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ABSTRACT

It is shown that recent, nondifferentiable optimization algorithms can be used for the solution of H^∞ formulated optimal design problems of linear, lumped, time-invariant, multivariable feedback systems, subject to various frequency and time-domain performance specifications.

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1. INTRODUCTION

Feedback is used to achieve various desirable properties in a control system, such as stability, disturbance attenuation, and low sensitivity to changes in the plant. Since these properties depend on the shape of various feedback loop responses, all control system design techniques are at least partially based on loop shaping. Currently one of the most popular and powerful techniques for shaping a *single* loop (or a composite loop which can be made to look like a single loop) is based on the *unconstrained* minimization of the weighted H^∞-norm of the corresponding transfer function matrix. A key element of the H^∞ technique is the use of the parametrization of stabilizing controllers described in [You.1], [Des.1], which makes all the transfer functions of the closed-loop feedback system in Fig.1 affine in the the controller parameter and hence leads to a *convex* unconstrained minimization problem in H^∞.

The H^∞ minimization technique was first proposed by Zames [Zam.1] who applied it to a scalar weighted sensitivity function, see also [Fra.2]. Since then at least three approaches to the solution of H^∞-norm *unconstrained* minimization of a weighted, affinely parametrized *transfer function matrix*, have emerged, see, e.g., [Fra.1], [Cha.1], [Glo.1] (for a survey see [Fra.3]). Software is either being written or is available for all of the methods cited above [Hel.1].

At present there is considerable interest in developing more powerful control system design techniques which permit simultaneous shaping of *several frequency and time domain responses*, some of which may be subject to constraints in the form of hard bounds. Such problems are beyond the scope of the techniques mentioned above.

In this paper we give a brief introduction to the new nondifferentiable optimization algorithms which were presented in [Pol.1] and show that they can be used in the H^∞ design of compensators, for feedback systems of the form in Fig. 1, which are required to shape both time and frequency domain responses. The use of our new algorithms requires the expansion of the H^∞ controller parameter in a series and the development of formulae for the *search direction maps* which are required by the algorithms. We show that our new optimization algorithms [Pol.1] can be used to get an arbitrarily good solution to the design problem and we give an example to illustrate the type of result our computations produce.

2. FORMULATION OF OPTIMAL COMPENSATOR DESIGN PROBLEM IN $\mathbb{R}_U(s)^{n_i \times n_o}$

We begin by transcribing a typical compensator design problem into a form suitable for solution by nondifferentiable optimization algorithms. Consider the feedback system $S(P,K)$ shown in Figure 1 where the plant has the state space representation

$$\dot{z} = Az + Be_2, \qquad y_2 = Cz + De_2, \tag{2.1}$$

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_i}$, $C \in \mathbb{R}^{n_o \times n}$, $D \in \mathbb{R}^{n_o \times n_i}$. We assume that (A,B) is stabilizable and (C,A) is detectable. Hence the plant transfer function $P(s) = C(sI - A)^{-1}B + D \in \mathbb{R}(s)^{n_o \times n_i}$, the space of matrices whose entries are proper rational functions. The compensator is specified only in transfer function form, with the entries of $K(s)$ in $\mathbb{R}(s)^{n_i \times n_o}$.

Let $\mathbf{U} \triangleq \{ s \in \mathbb{C} \mid \mathrm{Re}(s) > -\alpha_U \}$, with $\alpha_U \geq 0$, and let $\mathbb{R}_U(s)$ be the set of rational functions that are bounded and analytic in $\mathbf{U}$. We assume that the compensator K must ensure the internal stability [Cal.1, Vid.1] of the feedback system $S(P,K)$, i.e., that all the transfer functions in the feedback loop are in $\mathbb{R}_U(s)$. We make use of the following characterization of the class of stabilizing feedback compensators (see [Zam.1, You.1, Des.1]).

Theorem 2.1: Let $P = N_r D_r^{-1} = D_l^{-1} N_l$ be right and left coprime factorizations of P, with entries in $\mathbb{R}_U(s)$, and let U_r, V_r, N_l, U_l, V_l, with entries in $(\mathbb{R}_U(s))$ satisfy the Bezout identities $U_r N_r + V_r D_r = I_{n_i}$, $N_l U_l + D_l V_l = I_{n_o}$. Then, the set of stabilizing feedback compensators is given by

$$\mathbf{K} \triangleq \{ (V_r - RN_l)^{-1}(U_r + RD_l) \mid R \in \mathbb{R}_U(s)^{n_i \times n_o},\ \det(V_r - RN_l) \neq 0 \}. \quad \blacksquare \tag{2.2}$$

If we define the vectors of the Laplace transforms of inputs and outputs by $\hat{\underline{y}} = [\hat{e}_1, \hat{e}_2, \hat{z}_2]$ and $\hat{\underline{u}} = [\hat{u}_1, \hat{u}_2, \hat{d}_o, \hat{d}_s]$, and parametrize the compensator as in (2.2), then $\hat{\underline{y}}(s) = H(R)(s)\hat{\underline{u}}(s)$, where the stable input-output map $H(R)$ is given by

$$H(R) = \begin{bmatrix} -N_r R D_l + V_l D_l & N_r R N_l - N_r V_r & N_r R D_l - V_l D_l & N_r R D_l - V_l D_l \\ D_r R D_l + D_r U_r & -D_r R N_l + D_r V_r & -D_r R D_l - D_r U_r & -D_r R D_l - D_r U_r \\ N_r R D_l + N_r U_r & -N_r R N_l + N_r V_r & -N_r R D_l + V_l D_l & -N_r R D_l - N_r U_r \end{bmatrix}. \tag{2.3}$$

Referring to (2.3), we see that every transfer function in the feedback loop $S(P,K)$ is an *affine function* of the parameter R. Next we note that the requirement that $\det(V_r - RN_l) \neq 0$ is automatically satisfied for R in an open dense subset of $\mathbb{R}_U(s)^{n_i \times n_o}$ [Vid.1] and it is always satisfied when $P(s)$ is strictly proper. Hence one can at first ignore this requirement in finding a parameter $\hat{R} \in \mathbb{R}_U(s)$ which shapes the transfer functions in (2.3) and check at the end that it is not violated.

Frequency domain loop shaping requirements, such as command tracking and disturbance rejection, plant saturation avoidance and stability robustness, have the form

$$\sup_{\omega \in [0,\overline{\omega})} \{ \overline{\sigma}\, [H_{ij}(R)(j\omega)\,] - b_{ij}(\omega) \} \leq 0, \tag{2.4}$$

where $b_{ij}(\cdot)$ is a continuous bound function, $\overline{\omega}$ is large and $i \in \{1,2,3\}$, $j \in \{1,2,3,4\}$.

Time domain constraints, in the form of hard bounds on trajectories, cannot be expressed in the form (2.4) and must be dealt with directly, as we shall illustrate by example. Suppose we are required to confine the response of the *i–th* channel of the plant output to a unit step input in the *j–th* input channel within a prescribed envelope. This leads to a pair of inequalities of the form

$$\max_{t \in [0,t_f]} \{ \mathbf{L}^{-1}\{[H_{31}(R)]_{ij}(s)\, \frac{1}{s}\}(t) - \overline{b}_{ij}(t) \} \leq 0, \tag{2.5a}$$

$$\max_{t \in [0,t_f]} \{ \underline{b}_{ij}(t) - \mathbf{L}^{-1}[H_{31}(R)]_{ij}(s)\, \frac{1}{s}\}(t) \} \leq 0, \tag{2.5b}$$

where $-\overline{b}_{ij}(\cdot)$, $\underline{b}_{ij}(\cdot)$ are upper semicontinuous continuous bound functions and $\mathbf{L}^{-1}$ is the one-sided inverse Laplace transform operator.

More generally, we may wish the plant output to follow, within a tolerance, a given trajectory for a given command input. Let $\hat{u}_d(\cdot)$, $z_d(\cdot)$, be the Laplace transforms of the command input and the desired time-domain plant output and let $b_d(\cdot)$ be a lower semicontinuous, positive tolerance function. Then the tracking requirement can be expressed in the form

$$\max_{t \in [0,t_f]} \{ \| \mathbf{L}^{-1} \{ H_{31}(s)\hat{u}_d(s) \}(t) - z_d(t) \|_2 - b_d(t) \} \leq 0. \tag{2.6}$$

More complex expressions result when one attempts to ensure disturbance rejection and plant saturation avoidance in the time domain. Thus, disturbance rejection can be ensured by requiring that

$$\sup_{|d_0| \le 1} \; \sup_{t \in [0,\infty)} \mid L^{-1}\{ H_{33}(R)(s)\, \hat{d}_0(s) \}(t) \mid - b_o \le 0 , \tag{2.7a}$$

where $b_o > 0$. Note that (2.7a) is equivalent to

$$\bar{\sigma}[\lim_{|s| \to \infty} (H_{33}(s)] + \int_0^{\infty} \bar{\sigma}[L^{-1}\{ (H_{33}(s) - \lim_{|s| \to \infty} (H_{33}(s)\}(t)]dt - b_o \le 0 \quad , \tag{2.7b}$$

We see from the above that control system design specifications can be expressed in terms of inequalities involving three types of performance functions, all of which are defined on $\mathbb{R}_U(s)^{n_i \times n_o}$ and are real-valued.

(i) **Frequency domain performance functions.** These functions have the form

$$f^1(R) \; = \; \max_{\omega \in [0,\bar{\omega})} \{ \bar{\sigma}[(G_l R\, G_r - F)(j\omega)] - b_f(\omega) \} \tag{2.8}$$

where $G_l, R, G_r, F \in \mathbb{R}_U(s)$ and $b_f(\cdot)$ is a positive, bounded, lower semicontinuous function and $\bar{\omega} > 0$ is large.

(ii) **Time domain performance functions.** These functions have the form

$$f^2(R) \; = \; \max_{t \in [0,t_0]} \{ L^{-1}\{ e_j^T(G_l R\, G_r - F)(s) e_i \hat{u}(s) \}(t) - b_1(t) \} \tag{2.9a}$$

or

$$f^3(R) \; = \; \max_{t \in [0,t_0]} \{ \| L^{-1}\{ (G_l R\, G_r - F)(s)\hat{u}(s) \}(t) - d(t) \|_2 - b_2(t) \} \tag{2.9b}$$

where e_k denotes the $k\!-\!th$ unit vector, $G_l, R, G_r, F \in \mathbb{R}_U(s)$, $b_1(\cdot)$, $b_2(\cdot)$, and $d(\cdot)$ are lower semicontinuous functions, $b_1(\cdot)$, $b_2(\cdot)$ are positive and $\hat{u}(\cdot)$ is the Laplace transform of an input signal.

(iii) **Integral time domain performance functions.** These are of the form

$$f^4(R) \; = \; \bar{\sigma}[\lim_{|s| \to \infty} (G_l R\, G_r - F)(s)] \tag{2.10a}$$

$$+ \int_0^{\infty} \bar{\sigma}[L^{-1}\{(G_l R\, G_r - F)(s) - \lim_{|s| \to \infty} (G_l R\, G_r - F)(s)\}(t)]\, dt - b_I ,$$

where $G_l, R, G_r, F \in \mathbb{R}_U(s)$ and $b_I > 0$. In practice, one must replace the indefinite integral (2.10a) by

$$f^4(R) \; = \; \bar{\sigma}[\lim_{|s| \to \infty} (G_l R\, G_r - F)(s)] \tag{2.10b}$$

$$+\int_0^T \bar{\sigma}[\mathbf{L}^{-1}\{(G_l R G_r - F)(s) - \lim_{|s|\to\infty} (G_l R G_r - F)(s)\}(t)]\, dt - b_l \,,$$

for some large $T \in \mathbb{R}_+$.

Theorem 2.2 [Sal.1]: The functions $f^i : \mathbb{R}_U(s)^{n_i \times n_o} \to \mathbb{R}$, $i = 1,2,3,4$, defined in (2.8), (2.9a), (2.9b), (2.10a) and (2.10b), respectively, are convex. Furthermore, if the plant P is strictly proper, i.e., $D = 0$ in (2.1), then,

(i) $f^1 : (\mathbb{R}_U(s)^{n_i \times n_o}, \|\cdot\|_\infty) \to \mathbb{R}$, defined in (2.8), is Lipschitz continuous.

(ii) If $\bar{\sigma}[G_l(s)]\bar{\sigma}[G_r(s)]\, \|u(s)\|_2 = O(s^{-2})$, then the functions $f^2, f^3 : (\mathbb{R}_U(s)^{n_i \times n_o}, \|\cdot\|_\infty) \to \mathbb{R}$ defined in (2.9a), (2.9b), are Lipschitz continuous.

(iii) If $\bar{\sigma}[G_l(s)]\bar{\sigma}[G_r(s)] = O(s^{-2})$, then the function $f^4 : (\mathbb{R}_U(s)^{n_i \times n_o}, \|\cdot\|_\infty) \to \mathbb{R}$, defined in (2.10b), is given by

$$f^4(R) = \int_0^T \bar{\sigma}[\mathbf{L}^{-1}\{(G_l R G_r - F)(s)\}(t)]\, dt \tag{2.11}$$

and is Lipschitz continuous. ∎

Whenever the input $u(\cdot)$ in (2.9a). (2.9b) is such that $\|\hat{u}(s)\|_2 = O(s^{-1})$ (this includes steps, ramps, exponentials), strict properness of the plant implies that $\bar{\sigma}[G_l(s)]\bar{\sigma}[G_r(s)]\, \|\hat{u}(s)\|_2 = O(s^{-2})$ holds for all transfer functions $H_{ij}(R)$ in (2.3) except for $H_{21}(R)$, $H_{23}(R)$, $H_{24}(R)$. Also, whenever $P(s) = O(s^{-2})$, $\bar{\sigma}[G_l(s)]\bar{\sigma}[G_r(s)] = O(s^{-2})$ holds for all transfer functions in (2.3) except for $H_{21}(R)$, $H_{23}(R)$, $H_{24}(R)$.

We can ensure that $\bar{\sigma}[G_l(s)]\bar{\sigma}[G_r(s)] = O(s^{-2})$ is satisfied by all transfer functions in (2.3) in two ways. First, we can replace the domain of definition $\mathbb{R}_U(s)^{n_i \times n_o}$ of the functions $f^j(\cdot)$, $j = 1,2,3,4$ by $\{ Q \in \mathbb{R}_U(s)^{n_i \times n_o} \mid Q(s) = \frac{s_0^2}{(s+s_0)^2} R(s),\ R \in \mathbb{R}_U(s)^{n_i \times n_o} \}$, where $s_0 > \alpha_U$ is chosen to be much larger than the desired feedback system bandwidth. Second, we can weight the affine functions defined in (2.3) by low-pass, wide bandwidth filters, with a roll-off of at least 40 dB per decade. This will have the effect of attenuating high frequency inputs. Either of these methods will have little effect on the achievable performance of the feedback system.

Finally, one can also show Lipschitz continuity of $f^4(\cdot)$ in $(\mathbb{R}_U(s)^{n_i \times n_o}, \|\cdot\|_\infty)$ under the same assumptions as in Theorem 2.2, whenever, the region of analyticity U for which the coprime factorizations in $\mathbb{R}_U(s)^{n_i \times n_o}$ are obtained includes $\{ s \in \mathbb{C} \mid \mathrm{Re}(s) > -\varepsilon \}$ for some $\varepsilon > 0$. This conditions is usually satisfied in practice.

The optimal selection of the compensator parameter R can be formulated in several ways. Thus, suppose we are given a set of frequency and time domain performance functions $\psi^j(\cdot)$, $j \in \underline{m}$, where $\underline{m} = \{ 1, 2, \cdots, m \}$, with each ψ^j of the form of f^1, f^2, f^3 or f^4, and that we are required to obtain a compensator parameter $\hat{R}$ in the "desirable set" $\mathbf{F}$ defined by

$$\mathbf{F} \triangleq \{ R \in \mathbb{R}_U(s)^{n_i \times n_o} \mid \psi^k(R) \leq 0, k \in \underline{m} \} , \tag{2.12}$$

which might, in fact, be empty. Because our performance functions are convex and bounded from below, the problem

$$\mathbf{P}_1 : \quad \inf_{R \in \mathbb{R}_U(s)^{n_i \times n_o}} \max \{ \psi^k(R), k \in \underline{m} \} \tag{2.13}$$

must have finite value γ_1. Furthermore, $\gamma_1 < 0$ if and only if $\mathbf{F}$ is not empty, i.e., if the desired performance requirements are achievable. Our next observation is that if $\gamma_1 < 0$, then an appropriate unconstrained optimization algorithm will compute a parameter $R^* \in \mathbf{F}$ *in a finite number of iterations.*

Once one has obtained an $R^* \in F$, one may elect to tighten the performance requirements. This can be done, for example, by replacing an inequality $\psi^j(R) \leq 0$ with $\psi^j(R) + b^j \leq 0$, with $b^j > 0$. Another possibility is to add new performance functions to the set $\{ \psi^k(R), k \in \underline{m} \}$. In either case the set $\mathbf{F}$ and the problem $\mathbf{P}_1$ become redefined. If this new $\mathbf{F}$ is nonempty, an unconstrained optimization algorithm will produce a new parameter R^* which corresponds to a compensator with better performance.

Alternatively, once a compensator parameter is computed in the desirable set $\mathbf{F}$, one can define a weighted objective function $\psi^0(R) \triangleq \max_{j \in J} w^j \psi^j(R)$, where the ψ^j are performance functions that were omitted in the first round and use a phase II algorithm (see [Pol.1]) to solve to the problem

$$\mathbf{P}_2: \quad \min_{R \in \mathbb{R}_U(s)^{n_i \times n_o}} \{\psi^0(R) \mid \psi^k(R), k \in \underline{m}\}. \tag{2.14}$$

A phase II algorithm will reduce the value of $\psi^0(R)$ without violating the constraint $R \in \mathbf{F}$. It should be clear that various other alternatives are also possible, including carrying out complex trade-offs.

Computationally, one cannot deal with elements of $\mathbb{R}_U(s)^{n_i \times n_o}$ and hence the problems $\mathbf{P}_1, \mathbf{P}_2$ must somehow be discretized. Hence we propose to parametrize the parameter $R \in \mathbb{R}_U(s)^{n_i \times n_o}$ of the compensator $K(R)$ in terms of a vector $x \in \mathbb{R}^{n n_i n_o}$, with $n = 1,2,3, \cdots$, as follows:

We define the matrices $X_i \in \mathbb{R}^{n_i \times n_o}$, $i = 1,2,...,n$, by filling them in order, row-wise, with the components of x, i.e.,

$$[X_i]_{k,l} \triangleq [x]_{(i-1)n_i n_o + (k-1)n_o + l}, \; k \in \underline{n_i}, \; l \in \underline{n_0}. \tag{2.15}$$

Let $p \in \mathbb{R}_+$, then we define $R_n: \mathbb{R}^{n n_i n_o} \to \mathbb{R}_U(s)^{n_i \times n_o}$ by

$$R_n(x)(s) \triangleq \sum_{i=1}^{n} X_i \left(\frac{s - p + \alpha_U}{s + p + \alpha_U}\right)^{i-1}. \tag{2.16}$$

The parametrization (2.15), (2.16) has the following useful properties, the first of which follows from the Weierstrass approximation theorem:

Proposition 2.1 : The set $\{R_n(x) \mid x \in \mathbb{R}^{n n_i n_o}, n \in \mathbb{N}\}$ is dense in $(\mathbb{R}_U(s)^{n_i \times n_o}, \|\cdot\|_\infty)$, i.e., any proper rational transfer function with real coefficients that is analytic in U can be uniformly approximated arbitrarily closely by a polynomial of the form (2.16). ■

Proposition 2.2: The functions $f^k(R_n(\cdot))$, $k = 1,2,3,4$, obtained by composing $R_n(\cdot)$ defined in (2.16) with $f^k(\cdot)$, defined in (2.8), (2.9a), (2.9b) and (2.10b), respectively, are convex. ■

Theorem 2.3 [Sal.1] : Let $\{b_n\} \subset \mathbb{R}_+$ be such that $b_n \uparrow +\infty$, and let

$$\psi^{m+1}(R_n(x)) \triangleq \|R_n(x)\|_\infty - b_n. \tag{2.17}$$

Consider the sequences of problems

$$\mathbf{P}_{1,n}: \quad \min_{x \in \mathbb{R}^{n n_i n_o}} \{\psi(R_n(x)) \mid \psi^{m+1}(R_n(x)) \le 0\} \tag{2.18a}$$

where

$$\psi(R_n(x)) \triangleq \max_{j \in \underline{m}} \psi^j(R_n(x)), \quad n \in \mathbb{N}, \tag{2.18b}$$

and

$$\mathrm{P}_{2,n}: \quad \min_{x \in \mathbb{R}^{nn_in_o}} \{ \psi^0(R_n(x)) \mid \psi^j(R_n(x)) \le 0, j \in \underline{m+1} \}, \quad n \in \mathbb{N}. \tag{2.19}$$

Suppose that there exist $\tilde{R} \in \mathbb{R}_U(s)^{n_i \times n_o}$ and $\delta > 0$ such that $\psi^i(\tilde{R}) \le -\delta$ for all $i \in \underline{m}$. Then, for each $n \in \mathbb{N}$, there exist $\hat{x}_{1,n}, \hat{x}_{2,n} \in \mathbb{R}^{nn_in_o}$ which solve $\mathrm{P}_{1,n}, \mathrm{P}_{2,n}$, respectively, and

$$\lim_{n \to \infty} \psi(R_n(\hat{x}_{1,n})) = \gamma_1 \triangleq \inf_{R \in \mathbb{R}_U(s)^{n_i \times n_o}} \psi(R), \tag{2.20a}$$

$$\lim_{n \to \infty} \psi^0(R_n(\hat{x}_{2,n})) = \gamma_2 \triangleq \inf_{R \in \mathbb{R}_U(s)^{n_i \times n_o}} \{ \psi^0(R) \mid \psi^i(R) \le 0, \ i \in \underline{m} \}. \quad \blacksquare \tag{2.20b}$$

Thus it follows that an approximation to the solution of the original problems $\mathrm{P}_1, \mathrm{P}_2$ in $\mathbb{R}_U(s)^{n_i \times n_o}$ can be obtained by solving a sequence of convex problems $\mathrm{P}_{1,n}, \mathrm{P}_{2,n}$ in $\mathbb{R}^{nn_in_o}$.

3. SEMI-INFINITE OPTIMIZATION ALGORITHMS

We shall now give a brief introduction to our semi-infinite optimization algorithms as they apply to problem (2.18a). Algorithms for solving (2.18b) are quite similar in structure. For a full treatment the reader is referred to [Pol.1]. First, referring, to (2.18a), we simplify notation, by redefining $\psi^j(x)$ to be $\psi^j(R_n(x))$, with $x \in \mathbb{R}^N$ and $N \triangleq nn_in_o$, and $n \in \mathbb{N}$.

We shall develop an algorithm for solving (2.18a) by extension from differentiable steepest descent. For the sake of simplicity, we shall consider only the unconstrained problem

$$\min_{x \in \mathbb{R}^N} \psi(x), \tag{3.1}$$

and assume, for the moment, that $\psi(\cdot)$ is continuously differentiable. Then we have the following result.

Theorem 3.1 [Pol.1] : (i) If $\hat{x}$ is a local solution of (3.1) then

$$d\psi(\hat{x};h) \ge 0 \ \forall \ h \in \mathbb{R}^N \Longleftrightarrow 0 = \nabla\psi(\hat{x}); \tag{3.2a}$$

(ii) the search direction

$$h(x) \triangleq \nabla\psi(x) = \operatorname*{argmin}_{h \in \mathbb{R}^N} \{ \langle \nabla\psi(x), h \rangle + \tfrac{1}{2}\|h\|^2 \} \tag{3.2b}$$

is a continuous descent direction for $\psi(\cdot)$ at x. ■

Next suppose that $\psi(x) = \max_{j \in \underline{m}} \psi^j(x)$ and that the $\psi^j(\cdot)$ are continuously differentiable. Then we get the following extension of Theorem 3.1.

Theorem 3.2 [Pol.1] **:** (i) If $\hat{x}$ is a local solution of (3.1) then

$$d\psi(\hat{x};h) = \max_{j \in I(\hat{x})} d\psi^j(\hat{x};h) \geq 0 \;\forall\; h \in \mathbb{R}^N \Longleftrightarrow 0 \in \partial\psi(\hat{x}) = \underset{j \in I(\hat{x})}{\mathrm{co}} \{ \nabla\psi^j(\hat{x}) \}, \tag{3.3a}$$

where $\partial\psi(x)$ denotes the *generalized gradient* of $\psi(\cdot)$ at x and

$$I(x) \triangleq \{ j \in \underline{m} \mid \psi^j(x) = \psi(x) \}. \tag{3.3b}$$

(ii) The function

$$\theta(x) \triangleq \min_{h \in \mathbb{R}^N} \max_{j \in \underline{m}} \{ \psi^j(x) - \psi(x) + \langle \nabla\psi^j(x), h \rangle + \tfrac{1}{2}\|h\|^2 \} \tag{3.3c}$$

is continuous and satisfies (a) $\theta(x) \leq 0$ for all $x \in \mathbb{R}^N$, (b) $\theta(x) = 0$ if and only if (3.3a) holds.

(iii) The search direction

$$h(x) \triangleq \operatorname*{argmin}_{h \in \mathbb{R}^N} \max_{j \in \underline{m}} \{ \psi^j(x) - \psi(x) + \langle \nabla\psi^j(x), h \rangle + \tfrac{1}{2}\|h\|^2 \} \tag{3.3d}$$

is a continuous descent direction for $\psi(\cdot)$ at $x \in \mathbb{R}^N$ and satisfies $d\psi(x;h(x)) \leq \theta(x)$ for all $x \in \mathbb{R}^N$.

■

Theorem 3.3 [Pol.1] **:** Suppose that $\psi(x) = \max_{j \in \underline{m}} \psi^j(x)$ and that the $\psi^j(\cdot)$ are continuously differentiable in $\mathbb{R}^N$. Consider the algorithm defined by: $x_0 \in \mathbb{R}^N$ given,

$$x_{i+1} = x_i + \lambda_i h(x_i), \; i = 0,1,2,3,\ldots \tag{3.4a}$$

with

$$\lambda_i \in \operatorname*{argmin}_{\lambda > 0} \psi(x_i + \lambda h(x_i)), \tag{3.4b}$$

or the Armijo rule [Arm.1], with $\alpha, \beta \in (0,1)$,

$$\lambda_i = \max\{ \beta^k \mid k \in \mathbf{N}, \; \psi(x_i + \beta^k h(x_i)) - \psi(x_i) \leq \beta^k \alpha\theta(x_i) \}. \tag{3.4c}$$

Then any accumulation point $\hat{x}$ of $\{x_i\}_{i=0}^{\infty}$ satisfies the first order optimality condition $\theta(\hat{x}) = 0$.

Proof : We shall only give a proof for the simpler case (3.4b). First we note that $\{\psi(x_i)\}_0^\infty$ is monotone decreasing. Hence, if there is a convergent subsequence $x_i \underset{K}{\rightarrow} \hat{x}$, where $K \subset \mathbb{N}$, then we must have that $\psi(x_i) \rightarrow \psi(\hat{x})$ as $i \rightarrow \infty$. Now suppose that $\theta(\hat{x}) < 0$. Then we must have that $d\psi(\hat{x}; h(\hat{x})) \leq \theta(\hat{x}) < 0$. Hence there exists a $\hat{\lambda} > 0$ such that

$$\psi(\hat{x} + \hat{\lambda} h(\hat{x})) - \psi(\hat{x}) \; = \; -2\delta < 0 \,. \tag{3.5}$$

Hence, by continuity of ψ, there exists an i_o such that for all $i \geq i_o$, $i \in K$,

$$\psi(x_{i+1})) - \psi(x_i) \leq \psi(x_i + \hat{\lambda} h(x_i)) - \psi(x_i) \leq -\delta \,, \tag{3.6}$$

which implies that $\psi(x_i) \rightarrow -\infty$. Hence we have a contradiction and the theorem is proven. ∎

To extend the algorithm defined by (3.4a), (3.4b) to be applicable to problems of the form (2.18a), we proceed as follows. First we note that, by von Neumann's theorem [Ber.1],

$$\begin{aligned}
\theta(x) &\triangleq \min_{h \in \mathbb{R}^N} \max_{j \in \mathbf{m}} \{ \psi^j(x) - \psi(x) + \langle \nabla \psi^j(x), h \rangle + \tfrac{1}{2} \| h \|^2 \} \\
&= \max_{\mu \in \Sigma} \min_{h \in \mathbb{R}^N} \{ \textstyle\sum \mu^j \{ \psi^j(x) - \psi(x) + \langle \nabla \psi^j(x), h \rangle + \tfrac{1}{2} \| h \|^2 \} \} \\
&= - \min_{\xi \in \overline{G}\psi(x)} \{ \xi^0 + \tfrac{1}{2} \| \xi \|^2 \}
\end{aligned} \tag{3.7}$$

where $\Sigma \triangleq \{ \mu \in \mathbb{R}^m \mid \mu^j \geq 0, \sum_{j=1}^m \mu^j \; = \; 1 \}$, $\overline{\xi} \; = \; (\xi^0, \xi) \in \mathbb{R}^{N+1}$ and $\overline{G}\psi(x) \subset \mathbb{R}^{N+1}$ is defined by

$$\overline{G}\psi(x) \triangleq \underset{j \in \mathbf{m}}{\mathrm{co}} \{ \overline{\xi}^j(x) \} , \tag{3.8}$$

with $\overline{\xi}^j(x) \triangleq (\psi(x) - \psi^j(x), \nabla \psi^j(x))$. If we denote by $\overline{\xi}(x) \; = \; (\xi^0(x), \xi(x))$ the solution of (3.8), then we see that $h(x) \; = \; -\xi(x)$ holds.

Note that $\overline{G}\psi(\cdot)$ is a continuous (in the sense of Berge [Ber.1]) set valued *search direction finding map* which maps $\mathbb{R}^N$ into subsets of $\mathbb{R}^{N+1}$.

The above results can be extended to general, locally Lipschitz continuous functions $\psi : \mathbb{R}^N \rightarrow \mathbb{R}$. In this case, directional derivatives $d\psi(x; h)$ need not exist and they are replaced by *generalized directional derivatives* [Cla.1] defined by

$$d_o\psi(x;h) \triangleq \overline{\lim_{\substack{t\downarrow 0\\ y\to x}}} \frac{\psi(y+th)-\psi(y)}{t}. \tag{3.9a}$$

In addition, a *generalized gradient* [Cla.1] can be defined by

$$\partial\psi(x) \triangleq \{\xi \in \mathbb{R}^N \mid d_o\psi(x;h) \geq \langle \xi, h\rangle \quad \forall\, h \in \mathbb{R}^N\}. \tag{3.9b}$$

Quite analogously to (3.3a), we obtain that if $\hat{x}$ minimizes a locally Lipschitz continuous function $\psi : \mathbb{R}^N \to \mathbb{R}$, then $d_o\psi(\hat{x};h) \geq 0$ for all $h \in \mathbb{R}^N$ and $0 \in \partial\psi(\hat{x})$.

In the case of convex functions (such as the ones occuring in the control system design problem defined in the preceding section), the directional derivative always exists and is equal to the directional derivative, while the generalized gradient is equal to the subgradient [Cla.1].

The concept of search direction map generalizes as follows.

Definition 3.1 [Pol.1]: Let $\psi : \mathbb{R}^N \to \mathbb{R}$ be a locally Lipschitz continuous function. We shall say that $\overline{G}\psi : \mathbb{R}^N \to 2^{\mathbb{R}^{N+1}}$ is an *augmented convergent direction finding* (a.c.d.f.) map for $\psi(\cdot)$ if :

(a) $\overline{G}\psi(\cdot)$ is continuous in the sense of Berge [Ber.1] and $\overline{G}\psi(x)$ is convex for all $x \in \mathbb{R}^N$.

(b) For any $x \in \mathbb{R}^N$, if $\overline{\xi} = (\xi^0, \xi) \in \mathbb{R}^{N+1}$ is an element of $\overline{G}\psi(x)$, then $\xi^0 \geq 0$.

(c) For any $x \in \mathbb{R}^n$, a point $\overline{\xi} = (0,\xi)$ is an element of $\overline{G}\psi(x)$ if and only if $\xi \in \partial\psi(x)$. ■

Theorem 3.4 [Pol.1] : Suppose that $\psi : \mathbb{R}^N \to \mathbb{R}$ is l.L.c. and $\overline{G}\psi(\cdot)$ is an a.c.d.f. map for $\psi(\cdot)$. Then for any $x \in \mathbb{R}^N$,

(a) $0 \in \partial\psi(x) <= = => 0 \in \overline{G}\psi(x)$

(b) The functions $\theta : \mathbb{R}^n \to \mathbb{R}$ and $\overline{h} : \mathbb{R}^n \to \mathbb{R}^{n+1}$ defined by

$$\theta(x) \triangleq \min\{\xi^0 + \tfrac{1}{2}|\xi|^2 \mid \overline{\xi} \in \overline{G}\psi(x)\}, \tag{3.10a}$$

$$\overline{h}(x) = (h^0(x), h(x)) \triangleq -\mathrm{argmin}\ \{\xi^0 + \tfrac{1}{2}|\xi|^2 \mid \overline{\xi} \in \overline{G}\psi(x)\}, \tag{3.10b}$$

are both continuous and $\theta(x) = 0 <= = => 0 \in \partial\psi(x)$.

(c) The vector $h(x)$, is a descent direction for $\psi(\cdot)$ satisfying

$$d_o\psi(x;h(x)) \leq -\theta(x). \quad ■ \tag{3.10c}$$

The following theorem can be proved by an almost verbatim reproduction of the proof of Theorem 3.3.

Theorem 3.5 [Pol.1] : Suppose that $\psi : \mathbb{R}^N \to \mathbb{R}$ is l.L.c. and that $\overline{G}\psi(\cdot)$ is an a.c.d.f. map for it. Let $h(x)$ and $\theta(x)$ be defined by (3.10b), (3.10a), respectively, and consider the algorithm defined by: $x_0 \in \mathbb{R}^N$ given,

$$x_{i+1} = x_i + \lambda_i h(x_i), \quad i = 0,1,2,3,... \tag{3.11a}$$

with

$$\lambda_i \in \underset{\lambda > 0}{\operatorname{argmin}}\ \psi(x_i + \lambda h(x_i)), \tag{3.11b}$$

or the Armijo rule, with $\alpha, \beta \in (0,1)$,

$$\lambda_i = \max\{ \beta^k \mid k \in \mathbb{N},\ \psi(x_i + \beta^k h(x_i)) - \psi(x_i) \le \beta^k \alpha \theta(x_i) \}. \tag{3.11c}$$

Then any accumulation point $\hat{x}$ of $\{x_i\}_{i=0}^{\infty}$ satisfies the first order optimality condition $\theta(\hat{x}) = 0$. ■

Since all the functions in (2.18a) are convex, they are locally Lipschitz continuous, and hence the algorithm described in Theorem 3.5 is applicable to the case of problem (3.1) where $\psi(x) \triangleq \psi(R_n(x))$, *provided* we can produce a formula for $\overline{G}\psi(x)$ and *provided* we can compute the search direction $h(x)$ defined by (3.10b) and $h(x) = -\xi(x)$. The following two results give us a start.

Proposition 3.1 : (i) Suppose that for $j \in \underline{m}$, $\psi^j : \mathbb{R}^N \to \mathbb{R}$ are convex and that they have a.c.d.f. maps $\overline{G}\psi^j(x)$. Let $\psi(x) \triangleq \max_{j \in \underline{m}} \psi^j(x)$. For $j \in \underline{m}$, let $\overline{v}_j(x) \triangleq (\psi(x) - \psi^j(x), 0) \in \mathbb{R}^{N+1}$. Then $\psi(\cdot)$ is convex and

$$\overline{G}\psi(x) = \underset{j \in \underline{m}}{\operatorname{co}} \{ \overline{G}\psi^j(x) + \overline{v}_j(x) \} \tag{3.12a}$$

defines an a.c.d.f. map for $\psi(\cdot)$.

(ii) Suppose that $\phi : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ is continuous in (x,y) and convex in x, that its generalized gradient $\partial_x \phi(x,y)$ is continuous in y and that $Y \subset \mathbb{R}^M$ is compact, and that for each $y \in Y$, $\overline{G}\phi(x,y)$ is an a.c.d.f. map for $\phi(\cdot, y)$.

(a) If $\psi(x) \triangleq \max_{y \in Y} \phi(x,y)$, and $\overline{v}(x,y) \triangleq (\psi(x) - \phi(x,y), 0) \in \mathbb{R}^{N+1}$ for all $y \in Y$, then $\psi(\cdot)$ is convex and

$$\bar{G}\psi(x) = \underset{y \in Y}{\mathrm{co}}\{ \bar{G}\phi(x,y) + \bar{v}(x,y) \} \tag{3.12b}$$

defines an a.c.d.f. map for $\psi(\cdot)$.

(b) If $\psi(x) \triangleq \int_{y \in Y} \phi(x,y)dy$, then $\psi(\cdot)$ is convex and

$$\bar{G}\psi(x) = \int_{y \in Y} \bar{G}\phi(x,y)dy \tag{3.12c}$$

defines an a.c.d.f. map for $\psi(\cdot)$. ■

By analogy with (3.8), we get the following result.

Proposition 3.2 [Pol.1] : Suppose that $\phi : \mathbb{R}^N \times \mathbb{C}^M \to \mathbb{R}$ is is convex, that $Y \subset \mathbb{C}^M$ is sequentially compact, and that $\phi(x,y)$ is continuously differentiable in x and upper semicontinuous in y and that $\nabla_x\phi(x,y)$ is continuous in both arguments. If $\psi(x) \triangleq \max_{y \in Y} \phi(x,y)$, and $\bar{v}(x,y) \triangleq (\psi^j(x) - \phi(x), \nabla_x\phi^j(x,y)) \in \mathbb{R}^{N+1}$ with $y \in Y$, then

$$\bar{G}\psi(x) = \underset{y \in Y}{\mathrm{co}}\{ \bar{v}(x,y) \} \tag{3.13}$$

defines an a.c.d.f. map for $\psi(\cdot)$. ■

We are now ready to construct a.c.d.f. maps for the functions $f^j(R_n(\cdot))$, $j = 1,2,3,4$, defined in Section 2. We begin with f^1, defined in (2.8). First, we define $H(x,j\omega) \triangleq [G_lR_n(x)G_r - F)(j\omega)]$, and note that for any $\omega \in [0,\bar{\omega}]$,

$$\phi^1(x,\omega) \triangleq \bar{\sigma}[H(x,j\omega)] - b_f(\omega) = \max_{\substack{\|u\| = 1 \\ \|v\| = 1}} \langle u, [H(x,j\omega)]v \rangle - b_f(\omega), \tag{3.14a}$$

where u, v are complex vectors, $\langle a,b \rangle \triangleq \mathrm{Re}(a^*b)$ for all complex vectors, and $\|u\| \triangleq \langle u,u \rangle^{1/2}$. Hence, since for any u, v, ω, $\langle u, H(x,j\omega)v \rangle$ is differentiable in x, it follows from Proposition 3.2 that

$$\begin{aligned} \bar{G}\phi^1(x,\omega) \triangleq \mathrm{co}\{ \bar{v}(x,\omega) = (v^0(x,\omega), v(x,\omega)) \in \mathbb{R}^{N+1} \mid \\ v^0(x,\omega) = [\phi^1(x,\omega) - \langle u, [H(x,j\omega)]v \rangle], \\ v^i(x,\omega)) = \langle u, [\partial H(x,j\omega)/\partial x^i]v \rangle], \\ i = 1,2, \cdots, N, \ \|u\| = 1, \|v\| = 1 \} \end{aligned} \tag{3.14b}$$

is an a.c.d.f. map for $\phi^1(\cdot,\omega)$. It now follows from Proposition 3.1 that

$$\bar{G}f^1(R_n(x)) = \underset{\omega \in [0,\bar{\omega}]}{\mathrm{co}} \bar{G}\phi^1(x,\omega) \tag{3.14c}$$

is an a.c.d.f. map for $f^1(R_n(x))$.

Next we turn to $f^2(\cdot)$. Let $\phi^2(x,t) \triangleq \mathbf{L}^{-1}\{ e_j^T(G_l R_n(x) G_r - F)(s) e_i \hat{u}(s) \}(t) - b_1(t)]$. (The shifted output $\phi^2(x,t)$ and its gradient with respect to x are probably best computed by first constructing a realization for the system and then making use of the techniques described in [Wuu.1].) Then it follows from Proposition 3.2 that

$$\begin{aligned} \bar{G}f^2(R_n(x)) \triangleq \underset{t \in [0,t_0]}{\mathrm{co}} \{ \bar{v}(x,t) &= (v^0(x,t), v(x,t)) \in \mathbb{R}^{N+1} \mid \\ v^0(x,t) &= [f^2(R_n(x)) - \phi^2(x,t)] \\ v^i(x,t) &= \partial\phi^2(x,t)/\partial x^i ,\ i = 1,2, \cdots ,N \} \end{aligned} \tag{3.15}$$

is an a.c.d.f. map for $f^2(R_n(x))$ defined in (2.9a).

Since $\|y(x,t)\| = \max_{\|u\|=1} \langle u, y(x,t) \rangle$ an a.c.d.f. map for $f^3(R_n(x))$ has a very similar form to (3.14b). Finally, consider the simpler case of $f^4(\cdot)$ in (2.11). Let $\tilde{H}(x,t) \triangleq \mathbf{L}^{-1}\{ (G_l R_n(x) - F)(s) \}(t)$ and let $\phi^4(x,t) \triangleq \bar{\sigma}\tilde{H}(x,t)$. Then it follows from Proposition 3.2 that for every $t \geq 0$,

$$\begin{aligned} \bar{G}\phi^4(x,t) \triangleq \mathrm{co}\{ \bar{v}(x,t) &= (v^0(x,t), v(x,t)) \in \mathbb{R}^{N+1} \mid \\ v^0(x,t)) &= [\phi^4(x,t) - \langle u, [\tilde{H}(x,t)]v \rangle], \\ v^i(x,t)) &= \langle u, [\partial\tilde{H}(x,t)/\partial x^i]v \rangle], \\ i &= 1,2, \cdots ,N,\ \|u\| = 1,\ \|v\| = 1 \}, \end{aligned} \tag{3.16a}$$

and hence an a.c.d.f. map for $f^4(R_n(x))$ is obtained, via (3.12c):

$$\bar{G}f^4(x) = \int_0^T \bar{G}\phi^4(x,t)dt. \tag{3.16b}$$

The a.c.d.f. map for the function $\psi(\cdot)$ for our special case of problem (3.1) can now be obtained by making use of Proposition 3.1, (3.12a).

The above expressions for the a.c.d.f. maps are quite complex and hence we must address the issue as to whether the search direction finding problem (3.10b) is solvable. Problem (3.10b) can be solved by an algorithm evolved from the algorithm proposed by Gilbert [Gil.1], via [Wol.1] or [Hoh.1].

All these algorithms construct a sequence of contact points to the set $\overline{G}\psi(x)$. Contact points $\overline{\zeta}(\overline{\nu})$ are defined by a normal $\overline{\nu} \in \mathbb{R}^{N+1}$ and are given by the formula

$$\overline{\zeta}(\overline{\nu}) \triangleq \underset{\overline{\xi} \in \overline{G}\psi(x)}{\operatorname{argmax}} \langle \overline{\nu}, \overline{\xi} \rangle . \tag{3.17}$$

The evaluation of $\overline{\zeta}(\overline{\nu})$ is simplified by making use of the following result.

Proposition 3.3 : (a) Suppose that $\overline{G}\psi(x) = \underset{j \in \mathbf{m}}{\operatorname{co}} \overline{G}\psi^j(x)$. Then

$$\max_{\overline{\xi} \in \overline{G}\psi(x)} \langle \overline{\nu}, \overline{\xi} \rangle = \max_{j \in \mathbf{m}} \max_{\overline{\xi} \in \overline{G}\psi^j(x)} \langle \overline{\nu}, \overline{\xi} \rangle . \tag{3.18a}$$

(b) Suppose that $\overline{G}\psi^j(x) = \underset{y \in Y}{\operatorname{co}} \overline{G}\phi^j(x,y)$. Then

$$\max_{\overline{\xi} \in \overline{G}\psi^j(x)} \langle \overline{\nu}, \overline{\xi} \rangle = \max_{y \in Y} \max_{\overline{\xi} \in \overline{G}\phi^j(x,y)} \langle \overline{\nu}, \overline{\xi} \rangle . \tag{3.18b}$$

(c) Suppose that $\overline{G}\psi^j(x) = \int_0^T \overline{G}\phi^j(x,t)dt$. Then

$$\max_{\overline{\xi} \in \overline{G}\psi^j(x)} \langle \overline{\nu}, \overline{\xi} \rangle = \int_0^T \max_{\overline{\xi} \in \overline{G}\phi^j(x,t)} \langle \overline{\nu}, \overline{\xi} \rangle dt . \quad \blacksquare \tag{3.18c}$$

Hence we are left with considering the two special cases defined by (3.14b) and (3.15). We begin with (3.14b) and note that

$$\begin{aligned} \max_{\overline{\xi} \in \overline{G}\phi^1(x,\omega)} \langle \overline{\nu}, \overline{\xi} \rangle &= \max_{\substack{|u| = 1 \\ |v| = 1}} \{ \nu^0 \phi^1(x,\omega) + \langle u, [-\nu^0 H(x,j\omega) + \sum_{i=1}^{N} \nu^i \partial H(x,j\omega)/\partial x_i] v \rangle \} \\ &= \nu^0 \psi^1(x,\omega) + \overline{\sigma}[-\nu^0 H(x,j\omega) + \sum_{i=1}^{N} \nu^i \partial H(x,j\omega)/\partial x_i] , \end{aligned} \tag{3.19}$$

i.e., the contact function is evaluated by performing a singular value decomposition and picking up *any pair* of singular vectors corresponding to the maximum singular value.

Next, from (3.15),

$$\max_{\overline{\xi} \in \overline{G}f^1(R_n(x))} \langle \overline{\nu}, \overline{\xi} \rangle = \max_{t \in [0,t_0]} \{ \nu^0 f^2(R_n(x) - \nu^0 y(x,t) + \sum_{i=1}^{N} \nu^i \partial \phi^2(x,t)/\partial x^i \} . \tag{3.20}$$

The evaluation of (3.20) can be simplified, at least in the initial iterations, by making use of the observation that $\sum_{i=1}^{N} \nu^i \partial \phi^2(x,t)/\partial x^i \approx [\phi^2(x + \lambda \nu , t) - \phi^2(x,t)]/\lambda$, for λ small.

4. CONCLUSION

We have seen that at least in principle, the new semi-infinite optimization algorithms are applicable to the solution of optimal control system synthesis problems in an H^∞ setting. The state of our computational experience allows us to make only certain preliminary evaluations as to how well these methods will perform in practice. Our first empirical observation is that the optimal design problems in the H^∞ setting tend to be rather ill-conditioned. The reason for this seems to be that the functions $f^j(\cdot)$, defined by (2.8), (2.9a), (2.9b) and (2.10b), with $R = R_n(x)$, are of the form $g^j(A_jx + b_j)$, with the matrices A_j of low rank (singular). In addition, the co-prime factorizations of the plant seem to have a substantial influence on the problem conditioning. At present we are experimenting with two versions of a scaled algorithm. To form an idea of how we scale, consider the case where $\psi(x) \triangleq \max_{j \in \mathbf{m}} g^j(A_jx + b_j)$ and the $g^j(\cdot)$ are differentiable. Then the conceptually simpler version of the scaled algorithm computes the search direction according to

$$h(x) = \operatorname*{argmin}_{h \in \mathbb{R}^N} \max_{j \in \mathbf{m}} \{ g^j(A_jx + b_j) + \langle \nabla g^j(A_jx + b_j), h \rangle + \tfrac{1}{2}\|h\|^2_{Q_j} \}, \tag{4.1}$$

where

$$Q_j \triangleq [(A_j^T A_j)^+]^{-1} \tag{4.2}$$

(with D^+ denoting the pseudoinverse of D). The reader may recognize (4.1) as an extension of Newton's method. Since formula (4.1) requires the use of a gradient projection method for evaluation, it is somewhat costly. Hence we are using also a two step formula, which first solves (4.1) with all Q_j replaced by the identity matrix, as in (3.3d), and in the process obtains multipliers $\mu^j \geq 0$ such that $\sum_{j \in \mathbf{m}} \mu^j = 1$, and then solves (4.1) with all Q_j replaced by $\sum_{j \in \mathbf{m}} \mu^j Q^j$, which the reader may recognize as a form of sequential quadratic programming. These two evalutations are carried out by means of nearest point algorithms, such as the ones mentioned in the preceeding section (see [Gil.1, Hoh.1, Wol.1]). Our experiments with the second formula have been most encouraging and we will report on them, as well as on the details of our algorithm in a future paper. Our design experiments are being facilitated by the use of the DELIGHT.MIMO system [Wuu.2, Nye.1].

5. REFERENCES

[Arm.1] Armijo, L., "Minimization of Functions Having Continuous Partial Derivatives", *Pacific J. Math*, Vol. 16, pp. 1-3, 1966

[Cha.1] B. C. Chang and J. B. Pearson, "Optimal Disturbance Reduction in Linear Multivariable Systems", *IEEE Transactions on Automatic Control*, Vol. AC-29, pp. 880-887, October 1984.

[Cla.1] F. H. Clarke, Optimization and Nonsmooth Analysis, Wiley-Interscience, New York, 1983.

[Des.1] C.A. Desoer, R.W. Liu, J. Murray, and R. Saeks "Feedback System Design: The Fractional Representation Approach to Analysis and Synthesis", *IEEE Transactions on Automatic Control*, Vol. AC-25, pp. 399-412, 1980.

[Fra.1] B.A. Francis, J.W. Helton and G. Zames, "H^∞-Optimal Feedback Controllers for Linear Multivariable Systems", *IEEE Transactions on Automatic Control*, Vol AC-29, pp. 888-900, 1984.

[Fra.2] B.A. Francis and G. Zames, "On H^∞-Optimal Sensitivity Theory for SISO Feedback Systems", *IEEE Transactions on Automatic Control*, Vol. AC-29, pp. 9-16, 1984.

[Fra.3] B. A. Francis and A J. C. Doyle, "Linear Control Theory with an H^∞ Optimality Criterion" *SIAM Journal on Control and Optimization*, to appear.

[Gil.1] Gilbert, E. G., "An Interactive Method for Computing the Minimum of a Quadratic Form on a Convex Set", *SIAM Journal of Control*, Vol. 4, pp. 61-80, 1966

[Glo.1] K. Glover, "All Optimal Hankel-norm Approximations of Linear Multivariable Systems and their L^∞-Error Bounds", *International Journal of Control*, Vol. 39, No. 6, pp. 1115-1193, 1984.

[Hel.1] J. W. Helton, "Worst Case Analysis in the Frequency Domain: The H^∞ Approach to Control", *IEEE Transactions on Automatic Control*, Vol. AC-30, No. 12, pp. 1154-1170, 1985.

[Hoh.1] Hohenbalken, B. Von, "Simplical Decomposition in Nonlinear Programming Algorithms", *Mathematical Programming*, Vol. 13, pp. 49-68, 1977

[Nye.1] W. T. Nye, "DELIGHT: An Interactive System for Optimization-Based Engineering Design", *PhD. Thesis, Department of Electr. Engr. and Comp Sci.*, University of California, Berkeley, California, 1983.

[Pol.1] E. Polak, "On the Mathematical Foundations of Nondifferentiable Optimization in Engineering Design", *SIAM Review*, pp. 21-91, March 1987.

[Sal.1] S. E. Salcudean, "Algorithms for Optimal Design of Feedback Compensators", Ph.D. Thesis, Department of Electr. Engr. and Comp. Sci., University of California, Berkeley, California, December 1986.

[Vid.1] M. Vidyasagar, "Control Systems Synthesis: A Factorization Approach", MIT Press, Cambridge, MA, 1985.

[Wol.1] Wolfe, P., "Finding the Nearest Point in a Polytope", *Mathematical Programming*, Vol. 11, pp. 128-149, 1976.

[Wuu.1] T. L. Wuu, R. G. Becker and E. Polak, "A Diagonalization Technique for the Computation of Sensitivity Functions of Linear Time Invariant Systems", *IEEE Trans. on Automatic Control*, Vol. AC-31 No. 12, pp. 1141-1143, 1986.

[Wuu.2] S. Wuu, "DELIGHT.MIMO: An Interactive System for Optimization-Based Multivariable Control System Design", *PhD. Thesis*, Department of Electr. Engr. and Comp. Sci., University of California, Berkeley, California, December 1986.

[You.1] D.C. Youla, H. Jabr and J.J. Bongiorno, Jr., "Modern Wiener-Hopf Design of Optimal Controllers - Part II", *IEEE Transactions on Automatic Control*, Vol. AC-21, pp. 319-338, 1967.

[Zam.1] G. Zames, "Feedback and Optimal Sensitivity: Model Reference Transformations, Multiplicative Seminorms, and Approximate Inverses", *IEEE Transactions on Automatic Control*, Vol. AC-26, pp. 301-320, 1981.

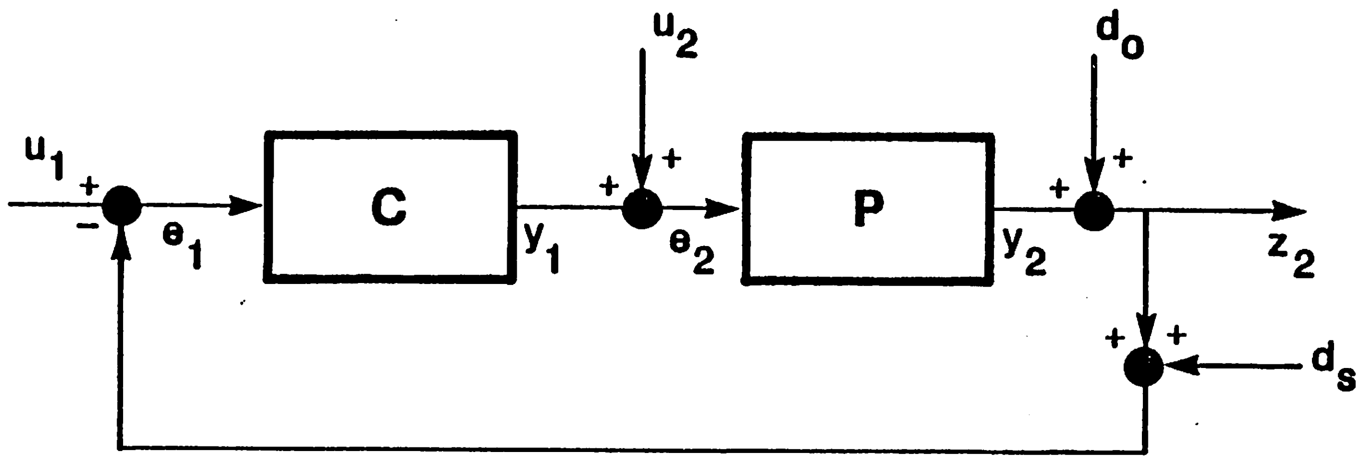

Fig. 1. Feedback System