MINIMAX ALGORITHMS FOR STRUCTURAL OPTIMIZATION

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MINIMAX ALGORITHMS FOR STRUCTURAL OPTIMIZATION

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ABSTRACT. In this paper we highlight the salient features of our recently developed theory for the construction of broad classes of nondifferentiable optimization algorithms. These algorithms can be used for the solution of a wide variety of unconstrained and constrained minimax problems, such as those occurring in the design of structures subjected to dynamic loads, floor planning and layout problems, control system and electronic circuit design.

1. INTRODUCTION

To motivate our discussion of minimax algorithms as a tool for the solution of structural design problems, let us consider an idealized base isolation problem. Thus, suppose that we are required to design a passive base isolation system for a structure that must be built on top of a metropolitan underground train station. Since the motions will be very small, the structure can be modeled as a linear second order differential equation of the form (see [5])

$$M\ddot{y}(t) + C(x)\dot{y}(t) + K(x)y(t) = BF(t) \tag{1.1}$$

where $y(t) \in \mathbb{R}^{3q}$ is a vector of floor displacements, with three components per floor (q is the number of floors), two for horizontal motion and one for vertical motion. Next, M, is a mass matrix, while $C(x)$ and $K(x)$ are damping and spring action matrices, which we assume to be continuously differentiable in $x \in \mathbb{R}^n$, the design parameter vector of the base isolation device. The three dimensional ground acceleration forces are described by the time dependent function $F(t) \in \mathbb{R}^3$, and B is the coupling matrix.

Now suppose that the passing trains leave several very specific and repeatable signatures, i.e., they produce a set of excitation functions $\{F_j(t)\}_{j=1}^{p}$, defined on the interval $[0,T]$, where T is the maximum duration of the disturbance caused by the trains. Assuming that the components of the designable parameter x must satisfy a constraint of the form $x \in \mathbf{X} \triangleq \{x \in \mathbb{R}^n \mid \underline{x}^i \leq x^i \leq \bar{x}^i,\ i = 1,\dots,n\}$, the base isolation design problem can be expressed in the form

$$\min_{x \in \mathbf{X}} \max_{j \in \mathbf{p}} \max_{t \in [0,T]} \|y(t;x,F_j)\| \ , \tag{1.2}$$

where $\mathbf{p} \triangleq \{1,2,\dots,p\}$, and $y(t;x,F_j)$ denotes the solution of (1.1) corresponding to the given value of the base isolation design vector x and ground motion $F_j(t)$.

Alternatively, we may have determined that the frequency spectrum of the disturbances is contained in an interval $[\omega',\omega'']$ and that its magnitude is bounded by a function $b(\omega) > 0$ on that interval. Then we can attempt to design the base isolation device using frequency domain techniques by solving the problem

$$\min_{x \in X} \max_{\omega \in [\omega', \omega'']} w(\omega) \| H(x, j\omega) \|, \tag{1.3}$$

where $w(\omega) = 1/b(\omega)$ and $H(x,j\omega)$ is the $3q \times 3$ complex valued transfer function matrix from the ground acceleration $F(t)$ to the floor displacement vector $y(t)$. Note that $\|H\|^2$ is the largest eigenvalue of the positive semidefinite matrix $H^* H$.

For more extensive treatments of modeling design problems as minimax optimization problems and numerical results, see [1], [3], and [11].

2. UNCONSTRAINED MINIMAX ALGORITHMS

We will develop a family of minimax algorithms by extension of the method of steepest descent which solves problems of the form

$$\min_{\mathbb{R}^n} f(x) \tag{2.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function. We begin by recalling the method of steepest descent and its convergence properties [9].

STEEPEST DESCENT ALGORITHM 2.1 :

Step 0 : Select an $x_0 \in \mathbb{R}^n$ and set $i = 0$.

Step 1 : Compute the *search direction*

$$h_i = -\nabla f(x_i) = \arg \min_{h \in \mathbb{R}^n} \{ f(x_i) + \langle \nabla f(x_i, h\rangle + \tfrac{1}{2}\|h\|^2 \} . \tag{2.2}$$

Step 2 : If $\nabla f(x_i) = 0$, stop. Else compute the *step size* $\lambda_i \in \lambda(x_i) \triangleq \arg\min_{\lambda \geq 0} f(x_i + \lambda h_i)$.

Step 3 : Set $x_{i+1} = x_i + \lambda_i h_i$, replace i by $i + 1$, and go to Step 1. ■

In practice one uses the Armijo step size rule [9] which is much more efficient, but somewhat harder to analyze than the one dimensional minimization rule, used in Step 2, above.

THEOREM 2.1 : If $\{x_i\}_{i=0}^{\infty}$ is an infinite sequence constructed by Algorithm 2.1, then every accumulation point $\hat{x}$ of $\{x_i\}_{i=0}^{\infty}$ satisfies $\nabla f(\hat{x}) = 0$.

PROOF : Suppose that $x_i \xrightarrow{K} \hat{x}$ as $i \to \infty$ and that $\nabla f(\hat{x}) \neq 0$. Then the directional derivative

$$df(\hat{x}; h(\hat{x})) = -\|\nabla f(\hat{x})\|^2 < 0 . \tag{2.3a}$$

Hence any $\hat{\lambda} \in \lambda(\hat{x})$ satisfies $\hat{\lambda} > 0$ and there exists a $\hat{\delta} > 0$ such that

$$f(\hat{x} + \hat{\lambda} h(\hat{x})) - f(\hat{x}) = -\hat{\delta} < 0 . \tag{2.3b}$$

Since $h(\cdot) = -\nabla f(\cdot)$ is *continuous* by assumption, the function $f(x + \hat{\lambda} h(x)) - f(x)$ is continuous in x and hence there exists an i_0 such that for all $i \in K$, $i \geq i_0$,

$$f(x_{i+1}) - f(x_i) \le f(x_i + \hat{\lambda} h(x_i)) - f(x_i) \le -\hat{\delta}/2 \, . \tag{2.3c}$$

Now, by construction, $\{f(x_i)\}_{i=0}^{\infty}$ is monotone decreasing and $f(x_i) \xrightarrow{K} f(\hat{x})$ as $i \to \infty$ by continuity of $f(\cdot)$. Therefore, we must have that $f(x_i) \to f(\hat{x})$ as $i \to \infty$. But this contradicts (2.3c). Hence $\nabla f(\hat{x}) = 0$ must hold. ■

Next let us examine the method of steepest descent geometrically, which requires the following notation. Given any function $g : \mathbb{R}^n \to \mathbb{R}$, we shall denote its level sets by $\mathrm{Lg}(\alpha)$, i.e., $\mathrm{Lg}(\alpha) \triangleq \{ x \in \mathbb{R}^n \mid g(x) \le \alpha \}$. Now, given a point x_i, we see that the method of steepest descent approximates the continuously differentiable function $f(x)$ by the quadratic function

$$q(x;x_i) \triangleq f(x_i) + \langle \nabla f(x_i), (x - x_i) \rangle + \tfrac{1}{2}\| x - x_i \|^2, \tag{2.4a}$$

and its level set

$$\mathbf{Lf}(f(x_i)) \triangleq \{ x \in \mathbb{R}^n \mid f(x) \le f(x_i) \}, \tag{2.4b}$$

by the "disk"

$$\mathbf{Df}(f(x_i);x_i) \triangleq \{ x \in \mathbb{R}^n \mid q(x;x_i) \le f(x_i) \}, \tag{2.4c}$$

which is tangent to $\mathbf{Lf}(f(x_i))$ at the point x_i. A minimizer of (2.1), $\hat{x}$, defines a "center" of $\mathbf{Lf}(f(x_i))$; $x_i - \nabla f(x_i)$, minimizes $q(x;x_i)$ and is the center of $\mathbf{Df}(f(x_i);x_i)$. The method of steepest descent treats the point $x_i - \nabla f(x_i)$ as an approximation to the point $\hat{x}$. Since this approximation is rather poor, the method of steepest descent performs a line search along the line passing through x_i and $x_i - \nabla f(x_i)$, according to (2.2c) to obtain a somewhat better approximation to $\hat{x}$, x_{i+1}, defined by (2.2d).

We can now consider the simplest minimax problem:

$$\min_{x \in \mathbb{R}^n} \max_{j \in \mathbf{m}} f^j(x), \tag{2.5a}$$

where the functions $f^j : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable and $\mathbf{m} \triangleq \{ 1,2,\ldots,m \}$. Let $\psi(x) \triangleq \max_{j \in \mathbf{m}} f^j(x)$, then (2.5a) becomes

$$\min_{x \in \mathbb{R}^n} \psi(x) \, , \tag{2.5b}$$

which is a nondifferentiable optimization problem. We need the following results [4].

THEOREM 2.2 : (a) For all $x,h \in \mathbb{R}^n$, the function $\psi(\cdot)$ has directional derivatives at x in the direction h which are given by

$$d\psi(x;h) \triangleq \lim_{t \downarrow 0} \frac{\psi(x + th) - \psi(x)}{t} = \max_{j \in \mathrm{I}(x)} \langle \nabla f^j(x), h \rangle, \tag{2.6a}$$

where $\mathrm{I}(x) \triangleq \{ j \in \mathbf{m} \mid f^j(x) = \psi(x) \}$.

(b) If $\hat{x}$ is a local minimizer for (2.5b), then the following *equivalent* statements hold:

(i) $$d\psi(\hat{x};h) \ge 0, \ \forall \ h \in \mathbb{R}^n \, , \tag{2.6b}$$

(ii) $$0 \in \mathrm{co}_{j \in \mathrm{I}(\hat{x})} \{ \nabla f^j(\hat{x}) \}, \tag{2.6c}$$

$$(iii) \quad 0 \in G\psi(\hat{x}) \triangleq \underset{j \in \mathbf{m}}{\mathrm{co}} \left\{ \begin{matrix} (f^j(\hat{x}) - \psi(\hat{x})) \\ \nabla f^j(\hat{x}) \end{matrix} \right\}, \tag{2.6d}$$

where "co" denotes the convex hull of the set in question. ∎

Next we make the observation that the level sets of the function $\psi(\cdot)$ are the intersection of level sets of the functions $f^j(\cdot)$, i.e.,

$$L\psi(\alpha)) \triangleq \{ x \in \mathbb{R}^n \mid \psi(x) \le \alpha) \} = \bigcap_{j \in \mathbf{m}} Lf^j(\alpha)) . \tag{2.7a}$$

Proceeding by analogy with the geometric interpretation of the method of steepest descent, given a point $x_i \in \mathbb{R}^n$, we approximate the level set $L\psi(\psi(x_i))$ by the intersection of the discs $Df^j(\psi(x_i))$ which approximate the level sets $Lf^j(\psi(x_i))$, i.e., by the set

$$D\psi(x_i, \psi(x_i)) \triangleq \bigcap_{j \in \mathbf{m}} Df^j(\psi(x_i))) = \{ x \in \mathbb{R}^n \mid q^j(x;x_i) \le \psi(x_i) \}, \tag{2.7b}$$

and we approximate the "center" $\hat{x}$ which solves (2.5) by the "center" $(x_i + h_i)$ of $D\psi(x_i, \psi(x_i))$ which solves the problem

$$\min_{x \in \mathbb{R}^n} \max_{j \in \mathbf{m}} q^j(x;x_i). \tag{2.8}$$

Adding a line search, we obtain the following extension of Algorithm 2.1.

MINIMAX ALGORITHM 2.2 :

Step 0 : Select a $x_0 \in \mathbb{R}^n$ and set $i = 0$.

Step 1 : Compute the *search direction*

$$h_i = \arg \min_{h \in \mathbb{R}^n} \max_{j \in \mathbf{m}} \{ f^j(x_i) + \langle \nabla f^j(x_i), h \rangle + \tfrac{1}{2}\|h\|^2 \} . \tag{2.9}$$

Step 2 : If $h_i = 0$, stop. Else compute the *step size* $\lambda_i \in \arg\min_{\lambda \ge 0} \psi(x_i + \lambda h_i)$.

Step 3 : Set $x_{i+1} = x_i + \lambda_i h_i$, replace i by $i + 1$, and go to Step 1. ∎

The search direction finding problem (2.9) is obviously much more difficult to solve than (2.2c). It is easiest to solve it in dual form. First, it is obvious that

$$\bar{\theta}(x_i) \triangleq \min_{h \in \mathbb{R}^n} \max_{j \in \mathbf{m}} \{ f^j(x_i) + \langle \nabla f^j(x_i, h \rangle + \tfrac{1}{2}\|h\|^2 \}$$

$$= \min_{h \in \mathbb{R}^n} \max_{\mu \in \Sigma} \{ \sum_{j=1}^{m} \mu^j \{ f^j(x_i) + \langle \nabla f^j(x_i, h \rangle + \tfrac{1}{2}\|h\|^2 \} , \tag{2.10a}$$

where $\Sigma \triangleq \{ \mu \in \mathbb{R}^m \mid \sum_{j=1}^{m} \mu^j = 1,\ \mu \ge 0 \}$. Next, making use of the von Neumann minimax theorem [2], we can interchange the min and max operations in (2.10a), to obtain that

$$\bar{\theta}(x_i) = \max_{\mu \in \Sigma} \min_{h \in \mathbb{R}^n} \sum_{j=1}^{m} \mu^j \{ f^j(x_i) + \langle \nabla f^j(x_i, h \rangle + \tfrac{1}{2}\|h\|^2 \} . \tag{2.10b}$$

Eliminating h from (2.10b) by unconstrained minimization, we obtain that

$$\bar{\theta}(x_i) = -\min_{\mu \in \Sigma} \{ \sum_{j=1}^{m} -\mu^j f^j(x_i) + \tfrac{1}{2}\| \sum_{j=1}^{m} \mu^j \nabla f^j(x_i) \|^2 \} . \tag{2.10c}$$

It now follows from the von Neumann minimax theorem that if $\mu_i \in \Sigma$ is *any* solution of (2.10b), then

$$h_i = -\sum_{j=1}^{m} \mu_i^j \nabla f^j(x_i) \tag{2.10d}$$

is the *unique* solution of (2.9). Since (2.10c) is easily solved by modern quadratic programming algorithms, such as [6], we see that the search direction h_i is readily computed.

Since it is easily shown that the search direction h_i is continuous [8], the following theorem can be proved by repeating the arguments for Theorem 2.1.

THEOREM 2.3 : If $\{x_i\}_{i=0}^{\infty}$ is an infinite sequence constructed by Algorithm 2.2, then every accumulation point $\hat{x}$ of $\{x_i\}_{i=0}^{\infty}$ satisfies $0 \in G\psi(\hat{x})$. ■

We are now ready to consider the general minimax problem

$$\min_{h \in \mathbb{R}^n} \max_{y \in Y} \phi(x,y) \ , \tag{2.11}$$

where $\phi : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$ and both $\phi(\cdot,\cdot)$ and $\nabla_x\phi(\cdot,\cdot)$ are continuous and $Y \subset \mathbb{R}^s$ is compact. First, we state an extension of Theorem 2.2.

THEOREM 2.4 : (a) For all $x,h \in \mathbb{R}^n$, the function $\psi(x) \triangleq \max_{y \in Y} \phi(x,y)$ has directional derivatives at x in the direction h which are given by

$$d\psi(x;h) \triangleq \lim_{t \downarrow 0} \frac{\psi(x+th) - \psi(x)}{t} = \max_{j \in Y(x)} \langle \nabla_x\phi(x,y), h \rangle \, , \tag{2.12a}$$

where $Y(x) \triangleq \{y \in Y \mid \phi(x,y) = \psi(x)\}$.

(b) If $\hat{x}$ is a local minimizer for (2.11), then the following *equivalent* statements hold:

(*i*) $$d\psi(\hat{x};h) \geq 0, \ \forall \ h \in \mathbb{R}^n \ , \tag{2.12b}$$

(*ii*) $$0 \in \operatorname{co}_{y \in Y(\hat{x})} \{ \nabla_x\phi(\hat{x},y) \} \tag{2.12c}$$

(*iii*) $$0 \in G\psi(\hat{x}) \triangleq \operatorname{co}_{y \in Y} \begin{Bmatrix} (\phi(\hat{x},y) - \psi(\hat{x})) \\ \nabla_x\phi(\hat{x},y) \end{Bmatrix} . \quad ■ \tag{2.12d}$$

If, as we have just done, we redefine $\psi(\cdot)$ by $\psi(x) = \max_{y \in Y} \phi(x,y)$, the formal extension of Algorithm 2.2 to this case is obvious and must be as given below.

MINIMAX ALGORITHM 2.3 :

Step 0 : Select $x_0 \in \mathbb{R}^n$ and set $i = 0$.

Step 1 : Compute the *search direction*

$$h_i = \arg \min_{h \in \mathbb{R}^n} \max_{y \in Y} \{\phi(x_i,y) + \langle \nabla_x\phi(x_i,y), h \rangle + \tfrac{1}{2}\|h\|^2\} \ . \tag{2.13}$$

Step 2 : If $h_i = 0$, stop. Else compute the *step size* $\lambda_i \in \arg\min_{\lambda \geq 0} \psi(x_i + \lambda h_i)$.

Step 3 : Set $x_{i+1} = x_i + \lambda_i h_i$, replace i by $i+1$, and go to Step 1. ■

Referring to [8], we see that the search direction h_i defined by (2.13) is continuous. Hence the following theorem can be proved by identical arguments as for Theorem 2.3.

THEOREM 2.5 : If $\{x_i\}_{i=0}^{\infty}$ is an infinite sequence constructed by Algorithm 2.3, then every accumulation point $\hat{x}$ of $\{x_i\}_{i=0}^{\infty}$ satisfies $0 \in G\psi(\hat{x})$. ■

The main question to be resolved is whether the search direction h_i can be computed. To this end, we begin by relating (2.10c) to (2.6d). We note that for any $x \in \mathbb{R}^n$, $G\psi(x) \subset \mathbb{R}^{n+1}$, and that $\bar{\xi} = (\xi^0, \xi)$ is an element of $G\psi(x)$ if and only if there exists a $\mu \in \Sigma$ such that $\xi^0 = \sum_{j=1}^m \mu^j [f^j(x) - \psi(x)]$ and $\xi = \sum_{j=1}^m \mu^j \nabla f^j(x)$. Hence (2.10c) can be rewritten in the form

$$\bar{\theta}(x_i) = -\min_{\bar{\xi} \in G\psi(x_i)} \{\xi^0 + \tfrac{1}{2}\|\xi\|^2\} \;, \tag{2.14a}$$

and the search direction h_i, defined by (2.2), is given by $h_i = -\xi(x_i)$, where $\xi(x_i)$ consists of the last n elements of the $(n+1)$ dimensional vector

$$\bar{\xi}(x_i) = (\xi^0(x_i), \xi(x_i)) = -\arg\min_{\bar{\xi} \in G\psi(x_i)} \{\xi^0 + \tfrac{1}{2}\|\xi\|^2\} \;. \tag{2.14b}$$

If we define $G\psi(x_i)$ by making use of (2.12d), and $\bar{\xi}(x_i)$ as in (2.14b), then the formula $h_i = -\xi(x_i)$ is also valid for Algorithm 2.3. The importance of this observation lies in the fact that $\bar{\xi}(x_i)$ can now be computed by means of a proximity algorithm. These algorithms are descendants of the Gilbert algorithm [7], see e.g., [10]. They depend on our ability to compute tangency points to the sets $G\psi(x)$, which are defined as solutions of the *contact* problem

$$\min\{\, \langle v, \bar{\xi} \rangle \mid \bar{\xi} \in G\psi(x) \,\} , \tag{2.15}$$

where v is any given direction.

The computation of these tangency points in structural design problems does not appear to pose any serious difficulty (see [8] for details). For the sake of completeness, we now state the simplest of these proximity algorithms for solving the problem (2.14a).

PROXIMITY ALGORITHM 2.4 [7]

Step 0: Select a $\bar{\xi}_0 = (\xi_0^0, \xi_0) \in G\psi(x_i)$ and set $k = 0$.

Step 1 : Set $v_k = [\partial/\partial\bar{\xi}](\xi_k^0 + \tfrac{1}{2}\|\xi_k\|) = (1, \xi_k)$.

Step 2 : Compute $\bar{\eta}_k \in G\psi(\bar{\xi}_i)$ such that $\langle v_k, \bar{\eta}_k \rangle = \min\{\, \langle v_k, \bar{\xi} \rangle \mid \bar{\xi} \in G\psi(x_i) \,\}$.

Step 3 : Compute $\bar{\xi}_{k+1} = (\xi_{k+1}^0, \xi_{k+1}) = \arg\min_{\lambda \in [0,1]} \|\bar{\xi}_k + \lambda(\bar{\eta}_k - \bar{\xi}_k)\|$.

Step 4 : Replace i by i+1 and go to step 1. ■

THEOREM 2.6 : The sequence $\{\xi_k\}_{k=0}^{\infty}$ constructed by Algorithm 2.4, converges to the search direction vector $-h_i$. ■

Algorithm 2.4 tends to converge very slowly. The version in [10] is considerably more complex, but it converges considerably faster.

Finally we turn to algorithms for the solution of constrained minimax problems, of the form

$$\min\{\ \psi^0(x)\mid \psi^j(x)\le 0,\ j\in \mathbf{m}\ \},\tag{2.16}$$

where, for $j = 0,1,2,\ldots,m$, $\psi^j(x) = \max_{y_j\in Y_j}\phi^j(x,y_j)$, $\phi^j : \mathbb{R}^n\times\mathbb{R}^{s^j}\to\mathbb{R}$, and both $\phi^j(\cdot,\cdot)$ and $\nabla_x\phi^j(\cdot,\cdot)$ are continuous and the subsets $Y_j\subset\mathbb{R}^{s^j}$ are compact.

Let $\delta > 0$, let $\psi_+(x) \triangleq \max_{j\in\mathbf{m}}\{0,\psi^j(x)\}$ and, for any $\bar{x}\in\mathbb{R}^n$, and let the parametrized function $F_{\bar{x}}(x)$ be defined by

$$F_{\bar{x}}(x)\triangleq\max\{\psi^0(x)-\psi^0(\bar{x})-\delta\psi_+(x),\psi^j(x),\ j\in\mathbf{m}\}\ .\tag{2.17}$$

We see that if $\hat{x}$ is a local minimizer for (2.16), then it must also be a local minimizer for the function $F_{\hat{x}}(x)$. Hence we deduce from Theorem 2.4 (b) the following result.

THEOREM 2.7 [4]: If $\hat{x}$ is a local minimizer for (2.16), then

$$0\in \mathrm{co}\left\{\bigcup_{j=0}^{m}\ \mathrm{co}_{y_j\in Y_j}\left\{\begin{pmatrix}\bar{\phi}^j(\hat{x},y)\\ \nabla_x\phi(\hat{x},y)\end{pmatrix}\right\}\right\},\tag{2.18}$$

where $\bar{\phi}^0(x,y) = \phi^0(x,y)-\psi^0(\hat{x})$, and $\bar{\phi}^j(x,y)=\phi^j(x,y)$ for all $j\in\mathbf{m}$. ■

Just as we obtained algorithms for unconstrained minimax problems by geometric extension of the method of steepest descent, we obtain algorithms for the solution of (2.16) from the following phase I - phase II generalization of the Huard *conceptual* method of centers (see [9]), below (c.f. (2.17)), where $\delta > 0$ is given.

CONCEPTUAL METHOD OF CENTERS 2.4 :

Step 0 : Select $x_0\in\mathbb{R}^n$ such that $\psi^j(x_0)\le 0$ for all $j\in\mathbf{m}$ and set $i = 0$.

Step 1 : Compute $x_{i+1} = \arg\min_{x\in\mathbb{R}^n} F_{x_i}(x)$.

Step 2 : Set $i = i + 1$ and go to Step 1. ■

It is easy to prove convergence of the above algorithm under the following simplifying assumption.

ASSUMPTION: (a) For every $\bar{x}\in\mathbb{R}^n$, the level sets $\mathrm{L}F_{\bar{x}}(\alpha)$ are compact. (b) for every $\bar{x}\in\mathbb{R}^n$ which is not a local minimizer of (2.16), $\gamma(\bar{x})\triangleq\min_{x\in\mathbb{R}^n}F_{\bar{x}}(x)<0$. ■.

THEOREM : If $\{x_i\}_{i=0}^{\infty}$ is an infinite sequence constructed by the conceptual method of centers, then every accumulation point $\hat{x}$ of $\{x_i\}_{i=0}^{\infty}$ is a local minimizer for (2.16).

PROOF : First, referring to [8], we conclude that $\gamma(\cdot)$ is continuous. Next, suppose that $\{x_i\}_{i=0}^{\infty}$ is an infinite sequence constructed by the conceptual method of centers, and that there exists an i_0 such that $\psi(x_+(x_{i_0}))\le 0$. Then $\psi(x_+(x_i))\le 0$ for all $i > i_0$ and $\psi^0(x_{i+1})-\psi^0(x_i)\le\gamma(x_i)<0$. If $\{x_i\}_{i=0}^{\infty}$ has an accumulation point $\hat{x}$ which is not a local minimum of (2.16), then $\gamma(\hat{x})<0$ and hence, by continuity, there exists an $i_1\ge i_0$ such that for all $i > i_0$ and the elements x_i of the subsequence which converges to $\hat{x}$,

$\psi^0(x_{i+1}) - \psi^0(x_i) \le \frac{1}{2}\gamma(\hat{x}) < 0$, which leads to the contradictory conclusion that $\psi^0(x_i) \to \infty$ and not to $\psi^0(\hat{x})$. Hence $\hat{x}$ must be a local minimizer.

The case where $\psi_+(x_i) > 0$ for all i, is dealt with similarly, using the function $\psi(x) \triangleq \max_{j \in \mathbf{m}} \psi^j(x)$. ■

To implement the conceptual method of centers, we simply replace Step 1, with a single iteration of Algorithm 2.3, and substitute in the convergence theorem the statement that accumulation points are local minima, by the statement that accumulation points are stationary.

3. CONCLUSION

We have presented a brief survey of the simplest minimax algorithms for engineering design. For a complete exposition as well as further references, the reader should consult [8].

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