NONLINEAR CONTROLLER DESIGN FOR FLIGHT CONTROL SYSTEMS

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Nonlinear Controller Design for Flight Control Systems*

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Abstract

There has been a great deal of excitement recently over the development of a theory for explicitly linearizing the input-output response of a nonlinear system using state feedback. One shortcoming of this theory is the inability to deal with *non-minimum phase* nonlinear systems. Highly maneuverable jet aircraft, such as the V/STOL Harrier, belong to an important class of a *slightly* non-minimum phase nonlinear systems. The non-minimum phase character of these aircraft is due in part to a slight coupling between rolling moments and lateral accelerations. In this paper, we show that, while straightforward application of the linearization theory to a non-minimum phase system results in a system with a linear input-output response but unstable internal dynamics, designing a feedback control based on a minimum phase approximation to the true system results in a system with desirable properties such as bounded tracking and asymptotic stability.

Introduction

There has been a great deal of excitement in recent years over the development of a rather complete theory for explicitly linearizing the input-output response of a nonlinear system using state feedback. This has been explicitly worked out in several papers, like those of [Por70], [SR72],[IKGM81]. Independently, a substantially identical synthesis technique was successfully implemented in several practical applications, such as flight control [MC80] and the control of rigid robots by the so-called computed torque method [Fre75]. The theory is now well developed and understood (see, for instance expository surveys in [Isi85], [BI88a] and [BI88b]).

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The chief drawbacks to making these considerable advances into a successful design methodology come from the fact that they rely on an exact cancellation of nonlinear terms and use a nonlinear version of pole-zero cancellation. Of course, the nonlinear pole-zero cancellation implicit in these techniques is only a problem when the cancellation is one involving unstable zero-dynamics (a precise definition of this concept for nonlinear systems was given in [BI88a]). In earlier work, we have discussed how to remedy the first deficiency by using parameter adaptive control (see [SI87]). In this paper, we discuss the second problem. We start with an example from flight control of a system which is slightly non-minimum phase: the dynamics of a vertical takeoff and landing aircraft, the Harrier. We show how a straightforward application of the linearization techniques results in a system with marginally stable unobservable modes resulting in undesirable closed loop system behavior. We then show that by neglecting certain, physically small, variables the approximate linearization results in much better performance.

Motivated by this example, we develop the rudiments of a theory of *approximate linearization* for *slightly non-minimum phase* systems.

Our primary goals in this project are twofold:

1. The robust nonlinear flight control of an important class of highly maneuverable aircraft.

2. The development of a practical methodology of nonlinear control for slightly non-minimum phase systems based on the recent advances in linearization. Slightly non-minimum phase nonlinear systems are to be thought of as the generalization of linear systems with far off right half plane zeros.

1 Modeling of Aircraft Dynamics

In general, the complete dynamics of an aircraft, taking into account flexibility of the wings and fuselage, the (internal) dynamics of the engine and control surface actuators, and the multitude of changing variables, are quite complex and somewhat unmanageable for the purposes of control. A more reasonable (and useful) approach is to consider the aircraft as a rigid body upon which a set of forces and moments act. Then, with r, R, and ω being the aircraft position, orientation, and angular velocity, respectively, the equations of motion can be written as

$$m\ddot{r} = Rf_a + mg \tag{1}$$

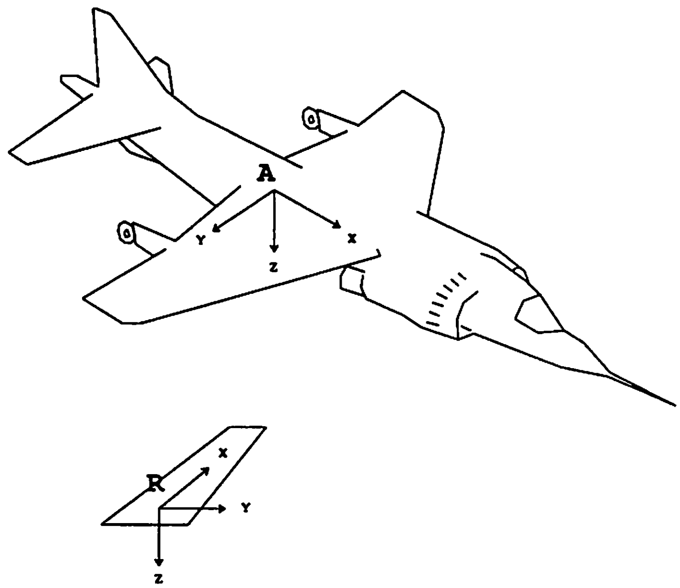

Figure 1: Aircraft coordinate systems

$$J\dot{\omega}_a = \tau_a - \omega_a \times J\omega_a \qquad (2)$$

$$\dot{R} = \omega \times R \qquad (3)$$

where f_a and τ_a are the force and moment acting on the aircraft expressed relative to the aircraft. Here, the a subscript means that a quantity is expressed with respect to the aircraft reference frame. Depending on the aircraft and its mode of flight, the forces and moments can be generated by aerodynamics (lift, drag, and roll-pitch-yaw moments), by momentum exchange (gross thrust vectoring and reaction controls to generate moments), or a combination of the two. The flight envelope of the aircraft is the set of flight conditions for which the pilot and/or the control system can effect the forces and moments needed to remain in the envelope and achieve the desired task.

1.1 Force and Moment Generation

For the sake of presentation, we will focus our attention on a particular aircraft, the YAV-8B Harrier produced by McDonnell Aircraft Company [McD82,McD83]. The Harrier is a single-seat transonic light attack V/STOL (vertical/short takeoff and landing) aircraft powered by a single turbo-fan engine. Figure 1 shows the aircraft with the coordinate frame, $\mathcal{A}$, attached at the (nominal) center of mass. The x-axis is directed forward toward the nose of the aircraft and is also known as the *roll* axis since positive rotation about the x-axis coincides with rolling the aircraft to the right (from the pilot's point of view). The y-axis is directed toward the right wing and is called the *pitch* axis (positive rotation is a pitch up). The z-axis is directed downward and is also known as the *yaw* axis (we yaw to the right about this axis).

Also shown in figure 1 is the (inertial) *runway* coordinate frame, $\mathcal{R}$. The x-, y-, and z-axes of the runway frame are often oriented in the north, east, and down (N-E-D) directions, respectively.

Four exhaust nozzles on the turbo-fan engine provide the gross thrust for the aircraft. These nozzles can be rotated from the aft position (used for conventional wing-borne flight) forward approximately 100 degrees allowing jet-borne flight and nozzle braking. The throttle and nozzle controls thus provide two degrees of freedom of thrust vectoring within the x-z plane of the aircraft. (If the line of action of the gross thrust does not pass through the object center of mass, then this thrust will also produce a net pitching moment.)

In addition to the conventional aerodynamic control surfaces (aileron, stabilator, and rudder for roll, pitch, and yaw moments, respectively), the Harrier also has a reaction control system (RCS) to provide moment generation during jet-borne and transition flight. Reaction valves in the nose, tail, and wingtips use bleed air from the high pressure compressor of the engine to produce thrust at these points and therefore moments (and forces) at the aircraft center of mass. The design of the aerodynamic and reaction controls provides complete (three degree of freedom) moment generation throughout the flight envelope of the aircraft. Since moments are often produced by applying a single force rather than a couple, a nonzero force (proportional to the moment) will usually be seen at the aircraft center of mass.

Using the throttle, nozzle, roll, pitch, and yaw controls we can produce (within physical limits) any moment and any force in the x-z plane of the aircraft. The function, $\mathcal{F}$, taking the control inputs,

$$c = (throttle, nozzle, roll, pitch, yaw)^T, \tag{4}$$

to the aircraft force and moment, $(f_a^T, \tau_a^T)^T$,

$$\begin{pmatrix} f_a \\ \tau_a \end{pmatrix} = \mathcal{F}(r, \dot{r}, R, \omega, c) \tag{5}$$

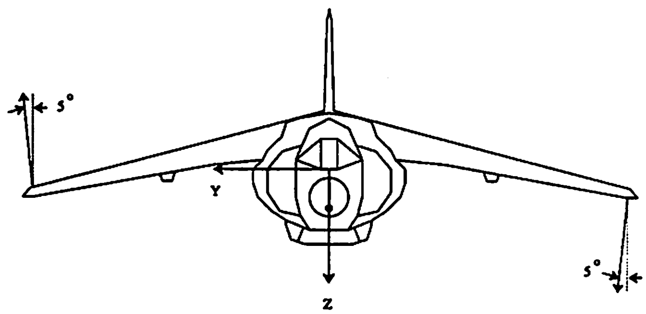

Figure 2: Reaction control system geometry

is complex and depends upon the state of the aircraft system (airspeed, altitude, etc.). The projection of this function taking the moments and the x and z components of force as outputs is one-to-one and hence invertible. That is, given a desired aircraft moment and (x-z plane) force that is achievable at the current aircraft state, there is a unique control input vector (throttle, nozzle, roll, pitch, yaw) that will produce that force and moment. Letting $u = (f_{ax}, f_{az}, \tau_a^T)^T$, this function can be written as

$$c = \mathcal{C}(r, \dot{r}, R, \omega, u) \tag{6}$$

where r, $\dot{r}$, R, and ω compose the aircraft state. Given the function $\mathcal{C}$, we are free to consider the desired moment and (x-z) force as the aircraft control input in place of the true control input. The idea of inverting the algebraic nonlinearities present in the system has been applied to real flight control problems [MC80].

It is now natural to ask what form the (state dependent) function taking u to f_{ay} will take. Since moments are produced by applying forces to the aircraft, one is hopeful that the resulting y-axis force at the aircraft center of mass will be a (state dependent) linear function of the moment acting on the aircraft. Note that this is not necessarily the case. For example, consider the generation of a right rolling moment (from the pilot's point of view) during jet-borne flight. Figure 2 shows the geometry of the wingtip reaction control valves. For small ranges of moment, the left reaction valve opens and blows downward creating a net upward force. Once the left reaction valve is fully open, the right reaction valve opens and blows upward which reduces the net upward force. In this case, there is a nonlinear coupling between the rolling moment and the force in the vertical (z-axis) direction of the aircraft. This case is easily reconciled, however, since we can directly affect the vertical (z-axis) force using the throttle and nozzle.

Clearly, forces in the x-z plane and moments about the y-axis will not contribute to y-axis forces. Thus we consider the y-axis forces generated by rolling (x-axis) and yawing (z-axis) moments. Yawing moments are generated by applying a force at the tail of the aircraft (by aerodynamic or reaction control methods). As long as this force is effectively applied at the same point regardless of the magnitude of the moment, there will be a (state dependent) linear relationship between the z-axis moment and the resulting y-axis force. The coupling between rolling (x-axis) moments and y-axis forces is more subtle and is the result of the geometry of the reaction control system. As figure 2 shows, the forces used to generate the rolling moment are not perpendicular to the y-axis of the aircraft. Thus, when a positive rolling moment is commanded, a negative force is generated in the y-axis direction (i.e., the airplane will initially accelerate to the left when it is commanded to go right). Also, as mentioned above, depending on the magnitude of the rolling moment, the right reaction valve could be actively blowing upward or be fully closed. Fortunately, the distance to and angle of the upward and downward reaction valve thrust vectors are equal. For this reason, the relationship between the rolling moment and y-axis force is linear.

We can now rewrite equations (1) and (2) as

$$\begin{pmatrix} m\ddot{r} \\ J\dot{\omega}_a \end{pmatrix} = \begin{pmatrix} mg \\ -\omega_a \times J\omega_a \end{pmatrix} + \begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix} Bu \tag{7}$$

where B is the (state dependent, 6-by-5) matrix providing the full vector of aircraft forces and moments given the *control* input, u. In particular, B has the form

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_{roll} & 0 & \beta_{yaw} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

where β_{roll} and β_{yaw} are the (scalar) functions giving the coupling between the roll and yaw moments and the y-axis force.

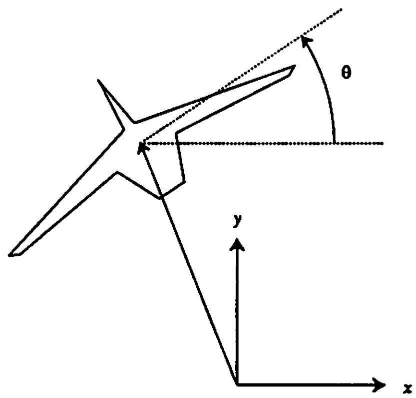

Figure 3: The planar vertical takeoff and landing (PVTOL) aircraft

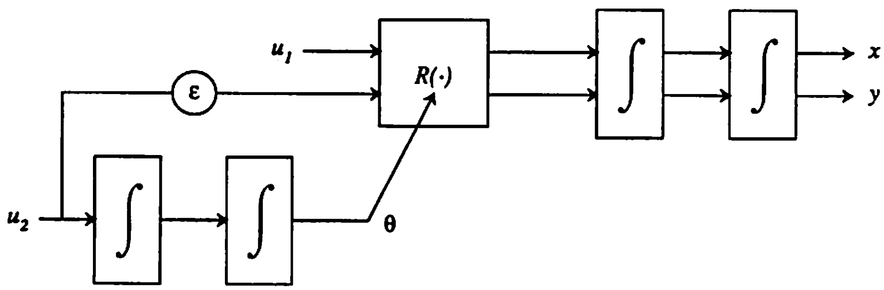

Figure 4: Block diagram of the PVTOL aircraft system

1.2 Simplification to a Planar Aircraft

It is particularly useful to consider a simple toy aircraft that has a minimum number of states and inputs but retains many of the features that must be considered when designing control laws for a real aircraft such as the Harrier. Figure 3 shows our prototype PVTOL (planar vertical takeoff and landing) aircraft. The aircraft state is simply the position, x, y, of the aircraft center of mass, the angle, θ, of the aircraft relative to the x-axis, and the corresponding velocities, $\dot{x}$, $\dot{y}$, $\dot{\theta}$. The control inputs, u_1, u_2, are the thrust (directed out the bottom of the aircraft) and the rolling moment.

The equations of motion for our PVTOL aircraft are given by

$$\left.\begin{array}{rcl}\ddot{x} &=& -\sin\theta u_1 + \epsilon\cos\theta u_2 \\ \ddot{y} &=& \cos\theta u_1 + \epsilon\sin\theta u_2 - 1 \\ \ddot{\theta} &=& u_2\end{array}\right\} \tag{9}$$

where '-1' is the gravitational acceleration and ϵ is the (small) coefficient giving the coupling between the rolling moment and the lateral acceleration of the aircraft. Note that $\epsilon > 0$ means that applying a (positive) moment to roll left produces an acceleration to the right (positive x). Figure 4 provides a block diagram representation of this dynamical system.

The PVTOL aircraft system is the natural restriction of V/STOL aircraft to jet-borne operation (e.g., hover) in a vertical plane. Our study of this simple planar model is but the first step in an ongoing project to understand and develop robust methods for the control of highly maneuverable aircraft systems.

2 Linearization by State Feedback

2.1 Exact Input-Output Linearization of the PVTOL Aircraft System

Consider the PVTOL aircraft system given by (9). Choosing x and y as the outputs to be controlled, we seek a (possibly dynamic) state feedback law of the form

$$u = a(z) + b(z)v \tag{10}$$

such that, for some $\gamma = (\gamma_1, \gamma_2)^T$,

$$\begin{array}{rcl}x^{(\gamma_1)} &=& v_1 \\ y^{(\gamma_2)} &=& v_2.\end{array} \tag{11}$$

Here, v is our new input and z is used to denote the entire state of the system (including compensator states, if necessary).

Proceeding in the usual way, we differentiate each output until at least one of the inputs appears. This occurs after differentiating twice and is given by (rewriting the first two equations of (9))

$$\begin{bmatrix}\ddot{x} \\ \ddot{y}\end{bmatrix} = \begin{bmatrix}0 \\ -1\end{bmatrix} + \begin{bmatrix}-\sin\theta & \epsilon\cos\theta \\ \cos\theta & \epsilon\sin\theta\end{bmatrix}\begin{bmatrix}u_1 \\ u_2\end{bmatrix}. \tag{12}$$

Since the matrix operating on u (the so-called *decoupling* matrix) is nonsingular (barely—its determinant is $-\epsilon$!), we can linearize (and decouple) the system by choosing the static state feedback law

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta \\ \frac{\cos\theta}{\epsilon} & \frac{\sin\theta}{\epsilon} \end{bmatrix} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right). \tag{13}$$

The resulting system is

$$\left.\begin{aligned} \ddot{x} &= v_1 \\ \ddot{y} &= v_2 \\ \ddot{\theta} &= \tfrac{1}{\epsilon}(\sin\theta + \cos\theta v_1 + \sin\theta v_2) \end{aligned}\right\} \tag{14}$$

This feedback law makes our input-output map linear, but has the unfortunate side-effect of making the dynamics of θ unobservable. In order to guarantee the internal stability of the system, it is not sufficient to look at input-output stability, we must also show that all internal (unobservable) modes of the system are stable as well.

The first step in analyzing the internal stability of the system (14) is to look at the *zero dynamics* [BI88a] of the system. The zero dynamics of a nonlinear system are the internal dynamics of the system subject to the constraint that the outputs (and, therefore, all derivatives of the outputs) are set to zero for all time.

Constraining the outputs and derivatives to zero by setting $v_1 = v_2 = 0$ (and using appropriate initial conditions), we find the zero dynamics of (14) to be

$$\ddot{\theta} = \frac{1}{\epsilon}\sin\theta. \tag{15}$$

Equation (15) is simply the equation of an undamped pendulum. Figure 5 shows the phase portrait ($\dot{\theta}$ vs θ) of the pendulum (15) with $\epsilon = 1$. The phase portrait for $\epsilon < 0$ is simply a π-translate of figure 5. Thus, for $\epsilon > 0$, the equilibrium point $(\theta, \dot{\theta}) = (0, 0)$ is unstable and the equilibrium point $(\pi, 0)$ is stable but not asymptotically stable and is surrounded by a family of periodic orbits with periods ranging from $2\pi\sqrt{\epsilon}$ to ∞. Outside of these orbits is a family of unbounded trajectories. Thus, depending on the initial condition, the aircraft will either rock from side to side forever or roll continuously in one direction (except at the isolated equilibria).

Nonlinear systems, such as (14), with zero dynamics that are not asymptotically stable are called *non-minimum phase*. Figure 6 shows the response

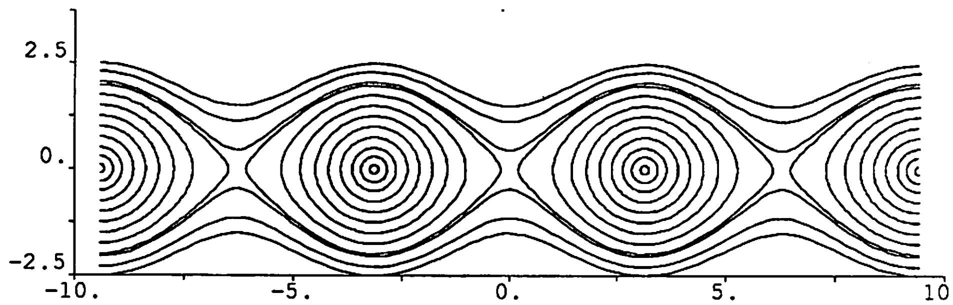

Figure 5: Phase portrait of an undamped pendulum ($\epsilon = 1$)

of the system (14) when (v_1, v_2) is chosen (by a stable feedback law) so that x will track a smooth trajectory from 0 to 1 and y will remain at zero. The bottom section of the figure shows snapshots of the PVTOL aircraft's position and orientation at 0.2 second intervals. From the phase portrait of θ (figure 6e), we see that the zero dynamics certainly exhibit pendulum like behavior. Initially, the aircraft rolls left (positive θ) to almost 2π. Then, it rolls right through four revolutions before settling into a periodic motion about the -3π equilibrium point. Since v_1 and v_2 are zero after $t = 5$, the aircraft continues rocking approximately $\pm\pi$ from the inverted position.

From the above analysis and simulations, it is clear that exact input-output linearization of a system such as (9) can produce undesirable results. The source of the problem lies in trying to control modes of the system using inputs that are weakly (ϵ) coupled rather than controlling the system in the way it was designed to be controlled and accepting a performance penalty for the parasitic (ϵ) effects. For our simple PVTOL aircraft, we should control the linear acceleration by vectoring the thrust vector (using moments to control this vectoring) and adjusting its magnitude using the throttle.

2.2 Approximate Linearization of the PVTOL Aircraft System using a Simplified Model

In the last section we (exactly) linearized input-output map of the PVTOL aircraft system (9). However, due to the small coupling between rolling moments and lateral acceleration, the linearized system had unstable zero dynamics. Thus, while the outputs (the x and y position) can be tracked perfectly, the internal behavior (the aircraft attitude) is not regulated and

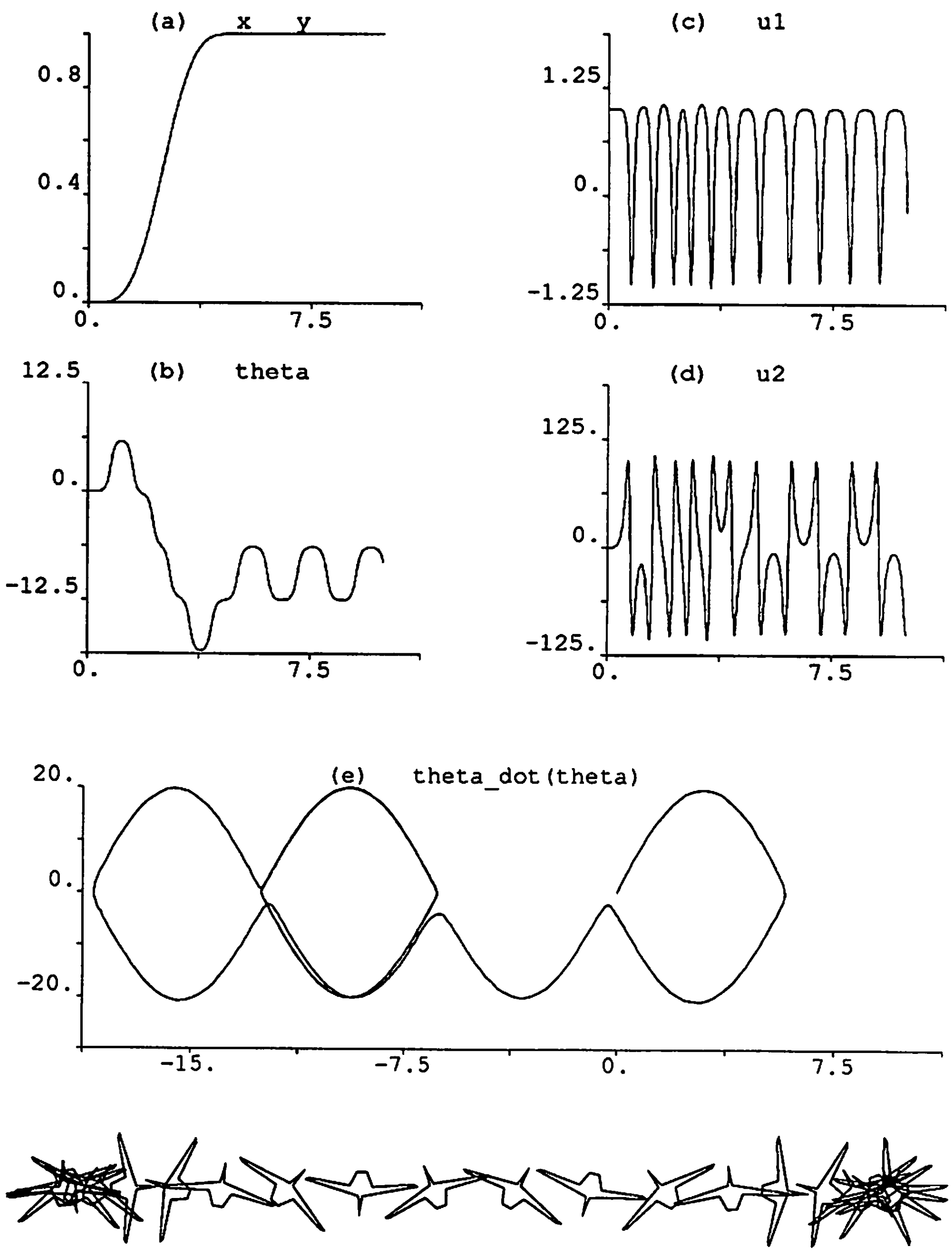

Figure 6: Response of non-minimum phase system to smooth *step* input

exhibits unstable behavior.

In this section, we propose controlling the system as if there were no coupling between rolling moments and lateral acceleration (i.e., $\epsilon = 0$). Using this approach to control the true system (9), we expect to see a loss of performance due to the *unmodeled* dynamics present in the system. In particular, we see that we can guarantee stable asymptotic tracking of constant velocity trajectories and bounded tracking for trajectories with bounded higher order derivatives.

We now *model* the PVTOL aircraft as ((9) with $\epsilon = 0$)

$$\left.\begin{array}{rcl} \ddot{x}_m &=& -\sin\theta u_1 \\ \ddot{y}_m &=& \cos\theta u_1 - 1 \\ \ddot{\theta} &=& u_2 \end{array}\right\} \tag{16}$$

so that there is no coupling between rolling moments and lateral acceleration. Differentiating the model system outputs, x_m and y_m, we get (analogous to (12))

$$\begin{bmatrix} \ddot{x}_m \\ \ddot{y}_m \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{17}$$

Now, however, the matrix multiplying u is singular which implies that there is no *static* state feedback that will linearize (16). Since u_2 comes into the system (16) through $\ddot{\theta}$, we must differentiate (17) at least two more times. Let u_1 and $\dot{u}_1$ be states (in effect, placing two integrators before the u_1 input) and differentiate (17) twice giving

$$\begin{bmatrix} x_m^{(4)} \\ y_m^{(4)} \end{bmatrix} = \begin{bmatrix} \sin\theta\dot{\theta}^2 u_1 - 2\cos\theta\dot{\theta}\dot{u}_1 \\ -\cos\theta\dot{\theta}^2 u_1 - 2\sin\theta\dot{\theta}\dot{u}_1 \end{bmatrix} + \begin{bmatrix} -\sin\theta & -\cos\theta u_1 \\ \cos\theta & -\sin\theta u_1 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ u_2 \end{bmatrix}. \tag{18}$$

The matrix operating on our new inputs, $(\ddot{u}_1, u_2)^T$, has determinant equal to u_1 and therefore is invertible as long as the thrust, u_1, is nonzero. This fact agrees well with our intuition since we know that no amount of rolling will affect the motion of the PVTOL aircraft if there is no thrust to effect an acceleration. Figure 7 shows a block diagram of the model system with u_1 and $\dot{u}_1$ considered as states. Note that each *input* must go through four integrators to get to the output. Thus, we linearize (16) using the *dynamic* state feedback law

$$\begin{bmatrix} \ddot{u}_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta \\ -\frac{\cos\theta}{u_1} & -\frac{\sin\theta}{u_1} \end{bmatrix} \left(\begin{bmatrix} -\sin\theta\dot{\theta}^2 u_1 + 2\cos\theta\dot{\theta}\dot{u}_1 \\ \cos\theta\dot{\theta}^2 u_1 + 2\sin\theta\dot{\theta}\dot{u}_1 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right)$$

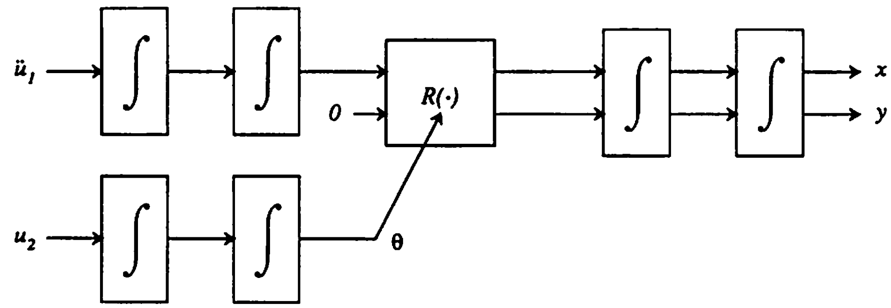

Figure 7: Block diagram of the augmented model PVTOL aircraft system

$$= \begin{bmatrix} \dot{\theta}^2 u_1 \\ -\frac{2\dot{\theta}\dot{u}_1}{u_1} \end{bmatrix} + \begin{bmatrix} -\sin\theta & \cos\theta \\ -\frac{\cos\theta}{u_1} & -\frac{\sin\theta}{u_1} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{19}$$

resulting in

$$\begin{bmatrix} x_m^{(4)} \\ y_m^{(4)} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{20}$$

Unlike the previous case (equation (14)), the linearized model system does not contain any unobservable (zero) dynamics. Thus, using a stable tracking law for v, we can track an arbitrary trajectory and guarantee that the (model) system will be stable.

Of course, the natural question that comes to mind is: will a control law based on the model system (16) work well when applied to the true system (9)? In the next section, we will show (in a more general setting) that, if ϵ is *small* enough, then the system will have reasonable properties (such as stability and bounded tracking).

How small is small enough? Figure 8 shows the response of the true system with epsilon ranging from 0 to 0.9 (0.01 is typical during jet-borne flight, i.e., hover, for the Harrier). As in section 2.1, the desired trajectory is a smooth lateral motion from $x = 0$ to $x = 1$ with the altitude (y) held constant at 0. The figure also shows snapshots of the PVTOL aircraft's position orientation at 0.2 second intervals for $\epsilon = 0.0, 0.1$, and 0.3. Since the snapshots were taken at uniform intervals, the spacing between successive pictures gives a clue of the aircraft velocity and acceleration. The computer graphics movie of the trajectories provides an even better sense of the system response.

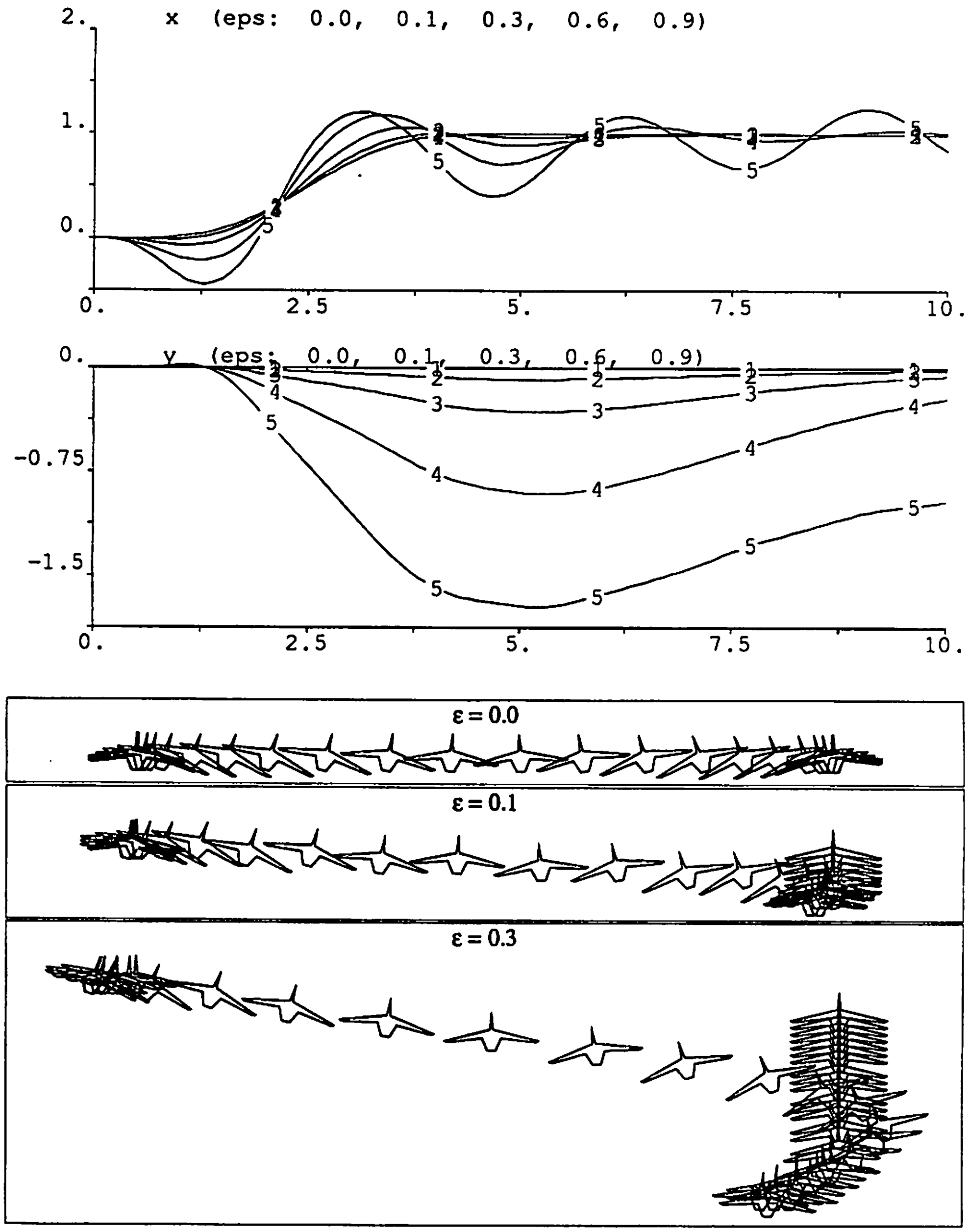

Figure 8: Response of the true PVTOL aircraft system under the approximate control

Interestingly, the x response is quite similar to the step response of a non-minimum phase linear system. Note that for ϵ less than approximately 0.6, the oscillations are reasonably damped. Although performance is certainly worse at higher values of ϵ, stability does not appear to be lost until ϵ is in the neighborhood of 0.9. A value of 0.9 for ϵ means that the aircraft will experience almost $1g$ (the acceleration of gravity) in the wrong direction when a rolling acceleration of one radian per second per second is applied. For the range of ϵ values that will normally be expected, the performance penalty due to approximation is small, almost imperceptible.

Note that, while the PVTOL aircraft system (9) with the approximate control (19) is stable for a large range of ϵ, this control allows the PVTOL aircraft to have a bounded but unacceptable altitude (y) deviation. Since the ground is hard and quite unforgiving and vertical takeoff and landing aircraft are designed to be maneuvered in close proximity to the ground, it is extremely desirable to find a control law that provides exact tracking of altitude if possible. Now, ϵ enters the system dynamics (9) in only one (state-dependent) direction. We therefore expect that one should be able to modify the system (by manipulating the inputs) so that the effects of the ϵ-coupling between rolling moments and aircraft lateral acceleration do not appear in the y output of the system.

Consider the decoupling matrix of the true PVTOL system (9) given in (12) as

$$\begin{bmatrix} -\sin\theta & \epsilon\cos\theta \\ \cos\theta & \epsilon\sin\theta \end{bmatrix}. \tag{21}$$

To make the y output independent of ϵ requires that the last row of this decoupling matrix be independent of ϵ. The only legal way to do this is by multiplication on the right (i.e., column operations) by a nonsingular matrix V which corresponds to multiplying the inputs by V^{-1}. In this case, we see that

$$\begin{bmatrix} -\sin\theta & \epsilon\cos\theta \\ \cos\theta & \epsilon\sin\theta \end{bmatrix} \begin{bmatrix} 1 & -\epsilon\tan\theta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta & \frac{\epsilon}{\cos\theta} \\ \cos\theta & 0 \end{bmatrix} \tag{22}$$

is the desired transformation. Defining new inputs, $\tilde{u}$, as

$$\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} = \begin{bmatrix} 1 & \epsilon\tan\theta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{23}$$

we see that (12) becomes

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -\sin\theta & \frac{\epsilon}{\cos\theta} \\ \cos\theta & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}. \tag{24}$$

Following the previous analysis, we set $\epsilon = 0$ and linearize the resulting approximate system using the dynamic feedback law

$$\begin{bmatrix} \ddot{\tilde{u}}_1 \\ \tilde{u}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta}^2 \tilde{u}_1 \\ -\frac{2\dot{\theta}\dot{\tilde{u}}_1}{\tilde{u}_1} \end{bmatrix} + \begin{bmatrix} -\sin\theta & \cos\theta \\ -\frac{\cos\theta}{\tilde{u}_1} & -\frac{\sin\theta}{\tilde{u}_1} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{25}$$

The true system inputs are then calculated as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & -\epsilon\tan\theta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}. \tag{26}$$

Figure 9 shows the response of the true system using the control law specified by equations (25) and (26) for the same desired trajectory. With this control law, our PVTOL aircraft maintains the altitude as desired and provides stable, bounded lateral (x) tracking for ϵ up to at least 0.6. Note, however, that the system is decidedly unstable for $\epsilon = 0.9$. Since we have forced the error into one direction (i.e., the x-channel), we expect the approximation to be more sensitive to the value of ϵ. In particular, compare the second column of the decoupling matrices of (12) and (24), i.e.,

$$\begin{bmatrix} \epsilon\cos\theta \\ \epsilon\sin\theta \end{bmatrix} \text{ and } \begin{bmatrix} \frac{\epsilon}{\cos\theta} \\ 0 \end{bmatrix}. \tag{27}$$

Notice that the first is simply ϵ times a bounded function of θ while the second contains ϵ times an unbounded function of θ (i.e., $1/\cos\theta$). Thus, for (24) with $\epsilon = 0$ to be a good approximation to (24) with non-zero ϵ requires that θ be bounded away from $\pm\pi/2$. This is not a completely unreasonable requirement since most V/STOL aircraft do not have a large enough thrust to weight ratio to maintain level flight with a large roll angle. Since the physical limits of the aircraft usually place constraints on the achievable trajectories, a control law analogous to that defined by (25) and (26) can be used for systems with small ϵ on reasonable trajectories.

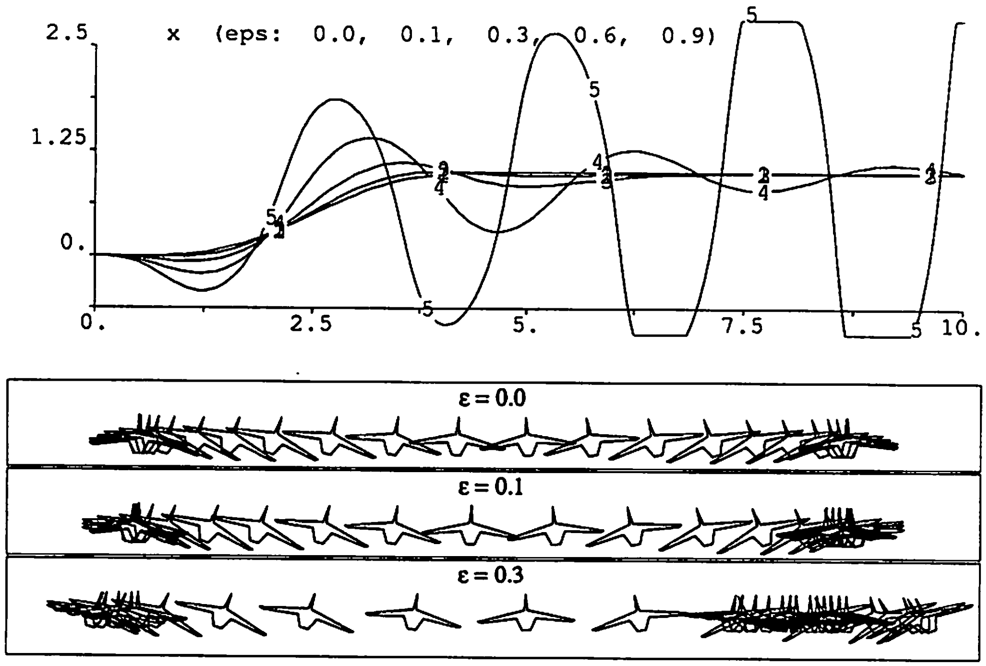

Figure 9: Response of the true PVTOL aircraft system under the approximate control with input transformation

3 A Formal Approach to the Control of Slightly Non-minimum Phase Systems

In this section we will take a more formal approach to the control of systems that are slightly non-minimum phase.

Consider the class of nonlinear systems of the form

$$\left.\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}\right\} \tag{28}$$

where $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}^m$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are smooth vector fields and $h : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth function with $h(0) = 0$.

In the sequel, we will assume that the origin is an equilibrium point of (28), i.e., $f(0) = 0$, and will consider x in an open neighborhood, U, of the origin, i.e., the analysis will be *local*. All statements that we make, such as the existence of certain diffeomorphisms, will be assumed merely to hold in U. Also, when we say that a function is zero, it vanishes on U, and when we say it is non-zero, we mean that it is bounded away from zero on U.

While we will not precisely define *slightly non-minimum phase* systems, the concept is easy enough to explain. The reader may wish to review the definition of the zero dynamics for non-linear systems (and the concept of minimum phase) in Isidori and Moog [IM87].

3.1 Single-Input Single-Output (SISO) Case

Consider first the single-input single-output (SISO) case. Suppose that $L_g h(x) = \epsilon\psi(x)$ for some scalar function $\psi(x)$ with $\epsilon > 0$ small. In other words, the *relative degree* of the system is one, but is very close to being greater than one. Here, $L_g h(x)$ is the *Lie derivative* of $h(\cdot)$ along $g(\cdot)$ and is defined to be

$$L_g h(x) = \frac{\partial h(x)}{\partial x} g(x). \tag{29}$$

Now, define two systems in *normal* form (see Byrnes and Isidori [BI88a]) using the following two sets of local diffeomorphisms of $x \in \mathbb{R}^n$

$$(\xi^T, \eta^T)^T = (\xi_1 := h(x), \eta_1(x), \ldots, \eta_{n-1}(x))^T, \tag{30}$$

and

$$(\tilde{\xi}^T, \tilde{\eta}^T)^T = (\tilde{\xi}_1 := h(x), \tilde{\xi}_2(x) := L_f h(x), \tilde{\eta}_1(x), \ldots, \tilde{\eta}_{n-2}(x))^T, \tag{31}$$

with

$$\frac{\partial \eta_i}{\partial x} g(x) = 0, \quad i = 1, \ldots, n-1 \tag{32}$$

and

$$\frac{\partial \tilde{\eta}_i}{\partial x} g(x) = 0, \quad i = 1, \ldots, n-2. \tag{33}$$

System 1 (true system)

$$\left.\begin{array}{rcl} \dot{\xi}_1 & = & L_f h(x) + L_g h(x) u \\ \dot{\eta} & = & q(\xi, \eta) \end{array}\right\} \tag{34}$$

System 2 (approximate system)

$$\left.\begin{array}{rcl} \dot{\tilde{\xi}}_1 & = & \tilde{\xi}_2 \\ \dot{\tilde{\xi}}_2 & = & L_f^2 h(x) + L_g L_f h(x) u \\ \dot{\tilde{\eta}} & = & \tilde{q}(\tilde{\xi}, \tilde{\eta}) \end{array}\right\} \tag{35}$$

Note that the system (34) represents the system (28) in *normal form* and the dynamics of $q(0, \eta)$ represent the *zero dynamics* of the system (28). System (35) does not represent the system (28), since in the $(\tilde{\xi}, \tilde{\eta})$ coordinates of (31), the dynamics of (28) are given by

$$\left.\begin{array}{rcl} \dot{\tilde{\xi}}_1 & = & \tilde{\xi}_2 + L_g h(x) u \\ \dot{\tilde{\xi}}_2 & = & L_f^2 h(x) + L_g L_f h(x) u \\ \dot{\tilde{\eta}} & = & \tilde{q}(\tilde{\xi}, \tilde{\eta}) \end{array}\right\}. \tag{36}$$

The system (28) is said to be *slightly* non-minimum phase *if the true system (34) is non-minimum phase but the approximate system (35) is minimum phase.* Since $L_g h(x) = \epsilon \psi(x)$, we may think of the system (35) as a perturbation of the system (34) ($\equiv$ (36)).

Of course, there are two difficulties with exact input-output linearization of (34):

- The input-output linearization requires a large control effort since the linearizing control is

$$u^*(x) = \frac{1}{L_g h(x)}(-L_f h(x) + v) = \frac{-L_f h(x) + v}{\epsilon \psi(x)}. \tag{37}$$

  This could present difficulties in the instance that there is saturation at the control inputs.

- If (34) is non-minimum phase, a tracking control law producing a linear input-output response may result in unbounded η states.

Our prescription for the *approximate* input-output linearization of the system (34) is to use the input-output linearizing control law for the approximate system (35); namely

$$u_a^* = \frac{1}{L_g L_f h(x)}(-L_f^2 h(x) + v) \tag{38}$$

where v is chosen depending on the control task. For instance, if y is required to track y_d, we choose v as

$$\begin{aligned} v &= \ddot{y}_d + \alpha_1(\dot{y}_d - \tilde{\xi}_2) + \alpha_2(y_d - \tilde{\xi}_1) &(39)\\ &= \ddot{y}_d + \alpha_1(\dot{y}_d - L_f h(x)) + \alpha_2(y_d - h(x)). &(40)\end{aligned}$$

Using (38) and (39) in (36) along with the definitions

$$\begin{aligned} e_1 &= \tilde{\xi}_1 - y_d \\ e_2 &= \tilde{\xi}_2 - \dot{y}_d \end{aligned} \tag{41}$$

yields

$$\left.\begin{aligned} \dot{e}_1 &= e_2 \qquad\qquad + \epsilon\psi(x)u_a^*(x) \\ \dot{e}_2 &= -\alpha_1 e_2 - \alpha_2 e_1 \\ \dot{\tilde{\eta}} &= \tilde{q}(\tilde{\xi}, \tilde{\eta}). \end{aligned}\right\} \tag{42}$$

The preceding discussion may be generalized to the case when the difference in the relative degrees between the true system and the approximate system is greater than one. For example, if

$$\begin{aligned} L_g h(x) &= \epsilon\psi_1(x) \\ L_g L_f h(x) &= \epsilon\psi_2(x) \\ &\ \vdots \\ L_g L_f^{\gamma-2} h(x) &= \epsilon\psi_{\gamma-1}(x) \end{aligned} \tag{43}$$

but $L_g L_f^{\gamma} h(x)$ is not of order ϵ, we define

$$(\tilde{\xi}^T, \tilde{\eta}^T) = (h(x), L_f h(x), \ldots, L_f^{\gamma-1} h(x), \tilde{\eta}^T)^T \in \mathbb{R}^n \tag{44}$$

and note that the true system is

$$\begin{aligned}
\dot{\tilde{\xi}}_1 &= \tilde{\xi}_2 + \epsilon\psi_1(x)u \\
&\vdots \\
\dot{\tilde{\xi}}_{\gamma-1} &= \tilde{\xi}_\gamma + \epsilon\psi_{\gamma-1}(x)u \\
\dot{\tilde{\xi}}_\gamma &= L_f^\gamma h(x) + L_g L_f^{\gamma-1} h(x)u \\
\dot{\tilde{\eta}} &= \tilde{q}(\tilde{\xi}, \tilde{\eta}).
\end{aligned} \tag{45}$$

The approximate (minimum phase) system (with $\epsilon = 0$) is given by

$$\begin{aligned}
\dot{\tilde{\xi}}_1 &= \tilde{\xi}_2 \\
&\vdots \\
\dot{\tilde{\xi}}_{\gamma-1} &= \tilde{\xi}_\gamma \\
\dot{\tilde{\xi}}_\gamma &= L_f^\gamma h(x) + L_g L_f^{\gamma-1} h(x)u \\
\dot{\tilde{\eta}} &= \tilde{q}(\tilde{\xi}, \tilde{\eta})
\end{aligned} \tag{46}$$

The approximate tracking control law for (46) is

$$u_a = \frac{1}{L_g L_f^{\gamma-1} h(x)}(-L_f^\gamma h(x) + y_d^{(\gamma)} + \alpha_1(y_d^{(\gamma-1)} - L_f^{\gamma-1}h(x)) + \cdots + \alpha_\gamma(y_d - y)). \tag{47}$$

The following theorem provides a bound for the performance of this control when applied to the true system.

Theorem 1 *Assume that*

- *the desired trajectory and its first $\gamma - 1$ derivatives (i.e., y_d, $\dot{y}_d$, $\ldots$, $y^{(\gamma-1)}$) are* bounded,
- *the zero dynamics of the approximate system (46) are locally* exponentially stable *and $\tilde{q}$ is locally Lipschitz in $\tilde{\xi}$ and $\tilde{\eta}$, and*
- *the functions $\psi(x)u_a(x)$ are locally Lipschitz continuous.*

Then, for ϵ sufficiently small, the states of the system (45) are bounded and the tracking error

$$|e_1| := |\tilde{\xi}_1 - y_d| \leq k\epsilon \tag{48}$$

for some $k < \infty$.

Proof See appendix. $\square$

When the control objective is stabilization and the approximate system has no zero dynamics we can do much better. In this case, one can show then that the control law that stabilizes the approximate system also stabilizes the original system.

Suppose that the approximate system has no zero dynamics, i.e.,

$$
\begin{aligned}
L_g h(x) &= \epsilon\psi_1(x) \\
L_g L_f h(x) &= \epsilon\psi_2(x) \\
&\vdots \\
L_g L_f^{n-2} h(x) &= \epsilon\psi_{n-1}(x)
\end{aligned}
\tag{49}
$$

Define

$$\tilde{\xi} = (h(x), L_f h(x), \ldots, L_f^{n-1} h(x))^T \in \mathbb{R}^n \tag{50}$$

and write the approximate system

$$
\begin{aligned}
\dot{\tilde{\xi}}_1 &= \tilde{\xi}_2 \\
&\vdots \\
\dot{\tilde{\xi}}_n &= L_f^n h(x) + L_g L_f^{n-1} h(x) u
\end{aligned}
\tag{51}
$$

and the stabilizing control law

$$u_s(x) = \frac{1}{L_g L_f^{n-1} h(x)}(-L_f^n h(x) - \alpha_1 \tilde{\xi}_{n-1} - \cdots - \alpha_n \tilde{\xi}_1) \tag{52}$$

$$= \frac{1}{L_g L_f^{n-1} h(x)}(-L_f^n h(x) - \alpha_1 L_f^{n-1} h(x) - \cdots - \alpha_n h(x)). \tag{53}$$

The true system in these coordinates is given by

$$
\begin{aligned}
\dot{\tilde{\xi}}_1 &= \tilde{\xi}_2 + \epsilon\psi_1(x)u \\
&\vdots \\
\dot{\tilde{\xi}}_{n-1} &= \tilde{\xi}_n + \epsilon\psi_{n-1}(x)u \\
\dot{\tilde{\xi}}_n &= L_f^n h(x) + L_g L_f^{n-1} h(x) u
\end{aligned}
\tag{54}
$$

Using $u_s(x)$ (from (52)) in (54) yields

$$\begin{bmatrix} \dot{\tilde{\xi}}_1 \\ \vdots \\ \dot{\tilde{\xi}}_{n-1} \\ \dot{\tilde{\xi}}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \\ -\alpha_n & -\alpha_{n-1} & \cdots & -\alpha_1 \end{bmatrix} \begin{bmatrix} \tilde{\xi}_1 \\ \vdots \\ \tilde{\xi}_{n-1} \\ \tilde{\xi}_n \end{bmatrix} + \epsilon \begin{bmatrix} \psi_1(x) \\ \vdots \\ \psi_{n-1}(x) \\ 0 \end{bmatrix} u_s(x). \tag{55}$$

Letting $\psi(x) = (\psi_1(x), \ldots, \psi_{n-1}(x), 0)^T$, we can state the following:

Theorem 2 *Suppose that $\psi(x)u_s(x)$ is Lipschitz in x and that $\psi(0)u_s(0) = 0$. Then, the system (55) is exponentially stable for ϵ sufficiently small.*

Proof See appendix. □

3.2 Generalization to MIMO Systems

We now consider MIMO systems of the form (28) which, for the sake of convenience, we rewrite as

$$\left.\begin{array}{rcl} \dot{x} & = & f(x) + g_1(x)u_1 + \cdots + g_m(x)u_m \\ y_1 & = & h_1(x) \\ & \vdots & \\ y_m & = & h_m(x) \end{array}\right\} \tag{56}$$

Let γ_i be the relative degree of the ith output, i.e., we need to differentiate y_i at least γ_i times before at least one of the inputs appears in the right hand side. Then, we have

$$y_i^{(\gamma_i)} = L_f^{\gamma_i} h_i + L_{g_1} L_f^{\gamma_i - 1} h_i u_1 + \cdots + L_{g_m} L_f^{\gamma_i - 1} h_i u_m \qquad i = 1, \ldots, m. \tag{57}$$

The decoupling matrix is defined to be $A(x) \in \mathbb{R}^{m \times m}$ with

$$A(x) = \begin{bmatrix} L_{g_1} L_f^{\gamma_1 - 1} h_1 & \cdots & L_{g_m} L_f^{\gamma_1 - 1} h_1 \\ \vdots & \ddots & \vdots \\ L_{g_1} L_f^{\gamma_m - 1} h_m & \cdots & L_{g_m} L_f^{\gamma_m - 1} h_m \end{bmatrix} \tag{58}$$

so that

$$\begin{bmatrix} y_1^{(\gamma_1)} \\ \vdots \\ y_m^{(\gamma_m)} \end{bmatrix} = \begin{bmatrix} L_f^{\gamma_1} h_1 \\ \vdots \\ L_f^{\gamma_m} h_m \end{bmatrix} + A(x) \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}. \tag{59}$$

If the decoupling matrix $A(x)$ is non-singular, the control law

$$u(x) = A(x)^{-1}\left(-\begin{bmatrix} L_f^{\gamma_1} h_1 \\ \vdots \\ L_f^{\gamma_m} h_m \end{bmatrix} + v\right) \tag{60}$$

with $v \in \mathbb{R}^m$ linearizes (and decouples) the system (56) resulting in

$$\begin{bmatrix} y_1^{(\gamma_1)} \\ \vdots \\ y_m^{(\gamma_m)} \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}. \tag{61}$$

To take up the ideas of Section 3.1, we will first consider the case when $A(x)$ is non-singular but is *close to being singular*, that is, its smallest singular value is uniformly small for $x \in U$. Definitions of zero dynamics for MIMO systems are considerably more subtle than those for SISO systems and the reader may wish to review them in [BI88a,BI88b] before proceeding further. Since $A(x)$ is close to being singular, i.e., it is close in norm to a matrix of rank $m-1$, we may transform $A(x)$ using elementary column operations to get

$$\bar{A}^0(x) = A(x)V^0(x) = \begin{bmatrix} \bar{a}^0_{\cdot 1}(x) & \cdots & \bar{a}^0_{\cdot, m-1}(x) & \epsilon \bar{a}^0_{\cdot m}(x) \end{bmatrix} \tag{62}$$

where each $\bar{a}^0_{\cdot i}$ is a column of $\bar{A}^0$. This corresponds to redefining the inputs to be

$$\begin{bmatrix} \bar{u}_1^0 \\ \vdots \\ \bar{u}_m^0 \end{bmatrix} = (V^0(x))^{-1} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}. \tag{63}$$

Now, the *normal* form of the system (56) is given by defining the following local diffeomorphism of $x \in \mathbb{R}^n$,

$$(\xi^T, \eta^T) = \begin{pmatrix} \xi_1^1 = h_1(x), \ldots, \xi_{\gamma_1}^1 = L_f^{\gamma_1 - 1} h_1(x), \\ \xi_1^2 = h_2(x), \ldots, \xi_{\gamma_2}^2 = L_f^{\gamma_2 - 1} h_2(x), \\ \vdots \\ \xi_1^m = h_m(x), \ldots, \xi_{\gamma_m}^m = L_f^{\gamma_m - 1} h_m(x), \\ \eta^T \end{pmatrix} \tag{64}$$

and noting that

$$
\left.\begin{array}{rcl}
\dot{\xi}_1^1 & = & \xi_2^1 \\
& \vdots & \\
\dot{\xi}_{\gamma_1}^1 & = & b_1(\xi,\eta) + \sum_{j=1}^{m-1} \bar{a}_{1j}^0 \bar{u}_j^0 + \epsilon \bar{a}_{1m}^0 \bar{u}_m^0 \\
\dot{\xi}_1^2 & = & \xi_2^2 \\
& \vdots & \\
\dot{\xi}_{\gamma_m}^m & = & b_m(\xi,\eta) + \sum_{j=1}^{m-1} \bar{a}_{mj}^0 \bar{u}_j^0 + \epsilon \bar{a}_{mm}^0 \bar{u}_m^0 \\
\dot{\eta} & = & q(\xi,\eta) + P(\xi,\eta)\bar{u}^0
\end{array}\right\} \tag{65}
$$

where $b_i(\xi,\eta)$ is $L_f^{\gamma_i} h_i(x)$ for $i = 1,\ldots,m$ in (ξ,η) coordinates. The zero dynamics of the system are the dynamics of the η coordinates in the subspace $\xi = 0$ with the linearizing control law of (60) (with $v = 0$) substituted, i.e.,

$$
\dot{\eta} = q(0,\eta) - P(0,\eta)(\bar{A}^0(0,\eta))^{-1} b(0,\eta). \tag{66}
$$

We will assume that (56) is non-minimum phase, that is to say that the origin of (66) is not stable.

Now, an approximation to the system is obtained by setting $\epsilon = 0$ in (65). The resultant decoupling matrix is singular and the procedure for linearization by (dynamic) state feedback (the so-called *dynamic extension* process) proceeds by differentiating (65) and noting that

$$
\dot{x} = f(x) + \bar{g}_1^0(x)\bar{u}_1^0 + \cdots + \bar{g}_m^0(x)\bar{u}_m^0 \tag{67}
$$

where

$$
\begin{bmatrix} \bar{g}_1^0(x) & \cdots & \bar{g}_m^0(x) \end{bmatrix} = \begin{bmatrix} g_1(x) & \cdots & g_m(x) \end{bmatrix} V^0(x). \tag{68}
$$

We then get

$$
\begin{aligned}
\begin{bmatrix} y_1^{(\gamma_1+1)} \\ \vdots \\ y_{m-1}^{(\gamma_{m-1}+1)} \\ y_m^{(\gamma_m+1)} \end{bmatrix} &= b^1(x,\bar{u}_1^0,\ldots,\bar{u}_{m-1}^0) + A^1(x,\bar{u}_1^0,\ldots,\bar{u}_{m-1}^0) \begin{bmatrix} \dot{\bar{u}}_1^0 \\ \vdots \\ \dot{\bar{u}}_{m-1}^0 \\ \bar{u}_m^0 \end{bmatrix} && (69) \\
&= b^1(x^1) + A^1(x^1)u^1 && (70)
\end{aligned}
$$

where

$$u^1 = (\dot{\bar{u}}_1^0, \ldots, \dot{\bar{u}}_{m-1}^0, \bar{u}_m^0)^T \tag{71}$$

is the *new* input and

$$x^1 = (x^T, \bar{u}_1^0, \ldots, \bar{u}_{m-1}^0)^T \tag{72}$$

is the *extended* state. Note the appearance of terms of the form $\dot{\bar{u}}_1^0, \ldots, \dot{\bar{u}}_{m-1}^0$ in (69). The system (69) is linearizable (and decouplable) if $A^1(x^1)$ is nonsingular. We will assume that the singular values of A^1 are all of order 1 (i.e., A^1 is uniformly nonsingular) so that (69) is linearizable. The normal form for the approximate system is determined by obtaining a local diffeomorphism of the states $x, \bar{u}_1^0, \ldots, \bar{u}_{m-1}^0$ ($\in \mathbb{R}^{n+m-1}$) given by

$$\begin{aligned}
&(\tilde{\xi}^T, \tilde{\eta}^T) = \\
&\quad (\ \tilde{\xi}_1^1 = h_1(x), \ldots, \tilde{\xi}_{\gamma_1}^1 = L_f^{\gamma_1 - 1} h_1(x), \tilde{\xi}_{\gamma_1+1}^1 = L_f^{\gamma_1} h_1(x) + \sum_{j=1}^{m-1} \bar{a}_{1j}^0 \bar{u}_j^0, \\
&\quad \tilde{\xi}_1^2 = h_2(x), \ldots, \tilde{\xi}_{\gamma_2}^2 = L_f^{\gamma_2 - 1} h_2(x), \tilde{\xi}_{\gamma_2+1}^2 = L_f^{\gamma_2} h_2(x) + \sum_{j=1}^{m-1} \bar{a}_{2j}^0 \bar{u}_j^0, \\
&\qquad \vdots \\
&\quad \tilde{\xi}_1^m = h_m(x), \ldots, \tilde{\xi}_{\gamma_m}^m = L_f^{\gamma_m - 1} h_m(x), \tilde{\xi}_{\gamma_m+1}^m = L_f^{\gamma_m} h_m(x) + \sum_{j=1}^{m-1} \bar{a}_{mj}^0 \bar{u}_j^0, \\
&\quad \eta^T \qquad)
\end{aligned} \tag{73}$$

Note that $\tilde{\xi} \in \mathbb{R}^{\gamma_1 + \cdots + \gamma_m + m}$ and $\tilde{\eta} \in \mathbb{R}^{n - \gamma_1 - \cdots - \gamma_m - 1}$ as compared with $\xi \in \mathbb{R}^{\gamma_1 + \cdots + \gamma_m}$ and $\eta \in \mathbb{R}^{n - \gamma_1 - \cdots - \gamma_m}$. With these coordinates, the *true* system

(56) is given by

$$\left.\begin{array}{lcl}
\dot{\tilde{\xi}}^1_1 & = & \tilde{\xi}^1_2 \\
 & \vdots & \\
\dot{\tilde{\xi}}^1_{\gamma_1} & = & \tilde{\xi}^1_{\gamma_1-1} + \epsilon \bar{a}^0_{1m}\bar{u}^1_m \\
\dot{\tilde{\xi}}^1_{\gamma_1+1} & = & b^1_1(\tilde{\xi},\tilde{\eta}) + a^1_{1\cdot}(\tilde{\xi},\tilde{\eta})u^1 \\
\dot{\tilde{\xi}}^2_1 & = & \tilde{\xi}^2_2 \\
 & \vdots & \\
\dot{\tilde{\xi}}^m_{\gamma_m} & = & \tilde{\xi}^m_{\gamma_m-1} + \epsilon \bar{a}^0_{mm}\bar{u}^1_m \\
\dot{\tilde{\xi}}^m_{\gamma_m+1} & = & b^1_m(\tilde{\xi},\tilde{\eta}) + a^1_{m\cdot}(\tilde{\xi},\tilde{\eta})u^1 \\
\dot{\tilde{\eta}} & = & \tilde{q}(\tilde{\xi},\tilde{\eta}) + \tilde{P}(\tilde{\xi},\tilde{\eta})u^1
\end{array}\right\} \tag{74}$$

In (74) above, $b^1_i(\tilde{\xi},\tilde{\eta})$ and $a^1_{i\cdot}(\tilde{\xi},\tilde{\eta})$ are the ith element and row of b^1 and A^1, respectively, in (69) above (in the $\tilde{\xi}$, $\tilde{\eta}$ coordinates). The approximate system used for the design of the linearizing control is obtained from (74) by setting $\epsilon = 0$. The zero dynamics for the approximate system are obtained in the $\tilde{\xi} = 0$ subspace by linearizing the approximate system using

$$u^1_*(\tilde{\xi},\tilde{\eta}) = -(A^1(\tilde{\xi},\tilde{\eta}))^{-1}\begin{bmatrix} b^1_1(\tilde{\xi},\tilde{\eta}) \\ \vdots \\ b^1_m(\tilde{\xi},\tilde{\eta}) \end{bmatrix}. \tag{75}$$

as

$$\dot{\tilde{\eta}} = \tilde{q}(0,\tilde{\eta}) + \tilde{P}(0,\tilde{\eta})u^1_*(0,\tilde{\eta}). \tag{76}$$

Note that the dimension of $\tilde{\eta}$ is *one less* than the dimension of η in (66). It would appear that we are actually determining the zero dynamics of the approximation to system (56) with dynamic extension—that is to say with integrators appended to the first $m-1$ inputs $\bar{u}^0_1$, $\bar{u}^0_2$, …, $\bar{u}^0_{m-1}$. While this is undoubtedly true, it has been shown in Byrnes and Isidori [BI88a] that the zero dynamics of systems are unchanged by dynamic extension. Thus, the zero dynamics of (76) are those of the approximation to system (56).

The system (56) is said to be *slightly non-minimum phase* if the equilibrium $\eta = 0$ of (66) is not asymptotically stable, but the equilibrium $\tilde{\eta} = 0$ of (76) is.

It is also easy to see that the preceding discussion may be iterated if it turns out that $A^1(\tilde{\xi},\tilde{\eta})$ has some small singular values. At each stage of the *dynamic extension* process $m-1$ *integrators* are added to the dynamics of the system and the act of approximation reduces the dimension of the zero dynamics by *one*. Also, if at any stage of this dynamic axtension process, there are two, three, ...singular values of order ϵ, the dynamic extension involves $m-2$, $m-3$, ... integrators.

If the objective is tracking, the approximate tracking control law is

$$u_a^1(\tilde{\xi},\tilde{\eta}) = (A^1(\tilde{\xi},\tilde{\eta}))^{-1}\left(-\begin{bmatrix} b_1^1(\tilde{\xi},\tilde{\eta}) \\ \vdots \\ b_m^1(\tilde{\xi},\tilde{\eta}) \end{bmatrix} + \begin{bmatrix} y_{d1}^{(\gamma_1+1)} + \alpha_1^1(y_{d1}^{(\gamma_1)} - \tilde{\xi}_{\gamma_1+1}^1) + \cdots + \alpha_{\gamma_1+1}(y_{d1} - \tilde{\xi}_1^1) \\ \vdots \\ y_{dm}^{(\gamma_m+1)} + \alpha_1^m(y_{dm}^{(\gamma_m)} - \tilde{\xi}_{\gamma_m+1}^m) + \cdots + \alpha_{\gamma_m+1}(y_{dm} - \tilde{\xi}_1^m) \end{bmatrix}\right) \tag{77}$$

with the polynomials

$$s^{\gamma_i+1} + \alpha_1^i s^{\gamma_i} + \cdots + \alpha_{\gamma_i+1}^i \qquad i = 1,\ldots,m \tag{78}$$

chosen Hurwitz.

The following theorem is the analog of Theorem 1 in terms of providing a bound for the system performance when the control law (77) is applied to the true system (56).

Theorem 3 *Assume that*

- *the desired trajectory y_d and the first γ_i+1 derivatives of its ith component are* bounded,
- *the zero dynamics (76) of the approximate system are locally* exponentially stable *and $\tilde{q} + \tilde{P}u_*^1$ is locally Lipschitz in $\tilde{\xi}$ and $\tilde{\eta}$, and*
- *the functions $\bar{a}_{im}^0 u_m^1$ are locally Lipschitz continuous for $i = 1,\ldots,m$.*

Then, for ϵ sufficiently small, the states of the system (74) are bounded and the tracking errors satisfy

$$\begin{array}{lcll} |e_1| & = & |\tilde{\xi}_1^1 - y_{d1}| & \leq k\epsilon \\ |e_2| & = & |\tilde{\xi}_1^2 - y_{d2}| & \leq k\epsilon \\ \vdots & & & \\ |e_m| & = & |\tilde{\xi}_1^m - y_{dm}| & \leq k\epsilon \end{array} \tag{79}$$

for some $k < \infty$.

Proof Similar to that of Theorem 1. □

As in the SISO case, the stronger conclusions of Theorem 2 can be stated when the control objective is stabilization and the approximate system has no zero dynamics.

Conclusion

In this paper, we have described the initial results of a research project on the application of techniques of exact linearization of nonlinear control systems to the flight control of vertical take-off and landing aircraft. We saw that the application of the theory to this example is not straightforward. In particular, the direct application of the theory yielded an undesirable controller. We remedied the situation by neglecting the coupling between the rolling moment input to the aircraft dynamics and the dynamics along the y axes.

The example of the vertical takeoff and landing aircraft is an example of a system which is slightly non-minimum phase. Thus,the exact linearization technique resulted in a system which was internally unstable. We generalized the lessons learned from this application to define, informally, *slightly non-minimum phase* systems and gave methods to *linearize* them *approximately.*

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Appendix

Proof of Theorem 1

Define the trajectory error, $e \in \mathbf{R}^{\gamma}$, to be

$$\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_\gamma \end{bmatrix} = \begin{bmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \\ \vdots \\ \tilde{\xi}_\gamma \end{bmatrix} - \begin{bmatrix} y_d \\ \dot{y}_d \\ \vdots \\ y_d^{(\gamma-1)} \end{bmatrix}. \tag{80}$$

Then, the system (45) with the approximate tracking control (47) may be expressed as

$$\begin{aligned} \begin{bmatrix} \dot{e}_1 \\ \vdots \\ \dot{e}_{\gamma-1} \\ \dot{e}_\gamma \end{bmatrix} &= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \\ -\alpha_\gamma & -\alpha_{\gamma-1} & \cdots & -\alpha_1 \end{bmatrix} \begin{bmatrix} e_1 \\ \vdots \\ e_{\gamma-1} \\ e_\gamma \end{bmatrix} + \epsilon \begin{bmatrix} \psi_1(x) \\ \vdots \\ \psi_{\gamma-1}(x) \\ 0 \end{bmatrix} u_a(x) \\ \dot{\tilde{\eta}} &= \tilde{q}(\tilde{\xi}, \tilde{\eta}) \end{aligned} \tag{81}$$

or, compactly,

$$\begin{aligned} \dot{e} &= Ae + \epsilon\psi(x)u_a(x) \\ \dot{\tilde{\eta}} &= \tilde{q}(\tilde{\xi}, \tilde{\eta}). \end{aligned} \tag{82}$$

Since the zero dynamics are assumed to be exponentially stable, a converse Lyapunov theorem implies the existence of a Lyapunov function (see, e.g., [Hah67]) $v_2(\tilde{\eta})$ for the system

$$\dot{\tilde{\eta}} = \tilde{q}(0, \tilde{\eta}) \tag{83}$$

satisfying

$$\begin{gathered} k_1|\tilde{\eta}|^2 \leq v_2(\tilde{\eta}) \leq k_2|\tilde{\eta}|^2 \\ \frac{\partial v_2}{\partial \tilde{\eta}}\tilde{q}(0, \tilde{\eta}) \leq -k_3|\tilde{\eta}|^2 \\ \left|\frac{\partial v_2}{\partial \tilde{\eta}}\right| \leq k_4|\tilde{\eta}| \end{gathered} \tag{84}$$

for some positive constants k_1, k_2, k_3, and k_4.

We first show that e and $\tilde{\eta}$ are bounded. To this end, consider as Lyapunov function for the error system (82)

$$v(e, \tilde{\eta}) = e^T Pe + \mu v_2(\tilde{\eta}) \tag{85}$$

where $P > 0$ is chosen so that

$$A^T P + PA = -I \tag{86}$$

(possible since $\dot{e} = Ae$ is stable) and μ is a positive constant to be determined later.

Note that, by assumption, y_d and its first $(\gamma-1)$ derivatives are bounded,

$$|\tilde{\xi}| \leq |e| + b_d, \tag{87}$$

the functions, $\tilde{q}(\tilde{\xi},\tilde{\eta})$ and $\psi(x)u_a(x)$ are locally Lipschitz with $\psi(0)u_a(0) = 0$,

$$|\tilde{q}(\tilde{\xi}^1,\tilde{\eta}^1) - \tilde{q}(\tilde{\xi}^2,\tilde{\eta}^2)| \leq l_q(|\tilde{\xi}^1 - \tilde{\xi}^2| + |\tilde{\eta}^1 - \tilde{\eta}^2|), \tag{88}$$

$$|2P\psi(x)u_a(x)| \leq l_u|x|, \tag{89}$$

and x is a local diffeomorphism of $(\tilde{\xi},\tilde{\eta})$,

$$|x| \leq l_x(|\tilde{\xi}| + |\tilde{\eta}|). \tag{90}$$

Using these bounds and the properties of $v_2(\cdot)$, we have

$$\begin{array}{rcl} \frac{\partial v_2}{\partial \tilde{\eta}}\tilde{q}(\tilde{\xi},\tilde{\eta}) & = & \frac{\partial v_2}{\partial \tilde{\eta}}\tilde{q}(0,\tilde{\eta}) + \frac{\partial v_2}{\partial \tilde{\eta}}(\tilde{q}(\tilde{\xi},\tilde{\eta}) - \tilde{q}(0,\tilde{\eta})) \\ & \leq & -k_3|\tilde{\eta}|^2 + k_4 l_q|\tilde{\eta}|(|e| + b_d). \end{array} \tag{91}$$

Taking the derivative of $v(\cdot,\cdot)$ along the trajectories of (82), we find

$$\begin{array}{rcl} \dot{v} & = & -|e|^2 + 2\epsilon e^T P\psi(x)u_a(x) + \mu\frac{\partial v_2}{\partial \tilde{\eta}}\tilde{q}(\tilde{\xi},\tilde{\eta}) \\ & \leq & -|e|^2 + \epsilon|e|l_u l_x(|e| + b_d + |\tilde{\eta}|) + \mu(-k_3|\tilde{\eta}|^2 + k_4 l_q|\tilde{\eta}|(|e| + b_d)) \\ & \leq & -(\frac{|e|}{2} - \epsilon l_u l_x b_d)^2 + (\epsilon l_u l_x b_d)^2 \\ & & -(\frac{|e|}{2} - (\epsilon l_u l_x + \mu k_4 l_q)|\tilde{\eta}|)^2 + (\epsilon l_u l_x + \mu k_4 l_q)^2|\tilde{\eta}|^2 \\ & & -\mu k_3(\frac{|\tilde{\eta}|}{2} - \frac{k_4 l_q b_d}{k_3})^2 + \mu\frac{(k_4 l_q b_d)^2}{k_3} \\ & & -(\frac{1}{2} - \epsilon l_u l_x)|e|^2 - \frac{3}{4}\mu k_3|\tilde{\eta}|^2 \\ & \leq & -(\frac{1}{2} - \epsilon l_u l_x)|e|^2 - (\frac{3}{4}\mu k_3 - (\epsilon l_u l_x + \mu k_4 l_q)^2)|\tilde{\eta}|^2 \\ & & +(\epsilon l_u l_x b_d)^2 + \mu\frac{(k_4 l_q b_d)^2}{k_3}. \end{array} \tag{92}$$

Define

$$\mu_0 = \frac{k_3}{4(\epsilon l_u l_x + \mu k_4 l_q)^2}. \tag{93}$$

Then, for all $\mu \leq \mu_0$ and all $\epsilon \leq min(\mu, \frac{1}{4 l_u l_x})$, we have

$$\dot{v} \leq -\frac{|e|^2}{4} - \frac{\mu k_3|\tilde{\eta}|^2}{2} + \frac{\mu(k_4 l_q b_d)^2}{k_3} + (\epsilon l_u l_x + \mu k_4 l_q)^2. \tag{94}$$

Thus, $\dot{v} < 0$ whenever $|\tilde{\eta}|$ or $|e|$ is large which implies that $|\tilde{\eta}|$ and $|e|$ and, hence, $|\tilde{\xi}|$ and $|x|$, are bounded. Using the boundedness of x and the continuity of $\psi(x)u_a(x)$, we see that

$$\dot{e} = Ae + \epsilon\psi(x)u_a(x) \tag{95}$$

is an exponentially stable linear system driven by an order ϵ input. Thus, we conclude that the tracking error, e, converges to a ball of order ϵ. $\square$

Proof of Theorem 2

The stabilized system (55) can be compactly written as

$$\dot{\tilde{\xi}} = A\tilde{\xi} + \epsilon\psi(x)u_s(x). \tag{96}$$

Choose as Lyapunov function $v = \tilde{\xi}^T P\tilde{\xi}$ with $A^T P + PA = -I$. Then, using the bounds analogous to (89) and (90), the derivative of v along trajectories of (96) is given by

$$\begin{array}{rcl} \dot{v} & = & -|\tilde{\xi}|^2 + 2\epsilon P\psi(x)u_s(x) \\ & <= & -(1-\epsilon l_u l_x)|\tilde{\xi}|^2. \end{array} \tag{97}$$

Thus, for all $\epsilon < \epsilon_0 := \frac{1}{l_u l_x}$, the system (96) is exponentially stable. $\square$