta\zeta_f}(\phi_{av})\phi_{av} \qquad \phi_{av}(0) = \phi_0 \tag{4.3.1.39}$$

$$\dot{P}_{av} = \varepsilon\lambda P_{av} - \varepsilon P_{av} R_{\zeta}(\phi_{av}) P_{av} \qquad P_{av}(0) = I \tag{4.3.1.40}$$

where $R_{\zeta\zeta_f}(\phi)$ and $R_\zeta(\phi)$ are defined respectively by

$$R_{\zeta\zeta_f}(\phi) = \frac{1}{2\pi c_0^*} \int_{-\infty}^{\infty} \hat{H}_{\zeta u}(\theta, j\omega)\, \hat{m}^*(j\omega)\, \hat{l}^*(j\omega)\, \hat{H}^*_{\zeta u}(\theta, j\omega)\, S_u(d\omega) \tag{4.3.1.41}$$

and

$$R_\zeta(\phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{H}_{\zeta u}(\theta, j\omega)\hat{H}^*_{\zeta u}(\theta, j\omega)\, S_u(d\omega)\ . \tag{4.3.1.42}$$

Let $SM(R_{\zeta\zeta_f}(\phi))$ denote the symmetric part of $R_{\zeta\zeta_f}(\phi)$. As indicated before, if the input u is SR, then both matrices $SM(R_{\zeta\zeta_f}(\phi))$ and $R_\zeta(\phi)$ are symmetric positive definite for all $\phi \in \Phi$. It can then be easily checked that $(\phi_{av}, P_{av})=(0, R_\zeta(0)^{-1})$ is an equilibrium point of (4.3.1.39)-(4.3.1.40).

Note that, in this modified scheme, $\frac{1}{c_0^*}\hat{m}\hat{l} - \frac{1}{2}$ is SPR. Further since Φ is compact, it follows that there exist non-zero positive constants $\beta_1, \beta_2, \beta_3, \beta_4$ such that for all $\phi \in \Phi$

$$\beta_1 I \le R_\zeta(\phi) \le \beta_2 I \tag{4.3.1.43}$$

and

$$\beta_3 I \le SM(R_{\zeta\zeta_f}(\phi) - \frac{1}{2}R_\zeta(\phi)) \le \beta_4 I \tag{4.3.1.44}$$

Rewrite (4.3.1.40), using the fact $\dot{P^{-1}} = -P^{-1}\dot{P}P^{-1}$,

$$\dot{P_{av}^{-1}} = -\varepsilon\lambda P_{av}^{-1} + \varepsilon R_\zeta(\phi_{av}) \qquad P_{av}^{-1}(0) = I \tag{4.3.1.45}$$

which together with (4.3.1.43) implies that whenever $\phi \in \Phi$

$$\min(1, \frac{\beta_1}{\lambda}) I \le P_{av}(t)^{-1} \le \max(1, \frac{\beta_2}{\lambda}) I . \tag{4.3.1.46}$$

Now we study the stability of the averaged differential equations (4.3.1.39)-(4.3.1.40) using the Lyapunov function $v: R^{2n} \times R^{2n \times 2n} \to R_+$:

$$v(\phi_{av}, P_{av}) = \frac{1}{2}\phi_{av}^T P_{av}^{-1} \phi_{av} \tag{4.3.1.47}$$

Denote

$$\alpha_1 := \frac{1}{2}\min(1, \frac{\beta_1}{\lambda}) \quad \text{and} \quad \alpha_2 := \frac{1}{2}\max(1, \frac{\beta_2}{\lambda}) \tag{4.3.1.48}$$

Then v satisfies

$$\alpha_1 \| \phi_{av} \|^2 \le v(\phi_{av}, P_{av}) \le \alpha_2 \| \phi_{av} \|^2 \tag{4.3.1.49}$$

$$\left\| \frac{\partial v}{\partial \phi_{av}} \right\| \le 2\alpha_2 \| \phi_{av} \|^2 := \alpha_3 \| \phi_{av} \|^2 \tag{4.3.1.50}$$

and using (3.3.2.13),

$$\left\| \frac{\partial v}{\partial P_{av}} \right\| \le 4n^2\alpha_2^2 \| \phi_{av} \|^2 := \alpha_4 \| \phi_{av} \|^2 \tag{4.3.1.51}$$

whenever $\phi_{av} \in \Phi$. The derivative of v along the trajectories of (4.3.1.39)-(4.3.1.40) can be estimated, using (4.3.1.44) and (4.3.1.49), by

$$\dot{v} + \varepsilon\lambda v = -\varepsilon\phi_{av}^T \left[R_{\zeta\zeta_f}(\phi_{av})) - \frac{1}{2} R_\zeta(\phi_{av}) \right] \phi_{av}$$

$$= -\varepsilon\phi_{av}^T \left[SM(R_{\zeta\zeta_f}(\phi_{av})) - \frac{1}{2} R_\zeta(\phi_{av}) \right] \phi_{av}$$

$$\leq -\varepsilon\beta_3 \| \phi_{av} \|^2 \leq -\varepsilon \frac{\beta_3}{\alpha_2} v \tag{4.3.1.52}$$

whenever $\phi \in \Phi$. It then follows from (4.3.1.49) and (4.3.1.52) that, $(0, R_\zeta(0)^{-1})$ is an exponentially stable equilibrium point of the averaged differential equations (4.3.1.39)-(4.3.1.40) with the rate of convergence at least $\frac{\varepsilon}{2}(\lambda + \frac{\beta_3}{\alpha_2})$. In conclusion, using Theorem 2.2.2.6, the bound on the rate of convergence of $\phi(t)$ of the original differential equation (4.3.1.37), for sufficiently small ε, is $\frac{\varepsilon}{2}(\lambda + \frac{\beta_3}{\alpha_2})$.

Remarks:

(1) Note that the above result is similar to that in subsection 3.3.2. Consequently, by a remark following that subsection, the rate of convergence of $\phi(t)$ for sufficiently small ε will actually be close to the interval:

$$[\, \frac{\varepsilon}{2}(\lambda + \frac{\beta_3}{\alpha_2}) \,,\, \frac{\varepsilon}{2}(\lambda + \frac{\beta_4}{\alpha_1}) \,] \tag{4.3.1.53}$$

(2) From (4.3.1.52),

$$\dot{v} + \varepsilon\lambda v = -\varepsilon\phi_{av}^T P_{av}^{-1/2} P_{av}^{1/2} \left[SM(R_{\zeta\zeta_f}(\phi_{av})) - \frac{1}{2} R_\zeta(\phi_{av}) \right] P_{av}^{1/2} P_{av}^{-1/2} \phi_{av}$$

$$\leq -2\varepsilon\lambda_{\min} \left[P_{av}^{1/2} \left[SM(R_{\zeta\zeta_f}(\phi_{av})) - \frac{1}{2} R_\zeta(\phi_{av}) \right] P_{av}^{1/2} \right] v \,. \tag{4.3.1.54}$$

When ϕ_{av} is sufficiently small, $R_\zeta(\phi)$ and $R_{\zeta\zeta_f}(\phi)$ are close to $R_{\zeta_m}(0)$ and $R_{\zeta_m\zeta_{mf}}(0)$ respectively, and, hence from (4.3.1.45), $P_{av}^{-1}(t)$ is close to $\frac{1}{\lambda} R_{\zeta_m}(0)$ so that (4.3.1.54) becomes

$$\dot{v} + 2\varepsilon\lambda v \leq -2\varepsilon\lambda\lambda_{\min}\left[R_{\zeta_m}(0)^{-1/2} SM(R_{\zeta_m\zeta_{mf}}(0)) R_{\zeta_m}(0)^{-1/2}\right] v \,. \tag{4.3.1.55}$$

Suppose that the Supp(u) is only a point spectrum, then it is shown in Lemma C2 that (4.3.1.55) can be approximated by

$$\dot{v} + 2\varepsilon\lambda v \leq \frac{2\varepsilon\lambda}{c_0^*} \min_{\omega\in \mathrm{Supp}(u)} \mathrm{Re}(\hat{m}(j\omega)\hat{l}(j\omega)) \,. \tag{4.3.1.56}$$

By a similar argument, it can be shown that

$$\dot{v} + 2\varepsilon\lambda v \geq -\frac{2\varepsilon\lambda}{c_0^*} \max_{\omega\in \mathrm{Supp}(u)} \mathrm{Re}(\hat{m}(j\omega)\hat{l}(j\omega)) \,. \tag{4.3.1.57}$$

Consequently, we can conclude that the rate of tail parameter convergence is close to the interval,

$$[\ \varepsilon\lambda\left[1+\min_{\omega\in \mathrm{Supp}(u)} \mathrm{Re}(\hat{m}(j\omega)\hat{l}(j\omega))\right],\ \varepsilon\lambda\left[1+\max_{\omega\in \mathrm{Supp}(u)} \mathrm{Re}(\hat{m}(j\omega)\hat{l}(j\omega))\right]\ ]$$

when ε is sufficiently small. Since, by choice, $\frac{1}{c_0^*}\mathrm{Re}\,\hat{m}(j\omega)\hat{l}(j\omega) - 1/2 > 0$ for all $\omega \in R$, the input u will not have a great effect on the bound on the rate of tail parameter convergence. This result is similar to that in the remark of subsection 3.3.2.

Case II: Relative Degree $n - m \geq 2$

Here we again assume that the high frequency gain k_p and, hence c_0^*, are known. The error signal used for the parameter update under the averaging analysis is of the form:

$$e_1 = \frac{1}{c_0^*}\,\hat{m}(s)\hat{l}(s)\,(\overline{\phi}^T\zeta - \varepsilon\zeta^T\zeta e_1) \tag{4.3.1.58}$$

for some $\varepsilon > 0$. The implementation of this scheme involves more dynamics than before. In the following, we will reconstruct all the dynamics involved in this scheme to facilitate our later analysis.

Let $\hat{l}(s)$ be a Hurwitz polynomial of order k where $k=n-m-1$ or $k=n-m$ for the relevant algorithm, described in subsection 4.2.1. Let (A_l, b_l, c_l) be a minimal realization of the transfer function $\hat{l}^{-1}(s)$, i.e.

$$\hat{l}^{-1}(s) = c_l^T (sI - A_l)^{-1} b_l \tag{4.3.1.59}$$

where $A_l \in R^{k \times k}$ is a Hurwitz matrix. If we define

$$\overline{A}_l := \text{diag}\left[A_l, \cdots, A_l\right] \in R^{(2n-1)k \times (2n-1)k} \tag{4.3.1.60}$$

$$\overline{B}_l := \text{diag}\left[b_l \cdots, b_l\right] \in R^{(2n-1)k \times (2n-1)} \tag{4.3.1.61}$$

$$\overline{C}_l := \text{diag}\left[c_l^T, \cdots, c_l^T\right] \in R^{(2n-1) \times (2n-1)k} \tag{4.3.1.62}$$

then the state space realization of $\overline{\zeta} = \hat{l}(s)^{-1}(\overline{w})$ is

$$\dot{x}_l = \overline{A}_l x_l + \overline{B}_l \overline{w} \qquad x_l \in R^{(2n-1)k} \tag{4.3.1.63}$$

$$\overline{\zeta} = \overline{C}_l x_l \,. \tag{4.3.1.64}$$

Similarly, $\hat{m}(s)\hat{l}(s)$, which appears in (4.3.1.58), also has a minimal realization (A_{ml}, b_{ml}, c_{ml}) so that

$$\hat{m}(s)\hat{l}(s) = c_0^* c_{ml}^T (sI - A_{ml})^{-1} b_{ml} \tag{4.3.1.65}$$

and a state $z \in R^n$ associated with it. Finally, a state space realization of (4.3.1.58), using (4.3.1.1) and (4.3.1.9), can be expressed as:

$$\begin{bmatrix} \dot{e} \\ \dot{x}_l \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A(\phi) & 0 & 0 \\ \overline{B}_l \overline{Q} & \overline{A}_l & 0 \\ 0 & b_{ml} \phi^T \overline{C}_l & A_{ml} \end{bmatrix} \begin{bmatrix} e \\ x_l \\ z \end{bmatrix}$$

$$- \varepsilon \begin{bmatrix} 0 \\ 0 \\ b_{ml} x_l^T \overline{C}_l^T \overline{C}_l x_l c_{ml}^T z \end{bmatrix} + \begin{bmatrix} b_m \phi^T \\ \overline{B}_l \\ 0 \end{bmatrix} \overline{w}_m \tag{4.3.1.66}$$

$$e_1 = [\, 0 \,, 0 \,, c_{ml}^T \,] \begin{bmatrix} e \\ x_l \\ z \end{bmatrix} \tag{4.3.1.67}$$

which is a nonlinear system with the form (2.2.2.90). Note that there are totally $(2n-1)(2+k)$ states here, in contrast with $3n-2$ states in the relative degree one case. Now if we define $\bar{A}(\phi)$ as the $(2n-1)(2+k)\times(2n-1)(2+k)$ matrix given in (4.3.1.66), then clearly for all $\phi\in\Phi$ all the eigenvalues of $\bar{A}(\phi)$ should satisfy a condition similar to (4.3.1.18). Hence the dynamics that govern the states of the adaptive system in the modified scheme are actually no different from (4.3.1.9)-(4.3.1.10) in the original scheme except for the extra term involving ε on the R.H.S. of (4.3.1.66).

On the other hand, the dynamics of the parameter error ϕ under averaging takes the form,

$$\dot{\bar{\phi}} = -\,\varepsilon\,\bar{\Gamma}(t)\,e_1\zeta \qquad \bar{\phi}(0)=\bar{\phi}_0 \tag{4.3.1.68}$$

where (i) $\bar{\Gamma}(t) = I$ for gradient algorithm, and

(ii) $\bar{\Gamma}(t) = \bar{P}(t)$ satisfying

$$\dot{\bar{P}} = \varepsilon\bar{P} - \varepsilon\,\overline{P\zeta\zeta^T P} \qquad \bar{P}(0)=I \tag{4.3.1.69}$$

for the least-squares with forgetting factor algorithm.

The system (4.3.1.66)-(4.3.1.67) again forms a mixed-time-scale system.

In applying averaging to this case, while we proceed with the same technique as given in the previous case, we set $\varepsilon=0$ in (4.3.1.66). This leads to the following expressions of e_1 and ζ in terms of ϕ (as if ϕ were constant):

$$e_1 = \frac{1}{c_0^*}\,\hat{m}(s)\,\hat{l}(s)\,(\phi^T\zeta) \tag{4.3.1.70}$$

$$\zeta = \hat{H}_{\zeta_u}(\theta,s)(u) \tag{4.3.1.71}$$

where $\theta=\phi+\theta^*$. By the similarity between (4.3.1.10) and (4.3.1.70), it can be easily seen that now the transfer function $\hat{m}\hat{l}$ and ζ play the roles of $\hat{m}$ and w in the relative degree one case. Using these as basic observations, the averaging analysis of the same parameter update algorithms discussed in the previous case will be simple.

(i) Gradient Algorithm:

The averaged dynamic equation for the slow variable $\overline{\phi}$ can be found to be:

$$\dot{\overline{\phi}}_{av} = -\varepsilon R_{\overline{\zeta}\zeta_f}(\overline{\phi}_{av})\overline{\phi}_{av} \qquad \overline{\phi}_{av}(0) = \overline{\phi}_0 \tag{4.3.1.72}$$

where

$$R_{\overline{\zeta}\zeta_f}(\overline{\phi}) = \lim_{T\to\infty} \frac{1}{T} \int_s^{s+T} \frac{1}{c_0^*} \zeta(t)\, \hat{m}(s)\hat{l}(s)(\zeta^T(t))\, dt$$

$$= \frac{1}{2\pi c_0^*} \int_{-\infty}^{\infty} \hat{H}_{\zeta_u}(\theta, j\omega)\, \hat{m}^*(j\omega)\hat{l}^*(j\omega)\, \hat{H}^*_{\zeta_u}(\theta, j\omega)\, S_u(d\omega) \tag{4.3.1.73}$$

which is similar to that in (4.3.1.20).

(ii) Least-Squares with Forgetting Factor Algorithm

In this case, the slow variables are ϕ and P. The averaged differential equations of these variables are of the form,

$$\dot{\overline{\phi}}_{av} = -\varepsilon P_{av}\, R_{\overline{\zeta}\zeta_f}(\overline{\phi}_{av})\overline{\phi} \qquad \overline{\phi}_{av}(0) = \overline{\phi}_0 \tag{4.3.1.74}$$

$$\dot{\overline{P}}_{av} = \varepsilon\lambda\overline{P}_{av} - \varepsilon\, \overline{P}_{av}\, R_{\zeta}(\overline{\phi}_{av})\overline{P}_{av} \qquad \overline{P}_{av}(0) = I \tag{4.3.1.75}$$

where $R_{\overline{\zeta}\zeta_f}(\phi)$ is similar to (4.3.1.69) but with $(\frac{1}{c_0^*}\hat{m}\hat{l} - \frac{1}{2})$ being SPR, and

$$R_{\zeta}(\overline{\phi}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{H}_{\zeta_u}(\theta, j\omega)\hat{H}^*_{\zeta_u}(\theta, j\omega)\, S_u(d\omega) \tag{4.3.1.76}$$

which is again similar to that in (4.3.1.42).

Since the averaged systems obtained in (4.3.1.72) and (4.3.1.74)-(4.3.1.75) are similar to those in (4.3.1.19) and (4.3.1.39)-(4.3.1.40), the conclusions obtained in the relative degree one can be directly applied here.

Remark: In the case where k_p is unknown, an augmented parameter θ_{2n+1} will have to be introduced so that the overall exponential stability will not be possible. However, if the analysis is only focused on the first $2n$ parameters θ, then the results will be exactly the same as those in the current case.

4.3.2 Input Error Direct Adaptive Control

In this subsection, we will first review some results on parameter convergence of the scheme discussed in subsection 4.2.2, and then analyze the adaptive system using averaging. The objective of this analysis is also to estimate the rate of parameter convergence.

Theorem 4.2.2.1 guarantees stability and error convergence of the adaptive system using either a normalized gradient algorithm plus projection as in (4.2.2.8) or a normalized least-squares with forgetting factor algorithm plus projection as in (4.2.2.9)-(4.2.2.10). These parameter update laws are the same as those for the adaptive identifier discussed in subsection 3.3.1-3.3.2 if one identifies the signal vector

$$\frac{v}{\sqrt{1+\gamma v^T v}} \tag{4.3.2.1}$$

with the signal vector w associated with the identifier. Therefore, if the signal vector (4.3.2.1) can be guaranteed to be PE, then exponential stability will be achieved. However, v is not exogenously specified in order to make such condition practical. The following theorem will provide conditions under which the system can be guaranteed to be exponentially stable.

Theorem 4.3.2.1: (Exponential Stability Under PE Condition)

Consider the input error direct adaptive control scheme in subsection 4.2.2. Let assumptions (A1)-(A6) and (DA7) be satisfied.

If the signal vector w_m is PE,

then the adaptive system is exponentially stable. In particular, this implies

$$\lim_{t \to \infty} \phi(t) = 0 \tag{4.3.2.2}$$

with exponential convergence. □

Proof: See Bodson and Sastry (1987).

As for the output error scheme, we will henceforth be only concerned with stationary signals to facilitate the averaging analysis. Note that the stable filter $\hat{l}(s)^{-1}$ is chosen such that $\hat{m}(s)\,\hat{l}(s)$ has a stable inverse. While applying averaging, an approach similar to that used for the same analysis in the output error scheme is taken, that is to freeze the parameters θ and to relate all signals inside the closed loop plant to input u via transfer functions which depend on θ.

Define $\hat{H}_{vu}(\theta, s)$ as the transfer function from input u to the signal vector v defined in (4.2.2.5), i.e.

$$\hat{H}_{vu}(\theta, s) = \begin{bmatrix} (\hat{m}(s)\,\hat{l}(s))^{-1} \\ \hat{l}(s)^{-1}\hat{F}_1(s)\,\hat{p}(s)^{-1} \\ \hat{l}(s)^{-1} \\ \hat{l}(s)^{-1}\hat{F}_2(s) \end{bmatrix} \hat{H}_{y_p u}(\theta, s) \tag{4.3.2.3}$$

then, using (4.3.1.13)-(4.3.1.15), $\hat{H}_{vu}(\theta, s)$ can be related to the transfer function $\hat{Q}_m(s)$ as follows:

$$\hat{H}_{vu}(\theta, s) = \hat{l}(s)^{-1}\frac{\hat{D}_m(s)}{\hat{D}_\phi(s)}\hat{Q}_m(s) \tag{4.3.2.4}$$

so that, for all $\phi \in \Phi$, the persistency of excitation of w_m will directly imply that of v. This fact is important to the following analysis.

(i) Normalized Gradient Algorithm Plus Projection

The study of averaging analysis will focus on the case where $\Gamma=\varepsilon I$, $\gamma=\varepsilon$ for some small $\varepsilon>0$, and for sufficiently small $\|\phi_0\|$ so that the projection mechanism can be neglected. Hence, using the fact that $e_2=\phi^T v$, the parameter update law becomes

$$\dot{\phi}=-\varepsilon\frac{vv^T}{1+\varepsilon v^T v}\phi \qquad \phi(0)=\phi_0\,. \tag{4.3.2.5}$$

In other words, in this analysis, only local properties will be of interest (so that $c_0(t)>c_{\min}$ for all $t>0$). When ε is small, the parameter ϕ characterizes the slow variables while variable e contains slow and fast components. As before, the averaged system of the slow variables ϕ can be found as

$$\dot{\phi}_{av}=-\varepsilon R_v(\phi_{av})\phi_{av} \qquad \phi_{av}(0)=\phi_0 \tag{4.3.2.6}$$

where $R_v(\phi)$ is defined by

$$\begin{aligned} R_v(\phi) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{H}_{vu}(\theta, j\omega)\, \hat{H}^*_{vu}(\theta, j\omega)\, S_u(d\omega) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{l}(j\omega)^{-1}|^2 \left|\frac{\hat{D}_m(j\omega)}{\hat{D}_\phi(j\omega)}\right|^2 \hat{Q}_m(j\omega)\, \hat{Q}^*_m(j\omega)\, S_u(d\omega) \end{aligned} \tag{4.3.2.7}$$

which is a positive (at least) semi-definite *symmetric* matrix. If the input u is SR, then, from previous arguments, v is PE and $R_v(\phi)>0$ for all $\phi\in\Phi$. Thus, using the similarity between (4.3.2.6) and (4.3.1.19), (3.3.1.3) the rate of parameter convergence of $\phi(t)$ can be estimated using the same method there.

(ii) Least-Squares with Forgetting Factor Algorithm Plus Projection

Performing an averaging analysis, with $g=\gamma=\varepsilon$, λ replaced by $\varepsilon\lambda$, and ϕ_0 assumed small enough so that $c_0(t)>c_{\min}$ for all $t>0$, on the following system:

$$\dot{\phi}=-\varepsilon P\frac{e_2 v}{1+\varepsilon v^T v} \qquad \phi(0)=\phi_0 \tag{4.3.2.8}$$

$$\dot{P} = \varepsilon P - \varepsilon P \frac{v v^T}{1 + \varepsilon v^T v} P \qquad P(0)=I \tag{4.3.2.9}$$

where ϕ and P constitute the slow variables when ε is small. The averaged version of their dynamics can be found to be

$$\dot{\phi}_{av} = -\varepsilon P_{av} R_v(\phi_{av}) \phi_{av} \qquad \phi_{av}(0) = \phi_0 \tag{4.3.2.10}$$

$$\dot{P}_{av} = \varepsilon P_{av} - \varepsilon P_{av} R_v(\phi_{av}) P_{av} \qquad P(0)=I\,. \tag{4.3.2.11}$$

When the input u is SR, as indicated before, $R_v(\phi)$ is positive definite for all $\phi \in \Phi$. Again, from the similarity between (4.3.2.10)-(4.3.2.11) and (4.3.1.39)-(4.3.1.40), (3.3.2.4)-(3.3.2.5), the same conclusions on the rate of parameter convergence of $\phi(t)$ can also be drawn.

4.4 Robustness to Unmodelled Dynamics and Bounded Output Disturbances

In this section, we will analyze the *robustness* properties of the model reference adaptive controller in the presence of unmodelled dynamics and bounded output disturbances. Both stability and output error convergence are re-examined, and conditions are provided to ensure these properties, so that the *reduced order adaptive controller* can be made practical.

Unlike adaptive identifiers, model reference adaptive controllers can cause several types of *instabilities* in the presence of unmodelled dynamics and output disturbances. In our later analysis using averaging, the study of such instability properties will only focus on *slow drift instability*, which is usually difficult to detect during operation.

As in the case of adaptive identification, the unmodelled dynamics are considered FDLT-I so that the overall plant can be represented by:

$$\hat{p}_u(s) = \hat{p}(s)(1 + \Delta\hat{p}(s)) = k_p \frac{\hat{n}_p(s)}{\hat{d}_p(s)} \frac{\hat{n}_u(s)}{\hat{d}_u(s)} = k_p \frac{\hat{n}_{pu}(s)}{\hat{d}_{pu}(s)} \tag{4.4.1}$$

where $\hat{p}(s)$ is the nominal plant transfer function of order n as described in section 4.2. The bounded output disturbance $d(t)$ is usually modeled in the system as a measurement noise, i.e.:

$$y_{pd}(t) = y_p(t) + d(t) \tag{4.4.2}$$

as shown in the Figure 4.4.1. In order to facilitate our analysis, we made the following assumptions additional to (A1)-(A7) and (DA7).

Assumptions:

(A8) $\hat{p}_u(s)$ is a strictly proper transfer function of order N where $N > n$.

(A9) $\hat{n}_u(s)$ is a Hurwitz polynomial.

(A10) Input u and bounded disturbance d are stationary signals with compact spectral support.

(A11) $\mathrm{Supp}(u) \cap \mathrm{Supp}(d) = \varnothing$

Remarks:

(1) Assumption (A9), which requires that the plant is minimum phase, is in general not always satisfied. However, it only serves to simplify the analysis and can be relaxed by assuming, instead, that Supp(u) does not contain the zero-frequency of $\hat{p}_u(s)$ (if any exists).

(2) In practice, the spectrum of the output disturbance is usually located in a frequency range which is either much higher or much lower than that in which the spectrum of the control command lies. This makes (A11) a reasonable assumption.

4.4.1 Matching Criterion

Let the parameter vector $\theta(t)$ be frozen at θ, then the adaptive system behaves like an LTI system so that y_{pd} and w can be related to the input u and disturbance d through linear transfer functions which depend on θ, i.e.:

$$y_{pd} = \hat{H}_{y_{pd}u}(\theta, s)(u) + \hat{H}_{y_{pd}d}(\theta, s)(d) \tag{4.4.1.1}$$

and

$$w = \hat{H}_{wu}(\theta, s)(u) + \hat{H}_{wd}(\theta, s)(d) \tag{4.4.1.2}$$

where transient responses due to initial conditions are omitted. These omissions, however, can be justified if the transfer functions are stable and there are no unstable hidden modes inside the closed loop plant. Denote

$$\hat{q}_u(s) = \begin{bmatrix} \hat{F}_1(s)\hat{p}_u^{-1}(s) \\ 1 \\ \hat{F}_2(s) \end{bmatrix} \tag{4.4.1.3}$$

then $\hat{H}_{wu}(\theta, s)$ and $\hat{H}_{wd}(\theta, s)$ can be related to $\hat{H}_{y_{pd}u}(\theta, s)$ and $\hat{H}_{y_{pd}d}(\theta, s)$ through the following:

$$\hat{H}_{wu}(\theta, s) = \begin{bmatrix} 1 \\ \hat{q}_u(s)\hat{H}_{y_{pd}u}(\theta, s) \end{bmatrix} \tag{4.4.1.4}$$

and

$$\hat{H}_{wd}(\theta, s) = \begin{bmatrix} 0 \\ \hat{q}_u(s)\hat{H}_{y_{pd}d}(\theta, s) \end{bmatrix} + \begin{bmatrix} 0 \\ -\hat{F}_1(s)\hat{p}_u(s)^{-1} \\ 0 \\ 0 \end{bmatrix}. \tag{4.4.1.5}$$

Recall that $u_p = \theta^T w$, and $\theta = [c_0, \bar{\theta}^T]^T$. Since

$$y_{pd} = y_p + d = \hat{p}_u(s)(u_p) + d \tag{4.4.1.6}$$

and

$$\hat{H}_{y_pu} = \hat{H}_{y_{pd}u} \qquad \text{and} \qquad \hat{H}_{y_pd} = \hat{H}_{y_{pd}d} - 1 \tag{4.4.1.7}$$

by substituting (4.4.1.2) and (4.4.1.4)-(4.4.1.5) into (4.4.1.6) and comparing it with (4.4.1.1) and (4.4.1.7), we can solve $\hat{H}_{y_pu}(\theta, s)$ and $\hat{H}_{y_pd}(\theta, s)$ explicitly in terms of θ as follows:

$$\hat{H}_{y_pu}(\theta, s) = \frac{c_0\hat{p}_u(s)}{1 - \bar{\theta}^T\hat{q}_u(s)\,\hat{p}_u(s)} \tag{4.4.1.8}$$

and

$$\hat{H}_{y_p d}(\theta, s) = \frac{(d_0 + D^T \hat{F}_2(s))\hat{p}_u(s)}{1 - \overline{\theta}^T \hat{q}_u(s)\hat{p}_u(s)} \tag{4.4.1.9}$$

Note that, in the absence of unmodelled dynamics and output disturbances, Lemma C1 guarantees perfect matching, i.e. there exists a unique $\theta^* \in R^{2n}$ such that:

$$\hat{H}_{y_p u}(\theta^*, s) = \hat{m}(s) \qquad \text{for all } s \in C \tag{4.4.1.10}$$

However, in general, this will not be possible from (4.4.1.8) nor will the following be possible:

$$\hat{H}_{y_p u}(\theta, j\omega) = \hat{m}(j\omega) \qquad \text{for all } \omega \in R \tag{4.4.1.11}$$

for any $\theta \in R^{2n}$. On the other hand, under the assumption (A10)-(A11), non-zero output disturbance d will always result in mismatch between output of the LT-I plant with fixed θ and that of the model no matter what the input u is. Here, we will pursue results similar to those for the adaptive identifiers of subsection 3.4.1, assuming $d \equiv 0$. In other words, in the following analysis, we will seek conditions to establish conditional matching, i.e. for some $\theta_0 \in R^{2n}$:

$$\hat{H}_{y_p u}(\theta_0, j\omega) = \hat{m}(j\omega) \qquad \text{for all } \omega \in \mathrm{Supp}(u) \tag{4.4.1.12}$$

The following lemma, like Lemma 3.4.1.1, will be fundamental to that condition.

Lemma 4.4.1.1:

Consider a transfer function $\hat{Q}_{mu}(s) \in C^{2n}$ defined by

$$\hat{Q}_{mu}(s) = \begin{bmatrix} 1 \\ \hat{m}(s)\,\hat{q}_u(s) \end{bmatrix}. \tag{4.4.1.13}$$

Then for any $k \in Z_+$, $1 \le k \le n$, there exists a set of $2k$ frequencies, $(\omega_1, \ldots, \omega_{2k})$, such that

$$\left[\hat{Q}_{mu}(j\omega_1), \cdots, \hat{Q}_{mu}(j\omega_{2k})\right] \tag{4.4.1.14}$$

form a linearly independent vector set. □

Proof: cf. Mason et al (1987).

Remark: Without unmodelled dynamics, $\hat{Q}_{mu}(s)$ is the same as $\hat{Q}_m(s)$ defined in (4.3.1.7).

Theorem 4.4.1.2: (Almost Always Matching Condition)

Consider the above adaptive control problem. Let assumptions (A1)-(A6) and (A8)-(A10) be satisfied.

Then for any $k \in Z_+$, $k \leq n$, there exists a subset $V_k \subset R^{2k}$ which is nowhere dense and measure zero such that the matching (4.4.1.12) is possible provided Supp(u) contains $2k$ points which form a $2k$ tuple not contained in V_k. □

Proof: Using (4.4.1.8), in the matching condition (4.4.1.12), we have

$$\hat{m}(j\omega) = \frac{\theta_{01}\hat{p}_u(j\omega)}{1 - \overline{\theta}_0^T \hat{q}_u(j\omega)\hat{p}_u(j\omega)}$$

$$= \theta_0^T \hat{p}_u(j\omega) \begin{bmatrix} 1 \\ \hat{m}(j\omega)\, \hat{q}_u(j\omega) \end{bmatrix}$$

$$= \theta_0^T \hat{p}_u(j\omega)\, \hat{Q}_{mu}(j\omega) \qquad \text{for all } \omega \in \text{Supp}(u) \tag{4.4.1.15}$$

where $\theta_0 = [\,\theta_{01}, \overline{\theta}_0^T\,]^T$. By the assumption (A9), (4.4.1.15) further leads to

$$\hat{m}(j\omega)\,\hat{p}_u(j\omega)^{-1} = \theta_0^T \hat{Q}_{mu}(j\omega) \qquad \text{for all } \omega \in \text{Supp}(u)\,. \tag{4.4.1.16}$$

Using Lemma 4.4.1.1. and a proof similar to that in Theorem 3.4.1.2, the conclusion will readily follow.

■

Remarks:

(1) The remark of Theorem 3.4.1.2 will also apply here.

(2) Note that the transfer function $\hat{H}_{y_p u}(\theta_0, s)$ may not be stable. In general, when the spectrum of the control input u lies in a lower frequency range, it is more likely that there exists a θ_0 that satisfies (4.4.1.12) and $\hat{H}_{y_p u}(\theta_0, s)$ is stable.

(3) Due to the difference between proofs of stability for identifiers and for controllers, a possible matching here will not imply stability of the adaptive system, like the statement of Theorem 3.4.1.3. In fact, when Supp(u) contains less than $2n$ points, some instabilities may arise (see subsection 4.4.4).

4.4.2 Tuned Model

In practice, matching between the output of the plant and that of the model will be hard to achieve especially when output disturbances exist. Besides, as indicated in the previous remark, consideration of the instability caused by a lack of richness of the input u will make such a match undesirable, except when this matching occurs only at a unique θ_0. In fact, a sufficiently rich control input will enhance the controller robustness.

In this subsection, similar to subsection 3.4.2, we will derive a model of the closed loop plant when matching is not possible. Two basic properties will be required of this model: stability as well as a good approximation of the reference model.

Consider a normalized cost function $J_n(\theta)$ defined by

$$J_n(\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\left| \hat{H}_{y_p u}(\theta, j\omega) - \hat{m}(j\omega) \right|^2}{\left| \hat{H}_{y_p u}(\theta, j\omega) \right|^2} S_u(d\omega)$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\left| \hat{H}_{y_p d}(\theta, j\omega) \right|^2}{\left| \hat{H}_{y_p u}(\theta, j\omega) \right|^2} S_d(d\omega) \,. \tag{4.4.2.1}$$

An interpretation of $J_n(\theta)$ can be extracted as follows. Define a pseudo error signal $e_*(\theta, t)$ by

$$e_*(\theta, t) := \hat{H}_{y_p u}(\theta, s)(u) + \hat{H}_{y_p d}(d) - y_m \,. \tag{4.4.2.2}$$

From (A9) and (4.4.1.8), $\hat{H}_{y_p u}(\theta, s)$ is also minimum phase. Thus, if $\hat{H}_{y_p u}(\theta, s)$ is stable (so is $\hat{H}_{y_p d}(\theta, s)$), then, from Lemma B2 and (A11), $J_n(\theta)$ can be expressed as

$$J_n(\theta) = \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} \left[\hat{H}_{y_p u}^{-1}(\theta, s)\,(e_*(\theta, t)) \right]^2 dt \,. \tag{4.4.2.3}$$

In words, $J_n(\theta)$ superficially represents the mean square power of the filtered signal $\hat{H}_{y_p u}(\theta, s)^{-1}(e_*(\theta, t))$. One, however, should note that the interpretation (4.4.2.3) would be more for analysis and less for practical purposes because $\hat{H}_{y_p u}(\theta, s)^{-1}\, \hat{m}(s)$ is in general not proper.

Important Remark:

Normally, in the presence of unmodelled dynamics, $\hat{H}_{y_p u}(\theta, s)$ is of order much higher than that of $\hat{m}(s)$, and the frequency gain of $\hat{H}_{y_p u}(\theta, j\omega)$ decreases faster than of $\hat{m}(j\omega)$ as ω increases. Thus,

$$\hat{H}_{y_p u}(\theta, j\omega) - \hat{m}(j\omega) \approx - \hat{m}(j\omega) \tag{4.4.2.4}$$

when θ is not properly chosen causing $\hat{H}_{y_p u}(\theta, j\omega)$ to start rolling off while ω lies in a mid-band region of $\hat{m}(j\omega)$. Consequently, if the cost function is defined such that there is no normalization or the normalization is against $\hat{m}(j\omega)$, it will not substantiate the good choice of θ in a situation just mentioned as much as otherwise. In optimization terms, we simply put a penalty on the higher frequency range.

Let Θ_s, similar to Θ defined in section 4.3, be a compact subset in the parameter space such that $\hat{H}_{y_p u}(\theta, s)$ is a stable transfer function for all $\theta \in \Theta_s$. A choice of the subset will certainly require some prior information about the plant, and in general this set will contain θ^*, the true parameter corresponding to the nominal plant. Now based on the cost function $J_n(\theta)$, we define *tuned parameters*, θ_T, by

$$\theta_T = \underset{\theta \in \operatorname{int}\Theta_s}{\operatorname{argmin}} J_n(\theta) \tag{4.4.2.5}$$

where $\operatorname{int}\Theta_s$ denotes the interior of Θ_s. This definition implies that θ_T also has to be one of the local minimizers of $J_n(\theta)$. Of course, θ_T may not exist if, for example, the spectral power of the input u and output disturbance d are concentrated in the high frequency spectrum, then local minimizers of $J_n(\theta)$ over R^{2n} may not lie in the set Θ_s.

Assumption:

(A12) Input u and output disturbance d of the adaptive system considered above are such that the tuned parameter θ_T defined in (4.4.2.5) exists.

Remarks:

(1) The assumption (A12), in fact, reflects the appropriateness of the order of the nominal plant. If the dominant poles of the nominal plant lie in the frequency spectrum of the control input and the output disturbance is small relative to the control input, then the tuned parameters θ_T should be close to θ^* to make (A12) reasonable. On the other hand, if the above is not the case, then the adaptive scheme starts with a bad model of the plant and a noisy environment, and, hence, a degraded performance of the control task should be expected.

(2) The closed loop plant with the fixed tuned parameters θ_T gives a pseudo plant called the *tuned plant*, and the pseudo error $e_*(\theta_T, t)$ will be called *tuned error*.

Under the assumption (A12), we can now obtain an expression of θ_T, similar to that in (3.4.2.9). Define

$$\hat{\rho}(s) := \hat{p}_u(s)\, \theta^T \hat{Q}_{mu}(s) - \hat{m}(s) \tag{4.4.2.6}$$

so that, by (4.4.1.4) and (4.4.1.3), we have

$$\hat{H}_{y_p u}(\theta, s) - \hat{m}(s) = \hat{p}_u(s)\, \theta^T \left[\hat{H}_{wu}(\theta, s) - \hat{Q}_{mu}(s)\right] + \beta(s)$$

$$= \hat{p}_u(s) \left[\hat{H}_{y_p u}(\theta, s) - \hat{m}(s)\right] \bar{\theta}^T \hat{q}_u(s) + \beta(s)$$

$$= \frac{\beta(s)}{1 - \hat{p}_u(s)\bar{\theta}^T \hat{q}_u(s)} \,. \tag{4.4.2.7}$$

Using expressions of $\hat{H}_{y_p u}(\theta, s)$ and $\hat{H}_{y_p d}(\theta, s)$ in (4.4.1.8)-(4.4.1.9), and (4.4.2.7), $J_n(\theta)$ defined in (4.4.2.1) can be more compactly written as

$$J_n(\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\beta(j\omega)}{c_0 \hat{p}_u(j\omega)} \right|^2 S_u(d\omega)$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{c_0^2} \left| d_0 + C^T \hat{F}_2(j\omega) \right|^2 S_d(d\omega) \tag{4.4.2.8}$$

$J_n(\theta)$ will be a quadratic function of parameter θ without the term c_0. Therefore, we consider the following parameter transformation:

$$\theta'_{01} = 1/c_0 \qquad \text{and} \qquad \bar{\theta}' = \bar{\theta}/c_0 \tag{4.4.2.9}$$

using the fact that the optimal c_0 should not be zero. Now choose Θ_s such that, for all $\theta \in \Theta_s$, $c_0 > 0$ (from (A2)), there exists a compact subset Θ'_s induced by the transformation. Moreover, this transformation is clearly a homeomorphism between Θ_s and Θ'_s. It then follows from (A12) and Rudin ((1976) p. 86) that $\theta'_T \in \operatorname{int}\Theta'_s$, where θ'_T is the image of θ_T under this transformation. Consequently, θ'_T satisfies

$$\frac{\partial}{\partial \theta'} J_n(\theta(\theta')) \Big|_{\theta' = \theta'_T} = 0 \tag{4.4.2.10}$$

and the tuned parameters, $\theta_T = \theta(\theta'_T)$, may be obtained. Denote

$$\hat{Q}'_{mu}(s) = \left[\hat{m}(s)\hat{p}_u(s)^{-1} \,,\, -\hat{m}(s)\hat{q}_u^T(s)\right]^T \tag{4.4.2.11}$$

and

$$\hat{Q}_d(s) = \left[0 \,,\, 0 \,,\, 1 \,,\, \hat{F}_2(s)\right]^T . \tag{4.4.2.12}$$

Then, using (4.4.2.6) and (4.4.2.8), we have

$$J_n(\theta(\theta')) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|1 - \theta'^T \hat{Q}'(j\omega)\right|^2 S_u(d\omega)$$

$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\theta'^T \hat{Q}_d(j\omega)\right|^2 S_d(d\omega) \tag{4.4.2.13}$$

which is a quadratic function of θ' so that, from (4.4.2.10), θ'_T satisfies

$$\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{Q}'_{mu}(j\omega)\,\hat{Q}'^{*}_{mu}(j\omega)\,S_u(d\omega) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{Q}_d(j\omega)\,\hat{Q}^{*}_d(j\omega)\,S_d(d\omega)\right]\theta'_T$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{Q}'(j\omega)\; S_u(d\omega) \tag{4.4.2.14}$$

The following theorem will provide conditions under which θ'_T (and hence θ_T) so obtained is uniquely defined.

Theorem 4.4.2.1: (Unique Tuned Parameters)

Consider the same adaptive control problem as given in subsection 4.4.1. Let assumptions (A1)-(A6) and (A8)-(A12) be satisfied.

Then there exists a subset $V_n \subset R^{2n}$ which is nowhere dense and of measure zero such that the tuned parameter vector θ_T is uniquely defined if Supp(u) contains at least $2n$ points which form a $2n$ tuple not contained in V_n. □

Proof: Using a proof similar to that of Lemma 4.4.1.1 and results of Theorem 4.4.1.2, it follows that, for any input u that satisfies conditions above, the matrix:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{Q}'(j\omega)\; \hat{Q}'^{*}(j\omega)\; S_u(d\omega) \tag{4.4.2.15}$$

is positive definite. Furthermore,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{Q}_d(j\omega)\,\hat{Q}_d^*(j\omega)\,S_d(d\omega) \tag{4.4.2.16}$$

is positive semi-definite and, hence, the conclusion readily follows.

■

Remarks:

(1) The theorem is similar to Theorem 3.4.2.1. It states that, for almost every properly chosen SR input u, the tuned parameters and, hence, the tuned model are uniquely determined.

(2) By assumption, the tuned model is stable. Since the model has a compatible structure with the closed loop plant, it will play a role similar to the reference model loop shown in Figure 4.1.1. A major function of the tuned model is to separate the plant and the reference model for the analysis. The theme will be that: *if* the closed loop plant converges to a neighborhood of the tuned model, and tuned model is close enough to the reference model, *then* the overall adaptive system is stable, and the output of the plant will be close to that of the reference model.

4.4.3 Reduced Order Controller

Due to the inevitable existence of unmodelled dynamics, the order of the adaptive controller will always be lower than that of the real plant. Yet the model reference adaptive scheme is designed based upon knowledge of the exact order and relative degree of the plant, and stability proofs heavily rely on these assumptions. Limitations of this scheme were first exposed after Rohrs et al (1982)(1985) showed several instances of instability due to the failure to satisfy that assumption as well as the assumption of no disturbances. A number of approaches have been proposed in the literature to robustify the scheme, such as, by Ioannou and Kokotovic (1984), Kreiselmeier and Anderson (1986), Narendra and Annaswamy (1986), Ioannou and Tsakalis (1986). While this is the case, one should expect the robustness of this adaptive algorithm due to *persistency of excitation* of the controlled system and consequently a proper choice of exogenous reference inputs to the system.

In this subsection, we use averaging as an approach to show robustness following from the *positive definite* PE condition (which will be clear in the sequel) on the reference input signals. The way we proceed is to use the notion of the tuned model defined in the previous subsection.

Consider the same adaptive control problem as described above but with the overall adaptive system redrawn in Figure 4.4.2. Denote by $\hat{m}_T(s)$ the tuned model transfer function, i.e.

$$\hat{m}_T(s) := \hat{H}_{y_p u}(\theta_T, s) . \tag{4.4.3.1}$$

Let y_T be the output of the tuned model and e_* be the tuned error so that

$$e_{*T} = e_*(\theta_T, t) = y_T - y_m . \tag{4.4.3.2}$$

The state space realization of the closed loop plant, similar to that given in subsection 4.2.1, can be written as

$$\begin{bmatrix} \dot{x}_p \\ \dot{v}_p^{(1)} \\ \dot{v}_p^{(2)} \end{bmatrix} = \begin{bmatrix} A_p & 0 & 0 \\ 0 & \Lambda & 0 \\ b\, c_p^T & 0 & \Lambda \end{bmatrix} \begin{bmatrix} x_p \\ v_p^{(1)} \\ v_p^{(2)} \end{bmatrix} + \begin{bmatrix} b_p \\ b \\ 0 \end{bmatrix} \theta^T w + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} d \tag{4.4.3.3}$$

$$y_p = [\, c_p^T , 0 , 0 \,] \begin{bmatrix} x_p \\ v_p^{(1)} \\ v_p^{(2)} \end{bmatrix} \tag{4.4.3.4}$$

where (A_p, b_p, c_p^T) is a minimal realization of the plant with dimension N $(> n)$ and

$$w = [\, u , v_p^{(1)T} , y_{pd} , v_p^{(2)T} \,]^T \tag{4.4.3.5}$$

Define

$$\theta_T = [\ c_{0T} \, , \, C_T^T \, , \, d_{0T} \, , \, D_T^T \ ]^T \tag{4.4.3.6}$$

then the state space realization of the tuned model loop can be written as: (by letting $\theta = \theta_T$ and $d \equiv 0$)

$$\begin{bmatrix} \dot{x}_T \\ \dot{v}_T^{(1)} \\ \dot{v}_T^{(2)} \end{bmatrix} = \begin{bmatrix} A_T + b_p d_{0T} c_p^T & b_p C_T^T & b_p D_T^T \\ b d_{0T} c_p^T & \Lambda + b C_T^T & b D_T^T \\ b c_p^T & 0 & \Lambda \end{bmatrix} \begin{bmatrix} x_T \\ v_T^{(1)} \\ v_T^{(2)} \end{bmatrix} + \begin{bmatrix} b_p \\ b \\ 0 \end{bmatrix} c_{0T} u \tag{4.4.3.7}$$

$$y_T = [c_p^T, 0, 0] \begin{bmatrix} x_T \\ v_T^{(1)} \\ v_T^{(2)} \end{bmatrix} . \tag{4.4.3.8}$$

In (4.4.3.7), let A_T be the $(N+2n-2)\times(N+2n-2)$ matrix, and b_T the $(N+2n-2)\times 1$ matrix; and in (4.4.3.8), c_T be the $1\times(N+2n-2)$ matrix. Then the tuned model transfer function, $\hat{m}_T(s)$, can be expressed in terms of (A_T, b_T, c_T^T),

$$\hat{m}_T(s) = c_{0T}\, c_T^T (sI - A_T)^{-1} b_T \tag{4.4.3.9}$$

It can be deduced from (A9) and the definition of tuned parameters that the tuned model is actually exponentially stable (there are no unstable unobservable modes).

Remark: If (A9) were to be relaxed, then the subset Θ_s would need the additional property that no unstable pole-zero cancellations occur in the closed loop transfer function for all $\theta \in \Theta_s$ in order to assure the exponential stability of the tuned model.

Define the state error e and the parameter ϕ as before, i.e.

$$e^T := [x_p^T, v_p^{(1)T}, v_p^{(2)T}] - [x_T^T, v_T^{(1)T}, v_T^{(2)T}] \tag{4.4.3.10}$$

and

$$\phi = \theta - \theta_T \tag{4.4.3.11}$$

then we have the dynamics of the state error e expressed by

$$\dot{e} = A_T e + b_T \phi^T w + \bar{b} d \tag{4.4.3.12}$$

$$e_{oT} := y_p - y_T \tag{4.4.3.13}$$

where

$$\bar{b} = [\ d_{0T} b_p^T \,,\, d_{0T} b^T \,,\, b^T\ ]^T \,. \tag{4.4.3.14}$$

In the following, we will apply averaging to analyze reduced order controllers using the output error and input error direct adaptive control schemes respectively.

4.4.3.1 Output Error Scheme

Here, for the sake of illustration, we will only consider the case where the relative degree of the nominal plant is one. The case for higher relative degrees can be similarly dealt with.

Using (4.4.3.2) and (4.4.3.13), the output error $e_o = y_{pd} - y_m$, which is used for the parameter update, can be expressed as

$$e_o = e_{oT} + d + e_{*T} \,. \tag{4.4.3.15}$$

In order to apply averaging here, we require that e_o and w be expressed in terms of the input u and disturbance d, assuming that the controller parameters θ are constant. As indicated in the remark after Theorem 4.4.2.1, an analogy can be drawn between the tuned model and the reference model in the nominal case so that, from (4.3.1.10) and (4.4.3.13), we have

$$e_{oT} = \frac{1}{c_{0T}} \hat{m}_T(s) \phi^T \hat{H}_{wu}(\theta, s)(u) + \hat{H}_{y_p d}(\theta, s)(d) \tag{4.4.3.16}$$

which together with (4.4.3.15) and (4.4.1.7) yields

$$e_o = \frac{1}{c_{0T}} \hat{m}_T(s) \phi^T \hat{H}_{wu}(\theta, s)(u) + \hat{H}_{y_{pd} d}(\theta, s)(d) + e_{*T} \tag{4.4.3.17}$$

Comparing (4.3.1.10) with (4.4.3.17), the exponential stability of the adaptive system will no longer be preserved due to the two extra terms on the R.H.S. of (4.4.3.17). However, if those two terms are sufficiently small relative to the first term, then BIBO stability can be expected. Before we examine different parameter update algorithms, we provide a theorem similar to Theorems 3.4.2.1 and 4.4.2.1 as a fundamental tool.

Theorem 4.4.3.1: (Almost Always PE Condition on w)

Consider the same adaptive control problem as given above. Let assumptions (A1)-(A6) and (A8)-(A12) be satisfied.

Then there exists a subset $V_n \subset R^{2n}$ which is nowhere dense and measure zero such that the signal vector w is PE if Supp(u) contains at least $2n$ points which form a $2n$ tuple not contained in V_n. $\square$

The proof is similar to that of Theorem 4.4.2.1 and, hence, will be omitted here.

Remark: The above theorem only provides sufficient conditions, in contrast to the necessary and sufficient condition given in Theorem 3.4.2.1, owing to possible richness from the disturbance d.

(i) Gradient Algorithm:

Consider the dynamics of parameter errors:

$$\dot{\phi} = -\varepsilon e_o w \qquad \phi(0) = \phi_0 \tag{4.4.3.18}$$

which with (4.4.3.12) form a mixed time scale system as before. so that the averaged system in the slow variable ϕ can be readily found as

$$\dot{\phi}_{av} = -\varepsilon R_{ww_f}(\phi_{av})\phi_{av} - \varepsilon g(\phi_{av}) \qquad \phi_{av}(0) = \phi_0 \tag{4.4.3.19}$$

where

$$R_{ww_f}(\phi) = \frac{1}{2\pi c_{0T}} \int_{-\infty}^{\infty} \hat{H}_{wu}(\theta, j\omega)\, \hat{m}_T^*(j\omega)\, \hat{H}_{wu}^*(\theta, j\omega)\, S_u(d\omega) \tag{4.4.3.20}$$

and

$$g(\phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\hat{m}_T^*(j\omega) - \hat{m}^*(j\omega)\right] \hat{H}_{wu}(\theta, j\omega)\, S_u(d\omega)$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{H}_{y_{pd}d}^*(\theta, j\omega)\, \hat{H}_{wd}(\theta, j\omega)\, S_d(d\omega)\,. \tag{4.4.3.21}$$

The following theorem will provide conditions under which the adaptive system using the gradient algorithm will remain BIBO stable. For our convenience, we will use $\Phi_s := -\theta_T + \Theta_s$ (a vector addition), which is a compact subset in R^{2n}.

Theorem 4.4.3.2: (BIBO Stability Theorem Using the Gradient Algorithm)

Consider the output error direct adaptive control problem described above. Let assumptions (A1)-(A6) and (A8)-(A12) be satisfied, θ_T be the tuned parameter defined by (4.4.2.5), and $R_{ww_f}(\phi)$ and $g(\phi)$ be defined in (4.4.3.20) and (4.4.3.21) respectively.

If the input u satisfies conditions of Theorem 4.4.3.1 and for sufficiently small $\delta > 0$

$$\max_{\phi \in \Phi_s} \| g(\phi) \| \le \min_{\phi \in \Phi_s} \lambda_{\min}\left[SM(R_{ww_f}(\phi))\right] \delta \tag{4.4.3.22}$$

then there exist $\gamma_1, \varepsilon_1 > 0$, $0 \le T < \infty$, and $\psi_1(\varepsilon) \in K$ such that

$$\| \phi(t) \| \le \psi_1(\varepsilon) + \gamma_1 \delta \qquad t \ge t_0 + T \tag{4.4.3.23}$$

for all $\varepsilon \le \varepsilon_1$, and for sufficiently small

condition ϕ_0. □

Proof: Consider the following Lyapunov function for the averaged differential equation (4.4.3.19):

$$v(\phi_{av}) = \frac{1}{2} \; \| \phi_{av} \|^2 . \tag{4.4.3.24}$$

By Theorem 4.4.3.1 and Lemma 3.3.2, $SM(R_{ww_f}(\phi))$ is positive definite for all $\phi \in \Phi_s$. Denote

$$\alpha := \min_{\phi \in \Phi_s} \lambda_{\min} \left[SM(R_{ww_f}(\phi)) \right] . \tag{4.4.3.25}$$

The derivative of v along the trajectories of (4.4.3.19) can be estimated using (4.4.3.25) and the condition in (4.4.3.22):

$$\dot{v} = -\, \varepsilon \phi_{av}^T \, SM(R_{ww_f}(\phi_{av})) \, \phi_{av} - \varepsilon \phi_{av}^T g(\phi_{av})$$

$$\leq -\, \varepsilon \alpha \, \| \phi_{av} \| \; (\| \phi_{av} \| \, - \delta) \tag{4.4.3.26}$$

whenever $\phi_{av} \in \Phi_s$. The conclusion then follows from Theorem 2.2.2.5.

■

Remarks:

(1) The condition (4.4.3.22) is crucial in determining the BIBO stability of the adaptive system when the parameter adaptation is slow. In the absence of unmodelled dynamics and output disturbances, $g(\phi) \equiv 0$ so that δ can also be chosen the unmodelled dynamics and the bounded output disturbances are mild, a reasonably small δ can also be found so that the stability of the system and, from (4.4.3.16), the closeness between the real plant and the tuned model are also guaranteed.

(2) In the absence of output disturbances, if u satisfies conditions in the theorem and, additionally, Supp(u) contains exactly $2n$ points, then it can be deduced from Theorem 4.4.2.1 that e_{*T} converges to zero exponentially (Callier and Desoer (1982) p. 127) and $\delta \equiv 0$ so that, from Theorem 2.2.2.3, the parameter errors ϕ converge to zero exponentially.

(ii) Least-Squares with Forgetting Factor Algorithm

Recall that the dynamics of the parameter errors are given by:

$$\dot{\phi} = -\varepsilon P e_o \zeta \qquad \phi(0) = \phi_0 \tag{4.4.3.27}$$

$$\dot{P} = \varepsilon \lambda P - \varepsilon P \zeta \zeta^T P \qquad P(0) = I \tag{4.4.3.28}$$

so that the averaged system can be found to be

$$\dot{\phi}_{av} = -\varepsilon P_{av} R_{\zeta\zeta_f}(\phi_{av})\,\phi_{av} - \varepsilon P_{av}\, g(\phi_{av}) \qquad \phi_{av}(0) = \phi_0 \tag{4.4.3.29}$$

$$\dot{P}_{av} = \varepsilon \lambda P_{av} - \varepsilon P_{av} R_{\zeta}(\phi_{av})\, P_{av} \qquad P_{av}(0) = I \tag{4.4.3.30}$$

where

$$R_{\zeta\zeta_f}(\phi) = \frac{1}{2\pi c_0^*} \int_{-\infty}^{\infty} \hat{H}_{\zeta u}(\theta, j\omega)\, \hat{m}_w^*(j\omega)\, \hat{l}^*(j\omega)\, \hat{H}_{\zeta u}^*(\theta, j\omega)\, S_u(d\omega) \tag{4.4.3.31}$$

and

$$R_{\zeta}(\phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{H}_{\zeta u}(\theta, j\omega)\, \hat{H}_{\zeta u}^*(\theta, j\omega)\, S_u(d\omega)$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{H}_{\zeta d}(\theta, j\omega)\, \hat{H}_{\zeta d}^*(\theta, j\omega)\, S_d(d\omega) \tag{4.4.3.32}$$

$$g(\phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\hat{m}_T^*(j\omega) - \hat{m}^*(j\omega)\right] \hat{l}^*(j\omega)\, \hat{H}_{\zeta u}(\theta, j\omega)\, S_u(d\omega)$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{H}_{y_p d}^*(\theta, j\omega)\, \hat{H}_{\zeta d}(\theta, j\omega)\, S_d(d\omega)\ . \tag{4.4.3.33}$$

The following theorem, similar to Theorem 4.4.3.2, provides conditions for BIBO stability.

Theorem 4.4.3.3: (BIBO Stability Using Least-Squares with Forgetting Factor Algorithm)

Consider the same adaptive control problem but use the modified scheme shown in Figure 4.2.2. Let assumptions (A1)-(A12) be satisfied, and $R_{\zeta\zeta_f}(\phi), R_{\zeta}(\phi), g(\phi)$ be defined as in (4.4.3.31)-(4.4.3.33).

If the input u satisfies the conditions of Theorem 4.4.3.1, and for sufficiently small $\delta > 0$

$$\max_{\phi \in \Phi_s} \| g(\phi) \| \le \min_{\phi \in \Phi_s} \lambda_{\min} \left[SM(R_{\zeta\zeta_f}(\phi)) - \frac{1}{2} R_{\zeta}(\phi) \right] \delta \qquad (4.4.3.34)$$

then there exist $\gamma_2, \varepsilon_2 > 0$, $0 \le T < \infty$, and $\psi_2(\varepsilon) \in K$ such that

$$\| \phi(t) \| \le \psi_2(\varepsilon) + \gamma_2 \delta \qquad t \ge t_0 + T \qquad (4.4.3.35)$$

for all $\varepsilon \le \varepsilon_2$, and for sufficiently small ϕ_0. □

Proof: The proof of this theorem is similar to that of Theorem 4.4.3.2. Construct a Lyapunov function for the averaged differential equations (4.4.3.29)-(4.4.3.30),

$$v(\phi_{av}, P_{av}) = \frac{1}{2} \phi_{av}^T P_{av}^{-1} \phi_{av} \,. \qquad (4.4.3.36)$$

Again from Theorem 4.4.3.1 and Lemma 3.3.2, $SM(R_{\zeta\zeta_f}(\phi))$ and $R_{\zeta}(\phi)$ are positive definite. Using a result established in subsection 4.3.1. CASE I (i.e. P_{av}^{-1} is bounded above and below whenever $\phi_{av} \in \Phi_s$), it follows that there exist non-zero positive constants α_1, α_2, α_3, α_4 such that

$$\alpha_1 \| \phi_{av} \|^2 \le v(\phi_{av}, P_{av}) \le \alpha_2 \| \phi_{av} \|^2 \qquad (4.4.3.37)$$

$$\left\| \frac{\partial v}{\partial \phi_{av}} \right\| \le \alpha_3 \| \phi_{av} \|^2 \qquad (4.4.3.38)$$

$$\left\| \frac{\partial v}{\partial P_{av}} \right\| \le \alpha_4 \| \phi_{av} \|^2 \qquad (4.4.3.39)$$

whenever $\phi_{av} \in \Phi_s$. Denote

$$\alpha_5 := \min_{\phi \in \Phi_s} \lambda_{\min}\left[SM(R_{\zeta\zeta_f}(\phi)) - \frac{1}{2}R_\zeta(\phi)\right]. \tag{4.4.3.40}$$

The derivative of v along the trajectories of (4.4.3.29)-(4.4.3.30) can be estimated as follows:

$$\dot v + \frac{\varepsilon\lambda}{2}v = -\varepsilon\phi_{av}^T\left[SM(R_{\zeta\zeta_f}(\phi_{av})) - \frac{1}{2}R_\zeta(\phi_{av})\right]\phi_{av} - \phi_{av}^T g(\phi_{av})$$

$$\leq -\varepsilon\alpha_5 \| \phi_{av} \| (\| \phi_{av} \| - \delta). \tag{4.4.3.41}$$

Thus, using Corollary 2.2.2.7, we may readily conclude the result.

■

4.4.3.2 Input Error Scheme

Recall from subsection 4.2.2 that the error signal used for the parameter update is e_2 defined in (4.2.2.6). If the parameters $\theta(t)$ are fixed at θ, then, using (4.2.2.7), we have

$$\begin{aligned} e_2 &= c_0(\hat{m}(s)\hat{l}(s))^{-1}(y_{pd} - y_m) \\ &= c_0(\hat{m}(s)\hat{l}(s))^{-1}(e_o) \end{aligned} \tag{4.4.3.42}$$

and

$$v = \hat{l}(s)^{-1}(w) + [\ (\hat{m}(s)\hat{l}(s))^{-1}(e_o)\ ,\ 0\ ,\ 0\ ,\ 0\ ] \tag{4.4.3.43}$$

so that

$$e_2 = \frac{c_0}{\phi_0}(\phi^T v - \hat{l}^{-1}(\phi^T w)) \tag{4.4.3.44}$$

where $\phi_0 = c_0 - c_{0T}$, and θ_T is the tuned parameters defined as before. Thus, using (4.4.3.17), we can solve e_2 explicitly in terms of u, d, and e_{*T}:

$$e_2 = \left[\frac{\frac{c_0}{c_{0T}}(\hat{m}(s)^{-1}\hat{m}_T(s))(\phi^T \hat{H}_{vu}(\theta, s))}{1 + \frac{\phi_0}{c_{0T}}\hat{m}(s)^{-1}\hat{m}_T(s)}\right](u)$$

$$+\left[c_0(\hat{m}(s)\hat{l}(s))^{-1}\hat{H}_{y_{pd}d}(\theta,s)\right](d)+\left[\frac{1}{1+\frac{\phi_0}{c_{0T}}\hat{m}(s)^{-1}\hat{m}_T(s)}\right](e_{*T})\,. \qquad (4.4.3.45)$$

The following corollary provides conditions, similar to that in Theorem 4.4.3.1, under which the PE condition on the signal vector v can be assured.

Corollary 4.4.3.4: (Almost Always PE Condition on v)

Consider the same adaptive control problem as given in Theorem 4.4.3.1. Let assumptions (A1)-(A6) and (A8)-(A12) be satisfied.

Then the signal vector v in (4.4.3.43) is PE if the input u satisfies conditions of Theorem 4.4.3.1. □

The proof is similar to that of Theorem 4.4.3.1 and, hence, is omitted here.

With these results, we now analyze the reduced order adaptive controller using both parameter update algorithms given in subsection 4.2.2.

(i) Normalized Gradient Algorithm:

For the case of averaging, consider the parameter update law of the form,

$$\dot{\phi}=-\varepsilon\frac{e_2 v}{1+\varepsilon v^T v} \qquad \phi(0)=\phi_0 \qquad (4.4.3.46)$$

Its averaged system is

$$\dot{\phi}_{av}=-\varepsilon R_{vv_f}(\phi_{av})\,\phi_{av}-\varepsilon g(\phi_{av}) \qquad \phi_{av}(0)=\phi_0 \qquad (4.4.3.47)$$

where

$$R_{vv_f}(\phi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{H}_{vu}(\theta,j\omega)\left[\frac{\frac{c_0}{c_{0T}}\hat{m}^{-1}(j\omega)\,\hat{m}_T(j\omega)}{1+\frac{\phi_0}{c_{0T}}\hat{m}^{-1}(j\omega)\,\hat{m}_T(j\omega)}\right]^{*}\hat{H}^{*}_{vu}(\theta,j\omega)\,S_u(d\omega) \qquad (4.4.3.48)$$

and

$$g(\phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{H}_{vu}(\theta, j\omega) \left[\frac{\hat{m}_T(j\omega) - \hat{m}(j\omega)}{1 + \frac{\phi_0}{c_{0T}} \hat{m}^{-1}(j\omega)\, \hat{m}_T(j\omega)} \right]^* S_u(d\omega)$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} c_0 \hat{H}_{vd}(\theta, j\omega)\, \hat{H}^*_{y_{pd}d}(\theta, s)\; S_d(d\omega) \qquad (4.4.3.49)$$

Note that the matrix $R_{vv_f}(\phi)$ defined in (4.4.3.48) is no longer symmetric like the one defined in (4.3.2.9) nor is it necessarily a positive definite matrix under the normal PE condition.

Since the system (4.4.3.47) is similar to that in (4.4.3.19), we will give a corollary to Theorem 4.4.3.2 to provide conditions for BIBO stability of the adaptive system.

Corollary 4.4.3.5:

Consider the same adaptive control problem as the one in Theorem 4.4.3.1 but using the input error scheme. Let assumptions (A1)-(A6), (DA7), and (A8)-(A12) be satisfied, $R_{vv_f}(\phi)$ and $g(\phi)$ be defined in (4.4.3.48), (4.4.3.49).

If the input u satisfies the conditions of Corollary 4.4.3.4 and there exists sufficiently small $\delta > 0$ such that

$$\max_{\phi \in \Phi_s} \| g(\phi) \| \le \min_{\phi \in \Phi_s} \lambda_{\min} \left[SM(R_{vv_f}(\phi)) \right] \delta \qquad (4.4.3.50)$$

then there exist $\gamma_3, \varepsilon_3 > 0$, $0 \le T < \infty$, and $\psi_3(\varepsilon) \in K$ such that

$$\| \phi(t) \| \le \psi_3(\varepsilon) + \gamma_3 \delta \qquad t \ge t_0 + T \qquad (4.4.3.51)$$

for all $\varepsilon \le \varepsilon_3$ and for sufficiently small ϕ_0. □

The proof is identical to that of Theorem 4.4.3.2 and, hence, is omitted.

Remarks:

(1) In the absence of unmodelled dynamics and output disturbances, $\hat{m}=\hat{m}_T$ (i.e. $\theta_T=\theta^*$) and, hence,

$$R_{vv_f}(\phi) = R_v(\phi)$$

is a symmetric positive semi-definite matrix, which becomes positive definite if u is SR. In the non-ideal case, if the input u satisfies conditions of Corollary 4.4.3.4, $\phi_0 / c_{0T} \ll 1$, and:

$$\hat{m}^{-1}(j\omega)\,\hat{m}_T(j\omega) \approx 1 \qquad \text{for all } \omega \in \mathrm{Supp}(u) \tag{4.4.3.52}$$

then $R_{vv_f}(\phi)$ will remain positive definite.

(2) Comparison between (4.4.3.20) and (4.4.3.48) suggests the following similarity: $R_{ww_f}(\phi)$ is the crosscovariance of w and $\hat{m}_T(s)(w)$, whereas $R_{vv_f}(\phi)$ is the crosscovariance of v and $\hat{m}_v(s)(v)$ where

$$\hat{m}_v(s) = \frac{\dfrac{c_0}{c_{0T}}\,\hat{m}(s)^{-1}\,\hat{m}_T(s)}{1+\dfrac{\phi_0}{c_{0T}}\,\hat{m}^{-1}(s)\,\hat{m}_T(s)}\,. \tag{4.4.3.53}$$

Since the positive definiteness of these matrices is crucial to the stability of the adaptive system, the spectrum of the control input u should not be too high so that $\hat{m}_T(j\omega)$ and $\hat{m}_v(j\omega)$ remain positive for all $\omega \in \mathrm{Supp}(u)$.

(ii) Normalized Least-Squares with Forgetting Factor Algorithm

The parameter update law in this case has the following form:

$$\dot{\phi} = -\,\varepsilon P \frac{e_2\, v}{1+\varepsilon v^T v} \qquad \phi(0)=\phi_0 \tag{4.4.3.54}$$

$$\dot{P} = \varepsilon P - \varepsilon P \frac{v\, v^T}{1+\varepsilon v^T v} P \qquad P(0)=I\,. \tag{4.4.3.55}$$

The averaged system of the slow variables ϕ and P is

$$\dot{\phi}_{av} = -\varepsilon P_{av} R_{vv_f}(\phi_{av})\phi_{av} - \varepsilon P_{av}\, g(\phi_{av}) \qquad \phi_{av}(0) = \phi_0 \tag{4.4.3.56}$$

$$\dot{P}_{av} = \varepsilon P_{av} - \varepsilon P_{av} R_v(\phi_{av}) P_{av} \qquad P_{av}(0) = I \tag{4.4.3.57}$$

where $R_{vv_f}(\phi)$ and $g(\phi)$ are defined in (4.4.3.48) and (4.4.3.49) respectively, and $R_v(\phi)$ is defined by

$$R_v(\phi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{H}_{vu}(\theta, j\omega)\, \hat{H}_{vu}^*(\theta, j\omega)\, S_u(d\omega)$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{H}_{vd}(\theta, j\omega)\hat{H}_{vd}^*(\theta, j\omega) S_d(j\omega)\,. \tag{4.4.3.58}$$

Due to the similarity to the averaged differential equations (4.4.3.29)-(4.4.3.30), we will state a corollary to Theorem 4.4.3.3 to provide conditions for BIBS stability.

Corollary 4.4.3.6:

Consider the same adaptive control problem as that considered in Corollary 4.4.3.5. Let $R_{vv_f}(\phi)$, $g(\phi)$ be defined in (4.4.3.48), (4.4.3.49), and $R_v(\phi)$ be defined in (4.4.3.58).

If the input u satisfies conditions of Corollary 4.4.3.4 and there exists sufficiently small $\delta>0$ such that

$$\max_{\phi\in\Phi_s} \| g(\phi) \| \le \min_{\phi\in\Phi_s} \lambda_{\min}\left[SM(R_{vv_f}(\phi)) - \frac{1}{2}R_v(\phi)\right]\delta \tag{4.4.3.59}$$

then there exist $\gamma_4, \varepsilon_4 > 0$, $0 \le T < \infty$, and $\psi_4(\varepsilon) \in K$ such that

$$\| \phi(t) \| \le \psi_4(\varepsilon) + \gamma_4\delta \qquad t \ge t_0 + T \tag{4.4.3.60}$$

for all $\varepsilon \le \varepsilon_0$, and for sufficiently small ϕ_0. □

The proof is identical to that of Theorem 4.4.3.3 and, hence, is omitted here.

Remark: Note that the condition in (4.4.3.59) is not an SPR condition, in contrast with condition (4.4.3.34). Under the assumption of no unmodelled dynamics and output disturbances we have, as indicated in the remark after Corollary 4.4.3.5,

$$R_{vv_f}(\phi)-\frac{1}{2}R_v(\phi)=\frac{1}{2}R_v(\phi)\ . \tag{4.4.3.61}$$

This, however, will not be the case in the non-ideal case. Yet, if the conditions in the remark after Corollary 4.4.3.5 are satisfied, then the L.H.S. in (4.4.3.61) will remain positive definite.

4.4.4 Slow-Drift Instability

To substantiate the importance of a proper choice of input signals as indicated in last subsection, we present *slow-drift instability*, one type of instability which appears in the adaptive system with unmodelled dynamics and bounded output disturbances when the adaptation is slowed down and the reference input u is not properly chosen. This type of an instability property will usually be detected only after a long period of operation time has elapsed. Roughly speaking, the parameter vector $\theta(t)$ fails to converge to a neighborhood of the fixed parameters, for example, tuned parameters, but rather drift in the parameter space until it finally reaches a region in which the closed loop system is unstable.

Definition 4.4.4.1: (Positive Definite PE Through a Stable Filter)

Consider a stable filter described by a transfer function $\hat{f}(s)$, and a signal vector $\underline{w}:R_+\rightarrow R^{2n}$. $\underline{w}$ is said to be *positive definite PE* through $\hat{f}(s)$ if there exist $\alpha_1,\ \alpha_2>0$ such that

$$\alpha_1\leq \mathrm{cov}(\underline{w},\hat{f}(s)(\underline{w})):=\lim_{T\rightarrow\infty}\frac{1}{T}\int_{t_0}^{t_0+T} w(t)\hat{f}(s)(w(t))\,dt\leq\alpha_2 \tag{4.4.4.1}$$

□

Remarks:

(1) The notation $\mathrm{cov}(\underline{w}, \hat{f}(s)(\underline{w}))$ is, in fact, the crosscovariance of $\underline{w}$ and $\hat{f}(s)(\underline{w})$, which has been defined in Definition 3.4.2.1.

(2) If the signal vector $\underline{w}$ can be described as

$$\underline{w} = \hat{H}_{\underline{w}u}(s)(u) \tag{4.4.4.2}$$

where the scalar signal u is stationary and has a power spectral measure $S_u(d\omega)$, then (4.4.4.1) can be expressed in its spectral form,

$$\mathrm{cov}(\underline{w}, \hat{f}(s)(\underline{w})) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{H}_{\underline{w}u}(j\omega) \hat{f}(j\omega) \hat{H}^{*}_{\underline{w}u}(j\omega)\, S_u(d\omega) . \tag{4.4.4.3}$$

It is clear that $\underline{w}$ is always positive definite PE if $\hat{f}(s)$ is SPR and $\underline{w}$ is PE.

Essentially, the positive definite PE condition is a key condition for the adaptive control algorithm. For example, in the ideal nominal case where the relative degree of plant is one, the error signal e_o is expressed by

$$e_o = \frac{1}{c_0^*} \hat{m}(s)(\phi^T w) . \tag{4.4.4.4}$$

When the gradient algorithm is used and the adaptation is slow, the parameter update law is "approximated" by its averaged system,

$$\dot{\phi}_{av} = -\,\varepsilon\, \mathrm{cov}(w, \hat{m}(s)(w))\, \phi_{av} . \tag{4.4.4.5}$$

Since by choice $\hat{m}(s)$ is SPR, $\mathrm{cov}(w, \hat{m}(s)(w))$ is positive definite whenever w is PE, which, in turn, implies the exponential stability of the adaptive system. On the other hand, in the presence of unmodelled dynamics, the error signal e_o may be expressed by

$$e_o \approx \frac{1}{c_{0T}} \hat{m}_T(s)\, (\phi^T w) \tag{4.4.4.6}$$

($\hat{m}_T(s)$ is the tuned model transfer function as defined in previous subsections) when the tuned model is close to the reference model. Hence the parameter update law can be "approximated" by

$$\dot{\phi}_{av} = -\,\varepsilon\,\mathrm{cov}(w, \hat{m}_T(s)(w))\;\phi_{av}\,. \tag{4.4.4.7}$$

Since the relative degree of $\hat{m}_T(s)$ is the same as that of $\hat{p}_u(s)$, which is, in this particular instance, usually greater than one, for sufficiently high frequencies ω, it is possible that

$$\mathrm{Re}\,\hat{m}_T(j\omega) < 0\,. \tag{4.4.4.8}$$

This implies, from Definition 4.4.4.1, that w fails to be positive definite PE through $\hat{m}_T(s)$. Consequently, the stability of both system (4.4.4.7) and, hence, the adaptive system are likely to be at stake. In the following, we will formalize the above arguments by using averaging to analyze the instability of the adaptive system with either of the two algorithms discussed above.

(i) Output Error Scheme:

Again, for illustration, we will only consider the case where the relative degree of the nominal plant is one. It will be easy to extend the these results to the cases with higher relative degrees.

Initially, we will proceed with some observations with the output disturbance $d \equiv 0$. Note that $y_p = \hat{p}_u(\theta^T w)$. When the parameter $\theta(t)$ varies sufficiently slowly and $\theta(t) \in \Theta_s$ at some time t, as suggested from (4.4.4.7), the gradient type parameter adaptation law is approximately:

$$\begin{aligned}
\dot{\theta} &\approx -\frac{\varepsilon}{2\pi}\int_{-\infty}^{\infty} \hat{p}_u^*(j\omega)\left[\theta^T \hat{H}_{wu}(\theta, j\omega) - \hat{m}(j\omega)\hat{p}_u^{-1}(j\omega)\right]^* \hat{H}_{wu}(\theta, j\omega)\, S_u(d\omega) \\
&= -\frac{\varepsilon}{2\pi}\frac{\partial}{\partial\theta}\int_{-\infty}^{\infty} \frac{1}{2}\hat{p}_u^*(j\omega)\left|\theta^T \hat{H}_{wu}(\theta, j\omega) - \hat{m}(j\omega)\hat{p}_u^{-1}(j\omega)\right|^2 S_u(d\omega) \\
&=: -\frac{\varepsilon}{4\pi}\frac{\partial}{\partial\theta} J_{og}(\theta)
\end{aligned} \tag{4.4.4.9}$$

where we neglect the terms $\theta^T \frac{\partial}{\partial\theta} \hat{H}_{wu}(\theta, j\omega)$ by assuming that: (i) $\theta(t)$ stays close to a set (or possibly a point) defined by

$$\Theta_{\min} := \left\{ \theta_{op} \mid \theta_{op} = \underset{\theta \in \Theta_s}{\operatorname{argmin}} \, J_{og}(\theta) \right\} \tag{4.4.4.10}$$

and (ii) $\theta^T \frac{\partial}{\partial\theta} \hat{H}_{wu}(\theta, j\omega)$ is much smaller relative to $\hat{H}_{wu}(\theta, j\omega)$ (elementwise) when the frequency ω is high.

Remark: In words, assumption (i) says: when $\theta(t)$ is close to $\Theta_{\min}$, i.e. the output of the closed loop plant tries to match that of the reference model for all frequencies in Supp(u). Since a better match can normally be achieved at low frequencies so that the effect of neglect could be diminished at low frequencies rather at high frequencies; assumption (ii) says: the rate of change of $\hat{H}_{wu}(\theta, s)$ with respect to θ is relatively much smaller than $\hat{H}_{wu}(\theta, s)$ itself at high frequencies.

When these assumptions hold, then the parameter update law optimizes the cost function $J_{og}(\theta)$, so that the steady state of $\theta(t)$ should be expected to be close to $\Theta_{\min}$, possibly until $\theta(t)$ starts to drift out of Θ_s.

Now consider a sufficiently low frequency ω_l. By Theorem 4.4.1.2 and the fact that the unmodelled dynamics become insignificant in the low frequency region, there almost always exists a $\theta_l \in \Theta_s$ such that:

$$\hat{H}_{y_p u}(\theta_l, j\omega_l) = \hat{m}(j\omega) \,. \tag{4.4.4.11}$$

But for a sufficiently high frequency ω_h, it may happen that

$$Re\, \hat{H}_{y_p u}(\theta, j\omega_h) < 0 \qquad \text{for all } \theta \in \Theta_s \tag{4.4.4.12}$$

which implies that

$$\hat{H}_{y_p u}(\theta, j\omega) \not\approx \hat{m}(j\omega) \qquad \text{for all } \theta \in \Theta_s . \tag{4.4.4.13}$$

Consider the following definition.

Definition 4.4.4.2: (Positive Real Frequency Set, Negative Real Frequency Set)

Consider a compact subset Θ_{sc} in the parameter space such that $\theta^* \in \Theta_{sc} \subset \Theta_s$ and $\Theta_{\min} \subset \Theta_{sc}$ (θ^* is the true parameter corresponding to the nominal plant). Ω_p and Ω_n are called positive real frequency set and negative real frequency set respectively if:

$$\Omega_p = \left\{ \omega \mid \mathrm{Re}\,\hat{H}_{y_p u}(\theta, j\omega) > 0 \;\;,\;\; \text{for all } \theta \in \Theta_{sc} \right\} \tag{4.4.4.14}$$

$$\Omega_n = \left\{ \omega \mid \mathrm{Re}\,\hat{H}_{y_p u}(\theta, j\omega) < 0 \;\;,\;\; \text{for all } \theta \in \Theta_{sc} \right\} \tag{4.4.4.15}$$

□

Remark: Clearly, the union of Ω_p and Ω_n does not cover the whole set of reals, R. However, in the context of study on instability, these two sets are our domain of interests.

Assumption:

(A13) Supp(u) $\cap\ \Omega_n$ contains $2n_1$ points, where $n_1 \geq 1$, and Supp(u) $\cap\ \Omega_p$ contains less than $2n$ points. The spectral magnitudes of frequency elements in Supp(u) $\cap\ \Omega_n$ are relatively smaller than of those in Supp(u) $\cap\ \Omega_p$.

Under this assumption, Theorem 4.4.1.2 and the remark following (4.4.4.9), (4.4.4.10) imply that there almost always exists a $\theta_0 \in \Theta_{sc}$ close to $\Theta_{\min}$ such that:

$$\hat{H}_{y_p u}(\theta_0, j\omega) \approx \hat{m}(j\omega) \qquad \text{for all } \omega \in \text{Supp}(u) \cap \Omega_p \tag{4.4.4.16}$$

On the other hand, from Definition 4.4.4.1, the signal vector w fails to satisfy the positive definite PE condition through $\hat{H}_{y_p\mu}(\theta_0, s)$. Under these conditions, we now study the behavior of the adaptive system through its averaged version, (compare with (4.4.3.19))

$$\dot{\phi}_{av} = -\,\varepsilon \operatorname{cov}(w, \hat{m}_0(s)(w))\,\phi_{av} + \varepsilon\, g(\phi_{av}) \tag{4.4.4.17}$$

where $\hat{m}_0(s) := \hat{H}_{y_p\mu}(\theta_0, s)$, and $\phi = \theta - \theta_0$. Using the fact in (4.4.4.16), $\operatorname{cov}(w, \hat{m}_0(s)(w))$ and $g(\phi)$ can be approximated by

$$\begin{aligned}\operatorname{cov}(w, \hat{m}_0(s)(w)) &= \frac{1}{2\pi\theta_{01}} \sum_{i=1}^{n_1} \Big[\hat{H}_{wu}(\theta, j\omega_i)\, \hat{m}_0(j\omega_i)\, \hat{H}^*_{wu}(\theta, j\omega_i) \\ &\quad + \hat{H}_{wu}(\theta, -j\omega_i)\, \hat{m}_0(-j\omega_i)\, \hat{H}^*_{wu}(\theta, -j\omega_i)\Big]\, t_i^2 \\ &= M_\theta (D_{SM} + D_{SK}) M_\theta^T \end{aligned} \tag{4.4.4.18}$$

where

$$M_\theta = \Big[\hat{H}_{wu}(\theta, j\omega_1), \hat{H}_{wu}(\theta, -j\omega_1), \cdots, \hat{H}_{wu}(\theta, j\omega_{n_1}), \hat{H}_{wu}(\theta, -j\omega_{n_1})\Big] \tag{4.4.4.19}$$

$$\begin{aligned} D_{SM} = \operatorname{diag}\Bigg[&\begin{bmatrix} \frac{t_1^2}{2\pi\theta_{01}} \operatorname{Re} \hat{m}_0(j\omega_1) & 0 \\ 0 & \frac{t_1^2}{2\pi\theta_{01}} \operatorname{Re} \hat{m}_0(j\omega_1) \end{bmatrix}, \cdots \\ &\cdots, \begin{bmatrix} \frac{t_{n_1}{}^2}{2\pi\theta_{01}} \operatorname{Re} \hat{m}_0(j\omega_{n_1}) & 0 \\ 0 & \frac{t_{n_1}{}^2}{2\pi\theta_{01}} \operatorname{Re} \hat{m}_0(j\omega_{n_1}) \end{bmatrix} \Bigg] \end{aligned} \tag{4.4.4.20}$$

$$D_{SK} = \operatorname{diag}\Bigg[\begin{bmatrix} 0 & -\frac{t_1^2}{2\pi\theta_{01}} \operatorname{Im} \hat{m}_0(j\omega_1) \\ \frac{t_1^2}{2\pi\theta_{01}} \operatorname{Im} \hat{m}_0(j\omega_1) & 0 \end{bmatrix}, \cdots$$

$$\cdots, \begin{bmatrix} 0 & -\frac{t_{n_1}^{\ 2}}{2\pi\theta_{01}} Re\,\hat{m}_0(j\omega_{n_1}) \\ \frac{t_{n_1}^{\ 2}}{2\pi\theta_{01}} \mathrm{Im}\,\hat{m}_0(j\omega_{n_1}) & 0 \end{bmatrix}\Bigg]\Bigg] \tag{4.4.4.21}$$

and

$$g(\phi) \approx \frac{1}{2\pi} \sum_{i=1}^{n_1} \Big[\hat{H}_{wu}(\theta, j\omega_i)\,(\hat{m}_0(j\omega_i) - \hat{m}(j\omega_i))^*$$

$$+ \hat{H}_{wu}(\theta, -j\omega_i)\,(\hat{m}_0(-j\omega_i) - \hat{m}(-j\omega_i))^* \Big]\, t_i^2$$

$$= M_\theta\, g_v \tag{4.4.4.22}$$

where

$$g_v := \begin{bmatrix} \frac{t_1^2}{2\pi} \mathrm{Re}\,(\hat{m}_0(j\omega_1) - \hat{m}(j\omega_1)) \\ \vdots \\ \frac{t_{n_1}^{\ 2}}{2\pi} \mathrm{Im}\,(\hat{m}_0(j\omega_{n_1}) - \hat{m}(j\omega_{n_1})) \end{bmatrix} \in R^{2n_1} \tag{4.4.4.23}$$

and θ_{01} is the first element of θ_0, $\omega_i \in \Omega_n$, $i = 1, \cdots, n_1$, t_i^2 is the spectral magnitude of the ith frequency element ω_i. Obviously, the matrix D_{SM} is a negative definite matrix by the definition of Ω_n and D_{SK} is a skew symmetric matrix. Such an approximation simply implies that the low frequency elements do not contribute as large a driving force to the parameter $\theta(t)$ as the high frequency elements do. This is reasonable since in general there is a subspace of R^{2n} such that for each θ belonging to that subspace $\hat{m}_0(j\omega)$ stays close to $\hat{m}(j\omega)$ for $\omega \in \mathrm{Supp}(u) \cap \Omega_p$ (due to lack of richness in $\mathrm{Supp}(u) \cap \Omega_p$). Such a property however does not hold for $\omega \in \mathrm{Supp}(u) \cap \Omega_n$. Hence these elements present more significant time varying factors so that the overall cost function $J_g(\theta)$ can be minimized.

By Corollary 4.4.3.4, we make the following assumption for the analysis.

Assumption:

(A14) For all $\phi \in \Phi_{sc} := -\theta_0 + \Theta_{sc}$, M_θ is of full column rank if $n_1 \leq n$, and $M_\theta^T M_\theta$ is non-singular if $n_1 > n$.

Now we define $\phi_M := M_\theta \, \phi$. The averaged dynamics of ϕ_M can be approximated using (4.4.1.17) and the remark after (4.4.1.10) (i.e. $\frac{\partial}{\partial \theta} \hat{H}_{wu}(\theta, j\omega)$ is neglected for $\omega \in \Omega_n$) so that

$$\dot{\phi}_{Mav} \approx -\varepsilon M_{\theta_{av}}{}^T M_{\theta_{av}} (D_{SM} + D_{SK}) \phi_{Mav} - \varepsilon M_{\theta_{av}}{}^T M_{\theta_{av}} g_v \, . \tag{4.4.4.24}$$

Now we are ready to state a theorem that provides conditions under which the adaptive system will be unstable.

Theorem 4.4.4.1: (Instability Theorem Using Gradient Algorithm)

Consider the foregoing adaptive scheme and let θ_0 be given as before.

If there exists a sufficiently small $\delta > 0$ such that

$$\| g_{av} \| \leq \delta \tag{4.4.4.25}$$

then there exist $\varepsilon_5 > 0$, small $r_{in} > 0$, such that $B_{r_{in}} \subset \Phi_{sc}$, and $\phi(t)$ will eventually leaves Φ_{sc} whenever $\varepsilon \leq \varepsilon_5$, and $\phi(t_1) \in \Phi_{sc} \mid B_{r_{in}}$ for some $t_1 \geq t_0$. □

Proof: Consider a Lyapunov function $v(\phi_{Mav})$ for the averaged differential equation (4.4.4.24):

$$v(\phi_{Mav}) = \frac{1}{2} \, \phi_{Mav}^T \, (M_{\theta_{av}}{}^T M_{\theta_{av}})^{-1} \phi_{Mav} \, . \tag{4.4.4.26}$$

Since Φ_{sc} is a compact set, by assumption, there exist $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1 \| \phi_{Mav} \|^2 \le v(\phi_{Mav}) \le \alpha_2 \| \phi_{Mav} \|^2 \tag{4.4.4.27}$$

Moreover, D_{SM} is negative definite so that we define

$$\alpha_3 := \lambda_{\min}(-D_{SM}) \tag{4.4.4.28}$$

Differentiating v along the trajectories of (4.4.4.24), using (4.4.4.28) and the hypothesis, we have

$$\begin{aligned} \dot{v} &= -\varepsilon \phi_{Mav}^T D_{SM} \phi_{Mav} + \varepsilon \phi_{Mav}^T g_v \\ &\ge \varepsilon \| \phi_{Mav} \| \ (\alpha_3 \| \phi_{Mav} \| - \delta) . \end{aligned} \tag{4.4.4.29}$$

Using Corollary 2.3.2.2, and the fact that

$$\| \phi_M \| \ge \sqrt{\lambda_{\min}(M_\theta^T M_\theta)} \ \| \phi \| \ge \frac{1}{\alpha_2} \| \phi \| \tag{4.4.4.30}$$

the conclusion then follows.

■

Remarks:

(1) In words, the theorem says that, under the above assumptions, especially (A13)-(A14), the adaptive system will undergo a slow-drift instability if there exists $t_1 \ge t_0$ such that $\phi(t_1) = \theta(t_1) - \theta_0$ is large enough. Note that g_{av} involves the pseudo error $e_*(\theta_0, t)$, which is defined in (4.4.4.23). Naturally, if this error signal is small so that g_{av} is small, then the instability is more likely to occur. This will be possible if the spectral magnitudes of frequency elements in Supp(u) $\cap\ \Omega_n$, t_i^2, $i = 1, \ldots n_1$, are relatively small. Violation of this condition, however, does not imply that instability is unlikely to occur. On the contrary, if t_i^2, $1 \le i \le n_1$, are sufficiently large, though the above approximation will not hold true, $\mathrm{cov}(w, \hat{m}_0(s)(w))$ still fail to be positive definite (in fact, it will be negative indefinite) so that, by a linearization argument, the system remains unstable (cf. Fu and Sastry (1987)).

(2) One should note that θ_0 in this theorem does not have to be a point on the trajectory of $\theta(t)$. This however does not imply that there always there exists $t_1 \ge t_0$ such that the condition

in the theorem can be satisfied. Yet, what makes that condition more tractable is the following.

$$\hat{H}_{y_p u}(\theta,s) - \hat{m}_0(s) = \frac{1}{\theta_{01}} \hat{m}_0(s)(\phi^T \hat{H}_{wu}(\theta,s)) \tag{4.4.4.31}$$

Since, for all $\omega \in \text{Supp}(u) \cap \Omega_n$, $\hat{H}_{y_p u}(\theta(t),j\omega) \approx \hat{m}(j\omega)$. Note $\hat{H}_{y_p u}(\theta(t_0),j\omega)$ is just a value when $t=t_0$, which by no means indicates a function of time t, and during the process of optimization, $\hat{H}_{y_p u}(\theta(t),j\omega)$ has to assume different values for each t (i.e. to use the time varying effect). Consequently, it may be expected that there exists $t_1 \geq t_0$ such that

$$\phi^T \hat{H}_{y_p u}(\theta(t_1),j\omega) \not\approx 0 \qquad \omega \in \text{Supp}(u) \cap \Omega_n \tag{4.4.4.32}$$

which, in turn, implies a possibility that

$$\| \phi_M(t_1) \| \geq r' > 0 . \tag{4.4.4.33}$$

(3) In the proof, if the term:

$$\phi_{Mav}^T \left(\frac{d}{dt}(M_{\theta_{av}}{}^T M_{\theta_{av}})\right)\phi_{Mav}$$

can not be neglected, then the trajectories of $\theta(t)$ may undergo only a local instability but global boundedness.

(4) The compact subset Θ_{sc} may not be too small in the parameter space. Moreover, this set can be very close to the unstable manifold of $\theta(t)$ so that, whenever $\theta(t)$ leaves Θ_{sc}, it is likely to be attracted into the unstable manifold and the adaptive system suddenly turns into a drastically unstable stage.

(5) It can be seen that the instability can occur even when the spectral magnitudes of the element in $\text{Supp}(u) \cap \Omega_n$ are very small due to the fact that the t_i^2 appears in both D_{SM} and g_v. In other words, if the system fails to satisfy the positive definite PE condition, then the system is extremely sensitive to the frequency elements in Ω_n.

(6) This result can clearly be extended to the case when $\text{Supp}(u) \cap \Omega_p$ contains more than 2n points and/or the output disturbances exist, so long the system fails to satisfy positive definite

PE condition.

As for the least-squares with forgetting factor algorithm, we make similar observations, as before, in the following.

Assume that $(\hat{p}_u \hat{l})^{-1}(y_m)$ exists (i.e. u is sufficiently smooth), and then define the cost function $J_\alpha(\theta, t)$ by

$$J_\alpha(\theta, t) := \int_0^t e^{-\varepsilon\lambda(t-\tau)} \left[\theta^T \zeta(\tau) - (\hat{p}_u \hat{l})^{-1}(y_m)(\tau)\right]^2 d\tau \qquad \lambda > 0 \qquad (4.4.4.34)$$

which is a function of time t. Note that, since $y_p = (\hat{p}_u \hat{l})(\theta^T \zeta)$ in the modified scheme, $J_\alpha(\theta, t)$ is, in fact, an exponentially discounted cost function. Let the set Θ_{sc} be given as before and define the minimizer of $J_\alpha(\theta, t)$ at time t by

$$\theta_0(t) := \operatorname{argmin}\left\{ J_\alpha(\theta, t) \mid \theta \in \Theta_s \right\} \qquad (4.4.4.35)$$

Now we neglect the fact that ζ depends on θ just as in the previous case and solve the following optimality equation,

$$\frac{\partial}{\partial\theta} J_\alpha(\theta, t) \Big|_{\theta = \theta_0(t)} = 0 \qquad (4.4.4.36)$$

to get

$$\theta_0(t) = \left[\varepsilon \int_0^t e^{-\lambda\varepsilon(t-\tau)} \zeta(\tau) \zeta(\tau)^T d\tau\right]^{-1} \left[\varepsilon \int_0^t e^{-\varepsilon\lambda(t-\tau)} (\hat{p}_u \hat{l})^{-1}(y_m)(\tau) \zeta(\tau) d\tau\right]$$

$$= \bar{P}(t) \left[\int_0^t e^{-\varepsilon\lambda(t-\tau)} (\hat{p}_u \hat{l})^{-1}(y_m)(\tau) \zeta(\tau) d\tau\right] \qquad (4.4.4.37)$$

By differentiating (4.4.4.37), we have

$$\dot{\theta}_0(t) = \dot{\tilde{P}}\left[\int_0^t e^{-\varepsilon\lambda(t-\tau)}(\hat{p}_u\hat{l}\,)^{-1}(y_m)\,\zeta(\tau)\,d\tau\right]$$

$$+\,\varepsilon\tilde{P}(\hat{p}_u\hat{l}\,)^{-1}(y_m)\,\zeta - \varepsilon\lambda\,\theta_0(t) \tag{4.4.4.38}$$

and

$$\dot{\tilde{P}} = \varepsilon\lambda\tilde{P} - \varepsilon\tilde{P}\zeta\zeta^T\tilde{P} \tag{4.4.4.39}$$

so that the dynamics of $\theta_0(t)$ is given by

$$\begin{aligned}\dot{\theta}_0 &= -\,\varepsilon\tilde{P}\zeta\zeta^T\theta_0 + \varepsilon\tilde{P}(\hat{p}_u\hat{l}\,)^{-1}(y_m)\,\zeta \\ &= -\,\varepsilon\tilde{P}(\hat{p}_u\hat{l}\,)^{-1}(y_p - y_m)\,\zeta \\ &= -\,\varepsilon\tilde{P}(\hat{p}_u\hat{l})^{-1}(e_o)\,\zeta\,.\end{aligned} \tag{4.4.4.40}$$

It can be seen that (4.4.4.39)-(4.4.4.40) is very similar to our regular least-squares with forgetting factor algorithm:

$$\dot{\theta} = -\,\varepsilon P e_o \zeta \qquad\qquad \theta(0) = \theta_0 \tag{4.4.4.41}$$

$$\dot{P} = \varepsilon\lambda P - \varepsilon P\zeta\zeta^T P \qquad\qquad P(0) = I \tag{4.4.4.42}$$

except that $\tilde{P}$ is a steady state of P, and the error signal used in (4.4.4.40) is $(\hat{p}_u\hat{l}\,)^{-1}(e_o)$ rather than e_o used in (4.4.4.41).

Heuristically speaking, despite the above differences, for sufficiently small ε, the parameter update law given in (4.4.4.41)-(4.4.4.42) tends to optimize a cost function of the output error e_o. As such, it allows us to analyze the system using the same setup as before but with the following averaged version:

$$\dot{\phi}_{av} = -\,\varepsilon P_{av}\,\mathrm{cov}(\zeta, \hat{m}_0\hat{l}(\zeta))\,\phi_{av} - \varepsilon P_{av}\,g(\phi_{av}) \tag{4.4.4.43}$$

$$\dot{P}_{av} = \varepsilon\lambda P_{av} - \varepsilon P_{av} R_\zeta(\phi_{av}) P_{av} \tag{4.4.4.44}$$

The difference, however, lies in the positive definite PE condition. To suit the analysis in this case, we modify Definition 4.4.4.2 as follows.

Definition 4.4.4.3: (Positive Real Frequency Set, Negative Real Frequency Set)

Ω_p and Ω_n are called positive real frequency set and negative real frequency set respectively if:

$$\Omega_p = \left\{ \omega \mid \frac{1}{c_{\max}} \mathrm{Re}\hat{H}_{y_p\mu}(\theta, j\omega) - \frac{1}{2} > 0 \,,\, \theta \in \Theta_{sc} \,,\, c_{\max} := \max_{\theta \in \Theta_{sc}} c_0 \right\} \tag{4.4.4.45}$$

$$\Omega_n = \left\{ \omega \mid \frac{1}{c_{\min}} \mathrm{Re}\hat{H}_{y_p\mu}(\theta, j\omega) - \frac{1}{2} < 0 \,,\, \theta \in \Theta_{sc} \,,\, c_{\min} := \min_{\theta \in \Theta_{sc}} c_0 \right\}. \tag{4.4.4.46}$$

Here, in addition to (A13)-(A14), we will make an additional assumption to guarantee the invertibility of $R_\zeta(\phi)$.

Assumption:

(A15) Supp(u) contains at least $2n$ points so that $\hat{H}_{wu}(\theta, s)(u)$ is PE for all $\theta \in \Theta_{sc}$.

Using (4.4.4.17)-(4.4.4.18) and (4.4.4.22) we can approximate (4.4.4.43) by

$$\dot{\phi}_{av} \approx -\varepsilon P_{av} M_{\theta_{av}} (D_{SM} + D_{SK}) M_{\theta_{av}}{}^T \phi_{av} - \varepsilon P_{av} M_{\theta_{av}} g_v \tag{4.4.4.47}$$

where M_θ is as defined in (4.4.4.19); D_{SM}, D_{SK}, and g_v are as defined in (4.4.4.20)-(4.4.4.21) and (4.4.4.23) but with $\hat{m}_0$ replaced by $\hat{m}_0 \hat{\ell}$; $R_\zeta(\phi)$ is as defined in (4.4.3.32). Again, we define $\phi_M := M_\theta \phi$ and use the approximation as before (neglect the rate of change of M_θ with respect to time t) so that the averaged dynamics of ϕ_M is given approximately by

$$\begin{aligned} \dot{\phi}_{Mav} \approx & -\varepsilon M_{\theta_{av}}{}^T P_{av} M_{\theta_{av}} (D_{SM} + D_{SK}) M_{\theta_{av}}{}^T \phi_{av} \\ & -\varepsilon M_{\theta_{av}}{}^T P_{av} M_{\theta_{av}} g_v \,. \end{aligned} \tag{4.4.4.48}$$

Furthermore, (4.4.4.44) can also be changed into the following form, using the same approximation,

$$\frac{d}{dt}(M_{\theta_{av}}{}^T P_{av} M_{\theta_{av}}) = \varepsilon\lambda M_{\theta_{av}}{}^T P_{av} M_{\theta_{av}} - \varepsilon M_{\theta_{av}}{}^T P_{av} R_\zeta(\phi_{av}) P_{av} M_{\theta_{av}} \qquad (4.4.4.49)$$

The following theorem, similar to Theorem 4.4.4.1, will summarize the condition for instability.

Theorem 4.4.4.2: (Instability Theorem Using Least-Squares with Forgetting Factor Algorithm)

Consider the above setup of the adaptive control problem and let θ_0 be given as before.

If there exists a sufficiently small $\delta > 0$ such that

$$\| g_{av} \| \leq \delta \qquad (4.4.4.50)$$

then there exist $\varepsilon_6, \lambda_0 > 0$ and small $r_{in} > 0$ such that $B_{r_{in}} \subset \Phi_{sc}$, and $\phi(t)$ will eventually leaves Φ_{sc} whenever $\varepsilon \leq \varepsilon_6$, $\lambda \leq \lambda_0$, and $\phi(t_1) \in \Phi_{sc} \mid B_{r_{in}}$ for some $t_1 \geq 0$. □

Proof: The proof will be similar to that of Theorem 4.4.4.1. Consider a Lyapunov function $v(\phi_{Mav})$ for the averaged differential equation (4.4.4.47):

$$v(\phi_{Mav}, P_{av}) = \frac{1}{2}\, \phi_{Mav}^T \,(M_{\theta_{av}}{}^T P_{av} M_{\theta_{av}})^{-1} \phi_{Mav} \qquad (4.4.4.51)$$

As in the proof of Theorem 4.4.3.3, due to (A15), P_{av}^{-1} is bounded above and below so that, again from (A14) and the compactness of Φ_{sc}, there exist $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1 \| \phi_{Mav} \|^2 \leq v(\phi_{Mav}, P_{av}) \leq \alpha_2 \| \phi_{Mav} \|^2 \qquad (4.4.4.52)$$

whenever $\phi_{av} \in \Phi_{sc}$. As in (4.4.4.28), we define

$$\alpha_3 := \lambda_{\min}\left[- D_{SM} + \frac{1}{2}\,\mathrm{diag}\left(\frac{t_1^2}{2\pi}, \cdot, \cdot, \cdot, \frac{t_{n_1}{}^2}{2\pi}\right)\right] \qquad (4.4.4.53)$$

which, by (4.4.4.45), is a positive number. Differentiating v along the trajectories of (4.4.4.48) and (4.4.4.49), we have

$$\dot{v} = -\frac{\varepsilon\lambda}{2} v - \varepsilon \phi_{Mav}^T D_{SM} \phi_{Mav} + \varepsilon \phi_{Mav}^T g_v + \frac{\varepsilon}{2} \phi_{Mav}^T (M_{\theta_{av}}{}^T P_{av} M_{\theta_{av}})^{-1}$$

$$\cdot (M_{\theta_{av}}{}^T P_{av} R_\zeta(\phi_{av}) P_{av} M_{\theta_{av}}) (M_{\theta_{av}}{}^T P_{av} M_{\theta_{av}})^{-1} \phi_{Mav} \qquad (4.4.4.54)$$

Note that

$$D_{SM} = (M_{\theta_{av}}{}^T P_{av} M_{\theta_{av}})^{-1} M_{\theta_{av}}{}^T P_{av} \left[M_{\theta_{av}} D_{SM} M_{\theta_{av}}{}^T \right]$$

$$\cdot P_{av} M_{\theta_{av}} (M_{\theta_{av}}{}^T P_{av} M_{\theta_{av}})^{-1} \qquad (4.4.4.55)$$

and

$$R_\zeta(\phi) = M_\theta^T \operatorname{diag}\left(\frac{t_1^2}{2\pi}, \cdots, \frac{t_{n_1}{}^2}{2\pi}\right) M_\theta + \bar{R}_\zeta(\phi) \qquad (4.4.4.56)$$

where $\bar{R}_\zeta(\phi)$ is a positive semi-definite matrix. Using these and (4.4.4.53), we then can estimate $\dot{v}$ in (4.4.4.54) to get

$$\dot{v} \geq \varepsilon \| \phi_{Mav} \| \left((\alpha_3 - \frac{\lambda\alpha_2}{2}) \| \phi_{Mav} \| - \delta \right). \qquad (4.4.4.57)$$

Thus, the conclusion follows from the proof of Theorem 4.4.4.1, Corollary 2.3.2.3.

■

(ii) Input Error Scheme

Again we assume $d \equiv 0$, i.e. disturbance-free case. As before, the results derived under such a condition can be clearly extended to the case where $d \neq 0$. From (4.4.3.43)-(4.4.3.44), for fixed θ, we can solve e_2 in terms of v and y_m as

$$e_2 = \frac{c_0 \hat{m}^{-1} \hat{p}_u}{1 + c_0 \hat{m}^{-1} \hat{p}_u} (\theta^T v - (\hat{p}_u \hat{?})^{-1} (y_m)) \qquad (4.4.4.58)$$

where we assume that y_m (or: u) is sufficiently smooth such that the term $(\hat{p}_u \hat{?})^{-1}(y_m)$ exists. Define a cost function J_{ig} by

$$J_{ig}(\theta) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{c_0 \hat{m}^{-1} \hat{p}_u}{1 + c_0 \hat{m}^{-1} \hat{p}_u} \left| \theta^T \hat{H}_{vu}(\theta,s) - (\hat{p}_u \hat{l})^{-1} \hat{m} \right|^2 S_u(d\omega) \qquad (4.4.4.59)$$

Similarly, we neglect the dependence of v on θ, and assume that θ is already close to where $\theta^T v - (\hat{p}\hat{l})^{-1}(y_m)$ is small so that, for sufficiently small ε, the normalized gradient algorithm, under this scheme, approximately follows:

$$\dot{\theta} \approx -\varepsilon \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{c_0 \hat{m}^{-1} \hat{p}_u}{1 + c_0 \hat{m}^{-1} \hat{p}_u} \frac{\partial}{\partial \theta} \left| \theta^T \hat{H}_v(\theta,s) - (\hat{p}_u \hat{l})^{-1} \hat{m} \right|^2 S_u(d\omega)$$

$$\approx -\varepsilon \frac{\partial}{\partial \theta} J_{ig}(\theta) \,. \qquad (4.4.4.60)$$

This suggests that, if the trajectories $\theta(t)$ is already close to the minizers of $J_{ig}(\theta)$, then it will continue following the path of optimizing the cost function $J_{ig}(\theta)$. It then follows from (4.4.3.42) that $\theta(t)$ also follows the path of optimizing the output error e_o. This conclusion is the same as that for the gradient algorithm using output error scheme. Obviously, it is not hard to extend such observations to the normalized least-squares with forgetting factor algorithm, following the same arguments as before. Consequently, similar results for both algorithms should be expected.

Since the input error scheme does not require the SPR condition, The instability arise under a condition slightly different from that using output error scheme. Let θ_0 and the subset Θ_{sc}, Φ_{sc} be given as before. Recall from (4.4.3.47) that the averaged dynamics of the parameter ϕ, replacing θ_T by θ_0, is the following:

$$\dot{\phi}_{av} = -\varepsilon \operatorname{cov}(v \,, \hat{f}(s)(v))\, \phi_{av} + \varepsilon\, g(\phi_{av}) \qquad (4.4.4.61)$$

where we repeat $\hat{f}(s)$ here:

$$\hat{f}(s) = \frac{\dfrac{c_0}{\theta_{01}} \hat{m}(s)^{-1} \hat{m}_0(s)}{1 + \dfrac{\phi_0}{\theta_{01}} \hat{m}^{-1} \hat{m}_0(s)} \qquad (4.4.4.62)$$

(note that $\phi_0 = c_0 - \theta_{01}$). Motivated by this, we modify Definition 4.4.4.2 to suit our later analysis.

Definition 4.4.4.4: (Positive Real Frequency Set, Negative Real Frequency Set)

Ω_p and Ω_n are called positive real frequency set and negative real frequency set respectively if:

$$\Omega_p := \left\{ \omega \mid \mathrm{Re}\left(\frac{\hat{m}(j\omega)}{\hat{H}_{y_p u}(\theta, j\omega)}\right) > 1 - \frac{c_0}{c_{\max}} , \; \theta \in \Theta_{sc} , \; c_{\min} = \min_{\theta \in \Theta_{sc}} c_0 \right\} \tag{4.4.4.63}$$

$$\Omega_n := \left\{ \omega \mid \mathrm{Re}\left(\frac{\hat{m}(j\omega)}{\hat{H}_{y_p u}(\theta, j\omega)}\right) < 1 - \frac{c_0}{c_{\min}} , \; \theta \in \Theta_{sc} , \; c_{\max} = \max_{\theta \in \Theta_{sc}} c_0 \right\} \tag{4.4.4.64}$$

□

Remarks:

(1) Rewrite (4.4.4.62) as

$$\hat{f}(j\omega) = \left[1 + \frac{\theta_{01}}{c_0} \left(\hat{m}(j\omega)\, \hat{m}_0^{-1}(j\omega) - 1 \right) \right]^{-1} . \tag{4.4.4.65}$$

Then it follows that, for all $\omega \in \Omega_p$, $\mathrm{Re}\hat{f}(j\omega) > 0$, whereas, for all $\omega \in \Omega_n$, $\mathrm{Re}\hat{f}(j\omega) < 0$.

(2) In the absence of unmodelled dynamics, $\theta_0 = \theta^*$ so that

$$\mathrm{Re}\left(\frac{\hat{m}(j\omega)}{\hat{m}_0(j\omega)}\right) \equiv 1 \qquad \text{for all } \omega \in R . \tag{4.4.4.66}$$

Hence all frequencies ω will be classified as members of Ω_p. However, under the assumption of unmodelled dynamics, $\hat{m}_0(s)$ has an order higher than that of $\hat{m}(s)$, which implies that $\hat{m}_0(j\omega)$ will not be *in phase* with $\hat{m}(j\omega)$ when ω is too high. This implies that ω may no longer lie in the set Ω_p but more likely in the set Ω_n. Consequently, under the assumptions (A13), the signal vector v will then fail to satisfy the positive definite PE condition.

Now we start with the averaged version of gradient algorithm in (4.4.3.61), which, using (4.4.3.49)-(4.4.3.50) and (4.4.4.18)-(4.4.4.23), can be approximated by

$$\dot{\phi}_{av} = -\varepsilon M_{\theta_{av}}{}^T (D_{SM}+D_{SK}) M_{\theta_{av}} \phi_{av} - \varepsilon M_{\theta_{av}} g_v \tag{4.4.4.67}$$

where M_θ is as defined in (4.4.4.19) but with $\hat{H}_{wu}(\theta,s)$ being replaced by $\hat{H}_{vu}(\theta,s)$, and D_{SM} and D_{SK} are as defined in (4.4.4.20)-(4.4.4.21) but with $\hat{m}_0$ being replaced by $\hat{f}(s)$ defined in (4.4.4.62). Due to its extreme similarity with that in (4.4.4.17), we will only state the results as a corollary to Theorem 4.4.4.1.

Corollary 4.4.4.3: (Instability Using Normalized Gradient Algorithm)

Consider the setup of adaptive control problem as given in Theorem 4.4.4.1.

If there exists sufficiently small $\delta > 0$ such that

$$\| g_{av} \| \leq \delta \tag{4.4.4.68}$$

then there exist $\varepsilon_7 > 0$ and small $r_{in} > 0$, such that $B_{r_{in}} \subset \Phi_{sc}$, and $\phi(t)$ will eventually leaves Φ_{sc} whenever $\varepsilon \leq \varepsilon_7$, and $\phi(t_1) \in \Phi_{sc} \,|\, B_{r_{in}}$ for some $t_1 \geq t_0$. □

The proof is identical to that in Theorem 4.4.4.1 and, hence, is omitted here.

For normalized least-squares with forgetting factor algorithm shown in (4.4.3.54)-(4.4.3.55), the instability can also be studied through it averaged system given in (4.4.3.56)-(4.4.3.57) as above. However, the remark after Corollary 4.4.3.6, we see that the Definition 4.4.4.3 will not be adequate here. Hence we again modify definitions of sets Ω_p and Ω_n to suit this case.

Definition 4.4.4.5: (Positive Real Frequency Set, Negative Real Frequency Set)

Ω_p and Ω_n are defined by

$$\Omega_p := \left\{ \omega \mid \mathrm{Re}\left(\frac{\hat{m}(j\omega)}{\hat{H}_{y_p u}(\theta, j\omega)}\right) \le 1 + \frac{c_0}{c_{\max}}, \ \theta \in \Theta_{sc}, \ c_{\min} = \min_{\theta \in \Theta_{sc}} c_0 \right\} \tag{4.4.4.69}$$

$$\Omega_n := \left\{ \omega \mid \mathrm{Re}\left(\frac{\hat{m}(j\omega)}{\hat{H}_{y_p u}(\theta, j\omega)}\right) \ge 1 + \frac{c_0}{c_{\min}}, \ \theta \in \Theta_{sc}, \ c_{\max} = \max_{\theta \in \Theta_{sc}} c_0 \right\} \tag{4.4.4.70}$$

□

Remark: The definition implies that, for all $\omega \in \Omega_p$, $\mathrm{Re}\hat{f}(j\omega) - 1/2 > 0$ and, for all $\omega \in \Omega_n$, $\mathrm{Re}\hat{f}(j\omega) - 1/2 < 0$.

Corollary 4.4.4.4: (Instability Using Normalized Least-Squares with Forgetting Factor Algorithm)

Consider the setup of the adaptive control problem as given in Theorem 4.4.4.2.

If there exists a sufficiently small $\delta > 0$ such that

$$\| g_{av} \| \le \delta \tag{4.4.4.71}$$

then there exist $\varepsilon_8, \lambda_0 > 0$ and small $r_{in} > 0$, such that $B_{r_{in}} \subset \Phi_{sc}$, and $\phi(t)$ will eventually leaves Φ_{sc} whenever $\varepsilon \le \varepsilon_8$, $\lambda \le \lambda_0$, and $\phi(t_1) \in \Phi_{sc} \mid B_{r_{in}}$ for some $t_1 \ge t_0$. □

The proof is identical to that of Theorem 4.4.4.2 and, hence, is omitted here.

4.5 Concluding Remarks

In this chapter, we reviewed the output error model reference direct adaptive scheme introduced by Narendra and Valavani (1978) and Narendra, Lin, and Valavani (1980), and the input error scheme introduced by Bodson and Sastry (1987). The two schemes examined here basically use the same structure for the controller but a different one for the parameter update laws. Identical properties shared by both schemes are summarized here: (i) the closed loop system remains stable and the output of the plant converges to that of the reference model (ii) if the signal vector w is PE, then the controller parameters converge to the true parameters exponentially. At this point, the input error scheme benefits owing to the relaxation of SPR condition, which is required in the output error scheme. A price, however, is paid in that the upper bound of the high frequency gain has to be known (assume $k_p > 0$).

Under the assumption that the control inputs are stationary signals, the adaptive system is categorized as a mixed-time scale system which is suitable for averaging analysis. By applying averaging results developed in Chapter 2, we obtain estimates of rates of parameter convergence. Results, here, are similar to those for the adaptive identifiers, and an example, using gradient algorithm, is given to illustrate the closeness of these approximations. Two by-products follow from this analysis: (i) the notion of optimizing rates of parameter convergence by choosing SR control input u, subject to some constrain, such that the smallest eigenvalue of $SM(R_{ww_f}(0))$ (tail convergence) is maximized, (ii) the notion of relaxation of SPR condition for establishing the stability proof. The former, later, evolves into a basic principle for synthesizing an optimal input u in Chapter 5, whereas the latter allows one to cope with the non-ideal case where unmodelled dynamics and output disturbances are brought in.

Due to the inevitable existence of unmodelled dynamics and/or bounded output disturbance associated with the plant, properties of model reference adaptive control schemes generally will no longer hold and the stability of the system is likely be at stake. Assuming FDLT-I unmodelled dynamics, results from Mason et al (1987) were extended to this control case so that the PE condition of w can still be related to the sufficient richness of the input u in an "almost always" fashion. These results then allow us to define tuned parameters and a tuned model through a frequency-domain interpretation, i.e. tuned parameters, θ_T, are the fixed values of the controller parameters θ that minimize the normalized mean squared power of tuned error, the output error between reference model and closed loop plant. Stability analysis will then be focused on the trajectories of controller parameters around tuned parameters. Under the assumptions that: (i) reference model is close enough to the tuned model, (ii) $SM(R_{ww_f}(\phi))$, where $\phi=\theta-\theta_T$ belongs to a compact set, is sufficiently positive definite, and (iii) the parameter adaptation is slow enough, the controller parameters will converge to a ball centered at the tuned parameters with a radius which is a sum of a class K function of the adaptation gain and a linear function of the mean squared power of tuned error. The conclusion on stability of the adaptive system and performance of the controller can readily be drawn.

While the reduced order controller works in the face of unmodelled dynamics and output disturbances under some benign assumptions, the crucial dependence on the positive definite PE condition should be emphasized. To substantiate the importance of this condition, a type of instability---slow-drift instability---resulting from the violation of that condition was under studied rigorously. Results there show that controller parameters will slowly drift, in the parameter space, out of a compact subset in which the tuned error is small, when the adaptation is slow enough and the spectral energy of the control input is not dominantly SR in the "low" frequency spectrum. In practice, when the adaptive system undergoes a slow-drift instability, the controller parameters, after they leave the compact subset, are very likely to be attracted into the unstable manifold so that the system will be driven into instability.

Chapter 5 Frequency-Domain Synthesis of Optimal Inputs for Adaptive Systems

5.1 Problem Statement

In this section, we formulate the input design problem of choosing proper inputs for use in SISO adaptive identification and model reference adaptive control schemes in the absence of unmodelled dynamics. Characterization of the optimal inputs is given in the frequency domain and is arrived at through the use of *averaging theory*. It has been shown in section 3.3 and 4.3 that exponential parameter convergence can be obtained in both adaptive identifiers and model reference adaptive controllers provided that the signal vector w defined in equations (3.2.6) and (4.2.1.2) is persistently exciting (PE). From the averaging analysis in section 3.3 and 4.3, we see that, when the adaptation gain is small enough, the dynamics of parameter evolution can be approximated by their averaged version, whose rates of convergence are especially easy to study. As has been pointed out in the conclusions of Chapter 3 and 4, these estimates suggest a means of optimizing the rates of parameter convergence.

We will primarily be interested in the "tail" rate of convergence and focus attention on the normalized gradient algorithm.

5.1.1 Adaptive Identifier

We consider an unknown plant, described by an SISO proper, stable transfer function,

$$\hat{p}(s) = k_p \frac{\hat{n}_p(s)}{\hat{d}_p(s)} \tag{5.1.1.1}$$

where $\hat{n}_p(s)$, $\hat{d}_p(s)$ are coprime monic polynomials, and $\hat{d}_p(s)$ is of a known degree n. The adaptive identifier of this plant has the same structure as the one discussed in section 3.2. The gradient algorithm, using averaging, is given by

$$\dot{\phi} = -\varepsilon e_o w \tag{5.1.1.2}$$

for some small $\varepsilon>0$ where $\phi=\theta-\theta^*$ is the parameter error, and $e_o=y_o-y_p$ is the output error.

The input design problem for an adaptive identifier is that of selecting an input u from an allowable class of signals (to be specified by the designer) in order that the rate of convergence of the parameter error ϕ may be optimized. There are various possible solutions to this problem. The solution pursued here is a frequency domain approach obtained by applying averaging theory to the parameter update law (5.1.1.2). It is shown in subsection 3.3.1 that bounds on the rate of parameter convergence can be assessed by studying the matrix $R_w(0)$ defined by

$$R_w(0) = \lim_{T\to\infty} \frac{1}{T} \int_s^{s+T} w(t)w(t)^T dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{Q}(j\omega)\,\hat{Q}^*(j\omega)\, S_u(d\omega) \tag{5.1.1.3}$$

when w is PE, where $\hat{Q}(s)$ is the transfer function given in (3.3.6). The bound on the rate of parameter convergence is simply $\varepsilon\lambda_{\min}(R_w(0))+o(\varepsilon)$ when ε is small enough.

The input design problem can therefore be put in the form of an optimization problem in which an input u is to be chosen from a class of signals to maximize the smallest eigenvalue of the average information matrix $R_w(0)$. Such a procedure is very reminiscent of the procedure indicated in Fedorov (1972), Goodwin and Payne (1977), and Mehra (1976) in the stochastic context for the design of input signals in parameter estimation. There, however, the objective is to achieve a greater accuracy of parameter estimates rather than a higher rate of parameter convergence.

Remark: Although averaging requires the adaptation gain ε be small enough so that the approximation will be close, in its application to adaptive identification, ε does not have to be too small. In fact, the estimates are reasonably close even for values of ε close to one. This fact can be seen from the simulation example given in subsection 3.3.1. Consequently, the optimal inputs obtained based on this criterion provide a reasonably fast parameter convergence rate.

5.1.2 Model Reference Adaptive Controller

In this subsection, we examine the optimal input design problem for both the input and output error direct model reference adaptive control schemes discussed in Chapter 4. As pointed out at the beginning of this section, the rate of parameter convergence of the linearized version of the adaptive system is the main focus here.

We consider an SISO plant with transfer function,

$$\hat{p}(s) = k_p \frac{\hat{n}_p(s)}{\hat{d}_p(s)} \tag{5.1.2.1}$$

where $\hat{n}_p(s)$ and $\hat{d}_p(s)$ are monic coprime polynomials of degree m and n respectively and k_p is the high frequency gain, satisfying assumptions (A1)-(A3) in section 4.2. The reference model is described by

$$\hat{m}(s) = k_m \frac{\hat{n}_m(s)}{\hat{d}_m(s)} \tag{5.1.2.2}$$

where $\hat{n}_m(s)$ and $\hat{d}_m(s)$ are monic but not necessarily coprime polynomials of degree m and n respectively (the same degrees as the corresponding plant polynomials), satisfying assumptions (A4)-(A5) in section 4.2. The controller structure for model reference direct adaptive control schemes is the same as the one discussed in section 4.2 as well. As has been shown in section 4.3, exponential parameter convergence can be achieved when the system is PE. Therefore the input design problem for a model reference adaptive controller, similar to that of an adaptive identifier, is to select an input u from a class of signals so as to optimize the rate of parameter convergence. As before, a frequency-domain approach through the application of averaging is the method adopted to solve the problem. In the following, we will have separate discussions on output and input error schemes.

5.1.2.1 Output Error Scheme

Using (4.2.1.9)-(4.2.1.10) and (4.3.1.1), as indicated above, the dynamics of state errors e of the adaptive system are linearized around $(e,\phi)=(0,0)$ to obtain

$$\dot{e} = A_m e + b_m \phi^T w_m \tag{5.1.2.3}$$

$$e_o = c_m^T e \,. \tag{5.1.2.4}$$

Here, for illustration, we will only consider the case where the relative degree of the plant is one. As pointed out in the remark before subsection 4.3.2, the case where the relative degree of the plant is greater than one can be dealt with similarly if one is only interested in the first $2n$ parameters of θ.

The parameter update law using gradient algorithm, under averaging, can also be linearized around $(e,\phi)=(0,0)$ to yield

$$\dot{\phi} = -\varepsilon e_o w_m \,. \tag{5.1.2.5}$$

It follows from the analysis in subsection 4.3.1 that estimates of the rate of parameter convergence of the above linearized system can be obtained by studying the matrix $SM(R_{w_m w_{mf}}(0))$ which, from (4.3.1.28), is given by

$$SM(R_{w_m w_{mf}}(0)) := \frac{1}{2\pi c_0^*} \int_{-\infty}^{\infty} \hat{Q}_m(j\omega)\ \mathrm{Re}\,\hat{m}(j\omega)\ \hat{Q}_m^*(j\omega)\ S_u(d\omega) \tag{5.1.2.6}$$

when w_m is PE, where $\hat{Q}_m$ is defined in (4.3.1.7). The bound on the rate of parameter convergence is simply $\varepsilon\left[\lambda_{\min} SM(R_{w_m w_{mf}}(0))\right] + o(\varepsilon)$ for sufficiently small ε.

The estimate of the bound is extremely similar to that of adaptive identifiers. This, again, allows us to formulate the input design problem here as an optimization problem in which an input u is chosen (subject to some constraints) so as to maximize the smallest eigenvalue of an symmetric positive definite matrix, just as the case of adaptive identifiers.

Remark: The comments on the smallness of ε stated in the remark in subsection 5.1.1 also apply here.

5.1.2.2 Input Error Scheme

The linearized version of the dynamics of state error e is the same as that given in (5.1.2.3)-(5.1.2.4). Since we are only concerned with the behavior of the system for small (e,ϕ), the projection mechanism used in the parameter update will not be necessary.

The parameter update law using normalized gradient algorithm, under averaging, takes the following form:

$$\dot{\phi} = -\varepsilon \frac{v_m v_m^T}{1+\varepsilon v_m^T v_m} \phi \tag{5.1.2.7}$$

where $v_m := \hat{r}^{-1}(s)(w_m)$. It is shown in subsection 4.3.2 that the bound on the rate of parameter convergence, similar to that of adaptive identifiers, is $\varepsilon \lambda_{\min}(R_{v_m}(0)) + o(\varepsilon)$ where

$$R_{v_m}(0) := \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{r}^{-1}(j\omega)|^2 \hat{Q}_m(j\omega) \hat{Q}_m^*(j\omega)\, S_u(d\omega) \tag{5.1.2.8}$$

when w_m is PE and ε is small enough.

In the same way, we can put the input design problem in the form of an optimization problem where the objective is to maximize the smallest eigenvalue of a symmetric positive definite matrix among a class of input signals.

5.2 Input Design Bases

In section 5.1, we see that the input design problem of an adaptive system can be formulated in terms of the optimization of the smallest eigenvalue of a positive definite symmetric matrix over a class of input signals. In this section, we make the problem more tractable by choosing the class of input signals to be *power-constrained*, by which we roughly mean that the average power of a signal $i(t) \in R$, defined as

$$\lim_{T \to \infty} \frac{1}{T} \int_{s}^{s+T} i^2(t)dt \tag{5.2.1}$$

with limit existing uniformly in $s \geq 0$, can be no greater than a fixed amount. In the following, more detailed definitions, based on Definition 3.3.2 and 3.3.3, are introduced to facilitate later development of the input design algorithm.

Definition 5.2.1: (Normalized Input Design (NID) over a Compact Support)

A normalized input design (NID) is defined on the spectral distribution function $F_u(\omega)$ which satisfies

$$\frac{1}{2\pi} F_u(\infty) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_u(d\omega) = 1 \,. \tag{5.2.2}$$

Let Ω be a compact subset in the frequency spectrum, symmetric about zero frequency, and $\omega_0 := \mathrm{argmax}\ \{\ \omega \mid \omega \in \Omega\ \}$. Then F_u is said to be a NID over Ω if

$$F(-\omega_o^-) = 0 \qquad\qquad F(\omega_o) = 1 \,. \tag{5.2.3}$$

□

Remarks:

(1) Note that the spectral function $F(\omega)$ can be identified with a positive measure, i.e.

$$\int_{\Omega} S(d\omega) = \int_{\Omega} dF(\omega) \tag{5.2.4}$$

In the sequel, we will frequently use $N(\Omega)$ to denote the set of all NID's over Ω, which can be expressed concisely as

$$N(\Omega) = \left\{ F \;\middle|\; F : \text{positive measure} \,,\ \frac{1}{2\pi} \int_{\Omega} dF(\omega) = 1 \right\}. \tag{5.2.5}$$

In particular, $N_D(\Omega)$, denoting a subset of $N(\Omega)$, contains all NID's with only discrete spectrum in Ω.

(2) In practice, stationary signals encountered in adaptive systems are usually bandlimited. In other words, it will suffice to choose Ω as the set $[-\omega_0, \omega_0]$ where $\omega_0 > 0$.

(3) $N(\Omega)$ is a convex set due to the fact that

$$(1-\alpha)F_1 + \alpha F_2 \in N(\Omega) \tag{5.2.6}$$

for all $F_1, F_2 \in N(\Omega)$, and $\alpha \in [0,1]$.

Definition 5.2.2: (**Normalized Average Information Matrix (NAIM)**)

A matrix G is said to be a normalized average information matrix if there exists a proper stable column transfer function $\hat{H} : C \to C^m$, a scalar strictly positive even function $g: R \to R_+$, and $F \in N(\Omega)$ such that

$$G = \frac{1}{2\pi} \int_{\Omega} g(\omega)\, \hat{H}(j\omega)\, \hat{H}^*(j\omega)\, dF(\omega) \tag{5.2.7}$$

Then, given $\hat{H}$ and g, such a matrix G will be denoted as $G(F)$ for the emphasis of its dependence on the specific NID F. □

Remarks:

(1) As indicated above, NAIM is always symmetric and at least positive semi-definite. The invertibility of such matrices relies on two factors: the structure of $\hat{H}(s)$ and the frequency support of F. Here, to suit our purposes, we will assume that the transfer function $\hat{H}$ is always such that, for some $F \in N(\Omega)$, the corresponding NAIM is invertible and, hence, positive definite. Also notice that such a condition is always satisfied in the nominal adaptive system.

(2) Again, in the sequel, we will use $M_{(\hat{H},g)}(\Omega)$ to denote the set of all NAIM's resulting from the column transfer function $\hat{H}(s)$, the scalar function $g(\omega)$, and all possible F's in $N(\Omega)$. It can be concisely expressed as

$$M_{(\hat{H},g)}(\Omega) = \left\{ G \;\middle|\; G = \frac{1}{2\pi} \int_{\Omega} g(\omega)\ \hat{H}(j\omega)\ \hat{H}^{*}(j\omega)\ d\mathrm{F}(\omega)\ ,\ \mathrm{F} \in N(\Omega) \right\}. \tag{5.2.8}$$

(3) In the special case where $\mathrm{F}(\omega)$ results from a single frequency sinusoidal input with frequency ω^{*}, the corresponding NAIM will be called the *point-input information matrix* (PIIM) and denoted as $G(\omega^{*})$.

For convenience, we will introduce the following notation.

Notation:

A function $\lambda_G : N(\Omega) \rightarrow R_{+}$ stands for the smallest eigenvalue of G, a NAIM, resulting from some $\mathrm{F} \in N(\Omega)$.

In the stochastic context, there are several parallels of these definitions. In fact, in the stochastic literature, the average information matrix that we used here is often referred to as the *Fisher information matrix* and is related to the error covariance matrix. To date, there are several existing results regarding the Fisher information matrix. Due to similarities between these matrices, in the following, we will only state some lemmas with proofs omitted (cf. Goodwin and Payne (1977), chapter 6, and Mehra (1976)).

Lemma 5.2.1: (Closed Convex Hull)

The set $M_{(\hat{H},g)}(\Omega)$ is the closed convex hull of all PIIM's corresponding to the same $\hat{H}$ and g, i.e.

$$M_{(\hat{H},g)}(\Omega) = Co \left\{ G(\omega) \;\middle|\; \omega \in \Omega \right\} \tag{5.2.9}$$

□

Lemma 5.2.2: (Matching Lemma)

For any $F_1 \in N(\Omega)$ with corresponding $G(F_1) \in M_{(\hat{\theta},g)}(\Omega)$, there always exists a $F_2 \in N_D(\Omega)$ containing no more than $\frac{m(m+1)}{2}+1$ distinct frequency elements ($m(m+1)+2$ spectral lines) such that

$$G(F_1) = G(F_2) \tag{5.2.10}$$

□

Lemma 5.2.3: (Optimizing Lemma)

The optimal normalized input design $F^* = \text{argmax} \{ \lambda_G(F) \mid F \in N(\Omega) \}$ exists, and contains no more than $\frac{m(m+1)}{2}$ distinct frequency elements (i.e. one less than that predicted by Lemma 5.2.2). □

Important Remark:

One can infer from Lemma 5.2.3 that, while designing optimal inputs for maximizing the smallest eigenvalue of the average information matrix, one can confine the search to sinusoidal inputs with only a finite number of frequencies.

5.3 Sequential Design Algorithm

In this section, we first derive some basic results on $\lambda_G(F)$ (the smallest eigenvalue of $G(F)$) using perturbation theory. Based on these results, a numerical algorithm for input design will be constructed later. The sequence $\{ \lambda_G(F^i) \}$ generated by the algorithm is then proved to converge to its global maximum.

Theorem 5.3.1: (Equivalence Theorem)

Consider some $F^* \in N(\Omega)$. Let $\lambda_G(F^*)$ be the smallest eigenvalue of $G(F^*)$ and v_i, $i = 1, \cdots, \gamma$ be the orthonormal eigenvectors associated with it.

Then the following three statements are equivalent.

$$(a)\ F^* = \operatorname{argmax} \left\{ \lambda_G(F) \mid F \in N(\Omega) \right\} \tag{5.3.1}$$

$$(b)\ \text{For all } F^o \in N(\Omega)\text{, with } \quad F^\alpha := (1-\alpha)F^* + \alpha F^o, \quad \alpha \in [0,1] \tag{5.3.2}$$

$$\frac{\partial}{\partial \alpha} \lambda_G(F^\alpha) \Big|_{\alpha=0} \leq 0 \tag{5.3.3}$$

$$(c)\ \lambda_G(F^*) \geq \underline{\sigma}_{max} \tag{5.3.4}$$

where

$$\underline{\sigma}_{max} = \max \left\{ \lambda(P^T G(F) P) \mid F \in N(\Omega) \right\} \tag{5.3.5}$$

and $P := \left[v_1, \cdots, v_\gamma \right]$. □

Proof: The way we proceed in the proof is to show (a) (b) are equivalent and then (b) (c) are equivalent.

(i) First of all, note that from (5.3.2)

$$G(F^\alpha) = (1-\alpha) G(F^*) + \alpha G(F^0) \tag{5.3.6}$$

and that, by perturbation theory, the smallest eigenvalue satisfies

$$\lambda_G(F^\alpha) = (1-\alpha) \lambda_G(F^*) + \alpha \underline{\sigma} + o(\alpha) \tag{5.3.7}$$

when α is small, where $\underline{\sigma}$ is defined by

$$\underline{\sigma} = \lambda(P^T G(F^0) P) \tag{5.3.8}$$

and P is the same as given above. It then follows that (a) implies (b) trivially. To show that (b) implies (a), we use a contradiction.

Suppose (b) is true but that there exists a $\hat{F} \neq F^*$ such that

$$\lambda_G(\hat{F}) > \lambda_G(F^*) \,. \tag{5.3.9}$$

Define F^α as

$$F^\alpha = (1-\alpha)F^* + \alpha\hat{F} \qquad\qquad \alpha \in [0,1] \,. \tag{5.3.10}$$

Then

$$G(F^\alpha) = (1-\alpha)\,G(F^*) + \alpha G(\hat{F}) \tag{5.3.11}$$

and its smallest eigenvalue satisfies

$$\lambda_G(F^\alpha) = (1-\alpha)\,\lambda_G(F^*) + \alpha\underline{\sigma} + o(\alpha) \tag{5.3.12}$$

when α is small, where $\underline{\sigma}$ is defined by

$$\underline{\sigma} = \underline{\lambda}(P^T G(\hat{F})\,P) \,. \tag{5.3.13}$$

Since, by definition, v_i , $i = 1, \cdots, \gamma$ are orthonormal vectors, one can easily show that

$$\underline{\sigma} \geq \lambda_G(\hat{F}) \,. \tag{5.3.14}$$

Further, with eq. (5.3.12) one can establish the following:

$$\frac{\partial}{\partial\alpha}\lambda_G(F^\alpha)\bigg|_{\alpha=0} = \underline{\sigma} - \lambda_G(F^*) \tag{5.3.15}$$

which along with (5.3.9) and (5.3.14) gives a contradiction. Hence, the implication is valid.

(ii) $(c) \Rightarrow (b)$

By hypothesis and definition of $\underline{\sigma}_{max}$, we have

$$\lambda_G(F^*) \geq \underline{\sigma}(F^0) \qquad\qquad \text{for all } \; F^0 \in N(\Omega) \tag{5.3.16}$$

where

$$\underline{\sigma}(F^0) = \underline{\lambda}(P^T G(F^0)\,P) \,. \tag{5.3.17}$$

With definition of F^α in (5.3.2), (5.3.16) then implies that

$$\frac{\partial}{\partial\alpha}\lambda_G(\mathrm{F}^\alpha)\Big|_{\alpha=0} = \underline{\sigma} - \lambda_G(\mathrm{F}^*) \leq 0 \tag{5.3.18}$$

(*b*) => (*c*)

This is more obvious to see since if $\underline{\sigma}_{max} > \lambda_G(\mathrm{F}^*)$, then there exists $\tilde{\mathrm{F}} \in N(\Omega)$ and F^α defined by

$$\mathrm{F}^\alpha = (1-\alpha)\mathrm{F}^* + \alpha\tilde{\mathrm{F}} \qquad \alpha \in [0,1] \tag{5.3.19}$$

such that

$$\frac{\partial}{\partial\alpha}\lambda_G(\mathrm{F}^\alpha)\Big|_{\alpha=0} = \underline{\sigma}(\tilde{\mathrm{F}}) - \lambda_G(\mathrm{F}^*) > 0\ . \tag{5.3.20}$$

Consequently, (b) and (c) are equivalent.

■

Remark: In the theorem, finding $\underline{\sigma}_{max}$ is generally less complex than finding $\lambda_{max} := \{ \lambda_G(\mathrm{F}) \mid \mathrm{F} \in N(\Omega) \}$ due to the smaller dimension of $P^T G(\mathrm{F}) P$ ($\gamma\times\gamma$ rather than the original $m\times m$ where $\gamma \leq m$). In fact, the most common and simplest case is where P consists of single vector, whence $P^T G(\mathrm{F}) P$ becomes purely a scalar. Thus, by Lemma 5.3.3, it is sufficient to compute $\underline{\sigma}_{max}$ by just using a one-line search optimization routine, i.e.

$$\underline{\sigma}_{max} = \max_{\omega \in \Omega} P^T G(\omega) P\ . \tag{5.3.21}$$

In general, the computation of optimal input designs, except in very simple cases, has to be done numerically. The following numerical algorithm will provide a tool by which such optimal input designs can be sought. The next theorem will show that this numerical algorithm will generate a convergent sequence $\{ \lambda_{min}(\mathrm{F}^i) \}$ whose limit point is the global optimum. By Lemma 5.3.2 and 5.3.3, this also shows that the sequence $\{ \mathrm{F}^i \}$ will converge to one of the global optima. Before we proceed, we start with some more notation for simplification in the sequel.

Notation:

(i) $N_D^k(\Omega)$, denoting a subset of $N_D(\Omega)$, contains NID's F_u whose frequency support consists of no more than $2k$ points in Ω (or u consists of no more than k sinusoidal components).

(ii) $P_i = \left[v_{i,1}, \cdots, v_{i,\gamma_i} \right]$ consists of orthonormal eigenvectors of $G(F^i)$ associated with the smallest eigenvalue $\lambda_G(F^i)$.

(iii) $\sigma^i_{max} = \max \{ \lambda(P_i^T G(F) P_i) \mid F \in N_D^{k_i}(\Omega) \}$ where $k_i = \gamma_i(\gamma_i+1)/2$.

Sequential Design Algorithm:

Data: $F^0 \in N_D^{k_0}(\Omega)$ is a feasible initial design.

Step 1: Set i = 0.

Step 2: Compute $\lambda_G(F^i)$ and find σ^i_{max}.

Step 3: If $\sigma^i_{max} \leq \lambda_G(F^i)$, then stop; else go to Step 4.

Step 4: Update the input design F^i by:

$$F^{i+1} = (1-\alpha_i)F^i + \alpha_i \hat{F}^i \qquad \alpha_i \in [0,1] \tag{5.3.22}$$

where $\hat{F}^i \in N_D^{k_i}(\Omega)$ is such that

$$\sigma^i_{max} = \lambda_G(P_i^T G(\hat{F}^i) P_i) . \tag{5.3.23}$$

Step 5: i = i + 1 and go to Step 2.

Remark: In Step 2, the procedure of finding σ^i_{max} is exactly the same as that of finding $\lambda_G(F^*)$, i.e. to go through Step 2 to Step 5 with some feasible initial design $F^{i_0} \in N_D^{k_i}(\Omega)$.

Theorem 5.3.2: (Convergence Theorem)

Consider the sequential design algorithm.

If the sequence $\{ \alpha_i \}$ is chosen such that

$$(a) \qquad \lim_{i \to \infty} \alpha_i = 0 \qquad \sum_{i=1}^{\infty} \alpha_i = \infty \qquad \alpha_i \in (0,1) \tag{5.3.24}$$

or

$$(b) \qquad \alpha_i = \mathrm{argmax} \; \{ \lambda_G((1-\alpha)F^i + \alpha \hat{F}^i) \mid \alpha \in [0,1] \} \tag{5.3.25}$$

then either the numerical algorithm terminates in some number of finite steps, say, k^* and

$$\lambda_G(F^{k^*}) = \lambda_G(F^*) \tag{5.3.26}$$

or

$$\lambda_G(F^i) \to \lambda_G(F^*) \qquad \text{as } i \to \infty \tag{5.3.27}$$

where F^* is an optimal input design as defined in (5.3.1). $\square$

Proof: If the algorithm terminates in step $k^* < \infty$, then by Equivalence Theorem and the algorithm itself, we can readily conclude that (5.3.26) is true. On the other hand, if the algorithm does not stop in finite steps, then the following proof will show that (5.3.27) is the case.

(a) Instead of showing that (5.3.27) is true, equivalently, we show that

$$\frac{\partial}{\partial \alpha} \lambda_G(F^{\alpha}_{(i)}) \Big|_{\alpha=0} \leq 0 \qquad \text{as } i \to \infty \tag{5.3.28}$$

where

$$F^{\alpha}_{(i)} := (1-\alpha) F^i + \alpha \hat{F}^i \tag{5.3.29}$$

as a result of Equivalence Theorem. Assume the contrary, i.e.

$$\frac{\partial}{\partial \alpha} \lambda_G(F^{\alpha}_{(i)}) \Big|_{\alpha=0} = \Delta > 0 \qquad \text{for all } i \geq 0 \,. \tag{5.3.30}$$

This implies

$$\lim_{i\to\infty} [\, \lambda_G(F^i) - \lambda_G(F^0) \,] = (\sum_{i=1}^{\infty} \alpha_i)\, \delta(\Delta) \tag{5.3.31}$$

where $\delta(\Delta) > 0$, which contradicts that $\{ \lambda_G(F^i) \}$ is a bounded sequence.

(b) By the Equivalence theorem, if F^i is not the optimal input design for all $i \in Z_+$, then

$$\frac{\partial}{\partial\alpha} \lambda_G(F^{\alpha}_{(i)}) \Big|_{\alpha=0} > 0 \tag{5.3.32}$$

which then implies that $\{ \lambda_G(F^i) \}$ is a monotonically increasing sequence which is bounded above. Hence, the sequence converges to a limit, say, $\lambda_G(\overline{F})$. We now show that $\lambda_G(\overline{F}) = \lambda_G(F^*)$, where F^* is assumed to be an optimal input design.

Assume a contradiction, i.e. $\lambda_G(\overline{F}) \neq \lambda_G(F^*)$. Again, by Equivalence Theorem, the gradient

$$\frac{\partial}{\partial\alpha} \lambda_G(F^{\alpha})\, |_{\alpha=0} = \Delta > 0 \tag{5.3.33}$$

where F^{α} is defined as

$$F^{\alpha} := (1-\alpha)\overline{F} + \alpha F \tag{5.3.34}$$

for some $F \in N(\Omega)$. This, in turn, implies

$$\lim_{i\to\infty} [\, \lambda_G(F^i) - \lambda_G(F^{i-1}) \,] = \delta(\Delta) > 0 \tag{5.3.35}$$

which contradicts the fact that the sequence converges. As a consequence,

$$\lim_{i\to\infty} \lambda_G(F^i) = \lambda_G(F^*) \,. \tag{5.3.36}$$

■

Remark: In fact, the numerical computation will usually stop in a finite number of iterations after a specified stopping criterion is satisfied. Thus, instead of constructing an optimal input design F^*, we normally obtain a suboptimal design F^{est} which can be made arbitrarily close to

F^* by having a different stopping criterion. Since the design F^{est} may have an undesirably large point spectrum, its approximation is usually considered. It is shown in Fedorov (1972) that, after an approximation, an acceptable rounded-off design, denoted F^{rnd}, can be obtained.

5.4 Application to Adaptive Systems

In this section, we will use the tools developed in section 5.3 to solve the input design problem formulated in section 5.2 for a general nominal adaptive system. Due to the fact that the solution will depend on the unknown plant, we will assume a prior estimate of the plant, similar to the Bayesian approach used in stochastic parameter estimation. To demonstrate these results, examples for both adaptive identifiers and model reference adaptive controllers are presented. However, for practical reasons, the final input design will only be a semi-optimal design F^{rnd} as mentioned in the preceding remark. In the following, separate treatments of the input design problem for an adaptive identifier and a model reference adaptive controller are given.

5.4.1 Adaptive Identifier

It is shown in subsection 5.1.1 that the bound on the rate of convergence of the parameter error vector ϕ is

$$\varepsilon \lambda_{\min}(R_w(0)) + o(\varepsilon)$$

when ε is small enough. Since the first term will dominate the second one for small ε, we will focus only on the first term within the context of our optimization problem. Now by choice of an input design over $N(\Omega)$, where Ω is preassigned to be in a frequency band $[-\omega_0, \omega_0]$ such that the bound is maximized, this problem falls into the same setup as given in section 5.3. By referring to (5.1.1.3) and Definition 5.2.2, we see that $g \equiv 1$ and $\hat{H} = \hat{Q}$. As has been proven in the Convergence Theorem, the numerical algorithm will then provide an optimal input design F_u^*, or, instead, a semi-optimal input design F_u^{rnd} in practice.

Example 5.4.1:

Consider the adaptive identification of a plant,

$$\hat{p}(s) = 2\frac{s+1}{s+3} . \tag{5.4.1.1}$$

Choose the filter of the compensator block to be $\frac{5}{s+5}$, and denote the parameter error vector $\phi=\theta-\theta^*=[\phi_1,\phi_2,\phi_3]^T$, where $\theta^*=[c_1^*,d_1^*,c_2^*]^T$ is found to be $[-1.6,0.4,2.0]^T$. As indicated above, we choose a prior guess of the plant as

$$\hat{p}^i(s) = 3\frac{s+2}{s+5} . \tag{5.4.1.2}$$

Moreover, since the plant is of first order, an initial input design:

$$\frac{1}{2\pi}\,F_u^0 = \frac{1}{2}\delta(\omega)+\frac{1}{4}\delta(\omega-2)+\frac{1}{4}\delta(\omega+2) \tag{5.4.1.3}$$

is considered. Also, by the locations of the pole and zero of the plant, a reasonable frequency search range $\Omega=[-10,10]$ is preassigned. Thus after applying sequential design algorithm, we obtain F_u^{rnd} shown below:

$$\begin{aligned}\frac{1}{2\pi}\,F_u^{rnd} &= 0.445\delta(\omega)+0.192\{\delta(\omega-2)+\delta(\omega+2)\}+0.0203\{\delta(\omega-3.52)+\delta(\omega+3.52)\,\} \\ &\quad + 0.00702\{\delta(\omega-3.80)+\delta(\omega+3.80)\}+0.00442\{\delta(\omega-4.29)+\delta(\omega+4.29)\} \\ &\quad + 0.0539\{\delta(\omega-4.43)+\delta(\omega+4.43)\} + 0.10\{\delta(\omega-10)+\delta(\omega+10)\} .\end{aligned} \tag{5.4.1.4}$$

Figure 5.4.1 shows the spectral distribution of F_u^{rnd}, and Figures 5.4.2 and 5.4.3 illustrate the difference in the convergence rates of parameter errors ϕ_1, ϕ_2, and the output error $e=y_o-y_p$ for the input designs F_u^0 and F_u^{rnd} respectively.

5.4.2 Model Reference Adaptive Controller

In this subsection, we will consider only the output error scheme since the input error scheme can be treated exactly the same way as the identifier. As was shown in subsection 5.1.2, the bound on the rate of parameter convergence is given by

$$\varepsilon\lambda_{\min}(SM(R_{w_m w_{mf}}(0))) + o(\varepsilon)$$

when ε is small. We will also optimize the first dominating term and neglect the second term in the context of the input design problem. Again, this problem fits into the setup given in section 5.3, and, by referring to (5.1.2.6) and Definition 5.2.2, we see that $g=\mathrm{Re}\,\hat{m}/c_0^*$, $\hat{H}=\hat{Q}_m$. Consequently, application of the sequential design algorithm will readily provide an optimal input design F_u^*, or a semi-optimal input design F_u^{rnd} in practice.

Example 5.4.2:

Consider the output error direct model reference adaptive control of a first order plant,

$$\hat{p}(s) = \frac{2}{s+1} \tag{5.4.2.1}$$

and a reference model,

$$\hat{m}(s) = \frac{3}{s+3} \,. \tag{5.4.2.2}$$

Denote the parameter error vector $\phi=\theta-\theta^*=[\phi_1,\phi_2]^T$ where the true value $\theta^*=[k_r^*,k_y^*]^T$ is $[1.5,-1]^T$. Unlike the case of identification, a prior estimate of the plant is not necessary. Since the plant is of first order, we consider the following initial input design:

$$\frac{1}{2\pi}\,F_u^0 = \frac{1}{2}\,\delta(\omega-1.5) + \frac{1}{2}\,\delta(\omega+1.5) \tag{5.4.2.3}$$

Also, the frequency search range Ω will be chosen to be the same as the one in the previous example. Then application of numerical algorithm provides the semi-optimal input design F_u^{rnd} as follows:

$$\frac{1}{2\pi}\,F_u^{rnd} = \frac{1}{2}\,\delta(\omega-2.46) + \frac{1}{2}\,\delta(\omega+2.46) \tag{5.4.2.4}$$

Figures 5.4.4 and 5.4.5 illustrate the difference of convergence rates of the parameter errors ϕ_1, ϕ_2, and the output error $e_o = y_p - y_m$ for the corresponding input designs, F_u^0 and F_u^{rnd}.

5.5 Design Guideline for Adaptive System with Unmodelled Dynamics

In section 5.4, we applied sequential design algorithm to both adaptive identifiers and model reference adaptive controllers with nominal plants; in which case the frequency search range Ω may be made as large as possible. However, adaptive identification and model reference adaptive control is usually performed in cases where the plants are contaminated by high frequency unmodelled dynamics. As a consequence, the choice of the frequency search range Ω becomes a relatively important factor for consideration in the context of input design.

In this section, we will first study some practical aspects in adaptive identification and model reference adaptive control, and then use these as a general guideline for choosing the frequency search range Ω. As would be expected, some prior information about plants and/or reference models is required, and will be discussed later in each subsection.

5.5.1 Adaptive Identification

Here, we consider the same setup for adaptive identification given in section 3.4, with the following additional assumptions.

Assumptions:

(A6) $|\hat{p}(j\omega)| \le L_i$ for some known $L_i > 0$, for all $\omega \in R$.

(A7) $|\Delta\hat{p}(j\omega)|_\infty \le \delta_i(\omega)$ for some known function $\delta_i(\omega)$ (usually small for small ω.

(A8) $\| \theta^* \| \le h_i$ for some known $h_i > 0$.

Remark: Assumption (A7) is very common in nonadaptive control for designing controllers, for example, [1]$S(P,C)$ control loop for *pole placement* shown in Figure 5.5.1. By the Nyquist criterion, if the nominal loop is stable and

$$|\Delta\hat{p}(j\omega)| \left| \frac{\hat{c}(j\omega)}{1+\hat{p}(j\omega)\,\hat{c}(j\omega)} \right| < 1 \qquad \text{for all } \omega \in R \qquad (5.5.1.1)$$

then the overall loop will remain stable.

From subsection 3.4.3, when the parameter adaptation is slow, for almost every input u with Supp(u) containing at least $2n$+1 points, the parameter vector $\theta(t)$ satisfies

$$\| \theta(t) - \theta_T \| \le \psi(\varepsilon) \qquad t \ge t_0 + T \qquad (5.5.1.2)$$

for some $T>0$, where θ_T is the tuned parameter given by (3.4.2.11) and $\psi(\varepsilon)$ is a class K function. In particular, if Supp(u) contains exactly $2n$+1 points, $\theta(t)$ converges to θ_T exponentially so that a unique transfer function $\hat{p}_T$ defined by (3.4.2.13) (tuned model transfer function) is obtained. Now if the controller shown in Figure 5.5.1 is to be designed based on the tuned model and the information (A7), the resulting loop may not be stable. Hence some confidence on the closeness between the tuned model, described by $\hat{p}_T$, and the true plant, described by $\hat{p}_u$, should be established before control is implemented. Following this, the loop that we are going to study instead is shown in Figure 5.5.2 where the bound on the perturbation part $|\hat{p}_u(j\omega) - \hat{p}_T(j\omega)|$ is left to be determined.

Remark: Note that the input u does not have to be such that Supp(u) contains exactly $2n$+1 points in order to get the tuned model transfer function $\hat{p}_T(s)$. By (5.5.1.2), if ε is small, $\theta(t)$ will stay within a neighborhood of θ_T so that, by some approximation, θ_T and, hence, $\hat{p}_T$ can be read off from the identifier.

Using Theorem 3.4.2.1 and 3.4.3.1, we consider an input u whose Supp(u) contains exactly $2n+1$ points such that the tuned parameter θ_T is well defined. It then follows that $\theta(t)$ converges to θ_T exponentially and thus

$$\hat{p}_u(j\omega) = \hat{p}_T(j\omega) \qquad \text{for all } \omega \in \text{Supp}(u) . \tag{5.5.1.3}$$

Let θ^* be the true parameter for the nominal plant as given in (3.2.9), and $\hat{Q}(s)$ be the transfer function given by (3.3.6) so that, from (3.2.7), we have $\hat{p}(s)=\theta^{*T}\hat{Q}(s)$. Using this and (3.4.2.13), the difference between $\hat{p}_u$ and $\hat{p}_T$ can be evaluated as follows:

$$\hat{p}_T - \hat{p}_u = \frac{\hat{C}_T + c_{(n+1)T}\hat{\Lambda}}{\hat{\Lambda} - \hat{D}_T} - (\theta^{*T}\hat{Q} + \Delta\hat{p})$$

$$= \frac{\hat{\Lambda}}{\hat{\Lambda} - \hat{D}_T}(\theta_T - \theta^*)^T \hat{Q} - \Delta\hat{p} \tag{5.5.1.4}$$

Assuming, without loss of generality, $\hat{p}_T(s)$ is stable, we have

$$|\hat{p}_T(j\omega) - \hat{p}_u| \le \left| \frac{\hat{\Lambda}(j\omega)}{\hat{\Lambda}(j\omega) - \hat{D}_T(j\omega)} \right| \, \| \hat{Q}(j\omega) \| \; \| \theta_T - \theta^* \| + |\Delta\hat{p}(j\omega)| . \tag{5.5.1.5}$$

All the bounds, except $\| \theta_T - \theta^* \|$, on the R.H.S. of (5.5.1.5) can be computed using knowledge of θ_T, (5.5.1.3), and assumption (A7). The difference between θ_T and θ^* can, however, be estimated through (3.4.2.11) as follows.

Let $\hat{Q}_u$ be the transfer function defined in (3.4.1.5) and denote

$$w^* := \hat{Q}(s)(u) \qquad \text{and} \qquad y_p^* := \hat{p}(s)(u) . \tag{5.5.1.6}$$

It then follows that

$$\theta^* = R_{w^*}(0)^{-1} R_{w^* y_p^*}(0) \tag{5.5.1.7}$$

where

$$R_{w^*}(0) = \frac{1}{2\pi} \sum_{i=0}^{n} \left[\hat{Q}(j\omega_i)\hat{Q}^*(j\omega_i) + \hat{Q}(-j\omega_i)\hat{Q}^*(-j\omega_i) \right] r_i^2 \tag{5.5.1.8}$$

$$R_{w^* y_p^*}(0) = \frac{1}{2\pi} \sum_{i=0}^{n} \left[\hat{Q}(j\omega_i)\hat{p}^*(j\omega_i) + \hat{Q}(-j\omega_i)\hat{p}^*(-j\omega_i) \right] r_i^2 \tag{5.5.1.9}$$

and t_i^2 is the spectral magnitude of the ith frequency element, in contrast with

$$\theta_T = R_w(0)^{-1} R_{wy_p}(0) \tag{5.5.1.10}$$

where

$$R_w(0) = \frac{1}{2\pi} \sum_{i=0}^{n} \left[\hat{Q}_u(j\omega_i)\hat{Q}_u^*(j\omega_i) + \hat{Q}_u(-j\omega_i)\hat{Q}_u^*(-j\omega_i) \right] t_i^2 \tag{5.5.1.11}$$

$$R_{wy_p}(0) = \frac{1}{2\pi} \sum_{i=0}^{n} \left[\hat{Q}_u(j\omega_i)\hat{p}_u^*(j\omega_i) + \hat{Q}_u(-j\omega_i)\hat{p}_u^*(-j\omega_i) \right] t_i^2 \,. \tag{5.5.1.12}$$

Since

$$\hat{Q}_u(s) = \hat{Q}(s) + \begin{bmatrix} 0 \\ \hat{F}_2(s)\Delta\hat{p}(s) \\ 0 \end{bmatrix} \tag{5.5.1.13}$$

$R_w(0)$ and $R_{wy_p}(0)$ are simply the ones perturbed from $R_{w^*}(0)$ and $R_{w^*y_p^*}(0)$ and can be represented as follows:

$$R_w(0) = R_{w^*}(0) + \Delta R_w(0) \tag{5.5.1.14}$$

$$R_{wy_p}(0) = R_{w^*y_p^*}(0) + \Delta R_{wy_p}(0) \,. \tag{5.5.1.15}$$

Note that $\| \Delta R_w(0) \|$, $\| \Delta R_{wy_p}(0) \|$ can all be estimated from assumptions (A6)-(A7). It then follows that the difference between θ_T and θ^* can now be estimated by

$$\| \theta_T - \theta^* \| \le \| R_w(0)^{-1} \| \left[\| \Delta R_{wy_p}(0) \| + \| \Delta R_w(0) \| \;\; \| \theta^* \| \right] \tag{5.5.1.16}$$

which is, in fact, computable since $R_w(0)$ can be computed using (5.5.1.3). This together with (5.5.1.5) leads to a bound on $|\hat{p}_T(j\omega) - \hat{p}_u(j\omega)|$.

There are two by-products of (5.5.1.5). The first is a measure of robustness as indicated in the first remark of this section; the second is a measure of performance, i.e. the closeness between transfer functions of the tuned model and of the true plant. If neither of these two should satisfy some least requirements, then another reference input u, satisfying the condition the same as the previous one (Supp(u) contains exactly $2n$+1 points), may have to be chosen again for the same identification task until the bound in (5.5.1.5) meets the requirements.

Remarks:

(1) A similar comment to that given in the remark after Figure 5.5.2 also applies here. (2) In some cases, there might be some output disturbance d, for example, the measurement noise, that contaminates the output of the plant. If $\mathrm{Supp}(u) \cap \mathrm{Supp}(d) = \varnothing$, which is similar to the assumption (A11) in Chapter 4, then it can be checked that Theorem 3.4.2.1 will still be true. It then can be deduced from the proof of Theorem 3.4.3.1 that $\theta \in L_\infty^{2n+1}$. The difference, however, is in that $\theta(t)$ may no longer converge to θ_T defined in (5.5.1.10). Yet, if the magnitude of disturbance d is small relative to that of the reference input u, then, when ε is small, (5.5.1.2) will imply that $\theta(t)$ may still be very close to θ_T so that a transfer function can be read off from the identifier. This will then allow us to treat this transfer function as if it were $\hat{p}_T$ and to evaluate (5.5.1.5) the same way as described above.

The above discussion is from the viewpoint of analysis. For design purposes, we would prefer to select proper inputs to achieve the same objectives as described above even before the identification is performed. However, the price one has to pay is that more conservative results are to be expected. Specifically, a prior guess of the tuned model has to be used for computing $R_w(0)^{-1}$ in (5.5.1.16). In the following, general input design guidelines for choosing proper inputs for adaptive identifiers is given based on these results.

General Input Design Guideline:

Data: Let $\delta^* > \max_{\omega \in R} \delta_i(\omega)$ be the *performance index*, i.e. for any identified transfer function $\hat{p}_T(s)$:

$$| \hat{p}_u - \hat{p}_T |_\infty := \max_{\omega \in R} | \hat{p}_u(j\omega) - \hat{p}_T(j\omega) | \le \delta^* \qquad (5.5.1.17)$$

$\Omega_0=[-\omega_0, \omega_0]$, for some $\omega_0>0$, is the initial choice of the frequency search range, and $\hat{p}_T^0$ be the prior guess of the tuned model transfer function.

Step 1: Set i = 0.

Step 2: Choose $\Omega_i=[-\omega_i, \omega_i]$, and the input u^i such that:

$$\text{Supp}(u^i) = \left\{ -\omega_i \, , - \frac{(n-1)}{n}\omega_i \, , \cdots , 0 \, , \cdots \frac{(n-1)}{n}\omega_i \, , \omega_i \right\}. \qquad (5.5.1.18)$$

Step 3: Compute the bound on the R.H.S. of (5.5.1.5), using the prior guess of the tuned model $\hat{p}_T^0$, and denote it σ_i. If $\sigma_i \le \delta^*$, then stop and go to Step 5; else goto Step 4.

Step 4: Chose $\omega_{i+1} < \omega_i$, and goto Step 1.

Step 5: Use Ω_i as the frequency search range and carry out the sequential design algorithm for finding semi-optimal inputs. Here, the prior guess of the plant will be replaced by $\hat{p}_T^0$.

Remarks:

(1) The prior guess of the tuned model, $\hat{p}_T^0$, can be updated while the identification process is running. This provides a more precise design procedure.

(2) Normally, the semi-optimal input design F_u^{rnd} generated by the numerical design algorithm contains more than $2n+1$ frequency elements; in which case the parameter $\theta(t)$ will not converge to the θ_T determined by F_u^{rnd}, but will oscillate around it. Therefore, the preferable input design should be a *two-phase design*. The phase I design is simply the semi-optimal input design F_u^{rnd} so that $\theta(t)$ converges to a neighborhood of θ_T quickly and stays within it. The phase II design is to include only $2n+1$ frequency elements through a *input reduction process* so that $\theta(t)$ converges to a new tuned parameter θ'_T which is close to the θ_T. However, the input reduction process requires some experience on the part of the designer.

Example 5.5.1:

Consider Example 5.4.1 but with the plant being changed into

$$\hat{p}_u(s) = \frac{2(s+1)}{(s+3)} \frac{30}{(s+30)} \tag{5.5.1.18}$$

where the additive unmodelled dynamics appears to be

$$\Delta\hat{p}(s) = \frac{2(s+1)}{(s+3)} \frac{-s}{(s+30)} \ . \tag{5.5.1.19}$$

Recall that the true parameter for the nominal plant is $\theta^* = [-1.6 \ , 0.4 \ , 2.0]^T$. If we now use F_u^{2nd} which was obtained in Example 5.4.1 as the phase I design, the tuned parameter θ_T is computed by (5.5.1.10) to be $[-1.531, 0.819, 1.650]^T$, which gives $\hat{p}_T(s)$ as

$$\hat{p}_T(s) = \frac{(1.650s+0.593)}{(s+0.905)} \ . \tag{5.5.1.20}$$

When t=20 (sec), the phase II design is started using the initial design F^0 which is used in Example 5.4.1, where the tuned parameter θ'_T can be read off from the identifier as: $[-1.443, 0.516, 1.767]^T$ so that $\hat{p}'_T(s)$ can, again, be computed as:

$$\hat{p}'_T(s) = \frac{(1.767s+1.613)}{(s+2.422)} \tag{5.5.1.21}$$

Figures 5.5.3 and 5.5.4 illustrate the difference of convergence rates of parameters c_1, d_1, and the output error $e_o = y_o - y_p$ resulting from the two-phase design and only a single design F_u^0 respectively. When the two-phase design is used, at roughly t=40 (sec), $\theta(t)$ appears to be very close to the tuned parameter value θ'_T, whereas $\theta(t)$ is still converging to the same tuned parameters after t=160 (sec) when only a single design F_u^0 is used. Figure 5.5.5 shows the Nyquist plots of the true plant, the nominal plant, and the tuned model. An interesting observation will be that, when the input contains frequency elements as low as dc and $2\,rad/$sec, the tuned model still approximates the true plant better than the nominal plant does.

5.5.2 Model Reference Adaptive Control

In this subsection, we consider the same setup for model reference adaptive control as described in section 4.4, with the following additional assumptions.

Assumptions:

(A16) $|\Delta\hat{p}(j\omega)| \le \delta_c(\omega)$ for some assumed known function bound $\delta_c(\omega)$, which is an increasing function of ω and is very small for small ω.

(A17) $\|\theta^*\| \le h_c$ for some known $h_c > 0$.

It is shown in subsection 4.4.4 that if the adaptive system fails to satisfy the positive definite PE condition, then the system is likely to undergo a slow drift of the controller parameters. Thus the objective for choosing proper frequency search range Ω here will be different from that in the case of adaptive identification. For the output error scheme, given the tuned parameters θ_T, Ω is to be chosen such that

$$\mathrm{Re}\,\hat{m}_T(j\omega) > 0 \qquad \text{for all } \omega \in \Omega\,. \tag{5.5.2.1}$$

Let θ_T be given in (4.4.3.6) and denote

$$\hat{C}_T(s) := C_T^T\hat{F}_1(s)\,\hat{\Lambda}(s) \qquad \text{and} \qquad \hat{D}_T(s) := (D_T^T\hat{F}_2(s)+d_{0T})\,\hat{\Lambda}(s)\,. \tag{5.5.2.2}$$

Referring to the controller structure shown in Figure 4.2.1, we define an nth order pseudo plant transfer function $\hat{p}_T$:

$$\hat{p}_T(s) = k_T\frac{\hat{n}_T(s)}{\hat{d}_T(s)} := (k_m/c_{0T})\frac{\hat{\Lambda}(s) - \hat{C}_T(s)}{(k_m/c_{0T})\,\hat{D}_T(s) + \hat{d}_m(s)} \tag{5.5.2.3}$$

which clearly is the identified plant transfer function if $\theta(t)$ should converge to θ_T. Define

$$\hat{Q}_T := \begin{bmatrix} 1 \\ \hat{F}_1\hat{m}\hat{p}_T^{-1} \\ \hat{m} \\ \hat{F}_2\hat{m} \end{bmatrix} \tag{5.5.2.4}$$

it then follows that

$$\theta_T^T \hat{Q}_T(s) = \hat{m}(s)\, \hat{p}_T(s)^{-1} \tag{5.5.2.5}$$

in contrast with the ideal nominal case

$$\theta^{*T} \hat{Q}_m(s) = \hat{m}(s)\hat{p}^{-1}(s) \tag{5.5.2.6}$$

where $\hat{Q}_m(s)$ is defined in (4.3.1.7). Hence we have

$$\begin{aligned}\hat{m}(s)\hat{p}_T(s)^{-1} - \hat{m}(s)\hat{p}(s)^{-1} &= \frac{\phi_T^T \hat{Q}_T(s)}{1-\hat{C}^*(s)/\hat{\Lambda}(s)} \\ &= \frac{\phi_T^T \hat{Q}_m(s)}{1-\hat{C}_T(s)/\hat{\Lambda}(s)}\end{aligned} \tag{5.5.2.7}$$

where $\phi_T := \theta_T - \theta^*$, and $\hat{C}^* := C^{*T} \hat{F}_1 \hat{\Lambda}$, similar to that in (5.5.2.2).

On the other hand, the tuned model transfer function, $\hat{m}_T(s)$, using (4.4.1.8) and (4.4.2.7), satisfies

$$\hat{m}_T(s) - \hat{m}(s) = \frac{1}{c_{0T}}\, \hat{m}_T\, (\, \theta_T^T \hat{Q}_{mu}(s) - \hat{m}(s)\hat{p}_u(s)^{-1}\,) \tag{5.5.2.8}$$

where $\hat{Q}_{mu}$ is given in (4.4.1.13). Since

$$\hat{Q}_m = \hat{Q}_{mu} + \left[0\,,\, \hat{F}_1 \hat{m}\, \hat{p}_u^{-1} \Delta\hat{p}\ ,\, 0\,,\, 0\right]^T \tag{5.5.2.9}$$

so that

$$\theta_T^T \hat{Q}_{mu} = \theta_T^T \hat{Q}_m - \frac{\hat{C}_T}{\hat{\Lambda}} \hat{m}\, \hat{p}_u^{-1} \Delta\hat{p} \tag{5.5.2.10}$$

which together with (5.5.2.8) leads to an expression of $\hat{m}_T$ as follows:

$$\begin{aligned}\hat{m}_T &= \hat{m} + \frac{1}{c_{0T}} \hat{m}_T \left[(\theta_T - \theta^*)^T \hat{Q}_m + \hat{m}\, (\hat{p}^{-1} - \hat{p}_u^{-1}) - \frac{\hat{C}_T}{\hat{\Lambda}} \hat{m}\, \hat{p}_u^{-1} \Delta\hat{p}\right] \\ &= \hat{m} + \frac{1}{c_{0T}} \hat{m}_T \left[\phi_T^T \hat{Q}_m + (1 - \frac{\hat{C}_T}{\hat{\Lambda}})\hat{m}\, \hat{p}_u^{-1} \Delta\hat{p}\right]\end{aligned}$$

$$= \hat{m} + \frac{1}{c_{0T}} \hat{m}_T \left[\frac{(1-\hat{C}_T/\hat{\Lambda})}{(1-\hat{C}^*/\hat{\Lambda})} \phi_T^T \hat{Q}_T + (1 - \frac{\hat{C}_T}{\hat{\Lambda}}) \hat{m} \hat{p}^{-1} \frac{\Delta \hat{p}}{1+\Delta \hat{p}} \right]$$

$$= \hat{m} \left[1 - \frac{1}{c_{0T}} \left[\frac{(1-\hat{C}_T/\hat{\Lambda})}{(1-\hat{C}^*/\hat{\Lambda})} \frac{\phi_T^T \hat{Q}_T}{(1+\Delta\hat{p})} + (1-\frac{\hat{C}_T}{\hat{\Lambda}}) \hat{m} \hat{p}_T^{-1} \frac{\Delta\hat{p}}{1+\Delta\hat{p}} \right] \right]^{-1} . \quad (5.5.2.11)$$

Using the expression of $\hat{p}_T$ in (5.5.2.3), the second term of the denominator in (5.5.2.11) can be further simplified as

$$\hat{\gamma} := \left[\frac{\hat{d}_T}{\hat{d}_m} (\frac{\Delta\hat{p}}{1+\Delta\hat{p}}) + \frac{k_T \hat{n}_T}{k_m \hat{n}_p} \frac{\phi_T^T \hat{Q}_T}{(1+\Delta\hat{p})} \right] . \quad (5.5.2.12)$$

Now restating criterion (5.5.2.1), our choice of frequency search range Ω should be such that

$$\max_{\omega \in \Omega} |\hat{\gamma}(j\omega)| < \tan(\frac{\pi}{2} - \max_{\omega \in \Omega} |\angle \hat{m}(j\omega)|) \quad (5.5.2.13)$$

to make sure that $\mathrm{Re}\,\hat{m}_T(j\omega) > 0$ for all $\omega \in \Omega$. This may not be possible even for sufficiently low frequencies due to the fact that the second term in (5.5.2.12) does not appear to be a function of the unmodelled dynamics (and, hence, can be made arbitrarily small for very low frequencies). However, if one is allowed to choose an input u that generates tuned parameters θ_T for the above analysis, then there always exists an Ω such that Supp $(u) \subset \Omega$, and (5.5.2.13) is satisfied.

Consider an input u whose Supp(u) contains exactly $2n$ points, then by Theorem 4.4.1.2 and 4.4.2.1, for almost every such input u, there exist unique tuned parameters θ_T that satisfy (4.4.1.12) which is repeated here

$$\hat{m}_T(j\omega) = \hat{m}(j\omega)$$

$$= \hat{p}_u(j\omega)\theta_T^T \hat{Q}_{mu}(j\omega) \qquad \text{for all } \omega \in \mathrm{Supp}(u) . \quad (5.5.2.14)$$

Assume that it is the case, then it follows from the remark after Theorem 4.4.3.2 that $\theta(t)$ converges to θ_T exponentially and, hence, the transfer function $\hat{p}_T$ defined in (5.5.2.3) can be computed. Denote

$$w_m := \hat{Q}_{mu}(s)(u) \qquad \text{and} \qquad w_m^* := \hat{Q}_m(s)(u) \quad (5.5.2.15)$$

$$u_p := \hat{m}_T(s)\hat{p}_u^{-1}(s)(u) \qquad \text{and} \qquad u_p^* := \hat{m}(s)\hat{p}^{-1}(s)(u) \tag{5.5.2.16}$$

It then follows from (5.5.2.6) and (5.5.2.14) that θ_T and θ^* can be expressed in a form similar to that in (5.5.1.7) and (5.5.1.10) respectively

$$\theta_T = R_{w_m}(0)^{-1}\, R_{w_m u_p}(0) \tag{5.5.2.17}$$

where

$$R_{w_m}(0) := \frac{1}{2\pi}\sum_{i=1}^{n}\left[\hat{Q}_{mu}(j\omega_i)\,\hat{Q}_{mu}^*(j\omega_i) + \hat{Q}_{mu}(-j\omega_i)\,\hat{Q}_{mu}^*(-j\omega_i)\right] t_i^2 \tag{5.5.2.18}$$

$$R_{w_m u_p}(0) := \frac{1}{2\pi}\sum_{i=1}^{n}\left[\hat{Q}_{mu}(j\omega_i)\,\hat{m}^*(j\omega_i)\,\hat{p}_u^*(j\omega)^{-1}\right.$$

$$\left. + \hat{Q}_{mu}(-j\omega_i)\,\hat{m}^*(-j\omega_i)\,\hat{p}_u^*(-j\omega_i)^{-1}\right] t_i^2 \tag{5.5.2.19}$$

and

$$\theta^* = R_{w_m^*}(0)^{-1}\, R_{w_m^* u_p^*}(0) \tag{5.5.2.20}$$

where

$$R_{w_m^*}(0) := \frac{1}{2\pi}\sum_{i=1}^{n}\left[\hat{Q}_m(j\omega_i)\,\hat{Q}_m^*(j\omega_i) + \hat{Q}_m(-j\omega_i)\,\hat{Q}_m^*(-j\omega_i)\right] t_i^2 \tag{5.5.2.21}$$

$$R_{w_m^* u_p^*}(0) := \frac{1}{2\pi}\sum_{i=1}^{n}\left[\hat{Q}_m(j\omega_i)\,\hat{m}^*(j\omega_i)\,\hat{p}^*(j\omega)^{-1}\right.$$

$$\left. + \hat{Q}_m(-j\omega_i)\,\hat{m}^*(-j\omega_i)\,\hat{p}^*(-j\omega_i)^{-1}\right] t_i^2 \,. \tag{5.5.2.22}$$

As before, the matrix $R_{w_m}(0)$ and the vector $R_{w_m u_p}(0)$ are the ones perturbed from $R_{w_m^*}(0)$ and $R_{w_m^* u_p^*}(0)$ respectively and can be represented as follows:

$$R_{w_m}(0) = R_{w_m^*}(0) + \Delta R_{w_m}(0) \tag{5.5.2.23}$$

$$R_{w_m u_p}(0) = R_{w_m^* u_p^*}(0) + \Delta R_{w_m u_p}(0) \tag{5.5.2.24}$$

Again, if the norms $\| \Delta R_{w_m}(0) \|$ and $\| \Delta R_{w_m u_p}(0) \|$ can be estimated, then, following from

(5.5.1.16), the difference between θ_T and θ^* can be estimated likewise as

$$\| \phi_T \| \leq \| R_{w_m}(0)^{-1} \| \left[\| \Delta R_{w_m u_p}(0) \| + \| \Delta R_{w_m}(0) \| \ \| \theta^* \| \right]. \tag{5.5.2.25}$$

However, from (5.5.2.9) and the fact $\hat{m}\hat{p}^{-1} = \hat{m}\hat{p}_u^{-1} + \hat{m}\hat{p}_u^{-1}\Delta\hat{p}$, the estimates of $\| \Delta R_{w_m}(0) \|$ and $\| \Delta R_{w_m u_p}(0) \|$ require knowledge of $\hat{m}(j\omega)\hat{p}_u(j\omega)^{-1}$ for all $\omega \in$ Supp(u) in addition to assumptions (A16). By referring to Figure 4.2.2, if $\theta(t)$ is fixed at θ_T, then it follows that

$$\begin{aligned} u_p &= \hat{m}_T \hat{p}_u^{-1}(u) \\ &= \left[\frac{c_{0T}}{1 - \hat{C}_T / \hat{\Lambda}} \right](u) + \left[\frac{\hat{m}_T \hat{D}_T / \hat{\Lambda}}{1 - \hat{C}_T / \hat{\Lambda}} \right](u). \end{aligned} \tag{5.5.2.26}$$

Therefore, using (5.5.2.14), we have

$$\hat{m}(j\omega)\hat{p}_u(j\omega)^{-1} = \frac{c_{0T} + \hat{m}(j\omega)\hat{D}_T(j\omega) / \hat{\Lambda}(j\omega)}{1 - \hat{C}_T(j\omega) / \hat{\Lambda}(j\omega)} \tag{5.5.2.27}$$

for all $\omega \in$ Supp(u) so that all the estimates of the norms on the R.H.S. of (5.5.2.25) and, hence, an estimate of $\| \phi_T \|$ can be obtained.

Now using the bound on ϕ_T, the frequency search range $\Omega = [-\omega_0, \omega_0]$ is sought such that (5.5.2.13) is satisfied and Supp (u) $\subset \Omega$. Difficulties may, however, arise from the unknown $\hat{n}_p$. As before, this can be replaced by a prior guess of the numerator of $\hat{p}$, similar to that in the case of adaptive identification. Again, as mentioned before, such an Ω may not exist for one choice of input u that satisfies the above requirements. But by such iterative procedure, we can eventually find an Ω and an input u that achieve the goal.

Remark: In general, the nominal control command signal usually consists of low frequency signals so that the unmodelled dynamics of the plant may not be excited significantly, and, hence, the tuned parameters θ_T should be close to the true parameters θ^* for the nominal plant. As such, the above analysis will provide a frequency band, which is generally much higher than that of the nominal control command signal. Any other signals whose frequency spectrum lies outside that band should be avoided as they may contaminate the nominal input, for example, by filtering.

For design rather than analysis, we prefer to select proper frequency search range to achieve the same objectives as given above even before the control task is performed. If this is the case, the prior guess of the tuned parameter θ_T (or $\hat{p}_T$) as well as that of $\hat{n}_p$, additional to assumptions (A16), is needed for computing all the bounds on the norms required in (5.5.2.13). In the following, a general input design guideline of choosing proper inputs for model reference adaptive controllers is given based on these results.

General Input Design Guideline:

Data: Let $\Omega_0=[-\omega_0,\omega_0]$, for some $\omega_0>0$, be the initial choice of the frequency search range, and θ_T^0, $\hat{n}_p^0$ be the prior guesses of tuned parameters and the numerator of the nominal plant respectively.

Step 1: Set i = 0.

Step 2: Choose $\Omega_i = [-\omega_i, \omega_i]$, and the input u^i such that

$$\text{Supp}(u^i)=\left\{-\omega_i, \frac{(n-1)}{n}\omega_i, \cdots, \frac{1}{(n-1)}\omega_i, \frac{1}{(n-1)}\omega_i \cdots \frac{(n-1)}{n}\omega_i, \omega_i\right\}.$$

Step 3: Compute the bound on the L.H.S. of (5.5.2.13), using the prior guesses $\hat{p}_T^0$ and $\hat{n}_p^0$, and denote it σ_i. If σ_i satisfies (5.5.2.13), then stop and go to Step 5; else go to Step 4.

Step 4: Chose $\omega_{i+1} < \omega_i$, and goto Step 1.

Step 5: Use Ω_i as the frequency search range and carry out the sequential design algorithm for finding semi-optimal inputs. There, the prior guess of the plant will be replaced by $\hat{p}_T^0$.

Remarks:

(1) As before, the prior guess $\hat{p}_T^0$ can be updated while the control process is running so that the design procedure becomes more precise.

(2) In practice, the suboptimal inputs generated from the above procedure generally will not be directly applicable to the adaptive system simply because the control task and, hence, the control command signals should be pre-specified. However, the suboptimal inputs serve to be good references for planning the control task so that both fast convergence and robustness can still be taken into account.

Example 5.5.2:

Consider Example 5.4.2 with the plant contaminated by a high frequency unmodelled pole $s=-20$, i.e.

$$\hat{p}_u(s) = \frac{2}{(s+1)}\,\frac{20}{(s+20)} \tag{5.5.2.28}$$

where the multiplicative unmodelled dynamics is seen to be

$$\Delta\hat{p}(s) = \frac{-s}{(s+20)}\ . \tag{5.5.2.29}$$

Recall that the true parameters for the nominal plant is $\theta^* = [\,1.5\,,-1\,]^T$, and the semi-optimal input design F_u^{rnd} obtained in that example contains only single frequency so that the two-phase design is not necessary. Suppose that we now use F_u^{rnd} to be the input design, then, from simulation, $\theta(t)$ converges to the tuned parameters $\theta_T = [\,1.575\,,-1.226\,]^T$, which leads to the following tuned model transfer function,

$$\hat{p}_{(s)} = \frac{63}{(s^2+21s+69.05)} \tag{5.5.2.30}$$

Figures 5.5.6 and 5.5.7 illustrate the difference of convergence rates of k_r and k_y using F_u^{rnd} and F_u^0 respectively. Note that, since the frequency elements in both designs are low, the unmodelled dynamics are attenuated and convergence rates observed in this example are not much different from those in Example 5.4.1. In Figure. 5.5.7, Nyquist plots of the reference model and of the tuned model are shown to indicate the closeness between two transfer functions.

5.6 Concluding Remarks

In this chapter, we have formulated the input design problem for adaptive systems, both identification systems and model reference adaptive control systems, in terms of a problem of optimization of the convergence rate of parameter errors. From Chapters 3 and 4, the analysis of parameter convergence clearly suggested an approach to this problem using averaging theory. The problem was thus recast in a form of maximization of the smallest eigenvalue of an average information matrix over a class of input signals. The problem formulation is very similar to that used in the stochastic literature (see, for example, Fedorov (1972) and Mehra (1974)) for parameter estimation in linear dynamical systems. However, their objective was to achieve a more accurate parameter estimate.

Under this formulation, the optimal inputs were characterized in the frequency domain and a sequential design algorithm was provided to attain these optima iteratively. These optimal inputs are found to be global maximizers of the smallest eigenvalue owing to a convexity property of the problem. However, the algorithm is more of an analysis tool than a design tool owing to the fact that the solution inevitably depends on the knowledge of the unknown plant. Yet, the design function can be achieved by replacing the unknown plant with an initial guess. Such a method is very common in the stochastic context and is referred to as a Bayesian approach, since it assumes a prior distribution of the parameters to be estimated. Examples of applications of this design algorithm to adaptive identifiers and controllers have been given to illustrate the results.

As indicated in Chapter 3 and 4, unmodelled dynamics exist in practice, and performance and/or stability of identifiers and controllers is crucially related to the tuned model. In subsections 3.4 and 4.4, the major dependence of these tuned models on the reference inputs was emphasized and a qualitative discussion was given. Here, under the same assumption of stationary inputs, a more quantitative study of the relationship between the frequency content of the reference input and the performance and/or stability of adaptive systems was performed. In the case of adaptive identification, the objective is to make the identified transfer function meet the requirements for the ultimate control task, whereas, in the control case, the objective is to make the system satisfy the positive definite PE condition. A product of this study is the determination of the frequency range that the spectral support of the reference input should lie in. This is then used as a frequency search range in the input design algorithm. Finally, based on these results, general input design guidelines were proposed for adaptive identifiers and controllers.

Chapter 6 Conclusion

In this thesis, we have presented averaging as a technique for the analysis and synthesis of both adaptive identification and control systems. A thorough analysis of adaptive systems using this technique was performed under ideal conditions as well as non-ideal conditions, where unmodelled dynamics and/or output disturbances were present. Also a synthesis procedure for generating reference inputs that maximize the rate of convergence of the adjustable parameters was proposed.

First, a full set of averaging results for adaptive systems was provided. Theorems were developed for exponential, partial exponential, and bounded stability, and also for instability of one- and two-time-scale dynamical systems. These results enabled us to relate several properties of nonautonomous systems to those of autonomous systems and hence simplify the analysis of the system. Adaptive systems, under the assumptions of stationary reference inputs and slow adaptation, satisfy the assumptions needed for these results.

Assuming ideal conditions, existing results have shown the stability and output convergence of the adaptive identification scheme and model reference direct adaptive control input and output error schemes. In particular, the adjustable parameters converge exponentially whenever the reference input is sufficiently rich. In this, averaging can be applied as an approximation method to obtain estimates of the convergence rates for different algorithms. Examples were given to illustrate the accuracy of these estimates.

The robustness of these schemes to unmodelled dynamics and/or output disturbances were examined. The PE condition of Boyd and Sastry (1984) was replaced by an almost always PE condition, and tuned models for adaptive identifiers and model reference adaptive controllers were precisely established for the analysis. Under the assumption of slow adaptation and persistency of excitation, the identifier parameters converge to a ball centered at the tuned parameters with a radius that is a class K function of the adaptation gains. Similarly, the controller parameters converge to a ball centered at the tuned parameters with a radius that is a sum of a class K function of the adaptation gain and a linear function of the mean squared power of the tuned error provided that: (i) the adaptation is slow, (ii) the tuned model is close enough to the reference model, and (iii) a sufficiently positive definite PE condition is satisfied.

Payoffs from this robustness analysis include:

(i) a better understanding of the behavior of the parameters under slow adaptation was obtained,

(ii) the importance of the spectral content of the reference input to the performance and/or stability was greatly emphasized,

(iii) relaxation of the SPR condition for the output error scheme of a model reference direct adaptive controller was suggested, and

(iv) a substantiation of the positive definite PE condition for both input and output error schemes of a model reference direct adaptive controller was given.

Finally, a frequency domain technique for the synthesis of reference inputs for adaptive systems was proposed. The idea is to select inputs subject to power constraints so as to maximize the rate of exponential convergence of the adjustable parameters under PE assumption. In the presence of unmodelled dynamics, a practical consideration of the range of input frequency content (spectral support) was provided supporting the qualitative conclusion (ii) given earlier.

APPENDIX A

Proof of Lemma 2.2.1.1:

Define:

$$w_\varepsilon(t,x)=\int_0^t d(\tau,x)e^{-\varepsilon(t-\tau)}d\tau \tag{A2.2.1.1}$$

and:

$$w_0(t,x)=\int_0^t d(\tau,x)d\tau \tag{A2.2.1.2}$$

From the assumptions:

$$\|w_0(t+t_0,x)-w_0(t_0,x)\|\leq\gamma(t).t \tag{A2.2.1.3}$$

for all $t,t_0\geq 0$, $x\in B_h$. Integrating (A2.2.1.1) by parts:

$$w_\varepsilon(t,x)=w_0(t,x)-\varepsilon\int_0^t e^{-\varepsilon(t-\tau)}w_0(\tau,x)d\tau \tag{A2.2.1.4}$$

Using the fact that:

$$\varepsilon\int_0^t e^{-\varepsilon(t-\tau)}w_0(t,x)d\tau=w_0(t,x)-w_0(t,x)\,e^{-\varepsilon t} \tag{A2.2.1.5}$$

(A2.2.1.4) can be rewritten as:

$$w_\varepsilon(t,x)=w_0(t,x)e^{-\varepsilon t}+\varepsilon\int_0^t e^{-\varepsilon(t-\tau)}(w_0(t,x)-w_0(\tau,x))d\tau \tag{A2.2.1.6}$$

and, using (A2.2.1.3):

$$\|w_\varepsilon(t,x)\|\leq\gamma(t)\,t\,e^{-\varepsilon t}+\varepsilon\int_0^t e^{-\varepsilon(t-\tau)}(t-\tau)\,\gamma(t-\tau)d\tau \tag{A2.2.1.7}$$

Consequently,

$$\|\varepsilon w_\varepsilon(t,x)\| \le \sup_{t'\ge 0} \gamma(\frac{t'}{\varepsilon})\, t' e^{-t'} + \int_0^\infty \gamma(\frac{\tau'}{\varepsilon})\, \tau' e^{-\tau'} d\tau' \qquad \text{(A2.2.1.8)}$$

Since, for some β, $\|d(t,x)\| \le \beta$, we also have that $\gamma(t) \le \beta$. Note that, for all $t' \ge 0$, $t' e^{-t'} \le e^{-1}$, and $t' e^{-t'} \le t'$, so that:

$$\|\varepsilon w_\varepsilon(t,x)\| \le \sup_{t'\in[0,\sqrt{\varepsilon}]} \left[\gamma(\frac{t'}{\varepsilon}) t' e^{-t'}\right] + \sup_{t'\ge\sqrt{\varepsilon}} \left[\gamma(\frac{t'}{\varepsilon}) t' e^{-t'}\right]$$

$$+ \int_0^{\sqrt{\varepsilon}} \gamma(\frac{\tau'}{\varepsilon})\, \tau' e^{-\tau'} d\tau' + \int_{\sqrt{\varepsilon}}^\infty \gamma(\frac{\tau'}{\varepsilon})\, \tau' e^{-\tau'} d\tau' \qquad \text{(A2.2.1.9)}$$

This, in turn, implies that

$$\|\varepsilon w_\varepsilon(t,x)\| \le \beta\sqrt{\varepsilon} + \gamma(\frac{1}{\sqrt{\varepsilon}})\, e^{-1} + \beta\frac{\varepsilon}{2} + \gamma(\frac{1}{\sqrt{\varepsilon}})(1+\sqrt{\varepsilon})\, e^{-\sqrt{\varepsilon}}$$

$$:= \xi_1(\varepsilon) \qquad \text{(A2.2.1.10)}$$

Clearly $\xi_1(\varepsilon) \in K$. From (A2.2.1.1), it follows that:

$$\frac{\partial w_\varepsilon(t,x)}{\partial t} - d(t,x) = -\varepsilon w_\varepsilon(t,x) \qquad \text{(A2.2.1.11)}$$

so that both (2.2.1.6) and (2.2.1.7) are satisfied.

If $\gamma(T) = a/T^r$, then the right-hand side of (A2.2.1.8) can be computed explicitly:

$$\sup_{t'\ge 0} a\varepsilon^r (t')^{1-r} e^{-t'} = a\varepsilon^r (1-r)^{1-r} e^{r-1} \le a\varepsilon^r \qquad \text{(A2.2.1.12)}$$

and, with Γ denoting the standard gamma function:

$$\int_0^\infty a\varepsilon^r (\tau')^{1-r} e^{-\tau'} d\tau' = a\varepsilon^r \Gamma(2-r) \le a\varepsilon^r \qquad \text{(A2.2.1.13)}$$

Defining $\xi_1(\varepsilon) = 2a\varepsilon^r$, the second part of the lemma is verified.

Q.E.D.

Proof of Lemma 2.2.1.2:

Define $w_\varepsilon(t,x)$ as in Lemma 2.2.1.1. Consequently,

$$\frac{\partial w_\varepsilon(t,x)}{\partial x}=\frac{\partial}{\partial x}\left[\int_0^t d(\tau,x)e^{-\varepsilon(t-\tau)}d\tau\right]=\int_0^t\left[\frac{\partial}{\partial x}d(\tau,x)\right]e^{-\varepsilon(t-\tau)}d\tau \tag{A2.2.1.14}$$

Since $\frac{\partial d(t,x)}{\partial x}$ is zero mean, and is bounded, Lemma 2.2.1.1 can be applied to $\frac{\partial d(t,x)}{\partial x}$, and inequality (2.2.1.6) of Lemma 2.2.1.1 becomes inequality (2.2.1.10) of Lemma 2.2.1.2. Note that since $\frac{\partial d(t,x)}{\partial x}$ is bounded, and $d(t,0)=0$ for all $t\geq 0$, $d(t,x)$ is Lipschitz. Since $d(t,x)$ is zero mean, with convergence function $\gamma(T)\|x\|$, the proof of Lemma 2.2.1.1 can be extended, with an additional factor $\|x\|$. This leads directly to (2.2.1.8) and (2.2.1.9) (although the function $\xi_1(\varepsilon)$ may be different from that obtained with $\frac{\partial d(t,x)}{\partial x}$, these functions can be replaced by a single $\xi_1(\varepsilon)$).

Q.E.D.

Proof of Lemma 2.2.1.3:

The proof proceeds in two steps.

Step 1: For ε sufficiently small, and for t fixed, the transformation (2.2.1.15) is a homeomorphism.

Apply Lemma 2.2.1.2, and let ε_1 such that $\xi_1(\varepsilon_1)<1$. Given $z\in B_r$, the corresponding x such that:

$$x=z-\varepsilon w_\varepsilon(t,z) \tag{A2.2.1.15}$$

may not belong to B_r. Similarly, given $x\in B_r$, the solution z of (A2.2.1.15) may not exist in B_r. However, for any x,z satisfying (A2.2.1.15), inequality (2.2.1.8) implies that:

$$(1-\xi_1(\varepsilon))\|z\|\leq\|x\|\leq(1+\xi_1(\varepsilon))\|z\| \tag{A2.2.1.16}$$

Define:

$$r'(\varepsilon) = r(1-\xi_1(\varepsilon)) \tag{A2.2.1.17}$$

and note that $r'(\varepsilon)\to r$ as $\varepsilon\to 0$.

We now show that:

(i) for all $z\in B_{r'}$, there exists a unique $x\in B_r$ such that (A2.2.1.15) is satisfied,

(ii) for all $x\in B_{r'}$, there exists a unique $z\in B_r$ such that (A2.2.1.15) is satisfied.

In both cases, $\|x-z\|\le\xi_1(\varepsilon)r$.

The first part follows directly from (A2.2.1.16), (A2.2.1.17). The fact that $\|x-z\|\le\xi_1(\varepsilon)r$ also follows from (A2.2.1.16), and implies that, if a solution z exists to (A2.2.1.15), it must lie in the closed ball U of radius $\xi_1(\varepsilon)r$ around x. It can be checked, using (2.2.1.10), that the mapping $F_x(z)=x-\varepsilon w_\varepsilon(t,z)$ is a contraction mapping in U, provided that $\xi_1(\varepsilon)<1$. Consequently, F has a unique fixed point z in U. This solution is also a solution of (A2.2.1.15), and since it is unique in U, it is also unique in B_r (and actually in R^n). For $x\in B_r$, but outside $B_{r'}$, there is no guarantee that a solution z exists in B_r, but if it exists, it is again unique in B_r. Consequently, the map defined by (A2.2.1.15) is well-defined. From the smoothness of $w_\varepsilon(t,z)$ with respect to z, it follows that the map is a homeomorphism.

Step 2: the transformation of variable leads to the differential equation (2.2.1.16)

Applying (A2.2.1.15) to the system (2.2.1.1):

$$\begin{aligned}(I+\varepsilon\frac{\partial w_\varepsilon}{\partial z})\dot z &= \varepsilon f_{av}(z)+\varepsilon(f(t,z,0)-f_{av}(z)-\frac{\partial w_\varepsilon}{\partial t})\\ &\quad+\varepsilon(f(t,z+\varepsilon w_\varepsilon,\varepsilon)-f(t,z,\varepsilon))\\ &\quad+\varepsilon(f(t,z,\varepsilon)-f(t,z,0))\\ &:=\varepsilon f_{av}(z)+\varepsilon p'(t,x,z,\varepsilon)\end{aligned} \tag{A2.2.1.18}$$

where, using the assumptions, and the results of Lemma 2.2.1.2:

$$\|p'(t,z,\varepsilon)\| \le \xi_1(\varepsilon)\|z\| + \xi_1(\varepsilon) l_1 \|z\| + \varepsilon l_2 \|z\| \tag{A2.2.1.19}$$

For $\varepsilon \le \varepsilon_1$, (2.2.1.10) implies that $(I+\varepsilon\frac{\partial w_\varepsilon}{\partial z})$ has a bounded inverse for all $t \ge 0$, $z \in B_r$. Consequently, z satisfies the differential equation:

$$\dot z = \left[I+\varepsilon\frac{\partial w_\varepsilon}{\partial z}\right]^{-1} (\varepsilon f_{av}(z)+\varepsilon p'(t,z,\varepsilon))$$

$$=\varepsilon f_{av}(z)+\varepsilon p(t,z,\varepsilon) \qquad z(0)=x_0 \tag{A2.2.1.20}$$

where:

$$p(t,z,\varepsilon)=\left[I+\varepsilon\frac{\partial w_\varepsilon}{\partial z}\right]^{-1}\left[p'(t,z,\varepsilon)-\varepsilon\frac{\partial w_\varepsilon}{\partial z} f_{av}(z)\right] \tag{A2.2.1.21}$$

and:

$$\|p(t,z,\varepsilon)\| \le \frac{1}{1-\xi_1(\varepsilon_1)}(\xi_1(\varepsilon)+\xi_1(\varepsilon)l_1+\varepsilon l_2+\xi_1(\varepsilon)l_{av})\|z\|$$

$$:=\psi_1(\varepsilon)\|z\| \tag{A2.2.1.22}$$

for all $t \ge 0$, $\varepsilon \le \varepsilon_1$, $z \in B_r$.

Q.E.D.

Proof of Lemma 2.2.1.6:

Applying Lemma 2.2.1.2, we see that there exists a class K function $\xi_1(\varepsilon)$ such that (2.2.1.38) is satisfied. Then by Lemma 2.2.1.1, it follows that

$$\|\varepsilon w_\varepsilon(t,x)\| \le \xi_1(\varepsilon)\|x\| + \bar\xi_1(\varepsilon) \tag{A2.2.1.23}$$

for some $\bar\xi_1(\varepsilon) \in K$. Denote:

$$h := 1 + \frac{\xi_1(\varepsilon_0)}{r} \qquad \text{(A2.2.1.24)}$$

It is clear that (2.2.1.36) and (2.2.1.37) will thus be satisfied.

Q.E.D.

Proof of Lemma 2.2.1.7:

This proof is similar to that of Lemma 2.2.1.3.

Step 1: Consider the transformation (2.2.1.39). Inequalities (2.2.1.36) implies that:

$$\|z\| - h\,\xi_1(\varepsilon)r \le \|x\| \le \|z\| + h\,\xi_1(\varepsilon)r \qquad \text{(A2.2.1.25)}$$

Define:

$$r'(\varepsilon) := r\,(1 - h\,\xi_1(\varepsilon)) \qquad \text{(A2.2.1.26)}$$

and let ε_1 be such that $h\,\xi_1(\varepsilon_1) < 1$ (this also implies $\xi_1(\varepsilon_1) < 1$). To show that claims (i) and (ii) in the proof of Lemma 2.2.1.3 will also hold here, it suffices to show that the mapping:

$$F_x(z) = x - \varepsilon w_\varepsilon(t, z) \qquad \text{(A2.2.1.27)}$$

is still a contraction mapping in B_r for all $x \in B_{r'}$. However, this fact directly follows from (2.2.1.38). Therefore, the transformation (2.2.1.39) is well defined.

Step 2: Applying Lemma 2.2.1.6, the conclusion (2.2.1.41) simply follows from (A2.2.1.18) and (A2.2.1.21).

Q.E.D.

Proof of Lemma 2.2.2.1:

We apply Lemma 2.2.1.2, and obtain a result similar to Lemma 2.2.1.3. Consider the transformation of variable:

$$x = z + \varepsilon w_\varepsilon(t, z) \qquad \text{(A2.2.2.1)}$$

with $\varepsilon \leq \varepsilon_1$. This transformation leads to:

$$\dot{z} = (I + \varepsilon \frac{\partial w_\varepsilon}{\partial z})^{-1} \varepsilon \Big\{ f_{av}(z) + (f(t,z,0,0) - f_{av}(z) - \frac{\partial w_\varepsilon}{\partial t})$$

$$+ (f(t,z+\varepsilon w_\varepsilon,0,0) - f(t,z,0,0))$$

$$+ (f(t,z+\varepsilon w_\varepsilon,0,\varepsilon) - f(t,z+\varepsilon w_\varepsilon,0,0))$$

$$+ (f(t,z+\varepsilon w_\varepsilon,y,\varepsilon) - f(t,z+\varepsilon w_\varepsilon,0,\varepsilon)) \Big\} \qquad \text{(A2.2.2.2)}$$

or:

$$\dot{z} = \varepsilon f_{av}(z) + \varepsilon p_1(t,z,\varepsilon) + \varepsilon p_2(t,z,y,\varepsilon) \qquad z(0)=x_0 \qquad \text{(A2.2.2.3)}$$

where:

$$\|p_1(t,z,\varepsilon)\| \leq \frac{1}{1-\xi_2(\varepsilon_1)} (\xi_2(\varepsilon) l_{av} + \xi_2(\varepsilon) + \xi_2(\varepsilon) l_1 + (1+\xi_2(\varepsilon)) l_5) \|z\|$$

$$:= \xi_2(\varepsilon) k_1 \|z\| \qquad \text{(A2.2.2.4)}$$

and:

$$\|p_2(t,z,y,\varepsilon)\| \leq \frac{1}{1-\xi_2(\varepsilon_1)} l_2 \|y\| := k_2 \|y\| \qquad \text{(A2.2.2.5)}$$

Q.E.D.

Proof of Lemma 2.2.2.4:

The proof is similar to that of Lemma 2.2.2.1. The result directly follows from Lemma 2.2.1.6 and (A2.2.2.2) in Lemma 2.2.2.1.

Q.E.D.

APPENDIX B

Lemma B1: (Unique Parameters for Adaptive Identifiers)

Consider an SISO plant described by (3.2.1) and an adaptive identifier with its structure shown in Fig. 3.2.1. Let assumptions (A1)-(A2) be satisfied.

Then there exists unique parameters $\theta^* \in R^{2n+1}$ such that, if the identifier parameters $\theta(t)$ is fixed at θ^*, the transfer function from the input u to the output of the identifier y_o is identical to that of the plant.

Proof: Denote $\theta = [\, C^T \, , D^T \, , c_{n+1} \,]^T$, and define:

$$\hat{C}(s) := C^T \hat{F}_1(s)\hat{\Lambda}(s)$$

$$= c_1 s^{n-1} + \cdots + c_n \tag{B3.2.1}$$

and:

$$\hat{D}(s) := D^T \hat{F}_2(s)\hat{\Lambda}(s)$$

$$= d_1 s^{n-1} + \cdots + d_n \tag{B3.2.2}$$

Let $\hat{H}_{y_o u}(\theta, s)$ be the transfer function from the input u to the output of the identifier y_o when identifier parameters $\theta(t)$ are fixed at θ. It then follows that:

$$\hat{H}_{y_o u}(\theta, s) = \frac{\hat{C}(s)}{\hat{\Lambda}(s)} + \frac{\hat{D}(s)}{\hat{\Lambda}(s)}\hat{p}(s) + c_{n+1} \tag{B3.2.3}$$

Let $\theta^* = [\, C^{*T} \, , D^{*T} \, , c^*_{n+1} \,]^T$ be such that:

$$\hat{C}^*(s) + c^*_{n+1} = k_p \, \hat{n}_p(s) \tag{B3.2.4}$$

and:

$$\hat{D}^*(s) = \hat{\Lambda}(s) - \hat{d}_p(s) \tag{B3.2.5}$$

where $\hat{C}^*(s)$ and $\hat{D}^*(s)$ are defined similarly as in (B3.2.1) and (B3.2.2). Then we have:

$$\hat{H}_{y_o u}(\theta^*, s) = \hat{p}(s) \tag{B3.2.6}$$

Now let any θ be such that:

$$\hat{H}_{y_o u}(\theta, s) = \hat{p}(s) \tag{B3.2.7}$$

and then, from (B3.2.3), θ must satisfy:

$$\frac{\hat{C}(s) + c_{n+1}\hat{\Lambda}(s)}{\hat{\Lambda}(s) - \hat{D}(s)} = \hat{p}(s) = k_p \frac{\hat{n}_p(s)}{\hat{d}_p(s)} \tag{B3.2.8}$$

Since $\hat{D}(s)$ and $\hat{C}(s)$ are of the order at most $n-1$, and $\hat{\Lambda}(s)$ is of the order n, it then follows from (B3.2.8) that the solution of the above equation must be unique. Consequently, from (B3.2.6), we can readily conclude that θ^* is the unique choice of parameters such that (B3.2.7) is satisfied.

Q.E.D.

Lemma B2: (Linear Filter Lemma)

Suppose that $y = \hat{H}(s)(u)$, where $\hat{H}(s): C \to C^m$ is a stable transfer function. Then y is stationary if u is stationary, and the power spectral measure $S_y(d\omega)$ is related to $S_u(d\omega)$ by the following:

$$S_y(d\omega) = \hat{H}(j\omega)\, S_u(d\omega)\, \hat{H}^*(j\omega) \tag{B3.2.9}$$

Proof: See Boyd and Sastry (1987).

APPENDIX C

Lemma C1: (Unique Parameters for Model Reference Adaptive Controllers)

Consider an SISO plant and a reference model described respectively by (4.2.1) and (4.2.2), and a model reference adaptive controller with its structure shown in Fig. 4.2.1. Let assumptions (A1)-(A5) be satisfied.

Then there exist unique parameters $\theta^* \in R^{2n}$ such that, if the controller parameters $\theta(t)$ are fixed at θ^*, then the transfer function from the input u to the output of the plant is identical to that of the reference model.

Proof: See Bodson (1987) (Ph.D. Dissertation, p. 52)

Lemma C2:

Let Supp(u) contains $2m$ points, where $m \geq n$.

Then

$$\lambda_{\min}\left[R_{\zeta_m}(0)^{-1/2}\, SM(R_{\zeta_m \zeta_{mf}}(0)\right] R_{\zeta_m}(0)^{-1/2}) \tag{C4.3.1.1}$$

$$\geq \frac{1}{c_0^*} \min_{\omega \in \mathrm{Supp}(u)} \mathrm{Re}\, \hat{m}(j\omega)\hat{l}(j\omega)$$

where $\zeta_m = \hat{l}(s)^{-1}(w_m)$.

Proof: By hypothesis and Theorem 4.3.1.3, ζ_m is PE so that $R_{\zeta_m}(0)$ and $R_{\zeta_m \zeta_{mf}}(0)$ are positive definite. Now from (4.3.1.42), we have:

$$R_{\zeta_m}(0) = \frac{1}{2\pi} \sum_{i=1}^{m} \left[\hat{H}_{\zeta_m u}(j\omega_i) \hat{H}^*_{\zeta_m u}(j\omega_i) + \hat{H}_{\zeta_m u}(-j\omega_i) \hat{H}^*_{\zeta_m u}(-j\omega_i) \right] t_i^2$$

$$= M_{\zeta_m}{}^T W^2 M_{\zeta_m} \tag{C4.3.1.2}$$

where t_i^2 is the spectral magnitude of the frequency element ω_i, and:

$$M_{\zeta_m} := \left[\hat{H}_{\zeta_m u}(j\omega_1), \cdots, \hat{H}_{\zeta_m u}(-j\omega_m) \right] \tag{C4.3.1.3}$$

$$W^2 = \mathrm{diag} \left[\mathrm{diag} \left(\frac{t_1^2}{2\pi}, \frac{t_1^2}{2\pi} \right), \cdots, \mathrm{diag} \left(\frac{t_m^2}{2\pi}, \frac{t_m^2}{2\pi} \right) \right] \tag{C4.3.1.4}$$

Similarly, from (4.3.1.41), we express $SM(R_{\zeta_m \zeta_{mf}}(0))$ as follows:

$$SM(R_{\zeta_m \zeta_{mf}}(0)) = \frac{1}{2\pi c_0^*} \sum_{i=1}^{m} \mathrm{Re}(\hat{m}(j\omega_i)\hat{l}(j\omega_i)) \left[\hat{H}_{\zeta_m u}(j\omega_i) \hat{H}^*_{\zeta_m u}(j\omega_i) \right.$$

$$\left. + \hat{H}_{\zeta_m u}(-j\omega_i) \hat{H}^*_{\zeta_m u}(-j\omega_i) \right] t_i^2$$

$$:= M_{\zeta_m}{}^T \, W \, D \, W \, M_{\zeta_m} \tag{C4.3.1.5}$$

where

$$D = \frac{1}{c_0^*} \mathrm{diag} \left[\mathrm{diag} \left[\mathrm{Re}\,\hat{m}(j\omega_1)\hat{l}(j\omega_1), \mathrm{Re}\,\hat{m}(j\omega_1)\hat{l}(j\omega_1) \right], \cdots \right.$$

$$\left. \cdots, \mathrm{diag} \left[\mathrm{Re}\,\hat{m}(j\omega_m)\hat{l}(j\omega_m), \mathrm{Re}\,\hat{m}(j\omega_m)\hat{l}(j\omega_m) \right] \right] \tag{C4.3.1.6}$$

Thus the matrix $R_{\zeta_m}(0)^{-1/2} SM(R_{\zeta_m \zeta_{mf}}(0)) R_{\zeta_m}(0)^{-1/2}$ in terms of W, D, and M_{ζ_m} as follows:

$$R_{\zeta_m}(0)^{-1/2} SM(R_{\zeta_m \zeta_{mf}}(0)) R_{\zeta_m}(0)^{-1/2}$$

$$= \left[M_{\zeta_m}{}^T W^2 M_{\zeta_m} \right]^{-1/2} \left[M_{\zeta_m}{}^T W D W M_{\zeta_m} \right] \left[M_{\zeta_m}{}^T W^2 M_{\zeta_m} \right]^{-1/2}$$

$$= \tilde{M}_{\zeta_m}{}^T \, D \, \tilde{M}_{\zeta_m} \tag{C4.3.1.7}$$

where

$$\tilde{M}_{\zeta_m} := W \, M_{\zeta_m} \left[M_{\zeta_m}{}^T W^2 M_{\zeta_m} \right]^{-1/2} \tag{C4.3.1.8}$$

Now let x be any unit vector in R^{2n}, i.e. $\|x\|=1$, and then, from (C4.3.1.7), we have:

$$\lambda_{\min}(D)\|\tilde{M}_{\zeta_m} x\|^2 \le \lambda_{\min}(R_{\zeta_m}(0)^{-1/2} SM(R_{\zeta_m \zeta_{mf}}(0)) R_{\zeta_m}(0)^{-1/2})$$

$$\le \lambda_{\max}(D)\|\tilde{M}_{\zeta_m} x\|^2 \qquad \text{(C4.3.1.9)}$$

which can further be simplified as:

$$\lambda_{\min}(D) \le \lambda_{\min}(R_{\zeta_m}(0)^{-1/2} SM(R_{\zeta_m \zeta_{mf}}(0)) R_{\zeta_m}(0)^{-1/2}) \le \lambda_{\max}(D) \qquad \text{(C4.3.1.10)}$$

From the definition of D in (C4.3.1.5), we have:

$$\lambda_{\min}(D) = \frac{1}{c_0^*} \min_{\omega_i \in \mathrm{Supp}(u)} \mathrm{Re}\,\hat{m}(j\omega_i)\hat{l}(j\omega_i) \qquad \text{(C4.3.1.11)}$$

thereby concluding the result.

Q.E.D.

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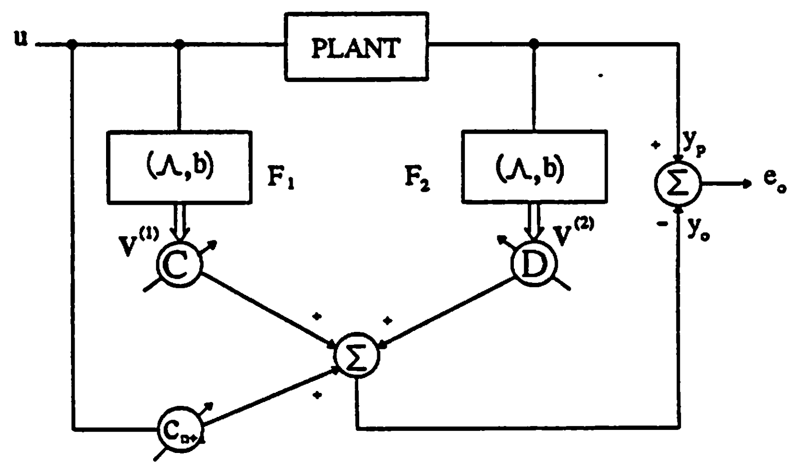
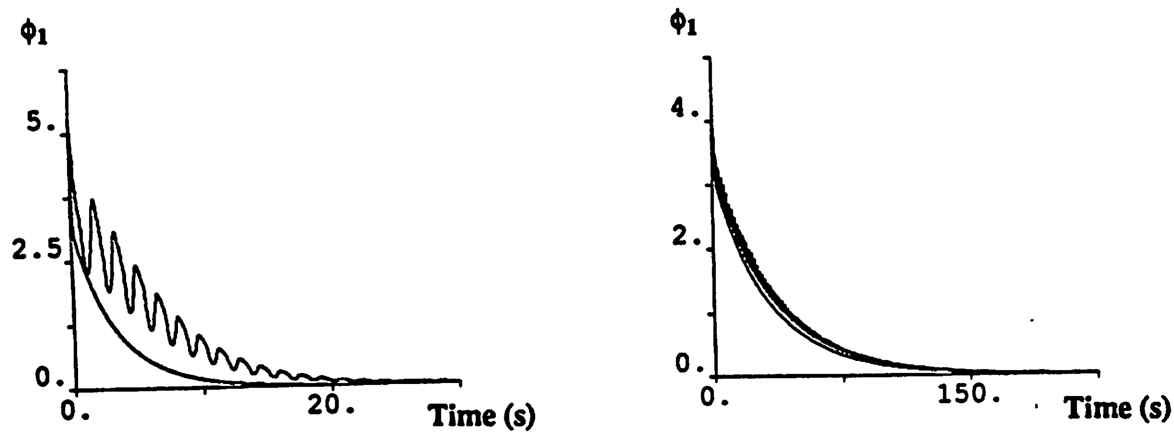


Figure 3.2.1 Adaptive Identifier



(a) $\varepsilon = 1$ (b) $\varepsilon = 0.1$

Figure 3.3.1 Parameter Error ϕ_1

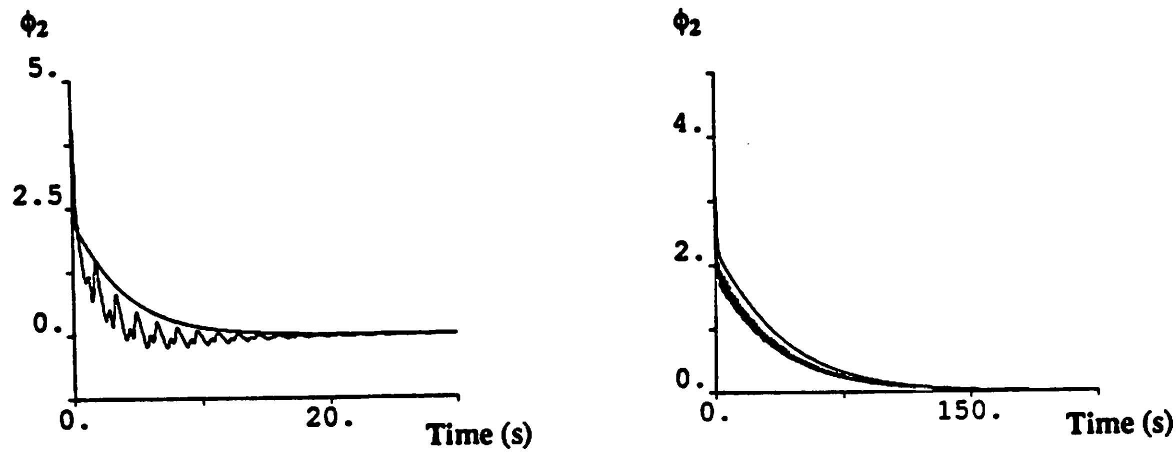


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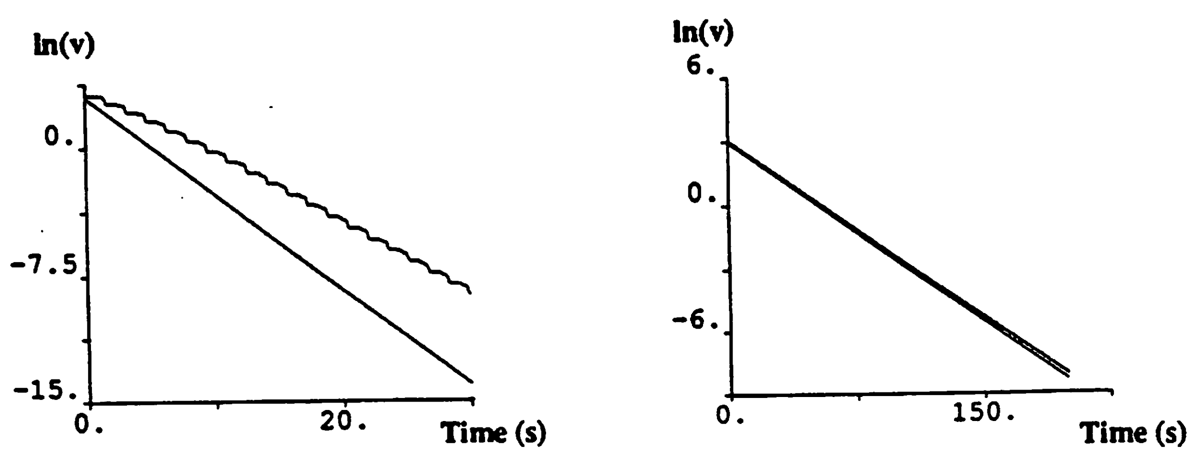
Figure 3.3.2 Parameter Error ϕ_2



(a) $\varepsilon = 1$ (b) $\varepsilon = 0.1$

Figure 3.3.3 Logarithm of the Lyapunov Function $v(\phi)$

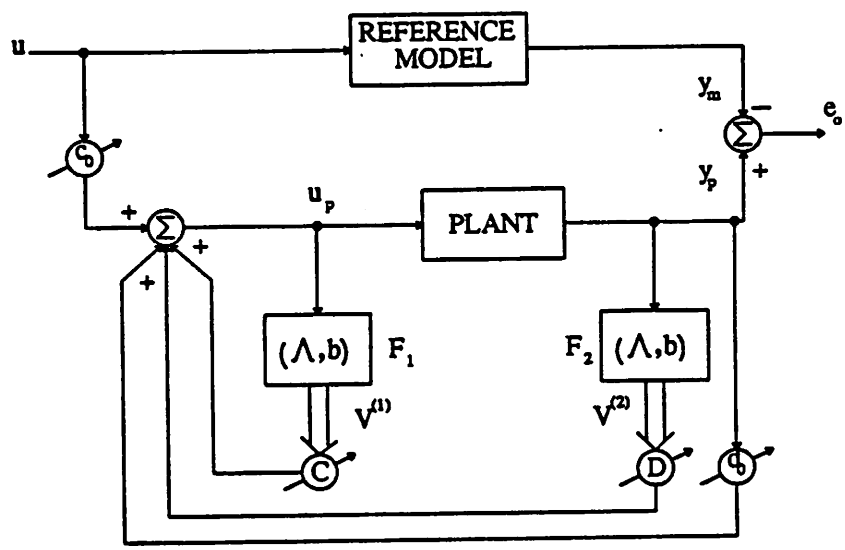


Figure 4.2.1 Model Reference Adaptive Controller

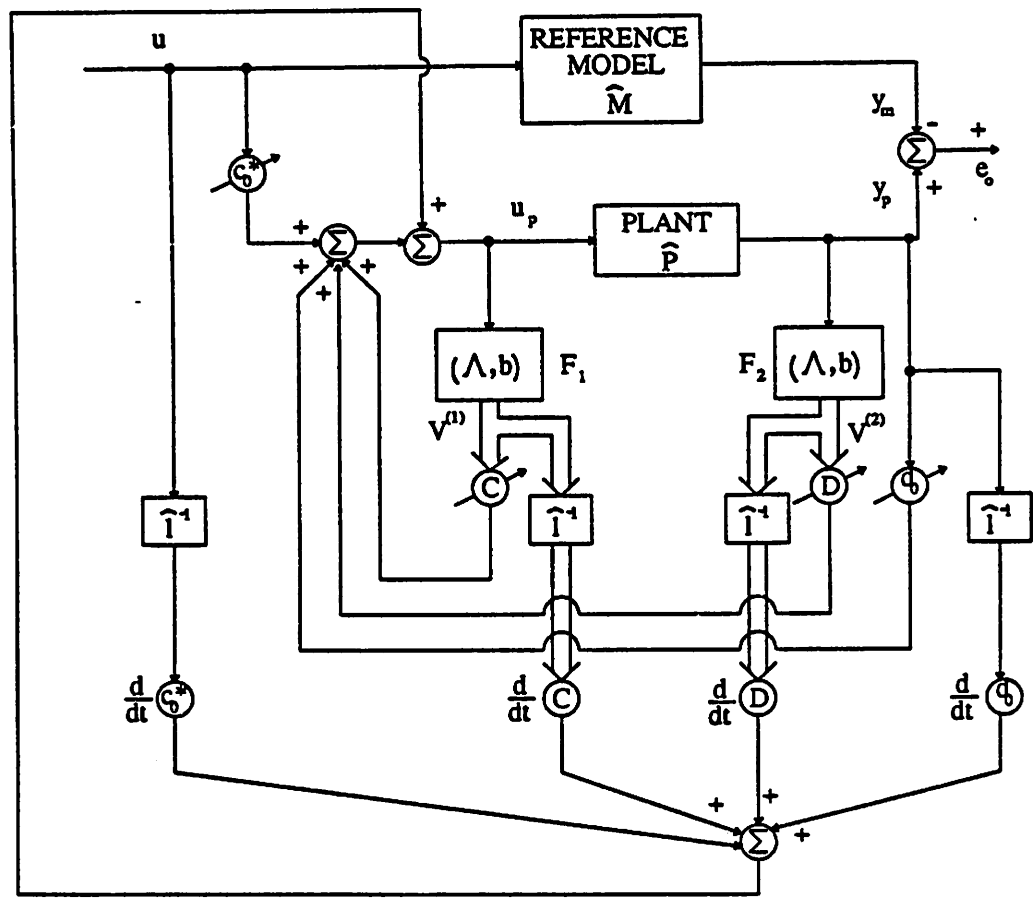


Figure 4.2.2 Modified Model Reference Adaptive Controller

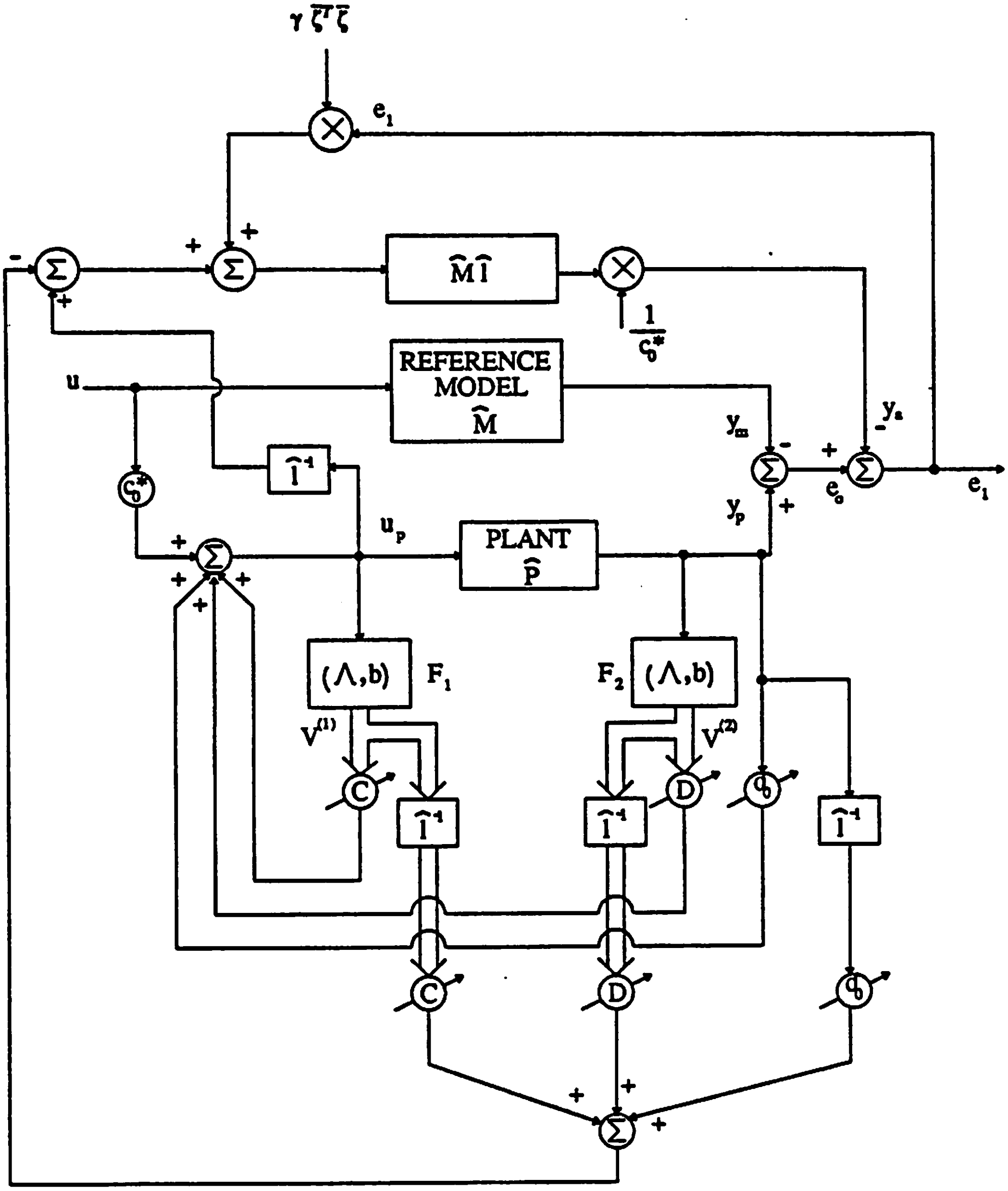
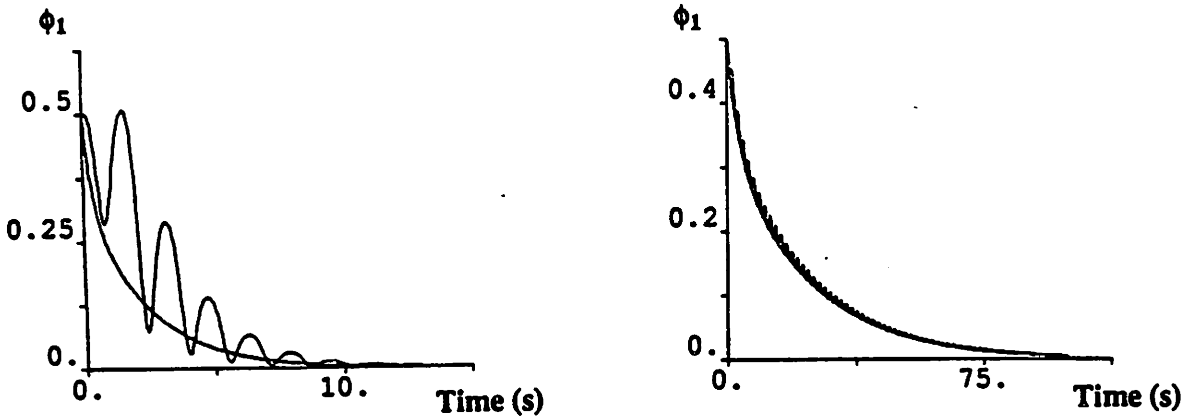
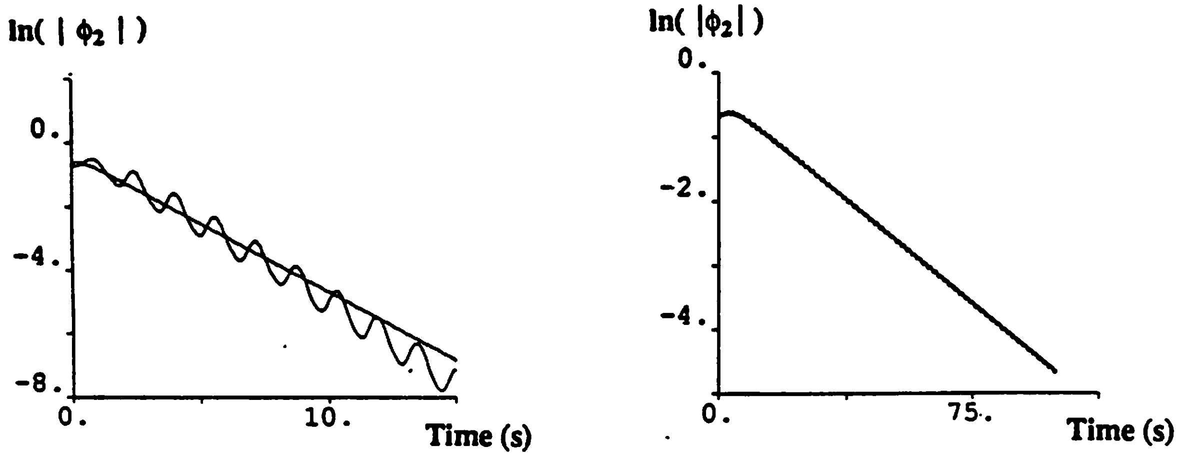


Figure 4.2.3 Modified Model Reference Adaptive Controller



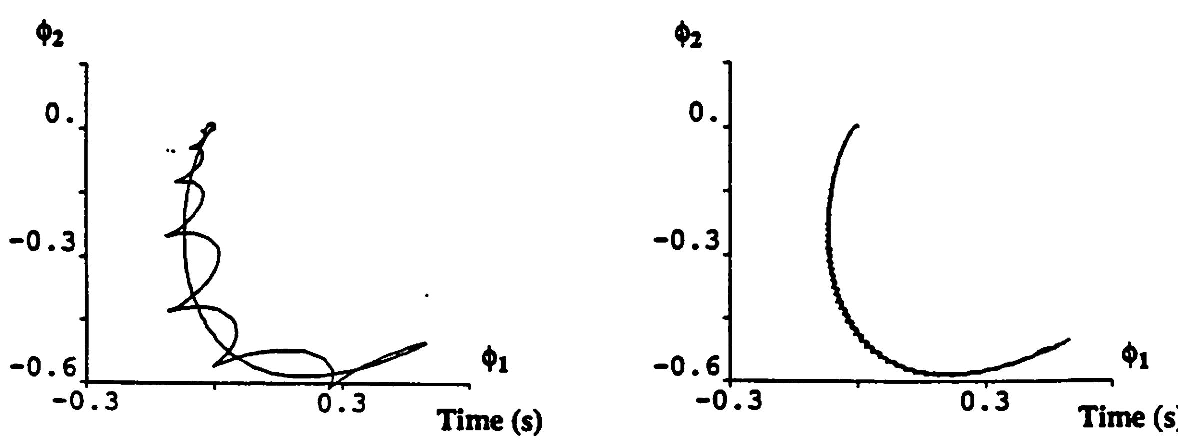
(a) $\varepsilon = 1$ (b) $\varepsilon = 0.1$

Figure 4.3.1 Parameter Error ϕ_1 ($u = \sin 2t$)



(a) $\varepsilon = 1$ (b) $\varepsilon = 0.1$

Figure 4.3.2 Logarithm of the Parameter Error $|\phi_2|$ ($u = \sin 2t$)



(a) $\varepsilon = 1$ (b) $\varepsilon = 0.1$

Figure 4.3.3 Phase Plot of the Parameter Errors $\phi_2(\phi_1)$ ($u = \sin 4t$)

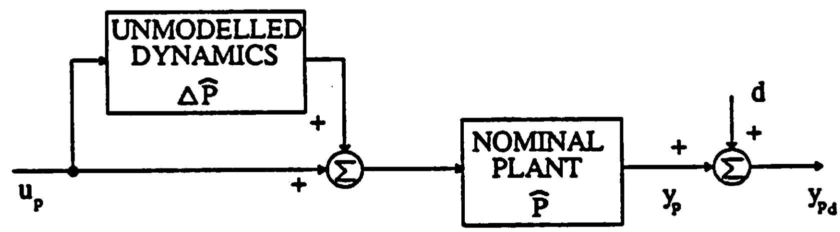


Figure 4.4.1 Unmodellde Dynamics and Output Disturbances

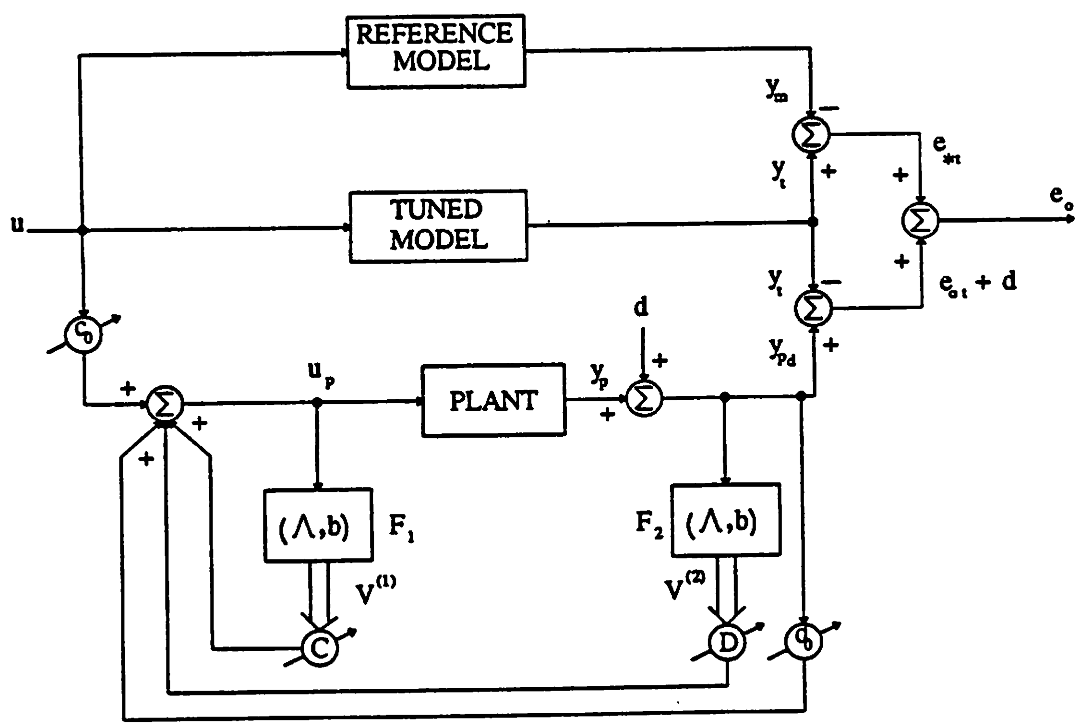


Figure 4.4.2 Reduced Order Controller

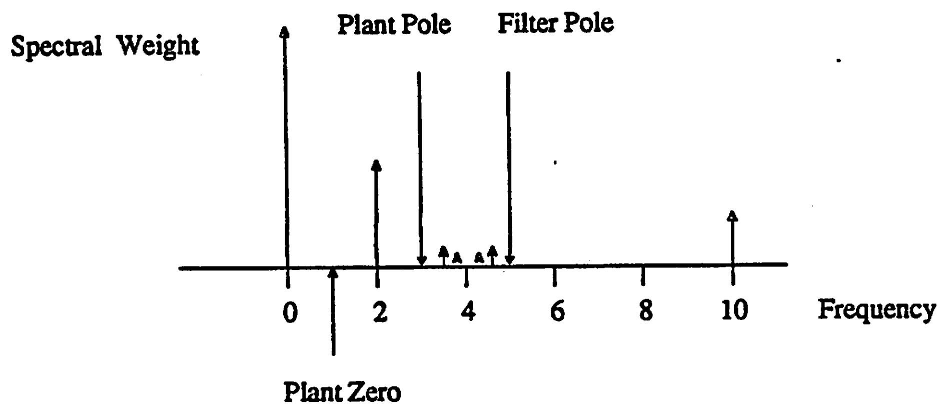
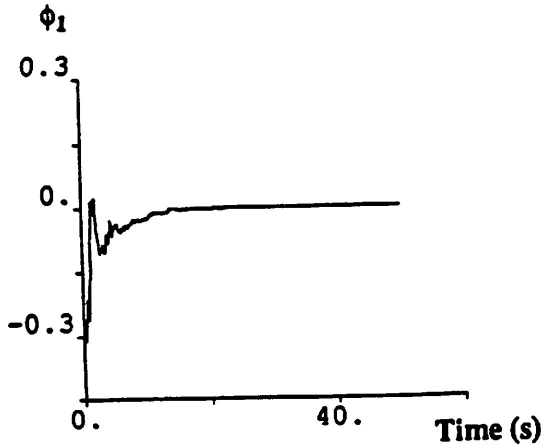
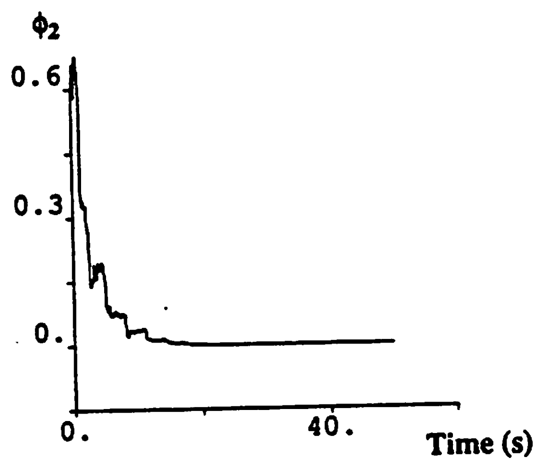


Figure 5.4.1 Spectral Distribution



(a) Parameter Error ϕ_1



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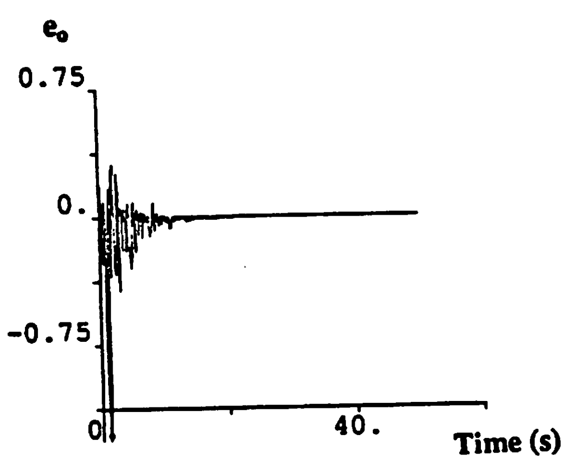
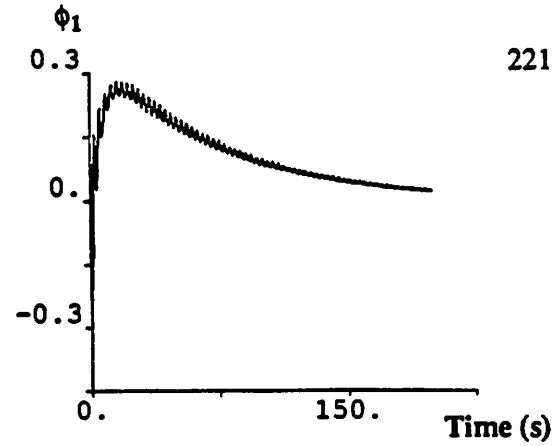
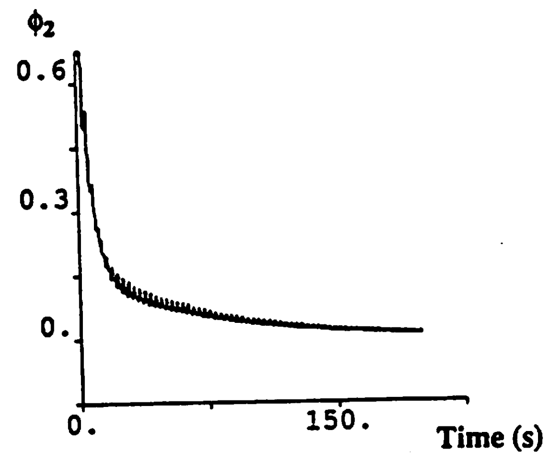


(c) Output Error e_o

Figure 5.4.2 Optimal Input Design



(a) Parameter Error ϕ_1



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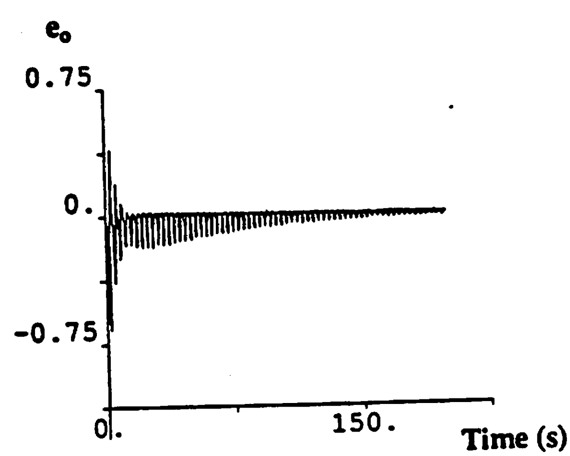
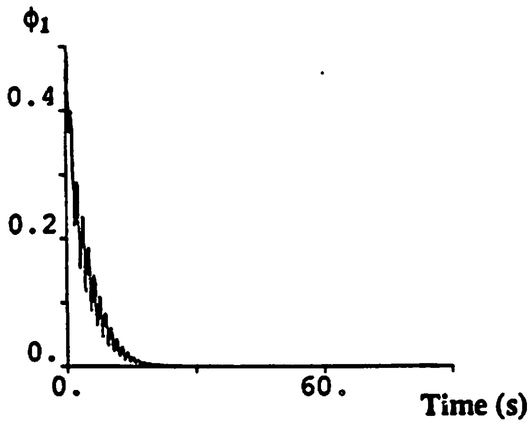
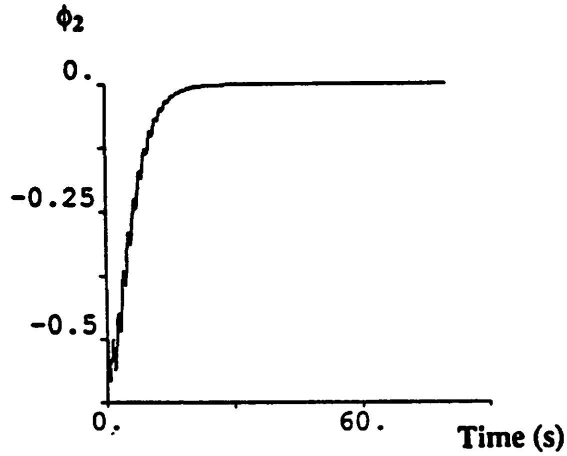


(c) Output Error e_o

Figure 5.4.3 Initial Input Design



(a) Parameter Error ϕ_1



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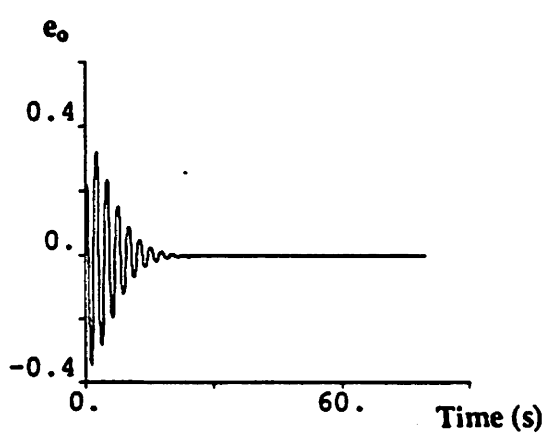
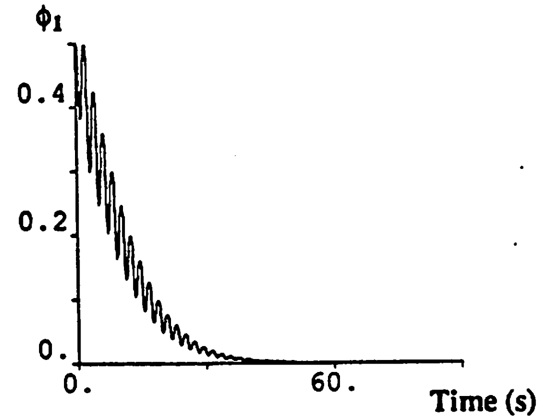
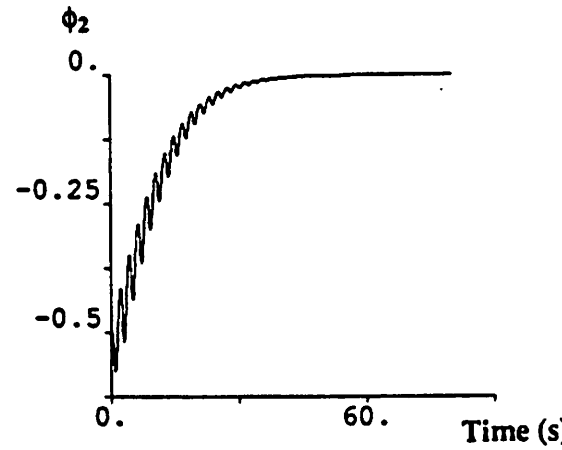


(c) Output Error e_o

Figure 5.4.4 Optimal Input Design



(a) Parameter Error ϕ_1



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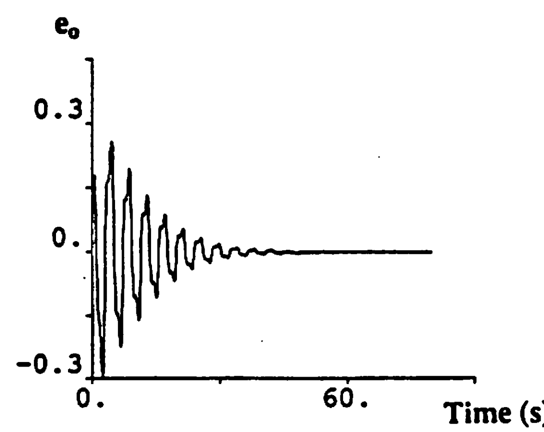


(c) Output Error e_o

Figure 5.4.5 Initial Input Design

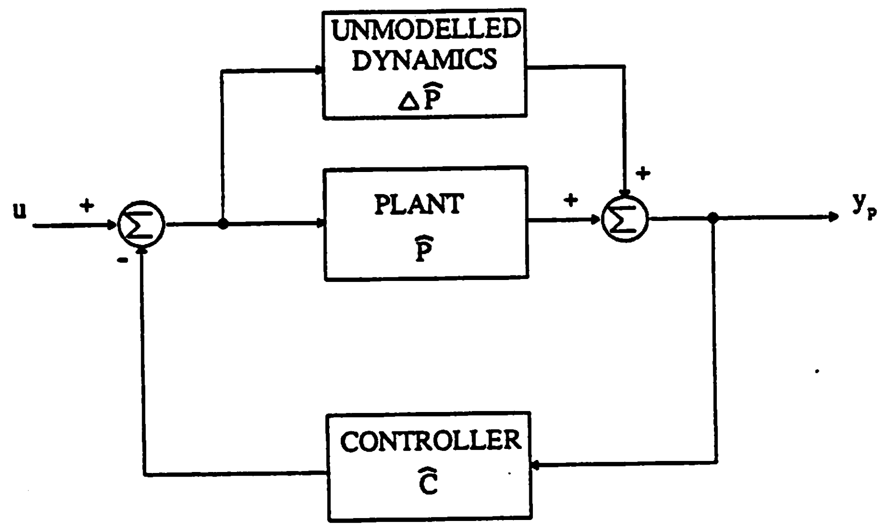


Figure 5.5.1 [1] S(P,C) Loop

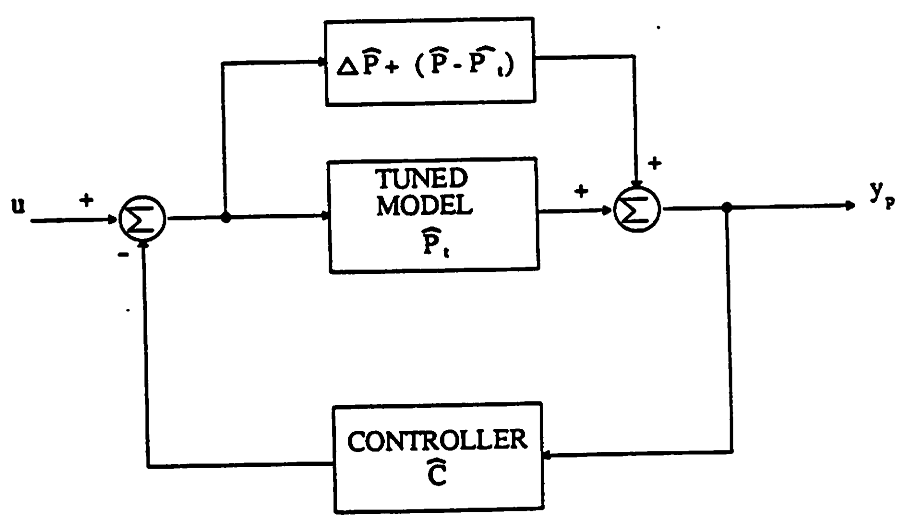
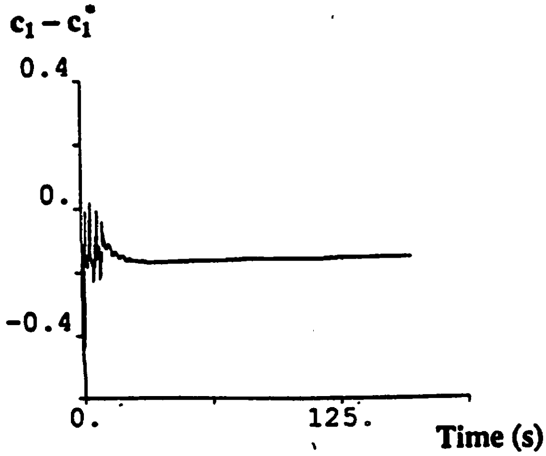
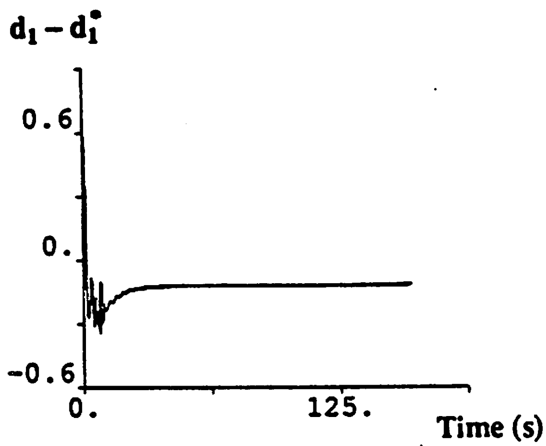


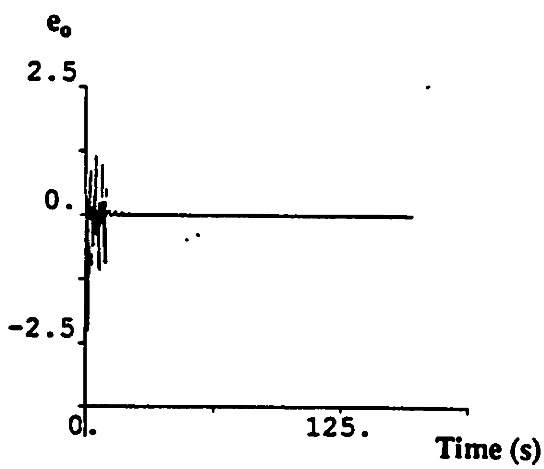
Figure 5.5.2 [1] S(P,C) Loop



(a) Parameter Error $c_1 - c_1^*$

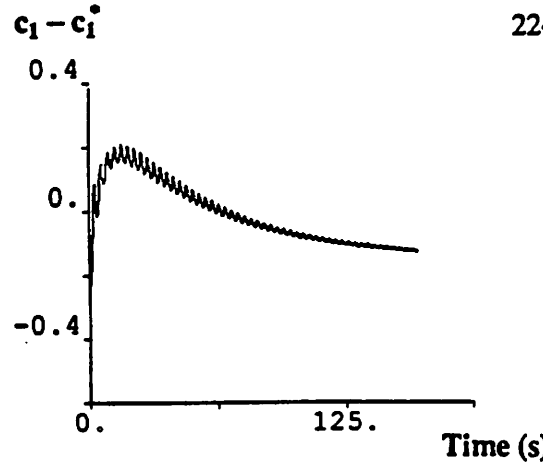


(b) Parameter $d_1 - d_1^*$

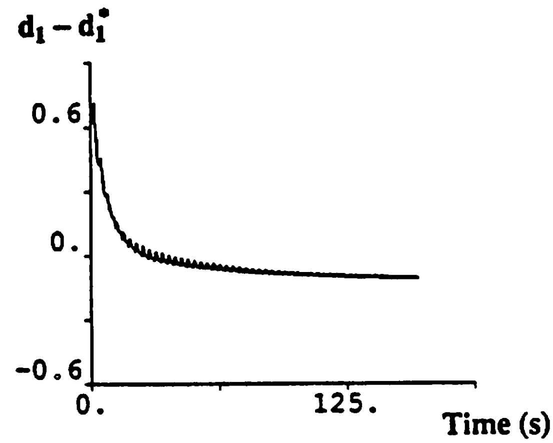


(c) Output Error e_o

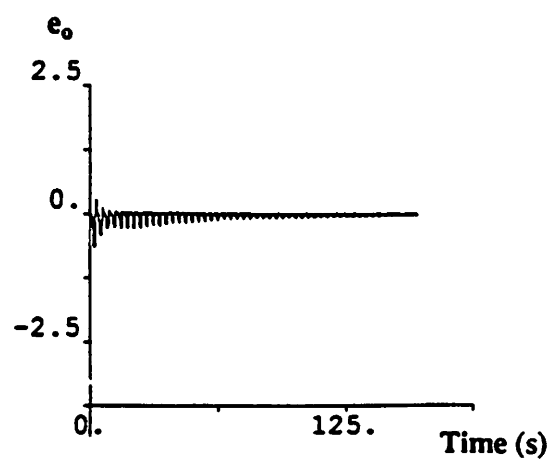
Figure 5.5.3 Two-Phase Input Design



(a) Parameter Error $c_1 - c_1^*$



(b) Parameter Error $d_1 - d_1^*$



(c) Output Error e_o

Figure 5.5.4 Single-Phase Input Design

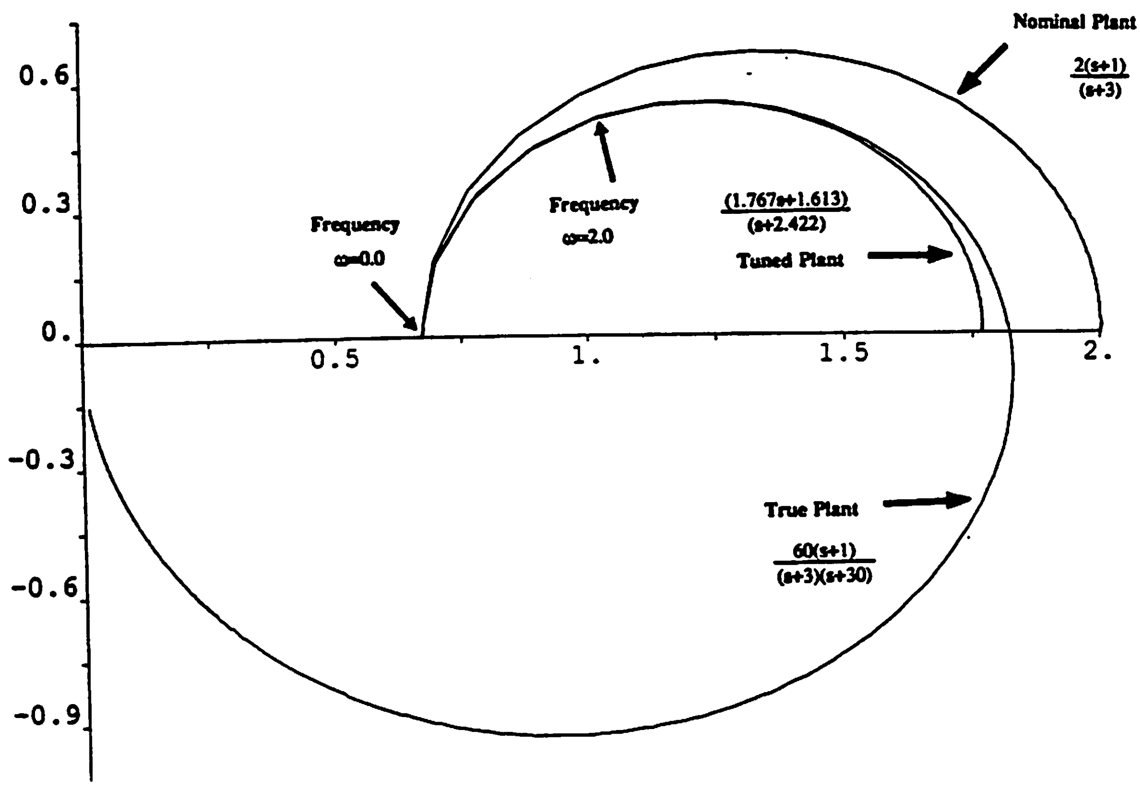
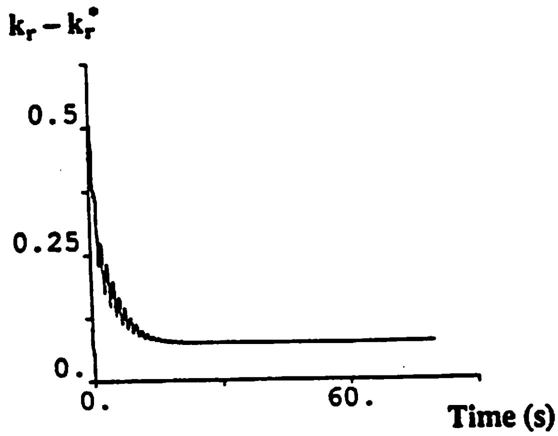
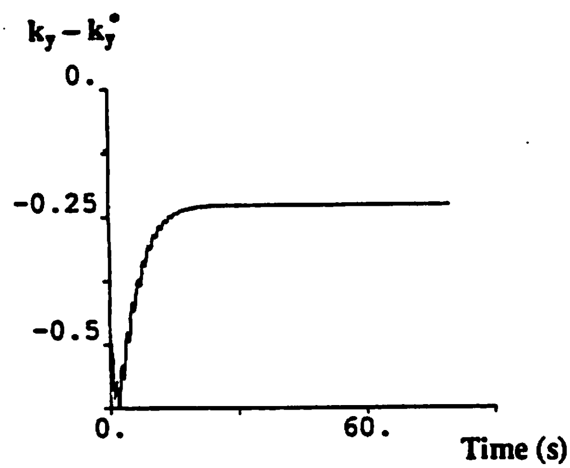


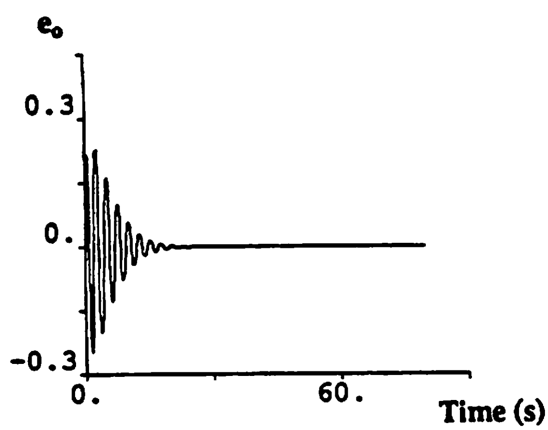
Figure 5.5.5 Nyquist Plots



(a) Parameter Error $k_r - k_r^*$

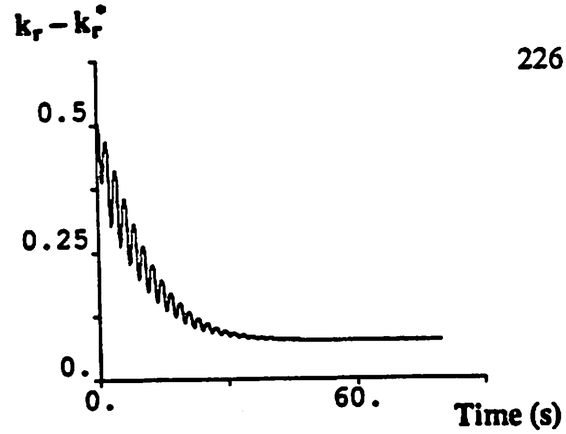


(b) Parameter Error $k_y - k_y^*$

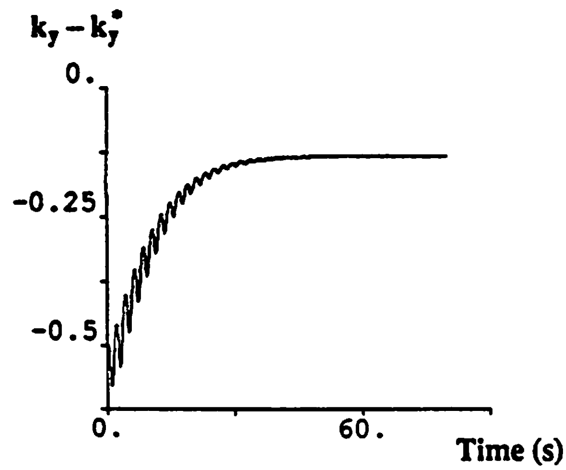


(c) Output Error e_o

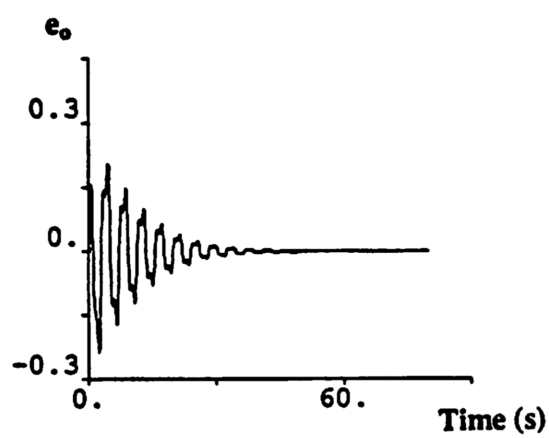
Figure 5.5.6 Optimal Input Design



(a) Parameter Error $k_r - k_r^*$



(b) Parameter Error $k_y - k_y^*$



(c) Output Error e_o

Figure 5.5.7 Initial Input Design

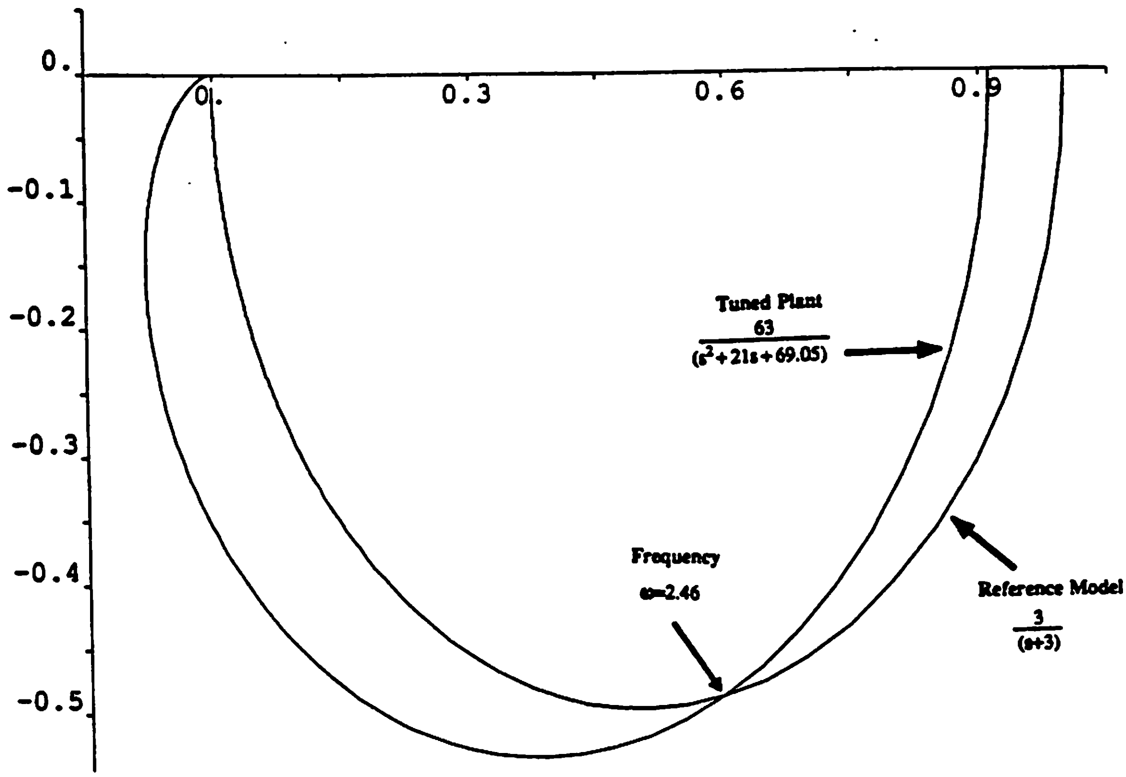


Figure 5.5.8 Nyquist Plots