RECENT RESULTS ON THE ASYMPTOTIC BEHAVIOR OF CONCURRENT PROCESSING SYSTEMS UNDER STATIONARY AND ERGODIC INPUTS

by

Nicholas Bambos and Jean Walrand

Memorandum No. UCB/ERL M88/78

14 December 1988

RECENT RESULTS ON THE ASYMPTOTIC BEHAVIOR OF CONCURRENT PROCESSING SYSTEMS UNDER STATIONARY AND ERGODIC INPUTS

by

Nicholas Bambos and Jean Walrand

Memorandum No. UCB/ERL M88/78

14 December 1988

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

RECENT RESULTS ON THE ASYMPTOTIC BEHAVIOR OF CONCURRENT PROCESSING SYSTEMS UNDER STATIONARY AND ERGODIC INPUTS

by

Nicholas Bambos and Jean Walrand

Memorandum No. UCB/ERL M88/78

14 December 1988

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

RECENT RESULTS ON THE ASYMPTOTIC BEHAVIOR OF CONCURRENT PROCESSING SYSTEMS UNDER STATIONARY AND ERGODIC INPUTS. †

Nicholas Bambos and Jean Walrand

Department of Electrical Engineering and Computer Sciences
and Electronics Research Laboratory
University of California, Berkeley CA 94720

ABSTRACT

This technical report contains recent results, extensions and improvements of the proofs of past results of the theory of stability, asymptotic behavior and performance of parallel processing systems with stationary and ergodic inputs, presented in the paper "On Stability and Performance of Parallel Processing Systems" by Bambos and Walrand ([1]).

1. Introduction.

For the description of the problem and the notation used in this technical report see [1]. The numbers in parentheses refer to the corresponding numbers in [1]. The discussion is essentially brief, but the details will be presented in the final version of [1] or in a published note.

2. A new result: The detailed structure; merging of sample-paths; strong stability.

a) The Detailed Structure.

At an arbitrary deterministic time, which without any loss of generality can be taken to be zero, we start processing the input N, according to the Processing Scheme described in Section 1 of [1], the system being initially empty.

Define the **Detailed Structure** $u_t, t \in \mathbb{R}_+$ to be the set of jobs, precedence constraints, and remaining processing times ("remaining" because some of them may have already started being serviced) of the jobs in the system at time t^+.

In order to construct a stationary regime for $u_t, t \in \mathbb{R}_+$, we use the following argument. Define $U_t^s, s < t, s, t \in \mathbb{R}$ to be the detailed structure (set of jobs and their remaining processing times) in the system at time t^+, given that we have started processing the input N at time s^+, and the system has been empty for all times before s^+. As $s \to -\infty$, it is easy to see that the set of jobs in the system, as well as their remaining processing times at time t^+ (t fixed) is an increasing sequence, so the limit

$$U_t = \lim_{s \to -\infty} u_t^s,$$

† Research supported in part by Pacific Bell and MICRO.

exists almost surely. Now, since $u_t^s(\theta_z N) = u_{t+z}^{s+z}(N)$, we have $U_t(\theta_z N) = U_{t+z}(N)$, so the process $\{U_t, t \in \mathbb{R}\}$ is stationary. Also, for $\gamma < 1$, since $M_t < \infty$ (see [1]), the set of jobs in U_t has only a finite number of elements. Thus, for $\gamma < 1$, the regime $\{U_t, t \in \mathbb{R}\}$ is well defined, "finite", and stationary.

b) Merging of the sample paths.

Lemma

If $\gamma < 1$, then there exists (pathwise) an almost surely finite random time $\tau^*(N) < \infty$, such that

$$U_t(N) = u_t(N), \quad \text{for every } t \in (\tau^*(N), \infty),$$

almost surely. That is, the two processes U_t and u_t merge pathwise after some finite random time (see [1] for background).

Proof:

Briefly speaking, on any fixed sample path, the process U_t differs from u_t in that in the first one there is an initial structure U_0 in the system at time 0, where in u_t there is none ($u_0 = 0$). We need to prove that the process $U_t(N)$ will eventually (in finite time) forget its initial structure and will not be distinguishable from $u_t(N)$ which had no initial structure.

Referring to the figure below,

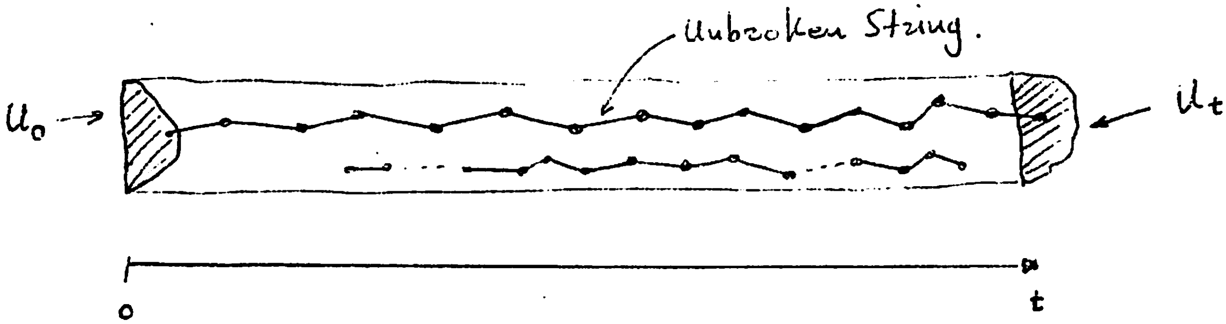

Figure.

we see that in order for the initial structure U_0 to be able to affect the structure U_t (thus not letting it forget the existence of its initial condition and thus differentiating it from u_t), there should be an "unbroken string" (see [1]) of jobs with its first job in U_0 and its last in U_t, such that each next job on this string arrives before the corresponding previous one has left (this is what we mean by "unbroken chain"). In this way, all the jobs on this string "feel" the effect of the initial structure U_0, and so its effect is transmitted by this "unbroken string" to the structure U_t. "Braking" of a "string" in $(0, t]$ occurs when the job on the string arrives at the system after the previous (the one that was supposed to block it) has left the system. Then, that job is not affected at all by the previous one.

But, if there exists an "unbroken string" in $(-d, t]$ ($-d$ is the time of arrival of the earliest arrived job in U_0), then $\xi_{-d,t} > t$ (otherwise the string would "brake"). Thus, if U_t never forgets its initial structure, then

$$\xi_{-d,t} > t, \text{ for every } t > 0,$$

so

$$\lim_{t \to \infty} [\frac{\xi_{-d,t}}{t}] = \gamma \leq 1,$$

which is a contradiction, since we have assumed that $\gamma < 1$.

Thus, every string "brakes" pathwise in finite time, so there exists an almost surely finite time $\tau^*(N) < \infty$, such that after this time $U_t(N)$ forgets its initial structure at time $t = 0$ and coincides with $u_t(N)$. This completes the proof of the the Lemma.

□

Corollary

If $\gamma < 1$, then there exists (pathwise) an almost surely finite random time $\tau^*(N) < \infty$, such that

$$C_t(N) = c_t(N), \text{ for every } t \in (\tau^*(N), \infty),$$

almost surely.

Proof:

Obvious by previous Lemma.

□

Analogous results hold for all the other quantities (m_t, w_t, n_t, see [1]).

c) Strong Stability.

The previous results allow us to strengthen the notion of stability by proving strong convergence of the process c_t to the process C_t uniformly over all Borel sets, where in [1] we had proven only weak convergence.

Theorem: Stability Condition.

1) If $\gamma < 1$, then, for any finite $K \in \{1, 2, \cdots\}$ and any $(x_1, x_2, \cdots x_K) \in \mathbb{R}^K$, the process $\{(c_{x_1+t}, c_{x_2+t}, \cdots c_{x_K+t}), t \in \mathbb{R}_+\}$ converges **strongly** (in **total variation**) to the random variable $(C_{x_1}, C_{x_2}, \cdots C_{x_K})$, as $t \to \infty$. That is, for any Borel subset B of $\mathbb{R}_+^K$, we have

$$\lim_{t \to \infty} P[(c_{x_1+t}, c_{x_2+t}, \cdots c_{x_K+t}) \in B] = P[(C_{x_1}, C_{x_2}, \cdots C_{x_K}) \in B],$$

Thus the system can be characterized as **Stable**.

2) If $\gamma > 1$, then $c_t \to \infty$ as $t \to \infty$, almost surely. Thus, the system can be characterized as Unstable.

Proof:

1) Since $\tau^*(N)$ is almost surely finite, $\lim_{t \to \infty} P[\tau^* > t] = 0$. Let $d_t(N) = (c_{x_1+t}(N), c_{x_2+t}(N), \cdots c_{x_K+t}(N))$ and $D_t(N) = (C_{x_1+t}(N), C_{x_2+t}(N), \cdots C_{x_K+t}(N))$. But,

we have $d_t(\mathrm{N}) = D_t(\mathrm{N})$ for every $t > \tau^*(\mathrm{N})$. Thus, $0 \leq |P[d_t \in B] - P[D_t \in B]| = |P[d_t \in B, t > \tau^*] + P[d_t \in B, t \leq \tau^*] - - P[D_t \in B, t > \tau^*] - P[D_t \in B, t \leq \tau^*]| \leq 2P[\tau^* > t]$, since the first and third term in the second part of the previous expression cancel each other out, and the two remaining are less than $P[\tau^* > t]$ each. Taking the limits in the above expression, and using the stationarity of the process C_t the result follows immediately.

2) This part of the proof is the same as in [1].

□

3. Extensions.

Briefly speaking, the results are extenbable to the case where the jobs have some random internal structure. That is, each job consists of many tasks, and there are precedence constraints between the tasks in each job (internal precedence constraints), forming an acyclic graph. There is also a static allocation scheme for each job that assigns the tasks to specific processors. There may also be precedence constraints between tasks of different jobs.

Given that the arrival times of the jobs, the jobs structure (internal precedence constraints and processing times of the tasks), and the blocking structure between tasks of different jobs, form a stationary and ergodic sequence, the results of [1] can be extended directly.

A simple but interesting example is the case of tandem queueing networks.

4. Simplification of the proof of Lemma 4 in [1].

The part of the proof after (13) in [1] can be substituted by the following relatively simpler argument (basically defining A_n in a different, simpler way):

Having fixed T as described in (13), define, for any $n \in \mathbb{Z}_+$, the quantity

$$A_n = \sup\{m \in \mathbb{Z}_+ : T < C_{t-kT}^{t-nT}, \text{ for every } k \in \{0, 1, 2, \cdots m\}\}, \tag{14'}$$

and $A_n = 0$, if $C_t^{t-nT} \leq T$. Because C_t^s is decreasing in s, A_n is a decreasing function of n, thus the limit $\lim_{n \to \infty} A_n$ exists.

We will now prove by contradiction that $\lim_{n \to \infty} A_n < \infty$. Indeed, suppose that $\lim_{n \to \infty} A_n = \infty$. Observe that, for any $k \in \{0, 1, \cdots A_n\}$ and any finite $n \in \mathbb{Z}_+$,

$$T < C_{t-kT}^{t-nT}, \tag{15'}$$

and taking into account the inequality (12), we have

$$C_{t-kT}^{t-nT} \leq [C_{t-(k+1)T}^{t-nT} - T]^+ + \xi_{t-(k+1)T,\, t-kT}. \tag{16'}$$

From (15') and (16'), we see that the operator $[\ ]^+$ must be positive for any $k \in \{0, 1, 2, \cdots A_n\}$, so

$$C_{t-kT}^{t-nT} \leq C_{t-(k+1)T}^{t-nT} - T + \xi_{t-(k+1)T,\, t-kT}, \quad k \in \{0, 1, 2, \cdots A_n\}. \tag{17'}$$

Also, by the definition of A_n, we have

$$C_{t-(A_n+1)T}^{t-nT} \leq T. \tag{18'}$$

Recursive application of (17') and use of (18') leads to

$$0 \le C_t^{t-nT} \le -A_n T + \sum_{k=0}^{A_n} \xi_{t-(k+1)T,\, t-kT}, \tag{19'}$$

so

$$T \le \frac{1}{A_n} \sum_{k=0}^{A_n} \xi_{t-(k+1)T,\, t-kT}. \tag{20'}$$

Letting $n \to \infty$, we have $A_n \to \infty$, and by Lemma 2 (in [1]) the right side of the above expression converges to $E[\xi_{t,t+T}]$. Thus, $T \le E[\xi_{t,t+T}]$, which contradicts (13).

In view of the above, we conclude that $\lim_{n \to \infty} A_n = A_* < \infty$. Therefore, for the limit $C_t = \lim_{n \to \infty} C_t^{t-nT}$, we have, according to the definition of A_n,

$$C_{t-(A_*+1)T} \le T \tag{21'}$$

so, as easily seen,

$$C_t \le T + \sum_{j \in \mathbb{Z}} \sigma_j \mathbf{1}\{t_j \in (t-(A_*+1)T, t]\} < \infty. \tag{22'}$$

The rest of the proof remains unchanged.

□

An important fact emerging in the proof of Lemma 4, which will be used later, is the following. For any $t \in \mathbb{R}$ and any $T \in \mathbb{R}_+$, such that $E[\xi_{t,t+T}] < T$, by (21'), there is a finite $k_0 \in \mathbb{Z}$, such that $C_{t-k_0T} \le T$. Applying repeatedly the same argument, we can construct an absolutely increasing sequence $\{k_n, n \in \mathbb{Z}_+\}$, such that

$$C_{t-k_nT} \le T, \quad \text{for every } n \in \mathbb{Z}_+. \tag{23'}$$

1. Simplification of the proof of Lemma 5 in [1].

The part of the proof after (29) in [1] can be substituted by the following simpler argument, in view of the simplification of the proof of Lemma 4.

Choose a finite $T \in \mathbb{R}_+$, such that $E[\xi_{t,t+T}] < T$ $(\gamma < 1)$. Because of (23'), there exists an absolutely increasing integer sequence $\{k_n, n \in \mathbb{Z}_+\}$, such that $C_{t-k_nT} \le T$, for every $n \in \mathbb{Z}_+$. Thus, by (29), we have

$$T \ge C_{t-k_nT} \ge t-(t-k_nT) = k_nT, \tag{30'}$$

for every $n \in \mathbb{Z}_+$, and $k_n \to +\infty$, which is a contradiction (impossible). Therefore, $M_t < \infty$, for any $t \in \mathbb{R}_+$.

The rest of the proof remains unchanged.

□

2. References.

[1] Bambos, N., Walrand, J. (1988). On Stability and Performance of Parallel Processing Systems. Journal of the Association of Computing Machinary. Submitted for publication.