Detecting Cusps and Inflection Points in Curves

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Abstract: In many applications it is desirable to analyze parametric curves for undesirable features like cusps and inflection points. Previously known algorithms to analyze such features are limited to cubics and in many cases are for planar curves only. We present a general purpose method to detect cusps in polynomial or rational space curves of arbitrary degree. If a curve has no cusp in its defining interval, it has a regular parametrization and our algorithm computes that.

In particular, we show that if a curve has a proper parametrization then the necessary and sufficient condition for the existence of cusps is given by the vanishing of the first derivative vector. We present a simple algorithm to compute the proper parametrization of a polynomial curve and reduce the problem of detecting cusps in a rational curve to that of a polynomial curve. Finally, we use the regular parametrizations to analyze for inflection points.

Keywords: Regular Curves, Proper Parametrization, Improper Parametrization, Geometric Continuity, Rational Curves, Cusps, Inflection Points

[1]Supported in part by David and Lucile Packard Fellowship and in part by National Science Foundation Presidential Young Investigator Award (# IRI-8958577).

1 Introduction

It is often desirable to analyze parametric curves, used in computer aided geometric design, for undesirable features like cusps and inflection points. The curves used are parametric curves, which may have a polynomial or rational parametrization. The coordinates for each point on the curve can be expressed as:

$$\mathbf{Q}(t) = (x, y, z) = \left(\frac{x(t)}{w(t)}, \frac{y(t)}{w(t)}, \frac{z(t)}{w(t)}\right), \quad w(t) \neq 0, \quad t \in (a, b)$$

where $x(t), y(t), z(t)$ and $w(t)$ are polynomials in $\mathbf{R}[t]$, the ring of all polynomials in t, whose coefficients are real numbers. If $w(t) = 1$, the curve has a polynomial parametrization, else a rational parametrization. The former will be referred as a *polynomial curve* and the latter as a *rational curve*.

Geometrically, the most common cusp is a discontinuity in the unit tangent vector (Fig. I). However this classification of cusps is not complete. There are higher order cusps which arise because some higher order derivatives of the curve are not defined at such points. Algebraically, cusps correspond to those points on the curve, in whose neighborhood the curve cannot be represented as a one-to-one and C^∞ bijective map with an open interval on the real line. In our case the curve, $\mathbf{Q}(t)$, is everywhere differentiable (as it has a polynomial or rational parametrization). Given such a curve and a point $\mathbf{p} = \mathbf{Q}(t_0)$, where $\mathbf{Q}'(t_0) \neq (0, 0, 0)$, it is always possible to define a one-to-one and C^∞ bijective map between the points in the neighborhood of $\mathbf{p}$ and an open real interval. Thus, the necessary condition for the existence of cusps is obtained by the vanishing of the first derivative vector. However, the vanishing of the first derivative vector is only a necessary and not a sufficient condition for the existence of cusps.

An inflection point occurs when the curvature vanishes. This formulation is based on the assumption that the curve is *regular*, i.e. the unit tangent vector is continuous, and the curve does not correspond to a straight line. Hence, it is necessary for us to make sure that a curve satisfies these assumptions, before we determine the inflection points. By finding cusps we can break the curve's defining interval into subintervals such that each of the resulting curve (defined with respect to the subinterval) is regular in its domain of definition.

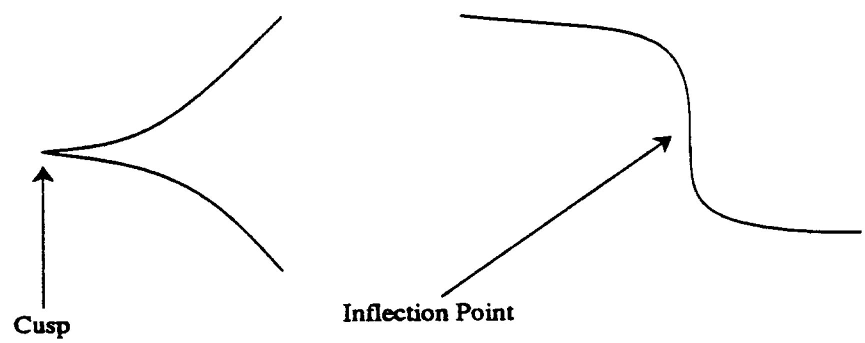

Fig. I

A single polynomial or rational function usually does not have enough freedom to represent a given curve; several such segments are used instead. Each segment has a polynomial or rational parametrization and associated with it is a finite interval, defining the domain of the parameter. To analyze for cusps and inflection points each segment is considered separately. We are only interested in determining these undesirable features for curves corresponding to the parameter values in the defining interval.

Previous work in this area has been mainly restricted to cubic curves. For cubic polynomial Bézier curves, geometric manipulations on the control polygon have been used in [Forrest '70; DeRose and Stone '89]. If the parametrization does not represent a straight line, then the necessary and sufficient condition for existence of cusps is the vanishing of the derivative vector. However, this analysis does not hold for rational cubic curves or higher degree polynomial or rational curves. The algebraic properties of the coefficients of the parametric polynomial curves and some geometric tests using B-spline control polygons have been used in [Wang '81] to detect these features. That analysis is limited to cubic polynomial planar B-splines. In [Liu '89], a method is presented to detect these features. However, no prior methods are knowns for detecting cusps or to check whether a given parametrization represents a regular curve, for arbitrary degree parametrizations.

A lot of results in differential geometry are based on the assumption that the curve is regular [Pogorelov '57; Stoker '69]. This assumption has also been used in defining the notions of geometric continuity of curves. There are two notions of geometric continuity and both make use of this assumption. The first is based on standard parametric continuity after a suitable reparametrization [DeRose '85]; the second on continuity of the Frenet Frame or higher order curvatures [Boehm '85; Boehm '87; Dyn and Micchelli '85]. The former notion has been expressed in terms of *beta-constraints* [DeRose '85]. These constraints have been used in designing *visually smooth* free form curves and surfaces [Barsky and Beatty '83]. However, their generalization in terms of continuously shaped beta-splines results in cusps for certain choice of shape parameters. The resulting parametrization of continuously shaped beta-splines consists of polynomials of degree eighteen. A similar occurrence of cusps was found for planar rational curves in [Manocha and Barsky '90], whose homogeneous representation consists of polynomials of degree eighteen as well. The offsets of many smooth rational curves may have cusps, too [Farouki and Neff '90; Hoffmann '89]. The only known way of detecting cusps in such high degree curves was by visual inspection, which is not accurate.

In this paper we present an algorithm to detect cusps in a curve. If the curve has no cusps, we obtain a regular parametrization of the curve in the given interval. The latter can be used to determine the inflection points. We initially present an algorithm for polynomial curves and later on reduce the problem of detecting cusps in a rational curve to that of a polynomial curve. The rest of the paper is organized in the following manner. In section 2, we specify the notation and present a mathematical formulation of the problem. In section 3, we make use of Sturm's sequences to check for the necessary condition for the existence of cusps. Section 4 specifies the class of functions which can be used for reparametrization, such that the resulting curve is regular. Section 5 shows that if a curve has a proper parametrization, then the necessary and sufficient condition for the existence of cusps is given by the vanishing of the derivative vector. In section 6, we present algorithms to compute proper parametrizations of polynomial curves and analyze them for cusps. Section 7 presents a complete algorithm to detect cusps and obtain regular parametrizations (if they exist) for polynomial curves and in section 8, we extend the algorithm for polynomial curves to rational curves. Moreover, we reduce the problem of detecting cusps in a rational curve to that of a polynomial curve. Finally, in section 9 we present an algorithm to determine inflection points on a curve.

All our algorithms are for space curves. Their reduction to plane curves is obvious.

2 Parametric Curves

In our applications a space curve is a vector valued function of the type

$$\mathbf{Q}(t) = (X(t), Y(t), Z(t)), \quad t \in [a, b]$$

where X(t), Y(t) and Z(t) are polynomial or rational functions. The set of all rational functions includes the polynomial functions. We restrict the use of the word *rational* in the following manner: a given function of the form $u(t)/w(t)$, where $u(t)$ and $w(t)$ are polynomial functions is not a rational function if $w(t)$ divides $u(t)$. Such functions are also referred to as *integral* functions to distinguish them from rational functions. We will use lower case letters to denote polynomial functions like $x(t), y(t), z(t)$ and upper case letters to denote rational functions like $X(t), Y(t), Z(t)$. The boldface letters are used to represent vector valued functions like q(t) or Q(t), where $\mathbf{q}(t)$ is a polynomial curve of the form $(x(t), y(t), z(t))$.

A rational curve also has a homogeneous representation of the type

$$\mathbf{Q}(t) = (x(t), y(t), z(t), w(t))$$

associated with it, where $X(t) = \frac{x(t)}{w(t)}, Y(t) = \frac{y(t)}{w(t)}, Z(t) = \frac{z(t)}{w(t)}$. Each polynomial $x(t), y(t)$ or $z(t)$ is assumed to have *power basis* representation. All Bézier, B-spline or Beta-spline curves can be converted into power basis representation. The degree of $\mathbf{q}(t)$ is the maximum of degrees of $x(t), y(t)$ and $z(t)$ and the degree of $\mathbf{Q}(t)$ is the maximum of the degrees of $x(t), y(t), z(t)$ and $w(t)$. We use the symbols **R** and **C** to denote the set of all real and complex numbers, respectively.

The class of curves that we are interested in is a subset of the family of curves defined by *analytic* parametrizations. In particular, each scalar component of the vector valued function $\alpha(t)$ has an analytic parametrization, i.e. it can be expressed as a *formal power series* [Walker '50]. An analytic parametrization $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ is said to be *reducible*, if each $\alpha_i(t) \in \mathbf{R}(t^r)'$ for some $r > 1$, where $\mathbf{R}(t^r)'$ represents the field of all formal power series with a finite number of terms of negative exponents and each term of the formal power series is of type $a(t^{kr})$, $a \in \mathbf{R}$ and k is an integer [Walker '50]. Else the parametrization is *irreducible*. An irreducible parametrization can be obtained by substituting s for t^r. In [Semple and Kneebone '59] the irreducible and reducible parametrizations are addressed as *minimal* and *redundant* parametrizations, respectively.

Lets consider analytic space curves. The following lemma is useful for constructing an equivalent parametrization of the space curves [Theorem 2.2, Chapter IV, Walker '50]

Lemma I: Any analytic space curve $\mathbf{Q}(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$, where

$$\alpha_1(t) = t^m(a_0 + a_1t + a_2t^2 + \ldots),$$
$$\alpha_2(t) = t^k(b_0 + b_1t + b_2t^2 + \ldots),$$
$$\alpha_3(t) = t^l(c_0 + c_1t + c_2t^2 + \ldots),$$
$$m > 0,\ k > 0,\ l > 0$$

has an equivalent parametrization of the form

$$\overline{\alpha}_1(s) = s^m,$$
$$\overline{\alpha}_2(s) = s^k(\overline{b}_0 + \overline{b}_1s + \overline{b}_2s^2 + \ldots),$$
$$\overline{\alpha}_3(s) = s^l(\overline{c}_0 + \overline{c}_1s + \overline{c}_2s^2 + \ldots).$$

Multiplicity: An analytic function $f(t)$ is said to have *multiplicity* m at $t = c$, if

$$f(c) = 0,$$
$$f^{(1)}(c) = 0,$$
$$\vdots$$
$$f^{(m)}(c) = 0,$$

where $f^{(i)}(c)$ denotes the i^{th} derivative of $f(t)$ at $t = c$.

Proper Parametrizations: In many cases a polynomial or rational curve can be identically described by a polynomial or rational parametrization, respectively, of lower degree. Such a curve is *improperly parametrized*, which means to every point on the curve there corresponds more than one parameter value. This hold almost everywhere if we extend the domain and range of the function to the complex numbers

$$\mathbf{q}: \ \mathbf{C} \rightarrow \mathbf{C}^3.$$

Curves which have a one-to-one relationship between parameter values and points on the curve (except for a finite number of points) are called *properly parametrized* curves. It is quite possible that there is a one-to-one relationship between the parameter values in the finite real range and the points on the curve, but the curve is still improperly parametrized. For rational parametric curves we show that the irreducible and reducible parametrizations correspond to proper and improper parametrizations, respectively. Another popular terminology for proper and improper parametrizations are *faithful* and *unfaithful* parametrizations, respectively. More on proper and improper parametrizations and algorithms to obtain a proper parametrization corresponding to improperly parametrized curves are given in [Semple and Kneebone '59; Sederberg '84; Sederberg '86].

2.1 Regular Curves

A regular curve is defined as the locus of points defined by a vector

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t)), \quad t \in [a, b],$$

where $\alpha_i(t)$ are analytic functions such that

1. $\alpha(t)$ has continuous second or third derivatives in the given interval.
2. $\alpha'(t)$, the derivative of $\alpha(t)$, is nowhere zero in the given interval.

A good introduction to regular curves and their properties is given in [Pogorelov '57; Stoker '69]. Given a regular curve, the following theorem tells us about an important characterization about the locus of points lying on such a curve:

Implicit Function Theorem: If $\mathbf{Q}(s)$ is a regular curve and

$$x = \beta_1(s) \quad y = \beta_2(s) \quad z = \beta_3(s), \quad s \in (c, d)$$

is its regular parametrization in the neighborhood of the point $\mathbf{Q}(s_0)$ corresponding to $s = s_0$. Suppose $\beta_1'(s_0) \neq 0$. Then in a sufficiently small neighborhood of the point $\mathbf{Q}(s_0)$, the curve $\mathbf{Q}(s)$ can be defined by means of the equations

$$y = \phi(x) \quad z = \psi(x),$$

where $\phi(x)$ and $\psi(x)$ are regular functions of x [Stoker '69].

In our applications the first condition of regular curves always hold, since $\alpha_i(t)$ are polynomial or rational functions. If the first derivative vector vanishes at some value of the parameter in the given interval, we show that there are two possibilities:

1. The curve has a *cusp*. One such case has been shown for a planar curve in Fig. II(a). Here
$$\mathbf{q}(t) = (t^2, t^3), \quad t \in (-1, 1)$$
$$\mathbf{q}'(0) = (0, 0)$$
2. Else the parametrization is *reducible*. In this case, the curve can be reparametrized such that the resulting parametrization represents a regular curve in the associated interval. For example, consider the planar curve (Fig. II(b))
$$\mathbf{q}(t) = (t^2 + t^3, t^6 + 2t^5 + t^4), \quad t \in (-1, 1)$$
$$\mathbf{q}'(0) = (0, 0)$$
Let
$$s = t^2 + t^3$$
then
$$\mathbf{q}(s) = (s, s^2), \quad s \in (0, 2)$$
which is a parabola and hence a regular curve in the associated interval.

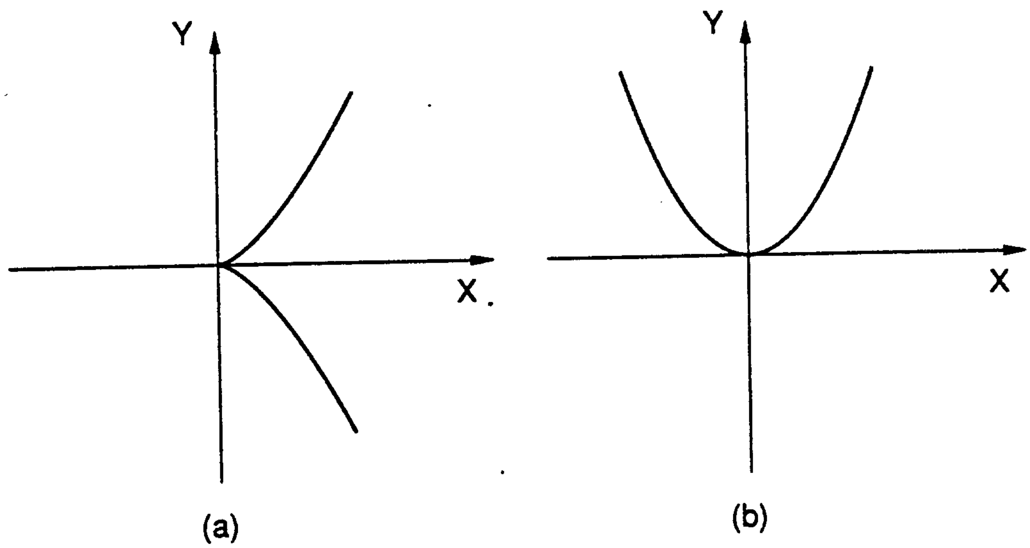

Fig. II

Definition: A point $\mathbf{Q}(t_0)$ on the curve $\mathbf{Q}(t)$ is called a *regular point* if the curve permits a regular parametrization $x = \beta_1(t), y = \beta_2(t), z = \beta_3(t)$, where $\beta_i(t)$ are analytic functions, in a neighborhood of t_0 satisfying the condition $x'^2 + y'^2 + z'^2 \neq 0$ at the point $\mathbf{Q}(t_0)$. But if such a parametrization does not exist, then $\mathbf{Q}(t_0)$ corresponds to a *cusp* on the curve [Pogorelov '57].

The fact that a regular parametrization does not exist implies that the vector valued function $\mathbf{Q}$ is not one-to-one and C^∞ with an open real interval in the neighborhood of that point. All points on analytic curves corresponding to a discontinuity in the unit tangent vector are cusps (such points also known an the *turning point*). Other examples of cusps include points like the origin on the curve $\mathbf{q}(t) = (t^3, t^5), \quad t \in (-1, 1)$ (shown in Fig. III). In this case the origin is not a turning point. Let us consider the implicit representation of this curve, given by $y = x^{5/3}$. The second derivative of the curve, $\frac{d^2y}{dx^2} = \frac{10}{9}x^{-1/3}$ is not defined at the origin and therefore, the curve has a cusp at the origin.

The implicit representation of a curve is an intrinsic representation of the curve. A point $\mathbf{p}$ corresponds to a cusp if some higher order derivative of the curve, defined with respect to the implicit representation, is not defined at $\mathbf{p}$.

It is possible that a curve has a cusp and is defined by a reducible parametrization. For example, the curve shown in Fig. II(a) can also be defined as

$$\mathbf{q}(t) = (t^6, t^9), \quad t \in (-1, 1).$$

This is a reducible parametrization and the corresponding irreducible parametrization is obtained by substituting s for t^3.

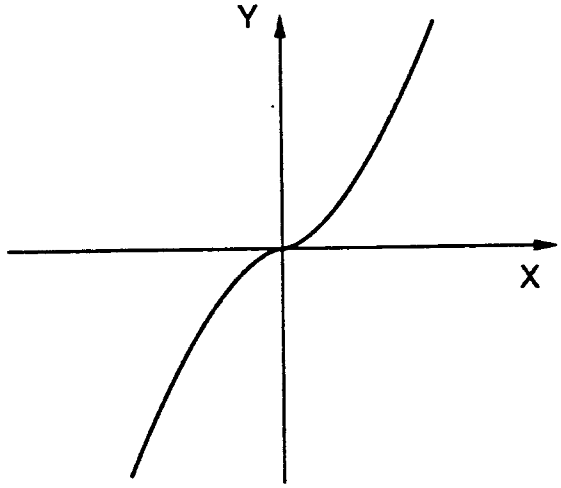

Fig. III

2.2 Rational Algebraic Curves

Every plane rational parametric curve can be represented as the zero set of a polynomial, $f(x, y)$. The latter representation corresponds to the implicit representation of the given curve. Given an algebraic plane curve of the form $f(x, y) = 0$, methods to detect cusps and inflection points are in [Semple and Kneebone '59; Walker '50]. In our case, the domain of the curve is restricted to a finite real interval. The procedure for detecting cusps and inflection points in plane curves consists of the following steps:

- Implicitize the given parametric curve. Algorithms to implicitize are given in [Hoffmann '89].
- Determine whether the implicit representation of the form, $f(x, y) = 0$, has any cusps or inflection points [Semple and Kneebone '59].
- Compute the parameter values corresponding to cusps and inflection points and check whether they lie in the domain of the parametrization. This corresponds to *inversion* [Hoffmann '89].

Our algorithm is simpler and easier to implement than to the above procedure.

An *algebraic* space curve is defined as the common intersection of two or more algebraic surfaces. Every rational parametric space curve can be represented as an algebraic space curve. Many algorithms in geometric modeling (including the ones to detect singular points) are restricted to algebraic curves which can be represented as the intersection of two algebraic surfaces. Furthermore the algebraic set defined by the intersection of such surfaces should be an irreducible set. It is not known at the moment whether all space curves can be represented as the intersection of two surfaces. Consider a non-planar, properly parametrized *cubic* rational parametric curve of the form, $\mathbf{Q}(t) = (x(t), y(t), z(t), w(t))$.

Using *Bezout's* theorem [Walker '50] one can show that the algebraic degree of $\mathbf{Q}(t)$ is three. The implicit representation of $\mathbf{Q}(t)$ is obtained by representing it as the intersection of three algebraic surfaces. It is possible that there are two algebraic surfaces, say $g_1(x, y, z, w)$ and $g_2(x, y, z, w)$ such that their intersection contains $\mathbf{Q}(t)$ as one of the many irreducible components or the intersection multiplicity of the surfaces along $\mathbf{Q}(t)$ is greater than one. Hence, it is difficult to use an implicitization based approach for detecting cusps in rational space curves.

3 Necessary Condition for Cusps

A necessary condition for the existence of cusps is given by the vanishing of the first derivative vector in the given interval. Given

$$\mathbf{Q}(t) = (x(t), y(t), z(t), w(t)), \quad w(t) \neq 0, \quad t \in [a, b], \tag{1}$$

a rational curve of degree n. The first derivative vector is represented as

$$\mathbf{Q}'(t) = (x'(t)w(t) - x(t)w'(t), y'(t)w(t) - y(t)w'(t), z'(t)w(t) - z(t)w'(t), (w(t))^2) \tag{2}$$

Let

$$g(t) = GCD(x'(t)w(t) - x(t)w'(t), y'(t)w(t) - y(t)w'(t), z'(t)w(t) - z(t)w'(t)) \tag{3}$$

denote the GCD (greatest common divisor) of the numerators of scalar components of the first derivative vector. The GCD of two or more polynomials is the polynomial of largest degree which exactly divides them. It is unique to within a scale factor. If the GCD of two or more polynomials is a constant, then the polynomials are relatively prime. The GCD of two or more polynomials can be found by using Euclid's algorithm [Brown '71]. Euclid's algorithm is a standard tool in computer algebra systems which uses exact arithmetic (or rational arithmetic) for polynomial division. It is impossible using finite precision arithmetic to test whether one polynomial with floating point coefficients divides the other [Brown '71]. Some heuristic approaches to implementing the algorithm in floating point are given in [Sederberg '84; Sederberg '86].

The first derivative vector vanishes at the roots of $g(t)$. If $g(t)$ is a constant, the given parametrization, $\mathbf{q}(t)$, represents a regular curve. Otherwise, $g(t)$ is a polynomial of degree $k \leq n - 1$ and it has up to k distinct roots over the field of complex numbers. However we are interested in knowing whether $g(t)$ has a real root in the interval $[a, b]$. For this we recommend Sturm sequences. More details on Strum sequences and their implementation are given in the appendix.

4 Reparametrization

In this section we consider a rational curve about a parameter value where its first derivative vector vanishes. We use a canonical representation of the curve about that parameter value. If the curve does not have a cusp corresponding to that parameter value, we develop constraints on the class of functions which can be used for reparametrizing, such that the resulting parametrization is a regular curve.

Given $\mathbf{Q}(t)$, a rational curve, and a parameter value $t_0 \in [a,b]$, where $\mathbf{Q}'(t_0) = (0,0,0)$. We obtain a parametrization $\mathbf{P}(s), s \in [a-t_0, b-t_0]$, such that $\mathbf{P}(0) = \mathbf{P}'(0) = (0,0,0)$. The new curve is obtained in the following manner:

- Reparametrize $\mathbf{Q}(t)$, by substituting $t = \alpha(s) = s + t_0$. Let

  $$\overline{\mathbf{P}}(s) = \mathbf{Q}(s+t_0) = (x(s+t_0), y(s+t_0), z(s+t_0), w(s+t_0)), \;\; s \in [a-t_0, b-t_0].$$

  The reparametrizing function is a bijective mapping

  $$\alpha : \; [a-t_0, b-t_0] \rightarrow [a,b].$$

- Let $\mathbf{P}(s) = \overline{\mathbf{P}}(s) - \overline{\mathbf{P}}(0)$. This is an affine transformation and it corresponds to a translation in the space. By applying chain rule we get

  $$\mathbf{P}'(s) = \frac{\overline{\mathbf{P}}(s)}{ds} = \mathbf{q}'(t)\frac{dt}{ds} = \mathbf{q}'(s+t_0).$$

Cusps in a curve are invariant under affine transformations. Thus, $\mathbf{P}(s)$ has a cusp at $s = 0$ if and only if $\mathbf{Q}(t)$ has a cusp at $t = t_0$. From now on we assume that we are analyzing a curve $\mathbf{Q}(t) = (X(t), Y(t), Z(t))$, $t \in [a,b]$, such that $\mathbf{Q}(0) = \mathbf{Q}'(0) = (0,0,0)$ and $0 \in [a,b]$. Our problem is that of determining whether $t = 0$ corresponds to a cusp or the curve has a regular parametrization in the neighborhood of $\mathbf{Q}(0)$. The fact that $\mathbf{Q}(0) = (0,0,0)$, $w(0) \neq 0$ and $\mathbf{Q}'(0) = (0,0,0)$ imply that $(x'(0), y'(0), z'(0)) = (0,0,0)$ and therefore $(X'(0), Y'(0), Z'(0)) = (0,0,0)$.

Without loss of generality, we assume that $X(t)$ has the minimum multiplicity at $t = 0$, among the three functions $X(t), Y(t)$ and $Z(t)$. Let its multiplicity be m. We know that $m \geq 2$. $X'(t)$'s multiplicity is $m-1$ at $t = 0$. Let $s = \alpha(t)$ be the reparametrizing function, as shown in Fig. IV.

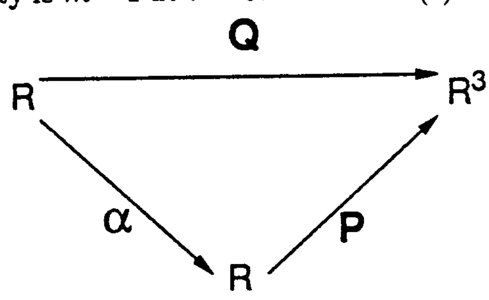

Fig. IV

Theorem I: $\mathbf{Q}(t)$ *does not have a cusp at* $t = 0$, *if and only if there exists an analytic function,* $s = \alpha(t)$, *whose multiplicity is exactly equal to* m *at* $t = 0$, *and there exists a regular parametrization*

$$\mathbf{P}(s) = (\beta_1(s), \beta_2(s), \beta_3(s))$$

where $\beta_i(s)$ *are analytic functions and*

$$\mathbf{P}(\alpha(t)) = \mathbf{Q}(t).$$

Proof: If $\mathbf{q}(t)$ does not have a cusp at $t = 0$ then there exists a reparametrizing function $s = \alpha(t)$[1], such that $\mathbf{P}(s)$ represents a regular parametrization of the curve in the neighborhood of the origin, where $\mathbf{P}(s) = \mathbf{P}(\alpha(t)) = \mathbf{Q}(t))$. $\alpha(t)$ is one-to-one in a small neighborhood around $t = 0$. Without loss of generality we assume that $\alpha(0) = 0$. By chain rule we obtain

$$\mathbf{Q}'(t) = \mathbf{P}'(s)\frac{ds}{dt} = \mathbf{P}'(s)\alpha'(t)$$

$$\Rightarrow \mathbf{P}'(s) = \frac{\mathbf{Q}'(t)}{\alpha'(t)},$$

since $\alpha'(t)$ is a scalar function.

$$\Rightarrow \mathbf{P}'(0) = \frac{\mathbf{Q}'(0)}{\alpha'(0)} = (\frac{X'(0)}{\alpha'(0)}, \frac{Y'(0)}{\alpha'(0)}, \frac{Z'(0)}{\alpha'(0)})$$

Let $\alpha'(t)$ have multiplicity k at $t = 0$. If $k \neq (m-1)$, there are two possibilities:

Case $k < (m-1)$: then

$$\mathbf{P}'(0) = \lim_{t \to 0} (\frac{X'(t)}{\alpha'(t)}, \frac{Y'(t)}{\alpha'(t)}, \frac{Z'(t)}{\alpha'(t)}) = (0, 0, 0).$$

Thus, $\mathbf{P}(s)$ is not a regular parametrization in the neighborhood of $s = 0$.

Case $k > (m-1)$: then

$$\beta_1'(0) = \lim_{t \to 0} (\frac{X'(t)}{\alpha'(t)}).$$

However this limit does not exist and the curve can not be regular in the neighborhood of $s = o$.

Thus $\alpha'(t)$ can be represented as:

$$\alpha'(t) = a_{m-1}t^{m-1} + a_m t^m + a_{m+1}t^{m+1} + \dots$$

$$\Rightarrow \alpha(t) = c + a'_{m-1}t^m + a'_m t^{m+1} + \dots$$

[1]The reason we have chosen a reparametrizing function of the type $s = \alpha(t)$ rather than $t = \gamma(s)$ is due to the ease of representation. In next section we will show the existence of a $\alpha(t) \in \mathbb{R}[t]$.

We have assumed that $c = 0$. Thus, $\alpha(t)$ has multiplicity m at $t = 0$.

Lets prove the other part of the theorem. If there exists an analytic function $s = \alpha(t)$, whose multiplicity at $t = 0$ is m, then $\alpha(t)$ can be represented as

$$\alpha(t) = a_m t^m + a_{m+1} t^{m+1} + \dots$$

Thus,

$$\beta_1'(0) = \lim_{t \to 0}\left(\frac{X'(t)}{\alpha'(t)}\right) \neq 0.$$

Therefore, $\mathbf{p}(s)$ represents a regular parametrization of the curve in the neighborhood of $s = 0$. Q.E.D.

The previous theorem tells us about the class of functions that can be used for substitution, such that the resulting parametrization represents a regular curve in the neighborhood of $t = 0$.

5 Necessary Condition for Reducible Parametrization

In this section we analyze a curve about a point where the derivative vector vanishes. If the point does not correspond to a cusp, we show that the given parametrization is improper.

Given $\mathbf{Q}(t) = (X(t), Y(t), Z(t)) = (\frac{x(t)}{w(t)}, \frac{y(t)}{w(t)}, \frac{z(t)}{w(t)})$ (which may or may not have a cusp at $t = 0$) as

$$\begin{aligned} x(t) &= a_0 t^m + a_1 t^{m+1} + \dots + a_{n-m} t^n \\ y(t) &= b_0 t^k + b_1 t^{k+1} + \dots + b_{n-k} t^n \\ z(t) &= c_0 t^l + c_1 t^{l+1} + \dots + c_{n-l} t^n, \\ w(t) &= d_0 + d_1 t + \dots + d_n t^n, \end{aligned} \tag{4}$$

where $k \geq m$, $l \geq m$, $m \geq 2$, $a_0 \neq 0$ and $d_0 \neq 0$.

Let us consider $X(t), Y(t)$ and $Z(t)$ as analytic functions and represent as formal power series. This can be obtained by taking the Taylor expansion of $\frac{1}{w(t)}$ about $t = 0$ and multiplying them with the corresponding numerators. According to Lemma I, $\mathbf{Q}(t)$ has an equivalent parametrization of the form

$$\begin{aligned} \overline{X}(s) &= s^m \\ \overline{Y}(s) &= s^k(\overline{b}_0 + \overline{b}_1 s + \dots) \\ \overline{Z}(s) &= s^l(\overline{c}_0 + \overline{c}_1 s + \dots) \end{aligned} \tag{5}$$

We make use of this equivalent parametrization, (5), in the following theorem:

Theorem II: *Given a space curve* $\mathbf{Q}(t)$ *of the form*

$$\begin{aligned} X(t) &= t^m \\ Y(t) &= a_0 t^{n_1} + a_1 t^{n_2} + \dots \\ Z(t) &= b_0 t^{p_1} + b_1 t^{p_2} + \dots \end{aligned}$$

where $n_1 \geq m$, $p_1 \geq m$ *and* $m \geq 2$. *The necessary and sufficient condition for the existence of a cusp at* $t = 0$ *is obtained if one of the* n_i*'s or* p_i*'s is not divisible by* m.

Proof[2]**:** *Necessity*

Let all the n_i's and p_j's be divisible by m. Put $(s = t^m)$[3]. The resulting parametrization $\mathbf{P}(s)$ is

$$\begin{aligned} X(s) &= s \\ Y(s) &= a_0 s^{k_1} + a_1 s^{k_2} + \dots \\ Z(s) &= b_0 s^{l_1} + b_1 s^{l_2} + \dots \end{aligned} \tag{6}$$

where $k_i = \frac{n_i}{m}$ and $l_i = \frac{p_i}{m}$. $\mathbf{P}(s)$ represents a regular parametrization (as $X'(0) = 1$). Thus, $\mathbf{Q}(t)$ does not have a cusp at $s = 0$.

Sufficiency

Suppose one of the n_i is not divisible by m. If $\mathbf{Q}(t)$ does not have a cusp at $t = 0$ then it can be reparametrized to have a regular parametrization at $t = 0$. By *implicit function theorem*, the curve can therefore be defined by $y = \phi(x)$ and $z = \psi(x)$ in a small neighborhood of the origin. Since $\phi(x)$ is a regular function, the coordinates of the curve are related by

$$y = \phi(x) = c_1 x + c_2 x^2 + \dots$$

in that small neighborhood. Substituting $x = x(t)$ and $y = y(t)$, we obtain

$$a_0 t^{n_1} + a_1 t^{n_2} + \dots = c_1 t^m + c_2 t^{2m} + c_3 t^{3m} + \dots$$

This identity implies that all n_j's are multiples of m, which is contrary to our assumption that m does not divide n_i. Thus, the curve has a cusp at $t = 0$.

Q.E.D.

A simple generalization of the last theorem results in the following sufficient condition:

Corollary I: *Given* $\mathbf{q}(t)$, *a curve defined as*

$$\begin{aligned} x(t) &= a_0 t^m + a_1 t^{m+1} + \dots + a_{n-m} t^n \\ y(t) &= b_0 t^k + b_1 t^{k+1} + \dots + b_{n-k} t^n \\ z(t) &= c_0 t^l + c_1 t^{l+1} + \dots + c_{n-l} t^n \end{aligned}$$

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[2]This proof is similar to the one used in Theorem 2.1, Chapter IV [Walker '50] for deciding when is a parametrization reducible.

[3]This is a function belonging to the class specified by Theorem I.

where $k \geq m$, $l \geq m$ and $m \geq 2$. A sufficient condition for the existence of cusp is obtained at $t = 0$ if m does not divide either k or l.

Corollary I provides us with a simple procedure to analyze whether $\mathbf{Q}(t)$ has a cusp at $t = 0$. However, it provides a sufficient condition only. Although theorem III is a complete method to detect cusps, it is difficult to express it computationally.

Theorem III: *If $\mathbf{Q}(t)$, as defined in (4) does not have a cusp at $t = 0$, then the curve is improperly parametrized.*

Proof: $\mathbf{Q}(t)$ has an equivalent parametrization, say $\mathbf{P}(s)$, of type (5). According to theorem II, $\mathbf{P}(s)$ can be reparametrized into $\mathbf{R}(u) = (\overline{X}(u), \overline{Y}(u), \overline{Z}(u))$, of type (6), where

$$\begin{aligned} \overline{X}(u) &= u \\ \overline{Y}(u) &= b_0 u^{k_1} + b_1 u^{k_2} + \dots \\ \overline{z}(Z) &= c_0 u^{l_1} + c_1 u^{l_2} + \dots \end{aligned}$$

$\mathbf{R}(u)$ is obtained by substituting $u = X(t)$ in $\mathbf{Q}(t)$. Consider $\mathbf{Q}$ and $\mathbf{R}$ as vector valued functions defining a curve in the complex space, i.e.

$$\mathbf{Q} : \ \mathbf{C} \rightarrow \mathbf{C}^3$$

$$\mathbf{R} : \ \mathbf{C} \rightarrow \mathbf{C}^3.$$

Both of them represent the same curve. Consider a point (X, Y, Z) on the curve and let $u = u_0$ be the parameter of $\mathbf{R}(u)$ corresponding to this point. Lets consider the equation $X(t) = u_0$. This equation can be expressed as

$$\frac{x(t)}{w(t)} = u_0,$$

$$\Rightarrow x(t) - u_0 w(t) = 0,$$

$$\Rightarrow a_0 t^m + \dots a_{n-m} t^n - u_0(d_0 + d_1 t + \dots + d_n t^n) = 0.$$

This equation has n roots over the complexes (counted properly with respect to multiplicity). For most values of u_0 the roots are distinct. Thus, almost all points on the curve $\mathbf{Q}(t)$ have n distinct preimages. The fact

$$n \geq m \geq 2$$

implies that $\mathbf{Q}(t)$ is an improper parametrization of the curve.

Q.E.D.

This theorem provides a necessary condition to determine whether a given parametrization is reducible. It is used in the following corollary:

Corollary II: *If a curve is properly parametrized, then the necessary and sufficient condition for the existence of a cusp is given by the vanishing of the derivative vector.*

Proof: follows from theorem III and the necessary condition for the existence of cusps.
Q.E.D.

In [DeRose and Stone '89] it is shown that the necessary and sufficient condition for the existence of a cusp in a non-degenerate cubic Bézier curve is given by the vanishing of the derivative vector. A cubic Bézier curve is considered a degenerate curve if its control vertices are collinear. This result also follows from corollary II. If a cubic Bézier curve (whose degree after converting to power basis is three) represents a straight line then it is improperly parametrized and its control vertices are, therefore collinear. Thus, for each non-degenerate cubic Bézier curve (and hence a properly parametrized curve) the necessary and sufficient condition for a cusp is given by the vanishing of the first derivative vector.

6 Proper Parametrizations

In algebraic geometry it is known that corresponding to every improperly parametrized rational curve there is a properly parametrized rational curve. An algorithm to compute the proper parametrization of an improperly parametrized rational curve is given in [Sederberg '86]. In this section we show that corresponding to every improperly parametrized polynomial curve there is a polynomial curve. We also present a simple algorithm to compute the proper parametrization of the polynomial curve.

A curve is said to have a proper parametrization if and only if there is a one-to-one relationship between the parameter values and the points on the curve. Every *planar* polynomial curve, say $\mathbf{p}(t)$, represents an algebraic curve of the type $f(x, y) = 0$, where $f(x, y)$ is a polynomial in x and y. Moreover, the algebraic curve so obtained is a *rational* algebraic curve, since it has corresponding a rational (or polynomial) parametrization. According to *Lüroth's theorem*, given a rational algebraic curve $f(x, y) = 0$, then there exist two rational functions $X(t)$ and $Y(t)$, where $t \in \mathcal{C}$, such that:

- For all but a finite set of $t \in \mathcal{C}$, $f(X(t), Y(t)) = 0$.
- With a finite number of exceptions, for every x_0,y_0 for which $f(x_0, y_0) = 0$, there is a unique $t \in \mathcal{C}$ such that $x_0 = X(t)$ and $y_0 = Y(t)$.

More details on Lüroth's theorem are given in [Archbold '47; Walker '50]. Since every rational space curve is birationally equivalent to a rational plane curve, there exists a properly parametrized rational space curve corresponding to every improperly parametrized rational space curve [Walker '50].

The following theorem highlights the class of properly parametrized rational curves which can be reparametrized into polynomial curves.

Theorem IV: *A properly parametrized rational curve* $\mathbf{Q}(t) = (x(t), y(t), z(t), w(t))$ *of degree* n *can be reparametrized into a polynomial curve if and only if*

- *$w(t)$ is a polynomial of the form $d_n(t-\beta)^n$. That is, it has a single root of multiplicity n.*
- *degree of $w(t)$ = maximum (degree of $x(t)$, degree of $y(t)$, degree of $z(t)$, degree of $w(t)$).*

Proof: [Manocha and Canny '90].

Q.E.D.

We use the result of previous theorem in the following theorem:

Theorem V: *Corresponding to every improperly parametrized polynomial curve, $\mathbf{q}(t) = (x(t), y(t), z(t))$, there is a properly parametrized polynomial curve, say $\mathbf{p}(s) = (\overline{x}(s), \overline{y}(s), \overline{z}(s))$ and moreover there is a polynomial function $g(t)$ such that*

$$\mathbf{p}(g(t)) = \mathbf{q}(t).$$

Proof: According to Lüroth's theorem, there is a properly parametrized rational curve $\mathbf{P}(u) = (\overline{X}(u), \overline{Y}(u), \overline{Z}(u))$, corresponding to $\mathbf{q}(t)$. If $\mathbf{P}(u)$ is not a polynomial curve, according to Theorem IV its homogeneous representation, $\mathbf{P}(u) = (\overline{x}(u), \overline{y}(u), \overline{z}(u), \overline{w}(u))$ satisfies the following properties:

- $\overline{w}(u)$ is a polynomial of the form $\overline{d}_n(u-\beta)^n$.
- degree of $\overline{w}(u)$ = maximum (degree of $\overline{x}(u)$, degree of $\overline{y}(u)$, degree of $\overline{z}(u)$, degree of $\overline{w}(u)$).

As a result, a linear reparametrization of the form

$$u = F(s) = \frac{1+s\beta}{s}$$

results in a polynomial curve

$$\mathbf{p}(s) = \mathbf{P}(F(s)) = (\overline{X}(F(s)), \overline{Y}(F(s)), \overline{Z}(F(s))) = (\overline{x}(s), \overline{y}(s), \overline{z}(s)).$$

and its degree is equal to that of $\mathbf{P}(u)$. The fact $F(s)$ is a one-to-one function implies that $\mathbf{p}(s)$ is a properly parametrized curve. Thus, every improperly parametrized polynomial curve can be represented as a properly parametrized polynomial curve.

Since $\mathbf{p}(s)$ and $\mathbf{q}(t)$ are polynomial parametrizations of the same curve, Lüroth theorem implies that they are related as

$$\mathbf{p}(G(t)) = \mathbf{q}(t),$$

where $G(t)$ is a rational function. Let us assume that $G(t)$ is not a polynomial function and can therefore be represented as

$$G(t) = \frac{g(t)}{h(t)}, \quad GCD(g(t), h(t)) = 1.$$

Let $\overline{x}(s)$ be a polynomial function of the form

$$\overline{x}(s) = \overline{a}_0 + \overline{a}_1 s + \ldots + \overline{a}_n s^n.$$

Since $\overline{x}(G(t)) = x(t)$ is a polynomial function,

$$\frac{\overline{a}_0(h(t))^n + \overline{a}_1(h(t))^{n-1}g(t) + \ldots + \overline{a}_n(g(t))^n}{(h(t))^n}$$

is a polynomial function, too. This happens if and only if every root of the denominator is a root of the numerator. Let $t = t_0$ be a root of $h(t)$. Since $GCD(g(t), h(t)) = 1$, the value of the numerator at this root $\overline{a}_n(h(t_0))^n \neq 0$. Thus, $x(G(t))$ is not a polynomial function and the proper and improper parametrizations of the polynomial curve are related by a polynomial reparametrization

$$\mathbf{p}(g(t)) = \mathbf{q}(t).$$

Q.E.D.

We use the result of the previous theorem in computing proper parametrization of an improperly parametrized polynomial curve. Let $\mathbf{q}(t) = (x(t), y(t), z(t))$, where each scalar polynomial has degree n, be an improperly parametrized curve and let its proper parametrization be denoted as $\mathbf{p}(s) = (\overline{x}(s), \overline{y}(s), \overline{z}(s))$, a curve of degree m. Let $d = \frac{n}{m}$. According to theorem V, there is a polynomial function $g(t)$ of degree d such that $\mathbf{p}(g(t)) = \mathbf{q}(t)$. Thus, every generic point on $\mathbf{Q}(t)$ has d preimages, with respect to the function $\mathbf{Q}(t)$. Let $\mathbf{Q}(t_0) = (x(t_0), y(t_0), z(t_0))$ be a non-singular point on the curve. We want to know about the other values of the parameter t, which according to $\mathbf{Q}(t)$ map to $\mathbf{Q}(t_0)$, *i.e.* the common roots of the equations

$$\begin{aligned} x(t) &= x(t_0), \\ y(t) &= y(t_0), \\ z(t) &= z(t_0). \end{aligned} \tag{7}$$

These common roots correspond to the roots of

$$h(t) = GCD(x(t) - x(t_0), y(t) - y(t_0), z(t) - z(t_0)) \tag{8}$$

and therefore, $h(t)$ is a polynomial of degree d. The following lemma shows the use of $h(t)$ in computing the proper parametrization.

Lemma II: $\mathbf{q}(t)$ *has a polynomial proper parametrization* $\mathbf{r}(u)$ *of degree* m *such that*

$$\mathbf{r}(h(t)) = \mathbf{q}(t),$$

where $h(t)$ *has been defined in (8).*

Proof: Any polynomial proper parametrization, say $\mathbf{p}(s)$, of $\mathbf{q}(t)$ is unique up to a linear reparametrization of the form $s = f(u) = au + b$. If $\mathbf{p}(g(t)) = \mathbf{q}(t)$, we can choose a proper parametrization $\mathbf{r}(u) = \mathbf{p}(au + b)$ such that

$$\mathbf{r}(g_1(t)) = \mathbf{q}(t),$$

where

$$g_1(t) = \frac{g(t) - b}{a}.$$

Another way to interpret this relationship is: Given $\mathbf{p}(s)$ and $g(t)$, it is possible to choose a reparametrizing function of the form $kg(t) + c$, where k and c are arbitrary scalars, such that there exists a proper parametrization $\mathbf{r}(u)$ and

$$\mathbf{p}(g(t)) = \mathbf{r}(kg(t) + c) = \mathbf{q}(t). \tag{9}$$

Let $\mathbf{p}(s)$ be the proper parametrization of $\mathbf{q}(t)$ (according to Theorem V) and let the preimage of $(x(t_0), y(t_0), z(t_0))$ be $s = s_0$. Since $\mathbf{p}(s)$ is properly parametrized, there is only one preimage. Let us analyze the equation

$$s_0 - g(t) = 0.$$

This equation has d roots. If $t = t_1$ is a root of the above equation then $\mathbf{p}(g(t_1)) = \mathbf{p}(s_0) = \mathbf{q}(t_0)$. As a result each root of $s_0 - g(t)$ is also a root of each of the set of equations in (7) and therefore, of $h(t)$. Since $s_0 - g(t)$ and $h(t)$ are both equations of degree d with the same roots

$$h(t) = k(s_0 - g(t)),$$

$$\Rightarrow g(t) = k_1 h(t) + c_1$$

where k_1 and c_1 are defined accordingly. This relation along with (9) implies that there exists a proper parametrization of the form $\mathbf{r}(u)$ such that $\mathbf{r}(h(t)) = \mathbf{q}(t)$.

Q.E.D.

Thus, given a non-singular point of the curve, we can compute $h(t)$ as highlighted in (8) and using the result of previous lemma compute a proper parametrization $\mathbf{r}(u)$ such that the following equality hold:

$$\mathbf{r}(h(t)) = \mathbf{q}(t).$$

We assumed that $\mathbf{q}(t_0)$ is not a multiple point because of the following reasons. Let us consider the case when $\mathbf{q}(t)$ is a properly parametrized curve. There are 2 cases:

- $\mathbf{q}(t_0)$ is a cusp. Thus, $\mathbf{q}'(t_0) = (0, 0, 0)$. As a result, $t = t_0$ is a root of multiplicity greater than one of the equations $x(t) = x(t_0)$, $y(t) = y(t_0)$ and $z(t) = z(t_0)$. Therefore, $h(t)$ is a polynomial of degree greater than one, leading us to the wrong conclusion that $\mathbf{q}(t)$ is an improperly parametrized curve. Even if $\mathbf{q}(t)$ is improperly parametrized, the degree of $h(t)$ is more than the number of preimages of any non-singular point on $\mathbf{q}(t)$.

- $\mathbf{q}(t_0)$ corresponds to a loop. Thus, $\mathbf{q}(t)$ has more than one *place* at $\mathbf{q}(t_0)$ [Manocha and Canny '90]. As a result there are at least two distinct values of the parameter t, which correspond to $\mathbf{q}(t)$ and thereby implying that $h(t)$ has degree greater than one. Rest of the argument is similar to the previous case.

Any rational curve of degree n has $d = \frac{(n-1)(n-2)}{2}$ singular points (counted properly with respect to multiplicity) [Walker '50]. Thus, the probability that any randomly chosen point corresponds to a non-singular point is 1. A deterministic algorithm for computing the proper parametrization is obtained by choosing $d+1$ random points, denoted t_i, and compute the corresponding $h_i(t)$ as shown in (8). $h(t)$ correspond to $h_i(t)$ of minimum degree, $1 \leq i \leq (d+1)$.

7 Polynomial Curves

In this section we present an algorithm for detecting cusps in polynomial curves and illustrate it with some examples.

There are two possible approaches to the problem of detecting cusps. In the first method we use the result of Lemma II to compute the proper parametrization of the polynomial curve and then check for the vanishing of the derivative vector of the properly parametrized curve. According to Corollary II, such points correspond to cusps.

In the second approach we compute the parameter values corresponding to the roots of the derivative vector. For each such root, we need to determine whether the corresponding on the curve is a cusp or is the parametrization reducible. According to Lemma II, the reparametrizing function $h(t)$, in (8), is unique up to a constant and multiplication by a scalar. This property along with Theorem I can be used to detect cusps by checking whether $h(t)$ is a function of the form

$$h_0 + h_m t^m + \ldots + h_n t^p,$$

where $h_m \neq 0$ and m is the minimum among the multiplicities of $x(t)$, $y(t)$ and $z(t)$ at $t = 0$ (such a curve is obtained after a linear reparametrization and a translation in space so that $m \geq 2$).

We recommend the first approach for the following reasons:

- In most applications of computer aided geometric design, it is desirable to compute the proper parametrizations, since they result in a low degree representation.
- It is possible that the parameter values corresponding to the vanishing of the first derivative vector cannot be represented as rational numbers. This can happen even if the given parametrization consist of polynomials with rational coefficients. In that case, we will be left with heuristic approaches to check whether the given parameter value corresponds to a cusp or is the parametrization reducible.
- In the second approach we analyze each parameter value separately, where the first derivative vector vanishes, whether it corresponds to a cusp or is the parametrization reducible. In the first approach each root of the first derivative vector of the proper parametrization corresponds to a cusp. No further analysis is required.

7.1 Algorithm

Given a polynomial space curve

$$\mathbf{q}(t) = (x(t), y(t), z(t)), \quad t \in [a, b],$$

of degree n, an algorithm to compute the proper parametrization and detect cusps is:

1. Choose a random parameter value $t = t_0$. Let $(x_0, y_0, z_0) = \mathbf{q}(t_0)$ and

   $$h(t) = GCD(x(t) - x_0, y(t) - y_0, z(t) - z_0)$$

   (x_0, y_0, z_0) should not be a multiple point of $\mathbf{q}(t)$. A simple probabilistic algorithm to ensure such a condition can be obtained by choosing a few random values of the parameter t and comparing the degrees of the corresponding $h(t)$'s (whether they are equal). A deterministic algorithm can be obtained by choosing $\frac{(n-1)(n-2)}{2} + 1$ distinct values of t, say t_i, and compute the corresponding $h_i(t)$. $h(t)$ correspond to the $h_i(t)$ of minimum degree.

2. Let $s = h(t)$, $m = degree(h(t))$. If $m = 1$ then the parametrization is proper else a proper parametrization $\mathbf{p}(s)$ has the form

   $$\overline{x}(s) = x_d s^d + \ldots + x_1 s + x_0,$$

   $$\overline{y}(s) = y_d s^d + \ldots + y_1 s + y_0,$$

   $$\overline{z}(s) = z_d s^d + \ldots + z_1 s + z_0,$$

   The coefficients x_i's, y_i's and z_i's can be determined by equating the powers of t in the vector equation $\mathbf{p}(h(t)) = \mathbf{q}(t)$ and $d = \frac{n}{m}$. The domain of the properly parametrized curve is given by $[u, v]$, where $u = h(a)$ and $v = h(b)$.

3. Let $\mathbf{p}(s)$, $s \in [u, v]$, be the proper parametrization of a polynomial space curve. Let

   $$g(s) = GCD(\overline{x}'(s), \overline{y}'(s), \overline{z}'(s)).$$

   Use the Sturm's sequence method to check whether $g(s)$ has any root in $[u, v]$. Each root of $g(s)$ corresponds to a cusp. If it has no roots then $\mathbf{p}(s)$ is a regular parametrization of the curve in $[u, v]$.

7.2 Numerical Examples

We illustrate the algorithm described above with a few examples. Consider the curve

$$\mathbf{q}(t) = (x(t), y(t)) = (t^2, t^3), \quad t \in (-1, 1).$$

Let $t_0 = 1$ and its image $\mathbf{q}(t_0) = (1,1)$ is a non-singular point. Thus, $h(t) = GCD(t^2 - 1, t^3 - 1) = t - 1$. Since $h(t)$ is a linear polynomial, $\mathbf{q}(t)$ represents a proper parametrization of the curve. Therefore, the origin corresponds to a cusp on $\mathbf{q}(t)$. It can be shown similarly for $\mathbf{q}(t) = (t^3, t^5)$ that the origin is a cusp.

Let us now consider the curve

$$\mathbf{q}(t) = (x(t), y(t)) = (t^2 + t^3, t^6 + 2t^5 + t^4), \quad t \in (-1, 1).$$

Let $t_0 = 2$ and its image $\mathbf{q}(2) = (12, 144)$ is not a cusp. In this case, $h(t) = GCD(t^2 + t^3 - 12, t^6 + 2t^5 + t^4 - 144) = t^2 + t^3 - 12$. Since $h(t)$ is a quadratic polynomial $\mathbf{q}(t)$ represents a improper parametrization and every point on $\mathbf{q}(t)$ has 2 preimages. The proper parametrization $\mathbf{p}(s) = (\overline{x}(s), \overline{y}(s))$ is obtained by equating

$$(\overline{x}(h(t)), \overline{y}(h(t))) = (t^2 + t^3, t^6 + 2t^5 + t^4).$$

Thus,

$$\mathbf{p}(s) = (s + 12, s^2 + 24s + 144), \quad s \in (-12, -10)$$

which is a regular parametrization in the given interval and hence the curve has no cusps.

8 Rational Curves

Given a rational curve

$$\mathbf{Q}(t) = (X(t), Y(t), Z(t)), \quad t \in (a, b)$$

whose corresponding homogeneous representation is given by

$$\mathbf{Q}(t) = (x(t), y(t), z(t), w(t)), \quad w(t) \neq 0, \quad t \in (a, b).$$

The algorithm for polynomial curves can be extended to rational curves. In particular we use the algorithm given in [Sederberg '86] to compute the proper parametrization of the rational curve and than check for the roots of the first derivative vector of the proper parametrization.

In many applications we might be only interested in determining whether a rational curve has any cusps and not computing a proper or regular parametrization. Let $\mathbf{Q}(t)$ be the rational curve (which may be improperly parametrized) and $t = t_0$ be a parameter value where the first derivative vector vanishes. Reparametrize $\mathbf{Q}(t)$ by substituting $t = s + t_0$. Let

$$\mathbf{P}(t) = \mathbf{Q}(t) - \mathbf{Q}(t_0).$$

Thus

$$\mathbf{P}(t_0) = \mathbf{P}'(t_0) = (0, 0, 0).$$

The transformation applied is similar to the one for polynomial curves. From now on we assume that we have been given a rational curve

$$\mathbf{Q}(t) = (x(t), y(t), z(t), w(t)), \quad w(t) \neq 0, \quad t \in [a, b] \tag{10}$$

such that $\mathbf{Q}(t_0) = \mathbf{Q}'(t_0) = (0,0,0)$. We need to determine whether $t = t_0$ corresponds to a cusp.

The representation in (10) implies that

$$(x(t_0), y(t_0), z(t_0)) = (0,0,0).$$

Moreover

$$\begin{aligned}(X'(t_0), Y'(t_0), Z'(t_0)) &= (x'(t_0)w(t_0) - x(t_0)w'(t_0), y'(t_0)w(t_0) - y(t_0)w'(t_0), \\ &\quad z'(t_0)w(t_0) - z(t_0)w'(t_0)) \\ &= (0,0,0).\end{aligned}$$

Let us look at one of the equalities, say $x'(t_0)w(t_0) - x(t_0)w'(t_0) = 0$. The fact that $x(t_0) = 0$ and $w(t_0) \neq 0$ implies that $x'(t_0) = 0$. Thus, the multiplicity of each of the three polynomials $x(t)$,$y(t)$ and $z(t)$ at $t = t_0$ is at least 2. Let us consider the polynomial curve obtained by setting $w(t) = 1$:

$$\mathbf{q}(t) = (x(t), y(t), z(t)), \quad t \in (a, b).$$

Lemma III: $\mathbf{Q}(t)$ *has a cusp at* $t = 0$ *if and only if* $\mathbf{q}(t)$ *has a cusp at* $t = t_0$.

Proof: Assume that $\mathbf{Q}(t)$ has a cusp at $t = t_0$ and $\mathbf{q}(t)$ does not have a cusp at $t = t_0$. Thus, there exist a reparametrizing function $s = f(t)$[4], such that $\overline{x}'(0) \neq 0$, where $\overline{x}(s) = x(t)$ and $x(t)$ has the minimum multiplicity (among $x(t)$,$y(t)$ and $z(t)$) at $t = t_0$. We can use the same function for reparametrizing the rational curve. In this case $\overline{w}(s)$, where $\overline{w}(s) = w(t)$ may not be a polynomial function. However it is still an analytic function and the curve $\mathbf{Q}(t)$ has a regular parametrization in the neighborhood of $t = t_0$ given by $(\overline{x}(s), \overline{y}(s), \overline{z}(s), \overline{w}(s))$. This is contrary to our assumption that $\mathbf{Q}(t)$ has a cusp at $t = t_0$. Therefore

$$\mathbf{Q}(t) \text{ has a cusp at } t = t_0 \Rightarrow \mathbf{q}(t) \text{ has a cusp at } t = t_0.$$

The other part of the proof can be shown in the same manner.

Q.E.D.

8.1 Algorithm

In this section we present an algorithm for detecting cusps in a rational curve by using the algorithm for polynomial curves. Given a rational curve

$$\mathbf{Q}(t) = (x(t), y(t), z(t), w(t)), \quad w(t) \neq 0, \quad t \in [a, b],$$

an algorithm to detect cusps is:

[4] f(t) is in fact a polynomial and belonging to a class of function specified by Theorem I.

1. Let

$$g(t) = GCD(x'(t)w(t) - x(t)w'(t), y'(t)w(t) - y(t)w'(t), z'(t)w(t) - (t)w'(t)).$$

2. Use Sturm sequence method to compute the roots of $g(t)$ lying in $[a, b]$. For each root of $g(t)$, say $t = t_0$, in the associated interval, do

   - Let

     $$\begin{aligned} \mathbf{P}(t) &= \mathbf{Q}(t) - \mathbf{Q}(t_0) = (\overline{x}(t), \overline{y}(t), \overline{z}(t), w(t)) \\ &= (w(t_0)x(t) - x(t_0)w(t), w(t_0)y(t) - y(t_0)w(t), w(t_0)z(t) - x(t_0)z(t), w(t)), \end{aligned}$$

     and

     $$\mathbf{p}(t) = (\overline{x}(t), \overline{y}(t), \overline{z}(t).$$

     Moreover, let m be the minimum of the multiplicities of $\overline{x}(t), \overline{y}(t)$ and $\overline{z}(t)$ at $t = t_0$. We know that $m \geq 2$.

   - Choose a random value of the parameter t (say $t = t_1$). Let $(x_1, y_1, z_1) = \mathbf{p}(t_1)$ and

     $$h(t) = GCD(\overline{x} - x_1, \overline{y} - y_1, \overline{z} - z_1).$$

     $\mathbf{p}(t)$ has a cusp at $t = t_0$ if and only if $h(t)$ is a polynomial of the form

     $$c + a_m(t - t_0)^m + a_{m+1}(t - t_0)^{m+1} + \ldots + a_n(t - t_0)^n,$$

     where $a_m \neq 0$. This holds if and only if

     $$h'(t_0) = 0; \quad h''(t_0) = 0; \quad \ldots \quad h^{m-1}(t_0) = 0$$

     and

     $$h^m(t_0) \neq 0.$$

     $\mathbf{P}(t)$ has a cusp at $t = t_0$ if and only if $\mathbf{p}(t)$ has a cusp at $t = t_0$.

The problem of detecting cusps in a rational curve has been reduced to that of a polynomial curve. If the rational curve has no cusps in its domain of definition, this algorithm does not compute the regular parametrization of the rational curve (assuming the given parametrization is not regular). Given a cubic rational curve, it involves computing the GCD of fourth order polynomials in step 1 and analyzing a cubic polynomial curve in step 2 for cusps at a given parameter value. Thus, the problem of detecting cusps in a cubic rational curve is equivalent to that of a fourth order polynomial curve (according to our procedure).

9 Inflection Points

In this section we present an algorithm to detect inflection points in rational curves and its restriction to polynomial curves is obvious. Given a regular parametrization

$$\mathbf{Q}(t) = (X(t), Y(t), Z(t)) \quad t \in [a, b],$$

whose homogeneous representation is

$$\mathbf{Q}(t) = (x(t), y(t), z(t), w(t)), \quad t \in [a, b].$$

The inflection points correspond to the parameter values where the curvature vanishes. For the space curve, $\mathbf{Q}(t)$, the curvature is given by:

$$\kappa(t) = \frac{\| \mathbf{Q}'(t) \times \mathbf{Q}''(t) \|}{\| \mathbf{Q}'(t) \|^3}.$$

Since the curve is regular in the given interval, the denominator term is nonzero. Compute the first and second derivatives vector of $\mathbf{Q}(t)$ and take their cross product. Let

$$\mathbf{R}(t) = \mathbf{Q}'(t) \times \mathbf{Q}''(t).$$

and its homogeneous representation be

$$\mathbf{R}(t) = (p(t), q(t), r(t), s(t)),$$

where $s(t)$ is some power of $w(t)$. We are only interested in the real values of the parameter $t \in [a, b]$, where the curvature vanishes. Let

$$g(t) = GCD[p(t), q(t), r(t)],$$

and use Sturm sequence method to compute the roots of $g(t)$ in $[a, b]$. They are the only parameter values in its domain of definition where the curvature vanishes.

If the original curve $\mathbf{Q}(t)$ has cusps in its domain, we compute the parameter values corresponding to cusps by the algorithm given in the previous section. Let the parameters values be $t_1, t_2, \ldots, t_k$. We break up our domain into the following $k + 1$ intervals:

$$[a, t_1), (t_1, t_2), (t_2, t_3), \ldots, (t_{k-1}, t_k), (t_k, b].$$

Within each interval the curve is regular. Apply the Sturm sequence method to determine the root of $g(t)$ in each interval separately. Each of those roots would correspond to an inflection point.

10 Conclusion

We presented algorithms to detect cusps and inflection points in parametric curves. The necessary condition for the existence of cusps, given by the vanishing of the derivative vector, has been known before. We showed that the vanishing of the derivative vector becomes necessary as well as a sufficient condition whenever the curve is properly parametrized. We showed that corresponding to every improperly parametrized polynomial curve there exists a properly parametrized polynomial curve and presented a simple algorithm to compute the latter. This algorithm is used to detect cusps in a polynomial curve and the problem of detecting cusps has been reduced to that of detecting cusps in a polynomial curve. Finally, we presented an algorithm to determine the inflection points in a regular curve. Much work needs to be done in trying to extend these methods to analyze parametric surfaces for such undesirable features.

11 Acknowledgment

We are grateful to Ray Sarraga for productive discussions.

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13 Appendix

Sturm Sequences In many applications, one needs to find the actual or approximate values of the real roots of the equation, say $f(t)$, with numerical methods. The aim here is distinct from that of finding an algebraical solution. If the equation has any rational roots, then they can be found or removed from the equation by using *Newton's method of Divisors*

[Uspensky '48]. Moreover multiple roots of an equation can be found by computing the GCD of $f(t)$ and $f'(t)$. A polynomial of the type

$$h(t) = \frac{f(t)}{GCD(f(t), f'(t))}$$

has no multiple roots. Thus, the main problem is that of finding the irrational roots to a desired accuracy. The usual procedure is [Collins '77]:

1. Find an interval which contains all the roots. Lower and upper bounds on the absolute value of a root are given in [Mignotte '82]. They can be used for forming such an interval.

2. Separate the roots. That is, we find intervals each of which contains a single root or a multiple root.

3. Take any interval which contains a single root. By dividing it into smaller and smaller intervals which contain the roots, we can determine the root of the polynomial up to a desired accuracy.

Sturm sequences are used to determine the number of real roots of a given polynomial in a real interval. They are used in steps (2) and (3) of the procedure given above.

13.0.1 Sturm's Functions

Let us look at the steps involved in Euclid's algorithm used for computing the GCD of two polynomials. If we are trying to compute GCD$(g(t), h(t))$, (assuming degree of $h(t)$ is less then or equal to that of $g(t)$), we perform polynomial division and obtain

$$g(t) = q(t)h(t) + r(t),$$

where $q(t)$ is the quotient and $r(t)$ is the remainder. The degree of $r(t)$ is less than the degree of $h(t)$. If $r(t) = 0$, then $q(t)$ is the GCD, else the procedure is called recursively as GCD$(h(t), r(t))$.

Consider $f(t)$ a polynomial and $f_1(t)$ as its first derivative. Let the operation of computing the GCD of $f(t)$ and $f_1(t)$ be performed with the following alteration:

The sign of each remainder is to be changed before it is used as a divisor. The sign of the last remainder is also to be changed.

Denote the *modified* remainders by $f_2(t), f_3(t), \ldots, f_r(t)$; then $f(t), f_1(t), f_2(t), \ldots, f_r(t)$ are called *Sturm's functions*. The collection as a whole is a *Sturm sequence* (in the order specified). Sturm's functions are related by the following equations:

$$f(t) \quad = \quad q_1(t)f_1(t) - f_2(t)$$

$$
\begin{aligned}
f_1(t) &= q_2(t)f_2(t) - f_3(t) \\
\vdots \quad & \quad \vdots \\
f_{r-2}(t) &= q_{r-1}(t)f_{r-1}(t) - f_r(t)
\end{aligned}
\tag{11}
$$

Sturm's Theorem: *If $f(t)$ is a polynomial and a, b are any real numbers ($a < b$), the number of distinct roots of $f(t) = 0$ which lie between a and b (any multiple root which may exist being counted once only) is equal to the excess of the number of changes of sign in the sequence of Sturm's functions*

$$f(t), f_1(t), f_2(t), \ldots, f_r(t)$$

when $t = a$ over the number of changes of sign in the sequence when $t = b$.

A good introduction to Sturm's functions and sequences is given in [Uspensky '48]. They have been used as a computational tool in the theory of reals [Ben-Or,Kozen and Reif '86].

Thus, given $g(t)$, in (3), we compute the Sturm's functions $g(t), g_1(t), g_2(t), \ldots, g_r(t)$ related in a manner shown in (11). Here $r \leq k$, the degree of $g(t)$. Given the interval $[a, b]$, associated with $\mathbf{q}(t)$ in (1), compute the number of sign changes in the sequence at $t = a$ and at $t = b$. The difference tells us the number of distinct real roots in the interval (a, b), (each multiple root being counted once). We can check whether a or b is a root of $g(t)$ by substitution and hence determine the number of roots of $g(t)$ in $[a, b]$. If it has no root then the parametrization represents a regular curve. Otherwise we need to compute each root by the procedure mentioned above up to a desired accuracy and determine whether it corresponds to a cusp. We are only interested in knowing the value of the root and not its multiplicity.

The complexity of computing the Sturm sequence is the same as that of computing the GCD of two polynomials. In case, the polynomials have rational coefficients, then we can perform exact arithmetic and compute the number of roots exactly. Otherwise, we suggest the use of heuristics given in [Sederberg '84; Sederberg '86] for polynomials with floating point coefficients.