ON THE OPTIMAL CONTROL OF SYSTEMS DESCRIBED BY EVOLUTION EQUATIONS

by

T. E. Baker and E. Polak

Memorandum No. UCB/ERL M89/113

26 September 1989
(Revised 20 October 1992)

cover page

ON THE OPTIMAL CONTROL OF SYSTEMS DESCRIBED BY EVOLUTION EQUATIONS

by

T. E. Baker and E. Polak

Memorandum No. UCB/ERL M89/113

26 September 1989
(Revised 20 October 1992)

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

TITLE PAGE

ON THE OPTIMAL CONTROL OF SYSTEMS DESCRIBED BY EVOLUTION EQUATIONS

by

T. E. Baker and E. Polak

Memorandum No. UCB/ERL M89/113

26 September 1989
(Revised 20 October 1992)

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

ON THE OPTIMAL CONTROL OF SYSTEMS DESCRIBED BY EVOLUTION EQUATIONS†

by

T. E. Baker and E. Polak

Department of Electrical Engineering
and Computer Science
University of California
Berkeley, Calif. 94720

ABSTRACT

We present a mathematical foundation for the algorithmic solution of free- and fixed-time optimal control problems with evolution equation dynamics, finite dimensional controls, and constraints on the controls and end points. In particular, *(i)* we develop expressions for the derivatives of the solutions of the evolution equations with respect to controls in $L_2^m[0,1]$ and to the final-time, *(ii)* we show that the solutions of the relaxed evolution equations have a certain kind of directional derivative, *(iii)* we develop algorithmic optimality conditions with respect to both ordinary and relaxed controls and the final-time, and *(iv)* we present an approximation theory which shows that finite dimensional, minimax and methods of centers type algorithms can be used to obtain arbitrarily good approximations to optimal controls for optimal control problems with evolution equation dynamics and various constraints.

KEY WORDS

optimal control algorithms, approximation theory, relaxed controls, optimality conditions, evolution equations.

† The research reported herein was sponsored in part by the National Science Foundation under grant ECS-8121149, the Air Force Office of Scientific Research grant AFOSR-83-0361, and the State of California MICRO Program.

1. INTRODUCTION

The results presented in this paper, dealing with the optimal control of evolution equations, were largely motivated by optimal slewing problems arising in the control of large, flexible aerospace structures and in the control of various earthbound mechanisms with flexible links, which are naturally modeled by coupled systems of partial differential equations. Since, in practice, only finite element plant models may be available (which are in the form of ordinary differential equations) and since it is much easier to work with a canonical system representation, we assume that the plant dynamics are in evolution equation form, which permits us to treat both cases in a unified manner.

The majority of optimal control algorithms (see, for example [May.1 - May.5, Pir.2, Teo.1, Teo.2, War.2 - War.4, Won.1]) are presented in *conceptual form*, i.e., the effects of numerical integration of the differential equations are ignored. In [Kle.1] we find an approximation theory for unconstrained optimal problems with ODE dynamics, in the form of an implementation of the method of steepest descent. More generally, this theory provides guidelines for adaptively increasing the precision of numerical integration so as to ensure that the numerical scheme retains the convergence properties of the conceptual one. It was later used by [Dun.1] to implement a conditional gradient method for optimal control problems with ODE dynamics. As far as optimal control problems with PDE dynamics are concerned, in [Gib.1, Gib.2, Gib.3], we find a detailed solution of the linear quadratic regulator problem, including conditions for the convergence of modal approximation schemes. However, for more general optimal control problems with PDE dynamics, the prevailing approach has been to use some method for constructing a *particular finite dimensional* approximating optimal control problem and then to solve this problem by some method or other, see, e.g. [Jun.1, Chu.1, Ben.1 Bur.1, Flo.1]. The relationship between the solutions and stationary points of the approximating optimal control problem and those of the original optimal control problem is not established in these papers.

In this paper, we deal with the numerical solution of optimal control problems not by *adaptive implementation* of conceptual algorithms, but by *adaptive diagonalization* which requires less restrictive assumptions and, in our experience, seems to produce more efficient computational schemes. In any diagonalization approach, an original optimal control problem, **P**, is decomposed into an infinite sequence of *finite dimensional* problems, $\mathbf{P}_n$, $n = 1, 2, 3, \ldots$ which are solvable by nonlinear programming or nonsmooth optimization algorithms. These problems $\mathbf{P}_n$ must satisfy the following minimal consistency condition. Since, in the absence of convexity, finite dimensional optimization algorithms can only be shown to compute stationary points, rather than optimal points, the problems $\mathbf{P}_n$ must be such that not only their solutions converge to a solution of **P**, but also their (first order) stationary points converge to a stationary point of **P**. Next, there is considerable empirical evidence to suggest that from a computational point of view, the most efficient approach is to proceed gradually, iterating towards a solution of a problem $\mathbf{P}_n$ until some test is satisfied and then carry over the last iterate as a starting point for problem $\mathbf{P}_{n+1}$, until the value of n is increased to some preassigned maximum value n^*, rather than to solve $\mathbf{P}_{n^*}$ directly. In an adaptive diagonalization scheme, we can expect to find tests which determine not only when the solution of problem $\mathbf{P}_n$ should be arrested, but also the next value of n, which may be larger than $n+1$. In return, as we will show later, the use of adaptive tests results in stronger convergence properties for the diagonalization method.

In developing an adaptive diagonalization scheme for the numerical solution of free- and fixed-time optimal control problems with evolution equation dynamics, finite dimensional controls, and constraints on the controls and end points, we had to deal with *(i)* the differentiability of solutions of PDEs with respect to controls, *(ii)* optimality conditions for optimal control problems, which relate to those used in finite dimensional nonlinear programming and nonsmooth optimization[1], *(iii)* relaxed control theory in a PDE setting, *(iv)* conditions on the numerical methods for integrating the dynamical equations, to ensure consistent discretization, and (v) tests for progressing from $\mathbf{P}_n$ to $\mathbf{P}_{n+1}$.

The results presented in this paper extend and generalize the results in [Kle.1, Wil.1]. In particular, the results in [Kle.1] do not apply to constrained problems and hence a new generation of tests had to be invented; furthermore, the results in [Kle.1, Wil.1] apply only to problems with ODE dynamics. Nor were algorithms for constrained minimax optimal control problems, such as those considered in this paper, addressed in [Kle.1, Wil.1].

In Section 2, we give a formulation of the problems that we will consider. In Section 3, we develop expressions for the derivatives of the solutions of the evolution equations with respect to controls in $(L_\infty^m[0\,,1]\,, \|\cdot\|_2)$ and the final-time, and we establish first order optimality conditions for minimax optimal control problems with control constraints and for optimal control problems with constraints on the control and inequality constraints on the final-point. In Section 4 we introduce relaxed controls extensions of the optimal control problems under consideration and develop appropriate extensions of the optimality conditions introduced in Section 3. In Section 5 we present our approximation theory, and our adaptive diagonalization schemes. We show that these can be combined with a finite dimensional minimax algorithm [Pir.1, Psh.1, Pol.1], and a new phase I - phase II method of feasible directions[Pol. 2] to obtain arbitrarily good approximations to optimal controls for optimal control problems with evolution equation dynamics and various constraints. In Section 6 we present computational examples.

[1] It should be clear that because the optimality conditions for finite dimensional problems are in terms of "weak variations", in the absence of convexity, stationary controls of finite dimensional approximations to an optimal control problem can only converge to a control satisfying a "weak" optimality condition. Hence the Maximum Principle is generally an inappropriate optimality condition within the particular numerical approximation framework considered in this paper.

2. FORMULATION OF OPTIMAL CONTROL PROBLEMS

Many optimal control algorithms, including the ones to be presented in this paper, are extensions of finite dimensional optimization algorithms that deal with problems defined in the Hilbert space $\mathbb{R}^n$. Now, the natural space for establishing differentiability of solutions of a differential equation with respect to m-dimensional controls is $L_\infty^m[0, 1]$. However, adoption of $L_\infty^m[0, 1]$ as the space for analysis leads to the somewhat awkward situation that the extensions of the finite dimensional algorithms do not appear to be natural, because they require that we also use the $L_2^m[0, 1]$ norm, $\|\cdot\|_2$, and $L_2^m[0, 1]$ scalar product, $\langle \cdot , \cdot \rangle_2$.

Fortunately, one can also establish differentiability of solutions of a differential equation with respect to controls in the Hilbert space $L_2^m[0, 1]$, provided that one imposes a *growth condition* on the velocity function, as we will do shortly. In the case of control constrained optimal control problems, such as the ones treated in this paper, the imposition of a growth condition on the velocity function does not restrict the class of problems that can be considered, and amounts to no more than a mathematically convenient device.

Finally, we recall that for any $u \in L_2^m[0, 1]$, $\|u\|_2 \triangleq [\int_0^1 \|u(t)\|^2 dt]^{1/2}$, and for any $u, v \in L_2^m[0, 1]$, $\langle u, v \rangle_2 \triangleq \int_0^1 \langle u(t), v(t) \rangle dt$, where $\|\cdot\|$ denotes the norm on $\mathbb{R}^m$ and $\langle \cdot , \cdot \rangle$ denotes the scalar product on $\mathbb{R}^m$.

We are now ready to proceed. For any $0 < \tau < \infty$, let $G(\tau)$ be the set of admissible controls defined by:

$$G(\tau) \triangleq \{ u \in L_2^m[0, \tau] \mid u(t) \in U , \text{ for almost all } t \in [0, \tau] \} , \tag{2.1}$$

where U is a compact convex subset of $\mathbb{R}^m$.

Let X denote a Hilbert space with inner product $\langle \cdot , \cdot \rangle_X$ and corresponding norm $\|\cdot\|_X$. Let $A : D(A) \to X$ be the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$; let $F : X \times \mathbb{R}^m \to X$ be a nonlinear operator that is Lipschitz continuous on bounded sets. We will consider dynamical systems of the following form:

$$\frac{d}{dt}\tilde{z}(t, \tilde{u}) = A\tilde{z}(t, \tilde{u}) + F(\tilde{z}(t, \tilde{u}), \tilde{u}(t)) , \quad \tilde{z}(0, \tilde{u}) = z_0 \in D(A) , \quad \tilde{u} \in G(\tau) . \tag{2.2a}$$

where $\tilde{z}(t, \tilde{u}) \in X$, for all $t \in [0, \tau]$.

Because the set $U \subset \mathbb{R}^m$ is compact, there exists a bound $b < \infty$ such that for all $v \in U$, $|v^i| \leq b$, $i = 1, 2, \ldots, m$. Hence, since our algorithms never violate the control constraint, we may assume without loss of generality that the operator F has the form $F(z, v) = \tilde{F}(z, SAT(v))$, where $SAT : \mathbb{R}^m \to \mathbb{R}^m$ is such $SAT(v) = (sat(v^1), sat(v^2), \ldots, sat(v^m))$, where for all $z \in \mathbb{R}$,

$$sat(z) = \begin{cases} z\,, & \text{if } |z| \le 2b\,, \\ sgn(z)(2b + 1 - e^{(2b-|z|)})\,, & \text{if } |z| \ge 2b\,. \end{cases} \tag{2.2b}$$

This *growth condition* allows us in Section 3 to postulate local Lipschitz continuity conditions that are independent of bounds on the control.

We will assume that (2.2a) has a unique *mild solution*, which is defined as follows (see [Paz.1]):

Definition 2.1. A function $\tilde{z}(\cdot\,, \tilde{u}) \in C([0\,, \tau]\,, X)$ is said to be a *mild solution* to (2.2a) if

$$\tilde{z}(t\,, \tilde{u}) = T(t)\tilde{z}_0 + \int_0^t T(t - s)F(\tilde{z}(s\,, \tilde{u})\,, \tilde{u}(s))ds\,. \tag{2.2c}$$

□

We can normalize[2] the final-time in *fixed-time* optimal control problems (originally defined on $[0\,, \tau]$) to be 1 and reduce *free-time* optimal control problems to fixed-time optimal control problems on the interval $[0\,, 1]$, by replacing (2.2a) by *scaled* dynamics, with the scaling parameter denoted by τ. Thus, with each $\tilde{u} \in G(\tau)$, we associate a $u \in G(1)$ defined by $u(t) \triangleq \tilde{u}(\tau t)$ for $t \in [0\,, 1]$. With each $\tilde{z} \in C([0\,, \tau]\,, X)$, we associate $z \in C([0\,, 1]\,, X)$ defined by $z(t) \triangleq \tilde{z}(t\tau)$ for all $t \in [0\,, 1]$. Then, the function $z(t\,, u\,, \tau) \triangleq \tilde{z}(t\tau\,, \tilde{u})$ is a mild solution of the differential equation

$$\frac{d}{dt}z(t\,, u\,, \tau) = \frac{d}{dt}\tilde{z}(t\tau\,, \tilde{u}) = \tau[A\tilde{z}(t\tau\,, \tilde{u}) + F(\tilde{z}(t\tau)\,, \tilde{u}(t\tau))]$$

$$= \tau[Az(t\,, u\,, \tau) + F(z(t\,, u\,, \tau)\,, u(t))]. \tag{2.2d}$$

Hence we abuse notation and let $G = G(1)$, and we replace the original dynamics (2.2a), with the scaled dynamics:

$$\frac{d}{dt}z(t\,, u\,, \tau) = \tau[Az(t\,, u\,, \tau) + F(z(t\,, u\,, \tau)\,, u(t))]\,, \quad z(0\,, u) = z_0 \in D(A)\,,\ t \in [0\,, 1]\,. \tag{2.2e}$$

Note that for any final-time $\tau > 0$, the operator τA generates the semigroup $\{T(\tau t)\}_{t \ge 0}$ and hence $z(t\,, u\,, \tau)$ is a mild solution of (2.2e) if

$$z(t) = T(\tau t)z_0 + \tau\int_0^t T(\tau(t - s))F(z(s)\,, u(s))ds\,. \tag{2.2f}$$

Next, for $j = 0\,, 1\,, 2\,, \ldots, q$, let $f^j : X \to \mathbb{R}$ be functions that are Lipschitz continuously differentiable on bounded sets. Then, for $j = 0\,, 1\,, 2\,, \ldots, q$, we define the functions $g^j : G \times (0\,, \infty) \to \mathbb{R}$ by $g^j(u\,, \tau) \triangleq f^j(z(1\,, u\,, \tau))$. The simplest problem that we will consider is

---

[2] Failure to normalize may lead to pathological computational results, see [Cul.1, Cul.2].

$$\textbf{MMP}: \inf \{ \max_{j \in \mathbf{q}} g^j(u, \tau) \mid u \in G, \tau \in [\tau_{min}, \tau_{max}] \}, \tag{2.3a}$$

where $\mathbf{q} \triangleq \{1, 2, \ldots, q\}$, and $0 < \tau_{min} \le \tau_{max} < \infty$. Note that when $\tau_{max} = \tau_{min}$, (2.3a) is a fixed-time problem, otherwise it is a free-time problem. In minimum time problems, τ_{min} is chosen to be very small and τ_{max} is chosen to be large, which ensures that the optimal value of the final-time, $\hat{\tau}$, is the minimum time.

We will also show that algorithms for solving **MMP** are trivially adapted to solving optimal control problems with control and end point inequality constraints, of the form

$$\textbf{CMP}: \inf \{ g^0(u, \tau) \mid \max_{j \in \mathbf{q}} g^j(u, \tau) \le 0, u \in G, \tau \in [\tau_{min}, \tau_{max}] \}. \tag{2.3b}$$

Our next task is to establish optimality conditions for the problems **MMP** and **CMP**.

3. OPTIMALITY CONDITIONS.

We begin with a few standard assumptions.

Assumption 3.1.

(i) The operator $F(\cdot, \cdot)$ is Frechet differentiable. We will denote its partial Frechet derivatives, with respect to z and u, by $\frac{\partial F}{\partial z}(z, u)$ and $\frac{\partial F}{\partial u}(z, u)$, respectively.

(ii) For all $u \in G$, and $\tau \in [\tau_{min}, \tau_{max}]$, a solution to (2.2f) exists.

(iii) There exists $b_1 \in (0, \infty)$ such that for all $t \in [0, 1]$, $u \in L_2^m[0, 1]$, and $\tau \in [\tau_{min}, \tau_{max}]$, $\|z(t, u, \tau)\|_X \le b_1$.

(iv) For every bounded set $S \subset X$, there exists $K_S < \infty$ such that for all $z, z' \in S$ and all $u, u' \in \mathbb{R}m$,

(a) $\|F(z', u') - F(z, u)\|_X \le K_S[\|z' - z\|_X + \|u' - u\|]$,

(b) $\|\frac{\partial F}{\partial z}(z', u') - \frac{\partial F}{\partial z}(z, u)\| \le K_S[\|z' - z\|_X + \|u' - u\|]$,

(c) $\|\frac{\partial F}{\partial u}(z', u') - \frac{\partial F}{\partial u}(z, u)\| \le K_S[\|z' - z\|_X + \|u' - u\|]$.

(v) The functions $f^j(\cdot)$, $j = 0, 1, 2, \ldots, q$, are Frechet differentiable; their Frechet differentials have the form $Df(z; \delta z) = \langle \nabla f^j(z), \delta z \rangle_X$, and their *gradients*, $\nabla f^j(\cdot)$, are Lipschitz continuous on the set $\{ z \in X \mid \|z\|_X \le b_1 \}$. □

The following assumption is needed only if the scaling parameter τ is allowed to vary.

Assumption 3.2. The semigroup generated by A, $\{T(t)\}_{t \ge 0}$, is an analytic semigroup. □

The following two results can be gleaned from [Paz.1].

Lemma 3.3. The semigroup $\{T(t)\}_{t \geq 0}$ generated by the operator A is analytic if and only if there exists a constant $C < \infty$ such that *(i)* $T(t)$ is differentiable in $t > 0$; *(ii)* $\frac{d}{dt}T(t) = AT(t)$; and *(iii)* $\| AT(t) \|_X \leq \frac{C}{t}$, for all $t > 0$. □

Since Lemma 3.3 implies local Lipschitz continuity of $T(t)$ for $t > 0$, it follows from Assumption 3.1 and Lemma 3.3, that the following must be true.

Lemma 3.4. There exists a $b_2 \in (0, \infty)$, such that for all $z, z' \in S \triangleq \{ z \in X \mid \| z \|_X \leq b_1 \}$, all $u, u' \in L_2^m[0, 1]$, all $\tau, \tau' \in [\tau_{\min}, \tau_{\max}]$, and all $t \in [0, 1]$:

(i) $\| \frac{\partial F}{\partial z}(z, u) \| \leq b_2$,

(ii) $\| \frac{\partial F}{\partial u}(z, u) \| \leq b_2$,

(iii) $\| \frac{\partial F}{\partial z}(z', u') - \frac{\partial F}{\partial z}(z, u) \| \leq b_2 [\| z' - z \|_X + \| u' - u \|]$,

(iv) $\| \frac{\partial F}{\partial u}(z', u') - \frac{\partial F}{\partial z}(z, u) \| \leq b_2 [\| z' - z \|_X + \| u' - u \|]$,

(v) $\| T(\tau' t) - T(\tau t) \| \leq b_2 | \tau' - \tau |$. □

In view of Assumption 3.1 and Lemma 3.4, it can be concluded from the Implicit Function Theorem in Banach spaces, as stated in [Lan.1, Ale.1], that the solutions, $z(t, u, \tau)$, with $t \in [0, 1]$, of (2.2f) are Lipschitz continuously Frechet differentiable with respect to (u, τ) on bounded sets, with the Frechet differential, $Dz(t, u, \tau; \delta u, \delta\tau) = \delta z(t)$, where $\delta z(t)$ is the solution of the variational equation:

$$\delta z(t) = \int_0^t \Big\{ T(\tau(t-s))\tau\big(\frac{\partial F}{\partial z}(z(s, u, \tau), u(s))\delta z(s) + \frac{\partial F}{\partial u}(z(s), u(s))\delta u(s)\big)$$

$$+ (T(\tau(t-s)) + \tau(t-s)AT(\tau(t-s)))F(z(s, u, \tau), u(s))\delta\tau \Big\} ds + tAT(\tau t)z_0\delta\tau . \quad (3.1a)$$

We give an independent proof of this fact in the Appendix.

Since by Assumption 3.1*(v)*, the gradients of the functions $f^j(\cdot)$ are Lipschitz continuous on bounded sets, we immediately obtain the following result.

Theorem 3.5.

(i) The functions $g^j : L_2^m[0, 1] \times [\tau_{\min}, \tau_{\max}] \to \mathbb{R}$, $j = 0, 1, 2, \ldots, q$, defined in Section 2, are Frechet differentiable in (u, τ), i.e., for all $u \in G$, $\tau \in [\tau_{\min}, \tau_{\max}]$, there exists a continuous linear functional $Dg^j(u, \tau) : L_2^m[0, 1] \times \mathbb{R} \to \mathbb{R}$, such that for any $u, u' \in L_2^m[0, 1]$, $\tau, \tau' > \tau_{\min}$

$$\lim_{\substack{\|u' - u\|_2 \to 0 \\ |\tau' - \tau| \to 0}} \frac{| g^j(u', \tau') - g^j(u, \tau) - Dg^j(u, \tau)(u' - u, \tau' - \tau) |}{(\|u' - u\|_2^2 + |\tau' - \tau|^2)^{½}} = 0 . \tag{3.1b}$$

(ii) There exist *gradients* $\nabla g^j : L_2^m[0, 1] \times [\tau_{min}, \tau_{max}] \to L_2^m[0, 1] \times \mathbb{R}$, $j = 0, 1, 2, \ldots, q$, $\nabla g^j(u, \tau) = (\nabla_u g^j(u, \tau), \nabla_\tau g^j(u, \tau))$, such that for all $u', u \in L_2^m[0, 1]$, $\tau', \tau \in [\tau_{min}, \tau_{max}]$,

$$Dg^j(u, \tau)(u' - u, \tau' - \tau) = \langle \nabla_u g^j(u, \tau), u' - u \rangle_2 + \nabla_\tau g^j(u, \tau)(\tau' - \tau) . \tag{3.1b}$$

(iii) The gradients $\nabla g^j(\cdot, \cdot)$ are Lipschitz continuous on bounded sets. □

We are finally ready to address the question of optimality conditions for the problems (2.3a), (2.3b). Because of algorithmic requirements, we chose a multiplier-free form for the optimality conditions. It is not difficult to show that these conditions are equivalent to standard optimality conditions involving multipliers. Thus, for problem (2.3a) we define the max function $\psi : G \times [\tau_{min}, \tau_{max}] \to \mathbb{R}$ and the corresponding *optimality function* $\theta_{MMP} : G \times [\tau_{min}, \tau_{max}] \to \mathbb{R}$ by

$$\psi(u', \tau') \triangleq \max_{j \in \mathbf{q}} g^j(u', \tau') , \tag{3.2a}$$

$$\theta_{MMP}(u', \tau') \triangleq \min_{(u, \tau) \in G \times [\tau_{min}, \tau_{max}]} \Big\{ ½\|u - u'\|_2^2 + ½|\tau - \tau'|^2$$

$$+ \max_{j \in \mathbf{q}} \{ g^j(u', \tau') - \psi(u', \tau') + \langle \nabla_u g^j(u', \tau'), u - u' \rangle_2 + \nabla_\tau g^j(u', \tau')(\tau - \tau') \} \Big\} . \tag{3.2b}$$

Referring to Proposition 5.5 in [Pol.1], we see that $\theta_{MMP}(u, \tau)$ is the obvious extension of an optimality function used in conjunction with first order algorithms for the solution of minimax problems in $\mathbb{R}^n$. Hence it is a correct optimality function to use in analyzing the convergence properties of *implementable* minimax algorithms for solving (2.3a), since such algorithms must construct finite dimensional approximations to (2.3a).

Theorem 3.6.

(i) The function $\theta_{MMP}(\cdot, \cdot)$ is well defined and continuous.

(ii) If $h_u(u', \tau') \in G - \{u'\}$, $h_\tau(u', \tau') \in [\tau_{min}, \tau_{max}] - \{\tau'\}$ are such that $(u' + h_u(u', \tau'), \tau' + h_\tau(u', \tau'))$ is a solution to the minimization problem (3.2b), then $h_u(\cdot, \cdot) : G \times [\tau_{min}, \tau_{max}] \to L_2^m[0, 1]$, and $h_\tau(\cdot, \cdot) : G \times [\tau_{min}, \tau_{max}] \to \mathbb{R}$ are unique and continuous.

Proof. With $\Sigma_q \triangleq \{ \mu \in \mathbb{R}^q \mid \sum_{j=1}^q \mu^j = 1, \mu \geq 0 \}$, and making use of the Fan minimax theorem [Fan.1], we obtain that

$$\theta_{MMP}(u,\tau) = \min_{(u',\tau') \in G \times [\tau_{min},\tau_{max}]} \max_{\mu \in \Sigma_q} \Bigg\{ ½\|u' - u\|_2^2 + ½|\tau' - \tau|^2$$

$$+ \sum_{j \in q} \mu^j \{ g^j(u,\tau) - \psi(u,\tau) + \langle \nabla_u g^j(u,\tau), u' - u \rangle_2 + \nabla_\tau g^j(u,\tau)(\tau' - \tau) \} \Bigg\}$$

$$= \max_{\mu \in \Sigma_q} \min_{(u',\tau') \in G \times [\tau_{min},\tau_{max}]} \Bigg\{ ½\|u' - u\|_2^2 + ½\|\tau' - \tau\|^2$$

$$+ \sum_{j \in q} \mu^j \{ g^j(u,\tau) - \psi(u,\tau) + \langle \nabla_u g^j(u,\tau), u' - u \rangle_2 + \nabla_\tau g^j(u,\tau)(\tau' - \tau) \} \Bigg\} \quad (3.3a)$$

The minimization with respect to (u',τ') in (3.3a) is decoupled. The minimization with respect to τ' is a simple, one dimensional quadratic problem. Because

$$½\|u' - u\|_2^2 + \sum_{j \in q} \mu^j \langle \nabla_u g^j(u,\tau), u' - u \rangle_2$$

$$= \int_0^1 \Big[½\|u'(t) - u(t)\|^2 + \sum_{j \in q} \mu^j \langle \nabla_u g^j(u,\tau)(t), u'(t) - u(t) \rangle \Big] dt, \quad (3.3b)$$

the minimizing u' for (3.3a) can constructed by minimizing the integrand pointwise in t in (3.3b). Consequently, $\theta_{MMP}(u,\tau)$ is well defined. Continuity now follows from the Maximum Theorem in [Ber.1]. Similarly, since the solution $(h^u(\cdot,\cdot), h_\tau(\cdot,\cdot))$ of the minimization problem (3.2b) is unique, it again follows from the Maximum Theorem that it is continuous. □

Theorem 3.7. Suppose that $(\hat{u},\hat{\tau}) \in G \times [\tau_{min},\tau_{max}]$ is an optimal solution to the problem **MMP** (2.3a). Then $\theta_{MMP}(\hat{u},\hat{\tau}) = 0$.

Proof. First, note that $\theta_{MMP}(\hat{u},\hat{\tau}) \le 0$ must hold. Hence, for the sake of contradiction, suppose that $\theta_{MMP}(\hat{u},\hat{\tau}) < 0$ and that (u^*,τ^*) is the corresponding solution of the minimization problem (3.2b). Then, for $\lambda \in [0,1]$, we must have that $\hat{u} + \lambda(u^* - \hat{u}) \in G$, $\hat{\tau} + \lambda(\tau^* - \hat{\tau}) \in [\tau_{min},\tau_{max}]$, and

$$\psi(\hat{u} + \lambda(u^* - \hat{u}), \hat{\tau} + \lambda(\tau^* - \hat{\tau})) - \psi(\hat{u},\hat{\tau}) = \max_{j \in q} \{ g^j(\hat{u},\hat{\tau}) - \psi(\hat{u},\hat{\tau})$$

$$+ \lambda(\langle \nabla_u g^j(\hat{u},\hat{\tau}), u^* - \hat{u} \rangle_2 + \nabla_\tau g^j(\hat{u},\hat{\tau})(\tau^* - \hat{\tau})) \} + o(\lambda)$$

$$\le \lambda \Big\{ ½\|u^* - \hat{u}\|_2^2 + ½|\tau^* - \hat{\tau}|^2 + \max_{j \in q} \{ g^j(\hat{u},\hat{\tau}) - \psi(\hat{u},\hat{\tau})$$

$$+ \langle \nabla_u g^j(\hat{u}, \hat{\tau}), u^* - \hat{u} \rangle_2 + \nabla_\tau g^j(\hat{u}, \hat{\tau})(\tau^* - \hat{\tau}) \} + \frac{o(\lambda)}{\lambda} \Big\}$$

$$\leq \lambda \{ \theta_{MMP}(\hat{u}, \hat{\tau}) + o(\lambda)/\lambda \}, \tag{3.4}$$

where $o(\lambda)/\lambda \to 0$ as $\lambda \to 0$. Hence there exists a $\hat{\lambda} \in (0, 1]$ such that $\psi(\hat{u} + \hat{\lambda}(u^* - \hat{u}), \hat{\tau} + \hat{\lambda}(\tau^* - \hat{\tau})) < \psi(\hat{u}, \hat{\tau})$, which is a contradiction. □

Under a convexity assumption, the above optimality condition becomes a necessary and sufficient condition. An examination of our definition of the functions $g^j(\cdot, \cdot)$ shows that they cannot be convex for free time problems. However, in the case of linear dynamics and fixed end time, the problem can become convex.

We can easily obtain an optimality condition for problem **CMP** (2.3b) from the one for **MMP** (2.3a) by making use of the following observation. Suppose that $(\hat{u}, \hat{\tau})$ is an optimal pair for **CMP**. Let $\Psi : G \times [\tau_{min}, \tau_{max}] \to \mathbb{R}$ be defined by

$$\Psi(u, \tau) = \max_{j \in \mathbf{q}} \max \{ g^0(u, \tau) - g^0(\hat{u}, \hat{\tau}), g^j(u, \tau) \}. \tag{3.5}$$

Then $\Psi(\hat{u}, \hat{\tau}) = 0$ and, for any (u, τ) sufficiently close to $(\hat{u}, \hat{\tau})$, $\Psi(u, \tau) \geq 0$. Hence $(\hat{u}, \hat{\tau})$ is a local minimizer for the function $\Psi(\cdot, \cdot)$. Therefore, referring to (3.2a), (3.2b), we define the *optimality function* $\theta_{CMP} : G \times [\tau_{min}, \tau_{max}] \to \mathbb{R}$ by

$$\theta_{CMP}(u', \tau') \triangleq \min_{(u, \tau) \in G \times [\tau_{min}, \tau_{max}]} \Big\{ \tfrac{1}{2}\| u - u' \|_2^2 + \tfrac{1}{2} | \tau - \tau' |^2$$

$$+ \max \{ -\psi(u', \tau')_+ + \langle \nabla_u g^0(u', \tau'), u - u' \rangle_2 + \nabla_\tau g^0(u', \tau')(\tau - \tau'),$$

$$g^j(u, \tau) - \psi(u, \tau)_+ + \langle \nabla_u g^j(u, \tau), u' - u \rangle_2 + \nabla_\tau g^j(u, \tau)(\tau' - \tau), \ j \in \mathbf{q} \} \Big\}, \tag{3.6}$$

where $\psi(u, \tau)_+ \triangleq \max \{ 0, \psi(u, \tau) \}$. Although the term $\psi(u, \tau)_+$ has no effect at feasible points and hence also at optimal points, it is introduced into the optimality function for algorithmic reasons. The following result should be obvious.

Theorem 3.8.

(i) The optimality function $\theta_{CMP}(\cdot, \cdot)$ is well defined and continuous.

(ii) If $h_u(u, \tau) \in G - \{u\}$, $h_\tau(u, \tau) \in [\tau_{min}, \tau_{max}] - \{\tau\}$ are such that $(u + h_u(u, \tau), \tau + h_\tau(u, \tau))$ is a solution to the minimization problem (3.6), then $h_u(\cdot, \cdot)$, $h_\tau(\cdot, \cdot)$ are unique,continuous functions.

(iii) Suppose that $(\hat{u}, \hat{\tau}) \in G \times [\tau_{min}, \tau_{max}]$ is an optimal solution to the problem **CMP** (2.3b). Then $\theta_{CMP}(\hat{u}, \hat{\tau}) = 0$. □

It is customary to add a constraint qualification to optimization problems with inequality constraints. The analog of the Slater constraint qualification [Sla.1] commonly used in nonlinear programming for problem **CMP** is as follows:

Assumption 3.9. We will assume that for all $(u, \tau) \in G \times [\tau_{min}, \tau_{max}]$ such that $\psi(u, \tau) \geq 0$, $\theta_{MMP}(u, \tau) < 0$. □

Assumption 3.9 is standard in phase I - phase II methods of feasible directions. It implies that the constraint violation function $\psi(\cdot, \cdot)$ has no local minimizers outside of feasible set $\{ (u, \tau) \in G \times [\tau_{min}, \tau_{max}] \mid \psi(u, \tau) \leq 0 \}$, nor on the set $\{ (u, \tau) \in G \times [\tau_{min}, \tau_{max}] \mid \psi(u, \tau) = 0 \}$, a fact that prevents phase I - phase II feasible directions algorithms from converging to infeasible points. Finally, under Assumption 3.9 and a convexity assumption, $\theta_{CMP}(u, \tau) = 0$ becomes both a necessary and sufficient condition of optimality.

4. OPTIMALITY CONDITIONS IN THE SPACE OF RELAXED CONTROLS

Since the closed unit ball in $L_2^m[0, 1]$ is not compact, there may be bounded sequences $\{ (u_i, \tau_i) \}_{i=0}^{\infty}$, generated by an algorithm in solving the problem **MMP** or **CMP**, which have no accumulation points in $L_2^m[0, 1]$, even when these problems do have solutions. However, as was established in [Ahm.1, Pap.1], such sequences always have accumulation points in the space of relaxed controls. Hence, it is common to show that all the accumulation points generated by algorithms for solving optimal control problems such as **MMP** and **CMP**, satisfy both a first order optimality condition in $L_2^m[0, 1]$ and the extension of these first order conditions to first order conditions for relaxed controls versions of **MMP** and **CMP**.

In order to define relaxed control versions of the problems **MMP** and **CMP**, we follow Warga [War.1], by defining $\bar{G}$, the relaxed controls closure of the set G, as follows:

$$\bar{G} = \{ \sigma : [0, 1] \rightarrow rpm(U) \mid \sigma \text{ is measurable} \} , \tag{4.1a}$$

where $rpm(U)$ denotes the set of Radon probability measures, topologized as in Chap 4, in [War.1]. In this topology, a sequence $\{ \sigma_i \}_{i=0}^{\infty} \subset \bar{G}$ converges to a $\sigma \in \bar{G}$ if and only if

$$\lim_{i \to \infty} \int_0^1 \int_U \phi(t, u)\sigma_i(t)(du)dt = \int_0^1 \int_U \phi(t, u)\sigma(t)(du)dt \ , \ \forall \phi \in L_1([0, 1], C(U)) . \qquad (4.1b)$$

The set $\bar{G}$ is sequentially compact. From our point of view, the most useful concept of continuity on $\bar{G}$ is that of sequential continuity. Hence all of our continuity statements, for functions defined on $\bar{G}$, are to be understood as sequential continuity statements, e.g., when we say that a function $\bar{g} : \bar{G} \to \mathbb{R}$ is continuous, we mean that for any sequence of relaxed controls $\{ \sigma_i \}_{i=0}^{\infty} \subset \bar{G}$ that converges to a $\sigma \in \bar{G}$, $\bar{g}(\sigma_i) \to \bar{g}(\sigma)$, as $i \to \infty$.

Next, we extend the map $z : G \times [\tau_{\min}, \tau_{\max}] \to C([0, 1], X)$ to $\bar{z} : \bar{G} \times [\tau_{\min}, \tau_{\max}] \to C([0, 1], X)$ by defining for each $\sigma \in \bar{G}$, $\bar{z}(\cdot, \sigma, \tau)) \in C([0, 1], X)$ to be the solution to

$$z(t) = T(\tau t)z_0 + \tau \int_0^t \int_U T(\tau(t - s))F(z(s), u)\sigma(s)(du)ds \ . \qquad (4.2)$$

Assuming that Assumption 3.1 holds, it can be shown that a mild solution to (4.2) exists, that it is unique, and that it is bounded by b_1, the bound introduced in Assumption 3.1*(ii)*. The simplest relation between the solutions of (2.2f) and (4.2) is as follows.

Proposition 4.1. If $\sigma \in \bar{G}$ is an ordinary control, i.e., there exists $u \in G$ such that $\sigma(t)(S) = \delta_{u(t)}(S)$ for all measurable sets $S \subset U$ and almost all $t \in [0, 1]$, then $z(t, \sigma, \tau) = z(t, u, \tau)$ for all $t \in [0, 1]$, where $\bar{z}(\cdot, \sigma, \tau)$ is the solution to (4.2) and $z(\cdot, u, \tau)$ is the solution to (2.2f). □

The following result follows by simple extension of results in [Pap.1].

Theorem 4.2. (Continuity of $\bar{z}(\cdot, \sigma, \tau)$ in (σ, τ)) If the sequence $\{ (\sigma_i, \tau_i) \}_{i=1}^{\infty} \subset \bar{G} \times [\tau_{\min}, \tau_{\max}]$, as $i \to \infty$, is such that $\sigma_i \to \sigma \in \bar{G}$, $\tau_i \to \tau$, as $i \to \infty$, then $\bar{z}(\cdot, \sigma_i, \tau_i) \to \bar{z}(\cdot, \sigma, \tau)$, as $i \to \infty$. □

With these preliminaries out of the way, we are ready to define the relaxed control versions of the problems **MMP**, **CMP**, defined in (2.3a), (2.3b). Thus, for $j = 0, 1, 2, \ldots, q$, we define $\bar{g}^j : \bar{G} \times [\tau_{\min}, \tau_{\max}] \to \mathbb{R}$ by $\bar{g}^j(\sigma, \tau) \triangleq f^j(\bar{z}(1, \sigma, \tau))$, and

$$\overline{\textbf{MMP}} : \quad \min \{ \max_{j \in \mathbf{q}} \bar{g}^j(\sigma, \tau) \mid \sigma \in \bar{G} \ , \ \tau \in [\tau_{\min}, \tau_{\max}] \} \ , \qquad (4.3a)$$

$$\overline{\textbf{CMP}} : \quad \min \{ \bar{g}^0(\sigma, \tau) \mid \max_{j \in \mathbf{q}} \bar{g}^j(\sigma, \tau) \le 0 \ , \ \sigma \in \bar{G} \ , \ \tau \in [\tau_{\min}, \tau_{\max}] \} . \qquad (4.3b)$$

Next, we need to obtain extensions of the optimality functions $\theta_{\mathbf{MMP}}(\cdot, \cdot)$ and $\theta_{\mathbf{CMP}}(\cdot, \cdot)$ for the problems $\overline{\textbf{MMP}}$, $\overline{\textbf{CMP}}$, with the property that these extensions assume the same values on ordinary controls as the functions $\theta_{\mathbf{MMP}}(\cdot, \cdot)$ and $\theta_{\mathbf{CMP}}(\cdot, \cdot)$. On the surface, it is not at all clear how to obtain a relaxed control version of $\theta_{\mathbf{MMP}}(\cdot, \cdot)$ or of $\theta_{\mathbf{CMP}}(\cdot, \cdot)$. However, this task becomes a lot easier if we observe (see Theorem 3.4) that the solution $(u(u', \tau') - u', \tau(u', \tau') - \tau')$ of the search direction finding problem (3.2b) defines a pair of continuous functions $(h_u(\cdot, \cdot), h_\tau(\cdot, \cdot))$. Hence we see that (3.2b) is equivalent to

$$\theta_{MMP}(\hat{u}\,,\hat{\tau}) \triangleq \min_{(h_u\,,h_\tau)\in C(G\times[\tau_{min}\,,\tau_{max}]\,,(G-\hat{u})\times([\tau_{min}\,,\tau_{max}]-\hat{\tau}))} \Big\{ \tfrac{1}{2}\|h_u(\hat{u}\,,\hat{\tau})\|_2^2 + \tfrac{1}{2}|h_\tau(\hat{u}\,,\hat{\tau})|^2$$

$$+ \max_{j\in \mathbf{q}} \{ g^j(\hat{u}\,,\hat{\tau}) - \psi(\hat{u}\,,\hat{\tau}) + \langle \nabla_u g^j(\hat{u}\,,\hat{\tau})\,, h_u(\hat{u}\,,\hat{\tau})\rangle_2 + \nabla_\tau g^j(\hat{u}\,,\hat{\tau}) h_\tau(\hat{u}\,,\hat{\tau}) \} \Big\} \quad (4.5a)$$

It is now clear that to obtain a relaxed control version of $\theta_{MMP}(\cdot\,,\cdot)$ we must first obtain a relaxed control version of the directional derivatives $\langle \nabla_u g^j(u\,,\tau)\,, h_u(u\,,\tau)\rangle_2 + \nabla_\tau g^j(u\,,\tau)(\tau' - \tau)$. Now, referring to (A.3) we see that $\delta z(t\,,u\,,\tau\,,\delta u\,,\delta\tau)$ is linear in $(\delta u\,,\delta\tau)$, and hence can be written as $\delta z(t\,,u\,,\tau\,,\delta u\,,\delta\tau) = \delta z_u(t\,,u\,,\tau\,,\delta u) + \delta z_\tau(t\,,u\,,\tau\,,\delta\tau)$, where $\delta z_u(t\,,u\,,\tau\,,\delta u) = \delta z(t\,,u\,,\tau\,,\delta u\,,0)$ and $\delta z_\tau(t\,,u\,,\tau\,,\delta\tau) = \delta z(t\,,u\,,\tau\,,0\,,\delta\tau)$. Consequently,

$$\langle \nabla_u g^j(u\,,\tau)\,, h_u(u\,,\tau)\rangle_2 = \langle \nabla f^j(z(1\,,u\,,\tau))\,, \delta z_u(1\,,u\,,\tau\,,h_u(u\,,\tau))\rangle_X\,, \quad (4.5b)$$

and

$$\nabla_\tau g^j(u\,,\tau)(\tau' - \tau) = \langle \nabla f^j(z(1\,,u\,,\tau))\,, \delta z_\tau(1\,,u\,,\tau\,,\tau' - \tau)\rangle_X\,. \quad (4.5c)$$

Hence, the relaxed control versions of (4.5b), (4.5c) appear to be

$$\langle \nabla f^j(\bar{z}(1\,,\sigma\,,\tau)\,, \overline{\delta z}_u(1\,,\sigma\,,\tau\,,h_u)\rangle_X\,, \quad (4.6a)$$

$$\langle \nabla f^j(\bar{z}(1\,,\sigma\,,\tau)\,, \overline{\delta z}_\tau(1\,,\sigma\,,\tau\,,\tau' - \tau)\rangle_X\,, \quad (4.6b)$$

where, with $h_u \in C([0\,,1]\times U\times[\tau_{min}\,,\tau_{max}]\,,\mathbb{R}^m)$ (i.e., its domain has been changed), $\overline{\delta z}_u(\cdot\,,\sigma\,,\tau\,,h_u)$ is the solution to

$$\overline{\delta z}(t) = \tau\int_0^t T(\tau(t-s))\int_U \Big\{ \frac{\partial F}{\partial z}(\bar{z}(s\,,\sigma\,,\tau)\,,u)\delta z(s) + \frac{\partial F}{\partial u}(\bar{z}(s\,,\sigma\,,\tau)\,,u)h_u(s\,,u\,,\tau)) \Big\}\,\sigma(s)(du)ds\,, \quad (4.6c)$$

and $\overline{\delta z}_\tau(\cdot\,,\sigma\,,\tau\,,\tau' - \tau)$ is the solution to

$$\overline{\delta z}(t) = \int_0^t\int_U \Big\{ \tau T(\tau(t-s))\frac{\partial F}{\partial z}(\bar{z}(s\,,\sigma\,,\tau)\,,u)\delta z(s)$$

$$+ (T(\tau(t-s)) + \tau(t-s)AT(\tau(t-s)))F(\bar{z}(s\,,\sigma\,,\tau)\,,u)\delta\tau \Big\}\,\sigma(s)(du)ds + tAT(\tau t)z_0\delta\tau\,. \quad (4.6d)$$

We will now show[3] that for any $\sigma\in\bar{G}$, $\lambda\in[-1\,,1]$, and a class of *search direction functions*

[3] A similar development for ODE's can be found in [Wil.1].

$h_u \in C([0,1] \times U \times [\tau_{\min}, \tau_{\max}], \mathbb{R}^m)$, $\lambda \overline{\delta z}_u(\cdot, \sigma, \tau, h_u)$ is a first order approximation, in λ, to $\bar{z}(\cdot, \sigma, \tau, \lambda, h_u) - \bar{z}(\cdot, \sigma, \tau)$, where (with some abuse of notation) $\bar{z}(\cdot, \sigma, \tau, \lambda, h_u)$ is the solution to

$$z(t) = T(\tau t)z_0 + \tau \int_0^t \int_U T(\tau(t-s))F(z(s), v + \lambda h_u(s, v, \tau))\sigma(s)(dv)ds \,. \tag{4.7}$$

We note that (4.6c) is the first variation of (4.7) along the curve in $\bar{G}$ defined by $\{ \rho(\cdot\,; \lambda, h_u) \mid \lambda \in [0,1] \}$, where

$$\rho(t\,; \lambda, h_u))(S) \triangleq \{ \sigma(t)(R),\ R \triangleq \{ v \in U \mid v + \lambda h_u(t, v, \tau) \in S \} \} \tag{4.8}$$

if $v + h_u(t, v, \tau) \in U$ for all $v \in U$ and almost all $t \in [0,1]$, otherwise ρ is undefined. It is easily seen that if ρ is well defined, then $\rho \in \bar{G}$ and $\bar{z}(1, \rho(\cdot, \lambda, h_u), \tau) = \bar{z}(1, \sigma, \tau, \lambda, h_u)$. Hence we introduce the following definition.

Definition 4.3. The *search direction function* $h_u \in C([0,1] \times U \times [\tau_{\min}, \tau_{\max}], \mathbb{R}^m)$ will be said to be *admissible* if $u' + h_u(t, u', \tau') \in U$ for all $u' \in U$ and almost all $t \in [0,1]$ and $\tau \in [\tau_{\min}, \tau_{\max}]$. We will denote by Γ the set of admissible search direction functions. □

Lemma 4.4. There exists an $L < \infty$ such that for any $h_u \in \Gamma$, $\sigma \in \bar{G}$, $\tau \in [\tau_{\min}, \tau_{\max}]$, $t \in [0,1]$ and λ sufficiently small, $\|\bar{z}(t, \sigma, \tau, \lambda, h_u) - \bar{z}(t, \sigma, \tau)\| \le L|\lambda|$.

Proof. Let

$$M_U \triangleq \max \{ \|u' - u''\| \mid u', u'' \in U \}. \tag{4.9a}$$

Since U is compact, $M_U < \infty$. Clearly, for every $h_u \in \Gamma$, $\|h_u(t, u', \tau')\| \le M_U$ for all $t \in [0,1]$, $u' \in U$, and $\tau' \in [\tau_{\min}, \tau_{\max}]$. Hence

$$\begin{aligned}\|\bar{z}(t, \sigma, \tau, \lambda, h_u) - \bar{z}(t, \sigma, \tau)\|_X &= \Big\| \int_0^t \int_U \tau T(\tau(t-s))[F(\bar{z}(s, \sigma, \tau, \lambda, h_u), u + \lambda h_u(s, u, \tau)) \\ &\quad - F(\bar{z}(s, \sigma, \tau), u)]\sigma(s)(du)ds \Big\|_X \\ &\le \tau_{\max} M \int_0^t K_S[\|\bar{z}(s, \sigma, \tau, \lambda, h_u) - \bar{z}(s, \sigma, \tau)\| + |\lambda| M_U]\, ds \,,\end{aligned} \tag{4.9b}$$

where M is a bound on $\|T(\tau(t-s))\|$, $s \in [0, t]$, as also used in the Appendix. Applying the Bellman-Gronwall inequality, we obtain that

$$\|\bar{z}(t, \sigma, \tau, \lambda, h_u) - \bar{z}(t, \sigma, \tau)\| \le L\,|\lambda| \,, \tag{4.9c}$$

where $L \triangleq MK_S M_U e^{\tau_{\max} MK_S}$ and K_S is defined as in Assumption 3.1*(iv)*. □

Lemma 4.5. There exists $d_1 < \infty$ such that for all $t \in [0,1]$, $\sigma \in \bar{G}$, $\tau \in [\tau_{\min}, \tau_{\max}]$, $h_u \in \Gamma$ and $\lambda \in [-1, 1]$,

$$\|\bar{z}(t,\sigma,\tau,\lambda,h_u)-\bar{z}(t,\sigma,\tau)-\lambda\bar{\delta z}_u(t,\sigma,\tau,h_u)\|_X \le d_1|\lambda|^2 . \tag{4.10}$$

Proof. Let $\Delta\bar{z}(t,\sigma,\tau,\lambda,h_u) \triangleq \bar{z}(t,\sigma,\tau,\lambda,h_u)-\bar{z}(t,\sigma,\tau)$. Then, with M_U as in (4.9a),

$$\|\Delta\bar{z}(t,\sigma,\tau,\lambda,h_u)-\lambda\bar{\delta z}_u(t,\sigma,\tau,h_u)\|_X \le \Big\|\int_0^t\int_U \tau T(\tau(t-s))\Big[F(\bar{z}(s,\sigma,\tau,\lambda,h_u),u$$

$$+\lambda h_u(s,u))-F(\bar{z}(s,\sigma,\tau),u)-\frac{\partial F}{\partial z}(\bar{z}(s,\sigma,\tau),u)\delta\bar{z}(s,\sigma,\tau,\lambda,h_u)$$

$$-\frac{\partial F}{\partial u}(\bar{z}(s,\sigma,\tau),u)\cdot\lambda h_u(s,u)\Big]\sigma(s)(du)ds\Big\|_X$$

$$\le \tau_{\max}M\int_0^t\int_U\Bigg[\int_0^1\Big\|\frac{\partial F}{\partial z}(\bar{z}(s,\sigma,\tau,\lambda,h_u)+r\Delta\bar{z}(s,\sigma,\tau,\lambda,h^u),u+r\lambda h_u(s,u))$$

$$-\frac{\partial F}{\partial z}(\bar{z}(s,\sigma,\tau),u)\Big\|dr\,\|\Delta\bar{z}(s,\sigma,\tau,\lambda,h_u)\|_X$$

$$+\int_0^1\Big\|\frac{\partial F}{\partial u}(\bar{z}(s,\sigma,\tau,\lambda,h_u)+r\Delta\bar{z}(s,\sigma,\tau,\lambda,h_u),u+r\lambda h_u(s,u))$$

$$-\frac{\partial F}{\partial u}(\bar{z}(s,\sigma,\tau),u)\Big\|dr\,|\lambda|M_U$$

$$+\Big\|\frac{\partial F}{\partial z}(\bar{z}(s,\sigma,\tau),u)\Big\|\,\|\Delta\bar{z}(s,\sigma,\tau,\lambda,h_u)-\bar{\delta z}_u(s,\sigma,\tau,\lambda,h_u)\|_X\Bigg]\sigma(s)(du)ds$$

$$\le \tau_{\max}M\int_0^t\int_U [K_S(\|\Delta\bar{z}(s,\sigma,\tau,\lambda,h_u)\|_X+|\lambda|M_U)\|\Delta\bar{z}(s,\sigma,\tau,\lambda,h_u)\|_X$$

$$+K_S(\|\Delta\bar{z}(s,\sigma,\tau,\lambda,h_u)\|_X+|\lambda|M_U)|\lambda M_U$$

$$+b_2|\lambda|M_U\|\Delta\bar{z}(s,\sigma,\tau,\lambda,h_u)-\delta\bar{z}_u(s,\sigma,\tau,\lambda,h_u)\|_X]\sigma(s)(du)ds\ , \tag{4.11a}$$

where b_2 is defined in Lemma 3.4. Since by Lemma 4.4 $\|\Delta\bar{z}(s,\sigma,\tau,\lambda,h_u)\|_X \le L|\lambda|$, it follows from the Bellman-Gronwall inequality that

$$\|\Delta\bar{z}(t,\sigma,\tau,\lambda,h_u)-\lambda\bar{\delta z}_u(t,\sigma,\tau,h_u)\|_X \le MK_S e^{Mb_2}(\|\Delta\bar{z}(s,\sigma,\tau,\lambda,h_u)\|+|\lambda|M_U)^2$$

$$\le d_1|\lambda|^2\ , \tag{4.11b}$$

where $d_1 = MK_S e^{Mb_2}(L+M_U)^2$. □

Proceeding in a similar manner, we can also prove the following, somewhat simpler result:

Lemma 4.6. There exists a $d_2 < \infty$ such that for all $h_u \in \Gamma$, $t \in [0, 1]$, $\sigma \in \bar{G}$, $\lambda \in [-1, 1]$, $\tau', \tau \in [\tau_{min}, \tau_{max}]$,

$$\| \bar{z}(t, \sigma, \tau, \lambda, h_u) - \bar{z}(t, \sigma, \tau) - \lambda \overline{\delta z}_\tau(t, \sigma, \tau, \tau' - \tau) \|_X \le d_2 |\lambda|^2 . \tag{4.12}$$

□

In addition, it is fairly easy to establish the following result:

Lemma 4.7. For any $t \in [0, 1]$, $\sigma \in \bar{G}$, $\tau \in [\tau_{min}, \tau_{max}]$, admissible h_u, and $\tau' \in [\tau_{min}, \tau_{max}]$, let $\overline{\delta z}(t, \sigma, \tau, h_u, \tau' - \tau)$ denote the solution to

$$\delta z(t) = \int_0^t \int_U \tau T(\tau(t-s)) \left\{ \frac{\partial F}{\partial z}(\bar{z}(s, \sigma, \tau), u) \delta z(s) + \frac{\partial F}{\partial u}(\bar{z}(s, \sigma, \tau), u) h_u(s, u, \tau) \right\} \sigma(s)(du) ds$$

$$+ \int_0^t \left\{ (T(\tau(t-s)) + \tau(t-s) A T(\tau(t-s))) F(\bar{z}(s, \sigma, \tau), u) \delta\tau \right\} ds + t A T(\tau t) z_0 \delta\tau . \tag{4.13}$$

Then *(i)* $\overline{\delta z}(t, \sigma, \tau, h_u, \tau' - \tau) = \overline{\delta z}_u(t, \sigma, \tau, h_u) + \overline{\delta z}_\tau(t, \sigma, \tau, \tau' - \tau)$, and *(ii)* $\overline{\delta z}(t, \sigma, \tau, h_u, \tau' - \tau)$ is continuous in $(t, \sigma, \tau, h_u, \tau')$. □

We are now ready to extend the optimality conditions in Section 3 to the relaxed optimal control problems $\overline{\textbf{MMP}}$, $\overline{\textbf{CMP}}$. We define the max function $\bar{\psi} : \bar{G} \times [\tau_{min}, \tau_{max}] \to \mathbb{R}$ and the optimality function $\bar{\theta}_{\textbf{MMP}} : \bar{G} \times [\tau_{min}, \tau_{max}] \to \mathbb{R}$ by

$$\bar{\psi}(\sigma, \tau) \triangleq \max_{j \in \mathbf{q}} \bar{g}^j(\sigma, \tau) , \tag{4.14a}$$

$$\bar{\theta}_{\textbf{MMP}}(\sigma, \tau) \triangleq \min_{(w, \tau') \in \Gamma \times [\tau_{min}, \tau_{max}]} \Bigg\{ \tfrac{1}{2} \int_0^1 \int_U \| w(t, u, \tau) \|^2 \sigma(t)(du) dt + \tfrac{1}{2} |\tau' - \tau|^2$$

$$+ \max_{j \in \mathbf{q}} \{ \bar{g}^j(\sigma, \tau) - \bar{\psi}(\sigma, \tau) + \langle \nabla f^j(\bar{z}(1, \sigma, \tau)), \overline{\delta z}_u(1, \sigma, \tau, w) \rangle_X$$

$$+ \langle \nabla f^j(\bar{z}(1, \sigma, \tau)), \overline{\delta z}_\tau(1, \sigma, \tau, \tau' - \tau) \rangle_X \} \Bigg\} . \tag{4.14b}$$

Making use of Lemmas 4.6 and 4.7, we get immediately the following extension of Theorems 3.4 and 3.5:

Theorem 4.8. *(i)* The function $\bar{\theta}_{\textbf{MMP}}(\cdot, \cdot)$ is well defined and continuous. *(ii)* Suppose that $(\hat{\sigma}, \hat{\tau}) \in \bar{G} \times [\tau_{min}, \tau_{max}]$ is an optimal solution to the problem $\overline{\textbf{MMP}}$ (4.3a). Then $\bar{\theta}_{\textbf{MMP}}(\hat{\sigma}, \hat{\tau}) = 0$. □

Similarly, we can define an extension, $\bar{\theta}_{\textbf{CMP}} : \bar{G} \times [\tau_{min}, \tau_{max}] \to \mathbb{R}$ of the optimality function $\theta_{\textbf{CMP}}(\cdot, \cdot)$ as follows:

$$\bar{\theta}_{CMP}(\sigma, \tau) \triangleq \min_{(w, \tau') \in \Gamma \times [\tau_{min}, \tau_{max}]} \Big\{ \tfrac{1}{2}\int_0^1 \int_U \| w(t, u, \tau)\|^2 \sigma(t)(du)dt + \tfrac{1}{2}|\tau' - \tau|^2$$

$$+ \max \{ -\bar{\psi}(\sigma, \tau)_+ + \langle \nabla f^0(\bar{z}(1, \sigma, \tau), \overline{\delta z}_u(1, \sigma, \tau, w)\rangle_X + \langle \nabla f^0(\bar{z}(1, \sigma, \tau), \overline{\delta z}_\tau(1, \sigma, \tau, \tau' - \tau)\rangle_X ,$$

$$\bar{g}^j(\sigma, \tau) - \bar{\psi}(\sigma, \tau)_+ + \langle \nabla f^j(\bar{z}(1, \sigma, \tau), \overline{\delta z}_u(1, \sigma, \tau, w)\rangle_X$$

$$+ \langle \nabla f^j(\bar{z}(1, \sigma, \tau), \overline{\delta z}_\tau(1, \sigma, \tau, \tau' - \tau)\rangle_X , \ j \in \mathbf{q} \} \Big\} , \tag{4.15}$$

where $\bar{\psi}(\sigma, \tau)_+ \triangleq \max \{ 0, \bar{\psi}(\sigma, \tau) \}$. We can now state the obvious extension of Theorem 3.6.

Theorem 4.9. *(i)* The function $\bar{\theta}_{CMP}(\cdot, \cdot)$ is well defined and continuous. *(ii)* Suppose that $(\hat{\sigma}, \hat{\tau}) \in \bar{G} \times [\tau_{min}, \tau_{max}]$ is an optimal solution to the problem $\overline{\mathbf{CMP}}$ (4.3b). Then $\bar{\theta}_{CMP}(\hat{\sigma}, \hat{\tau}) = 0$. □

We conclude this section with a rather obvious result that is essential in the analysis of optimal control algorithms:

Theorem 4.10. Suppose that $\sigma^* \in \bar{G}$ is an ordinary control, i.e., there exists a $u^* \in G$ such that $\sigma^*(t)(S) = \delta_{u^*(t)}(S)$ for all measurable sets $S \subset U$ and almost all $t \in [0, 1]$. Then

(i) For any $t \in [0, 1]$, $h_u \in \Gamma$, $\tau', \tau \in [\tau_{min}, \tau_{max}]$, $\overline{\delta z}(t, \sigma^*, \tau, h_u, \tau' - \tau) = \delta z(t, u, \tau, \delta u, \delta\tau)$, where $\delta u(t) = h_u(t, u^*(t), \tau)$ and $\delta\tau = \tau' - \tau$.

(ii) $\bar{\theta}_{MMP}(\sigma^*, \tau) = \theta_{MMP}(u^*, \tau)$, and $\bar{\theta}_{CMP}(\sigma^*, \tau) = \theta_{CMP}(u^*, \tau)$. □

Thus we see from Theorem 4.10 that when σ^* is an ordinary control, the stationary points of (3.2b) and (3.6) are also the stationary points of (4.14b) and (4.15), respectively.

5. APPROXIMATION THEORY

The numerical solution of optimal control problems such as **MMP** and **CMP** is impossible without some sort of discretization of the evolution equation (2.2f). We will now develop a theory for discretization of these problems. This theory depends on the convergence of the finite element method and on error bounds, such as those to be found in [Fuj.1, Fuj.2 , Fuj.3].

The use of a numerical method in integrating the evolution system (2.2f) results in the replacement of the set of admissible controls G by G_n, a compact, convex, finite-dimensional subset of G, and of the original functions $g^j : G \times [\tau_{min}, \tau_{max}] \to \mathbb{R}$ by approximating functions $g_n^j : G_n \times [\tau_{min}, \tau_{max}] \to \mathbb{R}$, where $n \in \mathbb{Z}_+$ is a precision control parameter. Thus, the use of numerical integration results in the replacement of the original problems **MMP** and **CMP** by approximations.

Hence, to establish an approximation theory, for each $n \in \mathbb{Z}_+$, we define the discretized problems $\mathbf{MMP}_n$ and $\mathbf{CMP}_n$ by

$$\mathbf{MMP}_n : \min \{ \max_{j \in \mathbf{q}} g_n^j(u, \tau) \mid u \in G_n, \ \tau \in [\tau_{\min}, \tau_{\max}] \}, \tag{5.1a}$$

$$\mathbf{CMP}_n : \min \{ g_n^0(u, \tau) \mid \max_{j \in \mathbf{q}} g_n^j(u, \tau) \le 0, \ u \in G_n, \ \tau \in [\tau_{\min}, \tau_{\max}] \}. \tag{5.1b}$$

To ensure that the functions $g_n^j(\cdot, \cdot)$ inherit the continuity and differentiability properties of the functions $g^j(\cdot, \cdot)$, we make the following reasonable assumptions:

Assumption 5.1.

(i) For all $n \in \mathbb{Z}_+$, the functions $g_n^j : G_n \times [\tau_{\min}, \tau_{\max}] \to \mathbb{R}$ are continuous.

(ii) For all $n \in \mathbb{Z}_+$, $j = 0, 1, 2, \dots, q$, and each $(u, \tau) \in G_n \times [\tau_{\min}, \tau_{\max}]$, there exists a gradient $\nabla g_n^j(u, \tau) = (\nabla g_{u,n}^j(u, \tau), \nabla g_{\tau,n}^j(u, \tau)) \in L_2^m[0, 1] \times \mathbb{R}$, such that for all $u' \in G_n$, $\tau' \in [\tau_{\min}, \tau_{\max}]$,

$$\lim_{\substack{\|u' - u\|_2 \to 0 \\ |\tau' - \tau| \to 0}} \frac{| g_n^j(u' \tau') - g_n^j(u, \tau) - (\langle \nabla_u g_n^j(u), v - u \rangle_2 + \nabla_\tau g_n^j(u, \tau)(\tau' - \tau)) |}{(\|u' - u\|_2^2 + |\tau' - \tau|^2)^{1/2}} = 0. \tag{5.2a}$$

(iii) There exists a Lipschitz constant[4] $L \in (0, \infty)$, such that for all $n \in \mathbb{Z}_+$, $j = 0, 1, 2, \dots, q$, $u', u \in G_n, \tau, \tau' \in [\tau_{\min}, \tau_{\max}]$,

$$\|\nabla g_n^j(u', \tau') - \nabla g_n^j(u, \tau)\|_2 \le L (\|u' - u\|_2^2 + |\tau' - \tau|^2)^{1/2}. \tag{5.2b}$$

(iv) For all $n \in \mathbb{Z}_+, G_n \subset G_{n+1}$.

(v) The closure of $\bigcup_{n \in \mathbb{Z}_+} G_n$ is G.

(vi) (Uniform Approximation Property.) For all $\varepsilon > 0$, there exists n_ε such that for all $n \ge n_\varepsilon$, $j = 0, 1, 2, \dots, q$, all $u \in G_n$, and all $\tau \in [\tau_{\min}, \tau_{\max}]$,

(a) $$| g^j(u, \tau) - g_n^j(u, \tau) | \le \varepsilon, \tag{5.2c}$$

(b) $$\|\nabla g^j(u, \tau) - \nabla g_n^j(u, \tau)\|_2 \le \varepsilon. \tag{5.2d}$$

□

Usually, when continuous dynamical equations are replaced by discrete dynamic equations, the resulting solutions inherit the continuity and differentiability properties of the original solutions, and hence satisfy Assumption 5.1*(i)* - *(iii)*. Assumption 5.1*(vi)* is satisfied at any particular u for any

[4] The existence of such a Lipschitz constant is a consequence of Assumption 3.1*(iv,v)*.

dynamics on which the finite element method converges. Thus the only thing one must verify is that the approximation is uniform on the finite dimensional set, G_n, as assumed.

Referring to Proposition 5.5 in [Pol.1], we see that the following analogs of Theorems 3.11 and 3.12 must hold for the problems $\mathbf{MMP}_n$, $\mathbf{CMP}_n$.

Theorem 5.2. For $n \in \mathbb{Z}_+$, let $\psi_n : G_n \times [\tau_{\min}, \tau_{\max}] \to \mathbb{R}$ and the corresponding *optimality function* $\theta_{\mathbf{MMP}_n} : G_n \times [\tau_{\min}, \tau_{\max}] \to \mathbb{R}$ be defined by

$$\psi_n(u', \tau') \triangleq \max_{j \in \mathbf{q}} g_n^j(u', \tau'), \tag{5.3a}$$

$$\theta_{\mathbf{MMP}_n}(u', \tau') \triangleq \min_{(u, \tau) \in G_n \times [\tau_{\min}, \tau_{\max}]} \Big\{ \tfrac{1}{2}\|u - u'\|_2^2 + \tfrac{1}{2}|\tau - \tau'|^2$$

$$+ \max_{j \in \mathbf{q}} \{ g_n^j(u', \tau') - \psi_n(u', \tau') + \langle \nabla_u g_n^j(u', \tau'), u - u' \rangle + \nabla_\tau g_n^j(u', \tau')(\tau\tau') \} \Big\}. \tag{5.3b}$$

Then,

(i) The optimality function $\theta_{\mathbf{MMP}_n}(\cdot, \cdot)$ is well defined and continuous.

(ii) If $h_u(u', \tau') \in G_n - \{u'\}$, $h_\tau(u', \tau') \in [\tau_{\min}, \tau_{\max}] - \{\tau'\}$ are such that $(u' + h_u(u', \tau'), \tau' + h_\tau(u', \tau'))$ is a solution to (5.3b), then $h_u(\cdot, \cdot)$, $h_\tau(\cdot, \cdot)$ are continuous functions.

(iii) Suppose that $(u'_n, \tau'_n) \in G_n \times [\tau_{\min}, \tau_{\max}]$ is an optimal solution to the problem $\mathbf{MMP}_n$. Then $\theta_{\mathbf{MMP}_n}(u'_n, \tau'_n) = 0$. □

Theorem 5.3. For $n \in \mathbb{Z}_+$, let $\psi_n(u, \tau)_+ \triangleq \max\{0, \psi_n(u, \tau)\}$, and let the corresponding *optimality function* $\theta_{\mathbf{CMP}_n} : G_n \times [\tau_{\min}, \tau_{\max}] \to \mathbb{R}$ be defined by

$$\theta_{\mathbf{CMP}_n}(u, \tau) \triangleq \min_{(u', \tau') \in G_n \times [\tau_{\min}, \tau_{\max}]} \Big\{ \tfrac{1}{2}\|u' - u\|_2^2 + \tfrac{1}{2}|\tau' - \tau|^2$$

$$+ \max\{ -\psi_n(u, \tau)_+ + \langle \nabla_u g_n^0(u, \tau), u' - u \rangle + \nabla_\tau g_n^0(u, \tau)(\tau' - \tau),$$

$$g_n^j(u, \tau) - \psi(u, \tau)_+ + \langle \nabla_u g_n^j(u, \tau), u' - u \rangle + \nabla_\tau g_n^j(u, \tau)(\tau' - \tau), \ j \in \mathbf{q} \} \Big\}. \tag{5.4}$$

Then

(i) The optimality function $\theta_{\mathbf{CMP}}(\cdot, \cdot)$ is well defined and continuous.

(ii) If $h_u(u, \tau)$, $h_\tau(u, \tau)$ are such that $u + h_u(u, \tau) \in G_n$, $\tau + h_\tau(u, \tau) \in [\tau_{\min}, \tau_{\max}]$ are a solution to (5.4), then $h_u(\cdot, \cdot)$, $h_\tau(\cdot, \cdot)$ are continuous functions.

(iii) Suppose that $(\hat{u}_n, \hat{\tau}_n) \in G_n \times [\tau_{\min}, \tau_{\max}]$ is an optimal solution to the problem **CMP**. Then $\theta_{\mathbf{CMP}}(\hat{u}_n, \hat{\tau}_n) = 0$. □

To simplify notation, we define $H \triangleq G \times [\tau_{\min}, \tau_{\max}]$, $H_n \triangleq G_n \times [\tau_{\min}, \tau_{\max}]$, and $\eta = (u, \tau)$, and, for any $\xi = (\xi_u, \xi_\tau) \in H$ and $\eta = (u, \tau) \in H$, we define $\langle \xi, \eta \rangle_H \triangleq \langle \xi_u, u \rangle_2 + \xi_\tau \tau$, and $\|\eta\|_H \triangleq (\|u\|_2^2 + |\tau|^2)^{1/2}$. Next, for any $\eta', \eta \in H$, we define

$$\hat{\psi}(\eta' - \eta \mid \eta) \triangleq \max_{j \in \mathbf{q}} \{ g^j(\eta) + \langle \nabla g^j(\eta), \eta' - \eta \rangle_H \} + \tfrac{1}{2}\|\eta' - \eta\|_H^2 , \tag{5.5a}$$

Next, for any $n \in \mathbb{Z}_+$, $\eta', \eta \in H_n$, we define

$$\hat{\psi}_n(\eta' - \eta \mid \eta) \triangleq \max_{j \in \mathbf{q}} \{ g_n^j(\eta) + \langle \nabla g_n^j(\eta), \eta' - \eta \rangle_H \} + \tfrac{1}{2}\|\eta' - \eta\|_H^2 , \tag{5.5b}$$

With this notation, we have that

$$\theta_{\mathbf{MMP}}(\eta) = \min_{\eta' \in H} \max_{j \in \mathbf{q}} \hat{\psi}^j(\eta' - \eta \mid \eta) - \psi(\eta) , \tag{5.5c}$$

$$\theta_{\mathbf{MMP}_n}(\eta) = \min_{\eta' \in H_n} \max_{j \in \mathbf{q}} \hat{\psi}_n^j(\eta' - \eta \mid \eta) - \psi_n(\eta) . \tag{5.5d}$$

Lemma 5.4. There exists a constant $K_1 < \infty$ such that for every $\varepsilon > 0$, there exists n_ε such that for all $n \geq n_\varepsilon$, and all $\eta', \eta \in H_n$,

$$| \hat{\psi}_n(\eta' - \eta \mid \eta) - \hat{\psi}(\eta' - \eta \mid \eta) | \leq K_1 \varepsilon . \tag{5.6a}$$

Proof. It follows from Assumption 5.1 that there exists an $n_\varepsilon \in \mathbb{Z}_+$ such that for all $n \geq n_\varepsilon$,

$$\hat{\psi}_n(\eta' - \eta \mid \eta) - \hat{\psi}(\eta' - \eta \mid \eta) \leq \max_{j \in \mathbf{q}} \{ g_n^j(\eta) - g^j(\eta) + \langle \nabla g_n^j(\eta) - \nabla g^j(\eta), \eta' - \eta \rangle_H \}$$

$$\leq [1 + K_H]\varepsilon , \tag{5.6b}$$

where $K_H = \max \{ \|\eta' - \eta\|_H \mid \eta', \eta \in H \}$. Reversing the roles of $\hat{\psi}_n(\eta' - \eta \mid \eta)$ and $\hat{\psi}(\eta' - \eta \mid \eta)$ we get the desired result. □

Theorem 5.5. There exists a constant $K_2 < \infty$ such that for every $\varepsilon \in (0, 1]$, there exists n_ε such that for all $n \geq n_\varepsilon$, and all $\eta \in H_n$,

$$| \theta_{\mathbf{MMP}_n}(\eta) - \theta_{\mathbf{MMP}}(\eta) | \leq K_2 \varepsilon . \tag{5.7}$$

Proof. For any $\eta \in H_n$, let

$$\xi(\eta) \triangleq \arg \min_{\eta' \in H} \hat{\psi}(\eta' - \eta \mid \eta) , \tag{5.8a}$$

$$\xi_n(\eta) \triangleq \arg \min_{\eta' \in H_n} \hat{\psi}_n(\eta' - \eta \mid \eta)\,, \tag{5.8b}$$

$$\tilde{\xi}_n(\eta) \triangleq \arg \min_{\eta' \in H_n} \|\eta' - \xi(\eta)\|_H\,. \tag{5.8c}$$

Let $\varepsilon > 0$ be given and let $n_\varepsilon \in \mathbb{Z}_+$ be defined as in Assumption 5.1*(vi)*. In view of Assumption 5.1*(v)*, there exists an $n'_\varepsilon \in \mathbb{Z}_+$, with $n'_\varepsilon \geq n_\varepsilon$, such that for any $n \geq n'_\varepsilon$ and any $\eta \in H$ there exists an $\eta_n \in H_n$ such that $\|\eta - \eta_n\|_H \leq \varepsilon$. Hence we obtain

$$\theta_{\mathbf{MMP}}(\eta) \leq \hat{\psi}(\xi_n(\eta) - \eta \mid \eta) - \psi(\eta)$$

$$\leq [\hat{\psi}_n(\xi_n(\eta) - \eta \mid \eta) - \psi_n(\eta)] + [\psi_n(\eta) - \psi(\eta)] + K_1\varepsilon \leq \theta_{\mathbf{MMP}_n}(\eta) + [K_1 + 1]\varepsilon\,; \tag{5.9a}$$

$$\theta_{\mathbf{MMP}_n}(\eta) \leq \hat{\psi}_n(\tilde{\xi}_n(\eta) - \eta \mid \eta) - \psi_n(\eta)$$

$$\leq \hat{\psi}(\tilde{\xi}_n(\eta) - \eta \mid \eta) - \psi(\eta) + [\psi(\eta) - \psi_n(\eta)] + K_1\varepsilon$$

$$\leq \hat{\psi}(\xi(\eta) - \eta \mid \eta) - \psi(\eta) + K'\|\xi(\eta) - \tilde{\xi}_n(\eta)\|_H$$

$$+ \frac{1}{2} \mid \|\xi(\eta) - \eta\|_H - \|\tilde{\xi}_n(\eta) - \eta\|_H \mid \|\xi(\eta) - \tilde{\xi}_n(\eta)\|_H + (K_1 + 1)\varepsilon$$

$$\leq \theta_{\mathbf{MMP}}(\eta) + K_2\varepsilon\,, \tag{5.9b}$$

where $K_2 = 1 + K_1 + K' + K''$, with $K' = \sup_{\eta \in H} \max_{j \in \mathbf{q}} \|\nabla g^j(\eta)\|_H$ and $K'' = \frac{1}{2}\sup_{\eta,\eta' \in H} \|\eta' - \eta\|_H$. The desired result now follows. □

The proof of the following result for problem **CMP** is quite similar to the one above and hence is omitted.

Theorem 5.6. There exists a constant $K_3 < \infty$ such that for every $\varepsilon > 0$, there exists n_ε such that for all $n \geq n_\varepsilon$, and all $\eta \in H_n$,

$$|\theta_{\mathbf{CMP}_n}(\eta) - \theta_{\mathbf{CMP}}(\eta)| \leq K_3\varepsilon\,. \tag{5.10}$$

□

The problems **MMP** and **CMP** are finite dimensional and hence can be solved with arbitrary precision using a finite dimensional minimax or nonlinear programming algorithm, respectively, such as any one of the following [Pol.1, Pol.3]. The first question we must answer is whether doing that is useful, i.e., we must establish whether our discretizations are consistent in an appropriate sense. The following pair of theorems gives an affirmative answer.

Theorem 5.7.

(i) Suppose that $\{(\hat{u}_n, \hat{\tau}_n)\}_{n=1}^{\infty}$ is a sequence of optimal solutions to the sequence of problems $\mathbf{MMP}_n$. Let $I \subset \mathbb{Z}_+$ be such that $\hat{u}_n \xrightarrow{I} \hat{\sigma} \in \bar{G}$ (in the sense of control measures (i.s.c.m.)) and $\hat{\tau}_n \xrightarrow{I} \hat{\tau} \in [\tau_{\min}, \tau_{\max}]$, as $i \to \infty$, then $(\hat{\sigma}, \hat{\tau})$ is an optimal solution of $\overline{\mathbf{MMP}}$.

(ii) Suppose that $\{(u^*_n, \tau^*_n)\}_{n=1}^{\infty}$, with $u^*_n \in G_n$ and $\tau^*_n \in [\tau_{\min}, \tau_{\max}]$, is such that

$$\theta_{\mathbf{MMP}_n}(u^*_n, \tau^*_n) \geq -\frac{1}{n}. \tag{5.11}$$

Let $I \subset \mathbb{Z}_+$ be such that $u^*_n \xrightarrow{I} \sigma^* \in \bar{G}$ (i.s.c.m.) and $\tau^*_n \xrightarrow{I} \tau^* \in [\tau_{\min}, \tau_{\max}]$, as $n \to \infty$, then $\bar{\theta}_{\mathbf{MMP}}(\sigma^*, \tau^*) = 0$.

Proof. *(a)* For the sake of contradiction, suppose that $(\hat{\sigma}, \hat{\tau})$ is not an optimal solution of $\overline{\mathbf{MMP}}$. Then there exists a pair (σ^{**}, τ^{**}), with $\sigma^{**} \in \bar{G}$ and $\tau^{**} \in [\tau_{\min}, \tau_{\max}]$ such that $\bar{\psi}(\sigma^{**}, \tau^{**}) < \bar{\psi}(\hat{\sigma}, \hat{\tau})$. Since $\bar{\psi}(\cdot, \cdot)$ is continuous, and $\hat{u}_n \in G_n$ is an ordinary control, we must have that $\bar{\psi}(\hat{u}_n, \hat{\tau}_n) \xrightarrow{I} \bar{\psi}(\hat{\sigma}, \hat{\tau})$. Hence, because of Assumption 5.1*(vi)*, we must also have that $\psi_n(\hat{u}_n, \hat{\tau}_n) \xrightarrow{I} \bar{\psi}(\hat{\sigma}, \hat{\tau})$. Now, by Assumption 5.1*(v)*, there exists a sequence $\{u'_n\}_{n \in I}$ such that $u'_n \xrightarrow{I} \sigma^{**}$ (i.s.c.m.), as $i \to \infty$. Hence because $\bar{\psi}(\cdot, \cdot)$ is continuous and because of Assumption 5.1*(vi)*, $\psi_n(u'_n, \tau^{**}) \xrightarrow{I} \bar{\psi}(\sigma^{**}, \tau^{**})$ which, for n sufficiently high, contradicts the optimality of the pairs $(\hat{u}_n, \hat{\tau}_n)$.

(b) This part follows directly from the continuity of the function $\bar{\theta}(\cdot, \cdot)$ and Theorem 5.5. □

We get a similar result for problem **CMP**, which we state without proof.

Theorem 5.8.

(i) Suppose that $\{(\hat{u}_n, \hat{\tau}_n)\}_{n=1}^{\infty}$ is a sequence of optimal solutions to the sequence of problems $\mathbf{CMP}_n$. Let $I \subset \mathbb{Z}_+$ be such that $\hat{u}_n \xrightarrow{I} \hat{\sigma} \in \bar{G}$ (i.s.c.m.) and $\hat{\tau}_n \xrightarrow{I} \hat{\tau} \in [\tau_{\min}, \tau_{\max}]$, as $n \to \infty$, then $(\hat{\sigma}, \hat{\tau})$ is an optimal solution of $\overline{\mathbf{CMP}}$.

(ii) Suppose that $\{(u^*_n, \tau^*_n)\}_{n=1}^{\infty}$, with $u^*_n \in G_n$ and $\tau^*_n \in [\tau_{\min}, \tau_{\max}]$ is such that

$$\theta_{\mathbf{MMP}_n}(u^*_n, \tau^*_n) \geq -\frac{1}{n}. \tag{5.12a}$$

$$\psi_n(u^*_n, \tau^*_n) \leq \frac{1}{n}. \tag{5.12b}$$

Let $I \subset \mathbb{Z}_+$ be such that $u^*_n \xrightarrow{I} \sigma^* \in \bar{G}$ (i.s.c.m.) and $\tau^*_n \xrightarrow{I} \tau^* \in [\tau_{\min}, \tau_{\max}]$, as $i \to \infty$, then

$\bar{\theta}_{MMP}(\sigma^* , \tau^*) = 0$ and $\bar{\psi}(\sigma^* , \tau^*) \le 0$. □

The computational scheme represented by Theorems 5.7 and 5.8 can be implemented as follows. An algorithm is applied to problem $\mathbf{MMP}_n$ (or $\mathbf{CMP}_n$), producing a sequence of iterates $(u_{n,i} , \tau_{n,i})$, $i = 0 , 1 , 2 , \dots , i_n$ which is arrested when (5.11) (or (5.12a) and (5.12b)) is satisfied. Then a new sequence, $(u_{n+1,i} , \tau_{n+1,i})$, $i = 0 , 1 , 2 , \dots$, is started for problem $\mathbf{MMP}_{n+1}$ (or $\mathbf{CMP}_{n+1}$), with $(u_{n+1,0} , \tau_{n+1,0}) = (u_{n,i_n} , \tau_{n,i_n})$. The main disadvantage of this scheme is that Theorems 5.7 and 5.8 deal only with a special subsequence of all the iterates computed, rather than with the whole sequence.

We will now show that it is possible to generalize the *algorithm implementation scheme* in [Kle.1] so as to obtain algorithms for solving **MMP** and **CMP**, with the property that *any* accumulation point of the computed sequence of iterates satisfies our first order optimality conditions. However, this requires that we strengthen Assumption 5.1*(vi)*, as follows.

Assumption 5.10. There exists a constant $\hat{K} < \infty$ such that for all $n \in \mathbb{Z}_+$, $j = 0 , 1 , 2 , \dots , q$, all $u \in G_n$, and $\tau \in [\tau_{\min} , \tau_{\max}]$,

$$|g^j(u , \tau) - g_n^j(u , \tau)| \le \frac{\hat{K}}{2^n} . \tag{5.13}$$

□

Referring to [Kle.1], we see that Assumption 5.10 is satisfied when *ordinary* differential equations are integrated numerically by a method of order at least one. It is shown in Section 6.5 of [Bak.1], making use of the results in [Fuj.1, Fuj.2, Fuj.3, [Ode.1]], that, when the finite element method is implemented using linear elements and Newmark's β method is used with $\beta = 0$, Assumption 5.10 is satisfied by the example treated in Section 6. We believe that it will also hold for many other cases as well.

For problem **MMP** we will extend a variant of the Pironneau-Polak-Pshenichnyi minimax algorithm (see [Pir.1, Pol.1, Psh.1]), which can be used for solving $\mathbf{MMP}_n$. To simplify proofs, we will use an exact line search step size rule; however, the results to follow remain valid also with an Armijo type step size rule (see [Pol.1, Psh.1] for step size rule). To simplify exposition, we resume the notation $\eta = (u , \tau)$, $H_n = G_n \times [\tau_{\min} , \tau_{\max}]$.

Minimax Algorithm 5.11 (Solves MMP).

Parameter. $\gamma \in (0 , 1)$.

Data. n_0, $\eta_0 \in H_{n_0}$.

Step 0. Set $i = 0$, $n(0) = n_0$.

Step 1. Compute the search direction,

$$h_i = h_{n(i)}(\eta_i) \triangleq \arg \min_{\eta \in H_{n(i)}} \max_{j \in \mathbf{q}} \left\{ g^j_{n(i)}(\eta_i) + \langle \nabla g^j_{n(i)}(\eta_i), \eta - \eta_i \rangle_H + \tfrac{1}{2}\|\eta - \eta_i\|^2_H \right\} - \eta_i \quad (5.14a)$$

Step 2. Compute the step size

$$\lambda_i \in \lambda_{n(i)}(\eta_i) \triangleq \arg \min_{\lambda \in [0,1]} \psi_{n(i)}(\eta_i + \lambda h_i) . \quad (5.14b)$$

Step 3. Set $\eta^* = \eta_i + \lambda_i h_i$.

If

$$\psi_{n(i)}(\eta^*) - \psi_{n(i)}(\eta_i) > -\frac{1}{2^{\gamma n(i)}} , \quad (5.14c)$$

replace $n(i)$ by $n(i) + 1$ and go to Step 1.

Else set $n(i+1) = n(i)$, $\eta_{i+1} = \eta_i + \lambda_i h_i$.

Step 4. replace i by $i + 1$ and go to Step 1. □

Note that (5.14c) causes the algorithm to increase precision when the decrease in cost becomes "unacceptably" small.

Theorem 5.12 Suppose that Algorithm 5.11 constructs a sequence $\{ \eta_i \}_{i=0}^{\infty}$. Then this sequence has accumulation points in $\bar{H} \triangleq \bar{G} \times [\tau_{\min}, \tau_{\max}]$, and every such accumulation point, $\hat{\eta}$, satisfies $\bar{\theta}_{MMP}(\hat{\eta}) = 0$.

Proof. First we note that since $\bar{H}$ is sequentially compact, the sequence $\{ \eta_i \}_{i=0}^{\infty}$ must have accumulation points in the relaxed controls topology. The rest of our proof is in three parts: *(a)* we will show that $n(i) \to \infty$ as $i \to \infty$, then *(b)* we will show that for any $\eta^* = (\sigma^*, \tau^*) \in \bar{H}$ such that $\bar{\theta}_{MMP}(\eta^*) < 0$, there exists an integer n^* and a $\delta^* > 0$, such that for all $n(i) \ge n^*$, if $\eta_i \in H_{n(i)}$ is sufficiently close to η^*, then $\psi_{n(i)}(\eta_{i+1}) - \psi_{n(i)}(\eta_i) \le -\delta^*$, and, *(c)* we will obtain a contradiction by showing that if the theorem is not true, then $\psi(\eta_i) \to -\infty$.

(a) Suppose that there exists integers i_0 and n_0 such that for all $i \ge i_0$, $n(i) = n_0$. Then we must have that $\psi_{n_0}(\eta_{i+1}) - \psi_{n_0}(\eta_i) \le -1/2^{\gamma n_0}$ for all $i \ge i_0$, which implies that $\psi_{n_0}(\eta_i) \to -\infty$ as $i \to \infty$. Since H_{n_0} is compact, this is impossible, and hence we conclude that $n(i) \to \infty$ as $i \to \infty$.

(b) Suppose that for $n \in \mathbb{Z}_+$, $\eta_n \in H_n$ and that $\eta_n \xrightarrow{i.s.c.m.} \eta^* \in \bar{H}$, as $n \to \infty$. Furthermore, suppose that $\bar{\theta}_{MMP}(\eta^*) \triangleq -8\delta^* < 0$. Since $\bar{\theta}_{MMP}(\cdot)$ is continuous, and since $\theta_{MMP}(\eta_n) = \bar{\theta}_{MMP}(\eta_n)$ for all $n \in \mathbb{Z}_+$, it follows that there is an integer n_0 such that for all $n \in \mathbb{Z}_+$, $n \ge n_0$, $\theta_{MMP}(\eta_n) \le -4\delta^*$. It now follows from Theorem 5.5 that there exists an integer $n_1 \ge n_0$ such that for all $n \in \mathbb{Z}_+$, $n \ge n_1$, $\theta_{MMP_n}(\eta_n) \le -2\delta^*$. Hence, for all $n \ge n_1$ and $\lambda \in [0, 1]$,

$$\psi_n(\eta_n + \lambda h_n(\eta_n)) - \psi_n(\eta_n) = \max_{j \in \mathbf{q}} \Big\{ g_n^j(\eta_n) - \psi_n(\eta_n) + \lambda \langle \nabla g_n^j(\eta_n), h_n(\eta_n) \rangle_H + \tfrac{1}{2} \| h_n(\eta_n) \|_H^2$$

$$+ \lambda \int_0^1 \langle \nabla g_n^j(\eta_n + s\lambda h_n(\eta_n)) - \nabla g_n^j(\eta_n), h_n(\eta_n) \rangle_H - \tfrac{1}{2} \| h_n(\eta_n) \|_H^2 \Big\}$$

$$\leq \lambda [\theta_{\mathrm{MMP}_n}(\eta_n) + \lambda L \| h_n(\eta_n) \|_H^2] \ , \tag{5.15a}$$

where L is as in (5.2b). Since the sets G_n are uniformly bounded, there exists a $b < \infty$ such that $\| h_n(\eta_i) \|_H \leq b$ for all $n \in \mathbb{Z}_+$. Hence it follows from (5.15a) that there exists a $\hat{\lambda} \in (0, 1]$, such that for all $n \geq n_1$,

$$\psi_n(\eta_n + \lambda_n(\eta_n) h_n(\eta_n)) - \psi_n(\eta_n) \leq \psi_n(\eta_n + \hat{\lambda} h_n(\eta_n)) - \psi_n(\eta_n) \leq -\hat{\lambda}\delta^* \ , \tag{5.15b}$$

which completes the second part of our proof.

(c) Now, by construction, we have that $\psi_{n(i)}(\eta_{i+1}) - \psi_{n(i)}(\eta_i) \leq -1/2^{\gamma n(i)}$, and hence, making use of Assumption 5.10,

$$\psi(\eta_{i+1}) - \psi(\eta_i) \leq -\frac{1}{2^{n(i)}}(2^{(1-\gamma)n(i)} - \hat{K}) \, . \tag{5.15c}$$

Hence, since $n(i) \to \infty$ as $i \to \infty$, there exists an i_o such that for all $i \geq i_o$, $\psi(\eta_{i+1}) - \psi(\eta_i) \leq 0$.

Now, for the sake of contradiction, suppose that the sequence $\{ \eta_i \}_{i=0}^{\infty}$ has an accumulation point $\hat{\eta} \in \bar{H}$ such that $\bar{\theta}_{\mathrm{MMP}}(\hat{\eta}) < 0$. Then there exists an infinite subset I of the positive integers such that $\eta_i \xrightarrow{I} \hat{\eta}$ (i.s.c.m.) as $i \to \infty$, and hence because $\bar{\psi}(\cdot)$ is continuous and $\psi(\eta_i) = \bar{\psi}(\eta_i)$, $\psi(\eta_i) \xrightarrow{I} \hat{\eta}$ as $i \to \infty$. Now, $\{ \psi(\eta_i) \}_{i=i_0}^{\infty}$ is monotone decreasing, and hence we conclude that $\psi(\eta_i) \to \bar{\psi}(\hat{\eta})$ as $i \to \infty$. Since $n(i) \to \infty$, it follows from *(b)* that there exist a $\hat{\delta} > 0$ and an integer i_1, such that for all $i \geq i_1, i \in I$, $\psi_{n(i)}(\eta_{i+1}) - \psi_{n(i)}(\eta_i) \leq -\hat{\delta} < 0$. Hence, for all $i \in I$,

$$\psi(\eta_{i+1}) - \psi_{n(i)}(\eta_i) \leq -\hat{\delta} + \frac{\hat{K}}{2^{n(i)}} \, . \tag{5.15d}$$

Since $n(i) \to \infty$ as $i \to \infty$, (5.15d) contradicts the convergence of the sequence $\{ \psi(\eta_i) \}_{i=0}^{\infty}$. This completes our proof. □

Two observations are in order at this point. First, it follows from (5.15c) that the cost sequence is eventually monotone decreasing. Since it is bounded, it must converge. Second, it can be deduced from the above proof that $\theta_{MMP_{n(i)}}(\eta_i) \to 0$, which implies in turn that $h_{n(i)} \to 0$. Hence, refering to Theorem

1.3.66 in [Pol.3], we conclude that if $\psi(\cdot)$ has only a finite number of stationary points, then the sequence of trajectories $\{ x^{\eta_i} \}$ must converge. Furthermore, if $\{ \eta_i \}$ has an accumulation point in the H topology, then the entire sequence $\{ \eta_i \}$ must converge to that point.

For problem **CMP**, we will extend the unified phase I - phase II method of feasible directions, using an Armijo step size rule, described in [Pol.2].

Algorithm 5.14

Parameters. $\gamma > 0, \alpha, \beta \in (0, 1)$.

Data . $n_0, \eta_0 \in H_{n_0}$.

Step 0. Set $i = 0$, $n(0) = n_0$.

Step 1. Compute the value of the *optimality function* $\theta_i = \theta_{\mathbf{CMP}_{n(i)}}(\eta_i)$, and the corresponding *search direction* $h_i = h_{n(i)}(\eta_i)$, where

$$\theta_{\mathbf{CMP}_{n(i)}}(\eta_i) \triangleq \min_{\eta \in H_{n(i)}} \Big\{ \tfrac{1}{2}\|\eta - \eta_i\|_H^2 + \max\{ -\psi_{n(i)}(\eta_i)_+ + \langle \nabla g_{n(i)}^0(\eta_i), \eta - \eta_i \rangle_H ,$$

$$g_{n(i)}^j(\eta_i) - \psi(\eta_i)_+ + \langle \nabla g_n^j(\eta_i), \eta - \eta_i \rangle_H , \ j \in \mathbf{q} \} \Big\} . \tag{5.16a}$$

$$h_{n(i)}(\eta_i) \triangleq \arg \min_{\eta \in H_{n(i)}} \Big\{ \tfrac{1}{2}\|\eta - \eta_i\|_H^2 + \max\{ -\psi_{n(i)}(\eta_i)_+ + \langle \nabla g_{n(i)}^0(\eta_i), \eta - \eta_i \rangle_H ,$$

$$g_{n(i)}^j(\eta_i) - \psi(\eta_i)_+ + \langle \nabla g_n^j(\eta_i), \eta - \eta_i \rangle_H , \ j \in \mathbf{q} \} \Big\} - \eta_i . \tag{5.16b}$$

Step 2. Compute the *step size* λ_i:

$$\lambda_i = \max\{ \beta^k \mid k \in \mathbf{N},\ F_{n(i)}(\eta_i + \beta^k h_i \mid \eta_i) \le \beta^k \alpha \theta_i \} , \tag{5.16c}$$

where, for $n \in \mathbb{Z}_+$, $\eta, \eta^* \in H_n$,

$$F_n(\eta \mid \eta^*) \triangleq \max\{ g_n^0(\eta) - g_n^0(\eta^*) - \psi_n(\eta^*)_+ , \psi_n(\eta) - \psi_n(\eta^*)_+ \} . \tag{5.16d}$$

Step 3. Set $\eta^* = \eta_i + \lambda_i h_i$.

If

$$F_{n(i)}(\eta^* \mid \eta_i) > -\frac{1}{2^{\gamma n(i)}} \tag{5.16e}$$

Replace $n(i)$ by $n(i) + 1$, and go to Step 1.

Else set $n(i+1) = n(i)$, $\eta_{i+1} = \eta_i + \lambda_i h_i$.

Step 4. Replace i by $i+1$ and go to Step 1. □

Theorem 5.15 Suppose that *(i)* for every $\eta \in \bar{H}$ such that $\bar{\psi}(\eta) \geq 0$, $\bar{\theta}_{MMP}(\eta) < 0$; and *(ii)* for every $n \in \mathbb{Z}_+$ and every $\eta \in H_n$ such that $\psi_n(\eta) \geq 0$, $\theta_{MMP_n}(\eta) < 0$. If Algorithm 5.14 constructs a sequence $\{ \eta_i \}_{i=0}^{\infty}$, then this sequence has accumulation points in $\bar{H} \triangleq \bar{G} \times [\tau_{min}, \tau_{max}]$, and every such accumulation point, $\hat{\eta}$, satisfies $\bar{\psi}(\hat{\eta}) \leq 0$, $\bar{\theta}_{CMP}(\hat{\eta}) = 0$.

Proof. First we note that since $\bar{H}$ is sequentially compact, the sequence $\{ \eta_i \}_{i=0}^{\infty}$ must have accumulation points in the relaxed controls topology. The rest of our proof is in three parts: *(a)* we will show that $n(i) \to \infty$ as $i \to \infty$, then *(b)* we will show that for any $\eta^* = (\sigma^*, \tau^*) \in \bar{H}$ such that $\bar{\theta}_{CMP}(\eta^*) < 0$, there exists an integer n^* and a $\delta^* > 0$, such that for all $n(i) \geq n^*$, and any η_i sufficiently close to η^*, $F_{n(i)}(\eta_{i+1} \mid \eta_i) \leq -\delta^*$, and *(c)* we will obtain a contradiction by showing that if the theorem is not true, then either $\psi(\eta_i) \to -\infty$ as $i \to \infty$ or $g^0(\eta_i) \to -\infty$ as $i \to \infty$.

(a) Suppose that there is a finite integer n^* such that $n(i) = n^*$ for all $i \geq i^*$, with $i^* < \infty$. Then the test (5.16e) fails to be satisfied for all $i \geq i^*$, and hence $F_{n^*}(\eta_{i+1} \mid \eta_i) \leq -(1/2^{n^*})^{\gamma}$ for all $i \geq i^*$. Without loss of generality, suppose that $\psi_{n^*}(\eta_{i^*}) \geq 0$ and that $\psi_{n^*}(\eta_i) > 1/n^*$ for all $i \geq i^*$. Then $\psi_{n^*}(\eta_{i+1}) - \psi_{n^*}(\eta_i)_+ \leq -(1/2^{n^*})^{\gamma}$ for all $i \geq i^*$, and hence there must exist an i_0 such that $\psi_{n^*}(\eta_i) \leq 0$ for all $i \geq i_0$. Furthermore, for $i \geq i_0$, we must also have that $g^0_{n^*}(\eta_{i+1}) - g^0_{n^*}(\eta_i) \leq -(1/2^{n^*})^{\gamma}$, which implies that $g^0_{n^*}(\eta_i) \to -\infty$ as $i \to \infty$. However, since H_n is compact and $g^0_{n^*}(\cdot)$ is continuous, this is clearly impossible, and we have a contradiction. Hence we must have that $n(i) \to \infty$ as $i \to \infty$.

(b) The proof of this part is quite similar to that of part *(b)* in the proof of Theorem 5.13, and is therefore omitted.

(c) For any η, $\eta^* \in H$, let $F_n(\eta \mid \eta^*)$ be defined by

$$F(\eta \mid \eta^*) \triangleq \max \{ g^0(\eta) - g^0(\eta^*) - \psi(\eta^*)_+ , \psi(\eta) - \psi(\eta^*)_+ \} . \tag{5.17a}$$

Then, because of the test (5.16e) and Assumption 5.10, we have that for all $i \in \mathbb{Z}_+$,

$$F(\eta_{i+1} \mid \eta_i) \leq -\frac{1}{2^{n(i)}} (2^{(1-\gamma)n(i)} - 2\hat{K}) . \tag{5.17b}$$

Since $\gamma \in (0, 1)$ and $n(i) \to \infty$ as $i \to \infty$, it follows that there is an $i_1 \in \mathbb{Z}_+$ such that

$$F(\eta_{i+1} \mid \eta_i) \leq 0 , \quad \forall i \geq i_1 , \tag{5.17c}$$

and hence, for all $i \geq i_1$,

$$\psi(\eta_{i+1}) - \psi(\eta_i)_+ \leq 0 , \tag{5.17d}$$

and

$$g^0(\eta_{i+1}) - g^0(\eta_i) - \psi(\eta_i)_+ \leq 0 \,. \tag{5.17e}$$

Now suppose that $\eta_i \overset{I}{\to} \hat{\eta} \in \bar{H}$ (i.s.c.m.) as $i \to \infty$ and that $\bar{\theta}(\hat{\eta}) < 0$. We distinguish between two possibilities:

(i) $\psi(\eta_i) > 0$ for all $i \geq i_1$. Then, by (5.17d), $\{ \psi(\eta_i) \}_{i=i_1}^{\infty}$ is a monotone decreasing sequence, and, since by continuity $\psi(\eta_i) \overset{I}{\to} \bar{\psi}(\hat{\eta})$ as $i \to \infty$, it follows that $\psi(\eta_i) \to \bar{\psi}(\hat{\eta})$ as $i \to \infty$. It now follows from *(b)* and Assumption 5.10 that there exist a $\hat{\delta}$ and an $i_2 \geq i_1$ such that for all $i \in I, i \geq i_2$,

$$\psi(\eta_{i+1}) - \psi(\eta_i) \leq F_{n(i)}(\eta_{i+1} \mid \eta_i) + \frac{2\hat{K}}{2^{n(i)}} \leq -\hat{\delta} + \frac{2\hat{K}}{2^{n(i)}} \,, \tag{5.17f}$$

which contradicts the fact that $\psi(\eta_i) \to \bar{\psi}(\hat{\eta})$ as $i \to \infty$. Hence we must have that $\bar{\theta}(\hat{\eta}) = 0$, and hence, by assumption, that $\bar{\psi}(\hat{\eta}) \leq 0$ also holds.

(ii) There exists an $i_3 \geq i_1$ such that $\psi(\eta_{i_3}) \leq 0$. then it follows for (5.17d) that $\psi(\eta_i) \leq 0$ for all $i \geq i_3$. Next, by (5.17e), $\{ g^0(\eta_i) \}_{i=i_3}^{\infty}$ is a monotone decreasing sequence, and, since by continuity $g^0(\eta_i) \overset{I}{\to} \bar{g}^0(\hat{\eta})$ as $i \to \infty$, it follows that $g^0(\eta_i) \to \bar{g}^0(\eta^*)$ as $i \to \infty$, (i.s.c.m.). It now follows again from *(b)* and Assumption 5.10 that there exists an $i_4 \geq i_1$ such that for all $i \in I, i \geq i_4$,

$$g^0(\eta_{i+1}) - g^0(\eta_i) \leq F_{n(i)}(\eta_{i+1} \mid \eta_i) \leq -\hat{\delta} + \frac{2\hat{K}}{2^{n(i)}} \leq -\hat{\delta}/2 \,, \tag{5.17g}$$

which contradicts the fact that $g^0(\eta_i) \to \bar{g}^0(\eta^*)$ as $i \to \infty$. Hence we must have that $\bar{\theta}(\eta^*) = 0$, which completes our proof. □

Again we can make some observations. First, it follows from (5.17f) that if the tail of the sequence $\{ \eta_i \}$ is infeasible, then the constraint violation function $\psi(\cdot)$ eventually decreases monotonically to zero. In this case, making use of (5.17b), one can conclude that either the cost sequence $\{ g^0(\eta_i) \}$ converges, or it has infinitely many accumulation points, a rather unlikely event. If the tail of the sequence $\{ \eta_i \}$ is feasible, then the the tail of the cost sequence is monotone decreasing, and hence, since it is bounded, it converges. Second, it can be deduced from the above proof that $\theta_{CMP_{n(i)}}(\eta_i) \to 0$, which implies in turn that $h_{n(i)} \to 0$. Hence, refering to Theorem 1.3.66 in [Pol.3], we conclude that if $\bar{\theta}_{CMP}(\cdot)$ has only a finite number of zeros, then the trajectory sequence $\{ x^{\eta_i} \}$ must converge. Furthermore, if $\{ \eta_i \}$ has an accumulation point in the H topology, then the entire sequence $\{ \eta_i \}$ must converge to that point.

6. COMPUTATIONAL RESULTS

We carried out three computational experiments involving the slewing motion of the hollow aluminum tube depicted in Figure 1. The tube is one meter long, has a cross sectional radius of 1.0 cm, and a thickness of 1.6 mm. Attached to one end of the tube is a mass of 1 kg, and attached to the other end is a shaft connected to a motor. To reduce the computational burden, we neglected small nonlinear terms, the coupling between the flexural and extensional vibrations, and assumed that the acceleration can be controlled, instead of assigning a mass to the shaft and assuming that the torque is controlled. These simplifications lead to a model in the form of the standard Euler-Bernoulli tube with Kelvin-Voigt visco-elastic damping:

$$m w_{tt}(t, x) + CI w_{txxxx}(t, x) + EI w_{xxxx}(t, x) - m\Omega^2(t) w(t, x) = -\frac{m}{1 + m/3} u(t) x \,,$$

$$t \in [0, \tau], \quad x \in [0, 1], \tag{6.1a}$$

with boundary conditions:

$$w(t, 0) = 0, \quad w_x(t, 0) = 0, \quad CI w_{txx}(t, 1) + EI w_{xx}(t, 1) = 0, \quad t \in [0, \tau], \tag{6.1b}$$

$$\Omega^2(t) w(t, 1) - w_{tt}(t, 1) - u(t) - CI w_{txxx}(t, 1) - EI w_{xxx}(t, 1) = 0, \quad t \in [0, \tau], \tag{6.1c}$$

$$\Theta_t(t) = \Omega(t), \quad t \in [0, \tau], \qquad \Omega_t(t) = u(t), \quad t \in [0, \tau], \tag{6.1d}$$

where $w(t, x)$ is the displacement of the tube from the *shadow tube* (which remains undeformed during the motion) due to bending as a function of time and distance along the tube; $u(t)$ is the acceleration produced by the motor, and $\Omega(t)$ is the resulting angular velocity (in radians per second), and $\Theta(t)$ is the angular displacement of the rigid body (in radians). The values for the parameters in (6.1a) - (6.1c) were chosen to be $m = .2815$ kg/m, $I = 1.005 \times 10^{-8} m^4$, $C = 6.89\times10^7$ pascals/sec.; $E = 6.89\times10^9$ pascals, as given in the CRC Handbook of Material Science. The tube is very lightly damped (0.1 per cent).

When time is normalized to the interval [0 , 1], the dynamics become:

$$m w_{tt}(t, x) + \tau CI w_{txxxx}(t, x) + \tau^2 EI w_{xxxx}(t, x) - \tau^2 m\Omega^2(t) w(t, x) = -\tau^2 \frac{m}{1 + m/3} u(t) x \,,$$

$$t \in [0, 1], \quad x \in [0, 1], \tag{6.2a}$$

with boundary conditions:

$$w(t, 0) = 0, \quad w_x(t, 0) = 0, \quad CI w_{txx}(t, 1) + \tau EI w_{xx}(t, 1) = 0, \quad t \in [0, 1], \tag{6.2b}$$

$$\tau^2\Omega^2(t) w(t, 1) - w_{tt}(t, 1) - \tau^2 u(t) - \tau CI w_{txxx}(t, 1) - \tau^2 EI w_{xxx}(t, 1) = 0, \quad t \in [0, 1] \,, \tag{6.2c}$$

$$\Theta_t(t) = \tau\Omega(t)\,, \quad t \in [0\,,1]\,, \qquad \Omega_t(t) = \tau u(t) \;\; t \in [0\,,1]\,. \tag{6.2d}$$

To transcribe these dynamics into the standard form (2.2a), we proceed as follows. First we define $\zeta(t) \in X \triangleq L_2([0\,,1]) \times \mathbb{R}$, and $\Phi : X \times \mathbb{R}^2 \to X$ by

$$\zeta(t) \triangleq \begin{bmatrix} w(t\,,x) \\ w(t\,,1) \end{bmatrix}\,, \quad \Phi(\zeta(t)\,, u(t)\,, \Omega(t)) \triangleq \tau^2 \begin{bmatrix} \Omega^2(t) w(t\,,x) - u(t)x/(1+m/3) \\ \Omega^2(t) w(t\,,1) - u(t) \end{bmatrix}\,. \tag{6.2e}$$

Next we define the operators A_1 and Q, and their respective domains $D(A_1)$ and $D(Q)$ as follows:

$$D(A_1) \triangleq \{\, \zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \in X \mid \zeta_{1xxxx} \in L_2([0\,,1]), \zeta_1(0) = \zeta_{1x}(0) = \zeta_{1xx}(1) = 0\,, \zeta_1(1) = \zeta_2 \,\}\,, \tag{6.2f}$$

$A_1 : D(A_1) \to X$ is defined by

$$A_1 \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \tau^2 \begin{bmatrix} \frac{\mathrm{EI}}{\mathrm{m}} \zeta_{1xxxx}(\cdot) \\ \mathrm{EI}\zeta_{1xxx}(1) \end{bmatrix}\,, \tag{6.2g}$$

and with $D(Q) \triangleq D(A_1)$, $Q : D(Q) \to X$ is defined by $Q \triangleq \frac{C}{\tau E} A_1$. Then (6.2a-c) can de written in the form

$$\zeta_{tt} + Q\,\zeta_t + A_1 \zeta = \Phi(\zeta\,, u\,, \Omega)\,. \tag{6.2h}$$

It is shown in Section 6.4 and Appendix II in [Bak.1], that Φ is a an operator that is Lipschitz continuous on bounded sets, and that A_1 and Q satisfy the assumptions in [Gib.1] needed to derive the infinitesimal generator of a contraction semigroup. We give a brief outline of this derivation, see [Gib.1] for the details. First, we define the space $V \triangleq D(A_1^{1/2}) \times X$, so that if $y = (y_1\,, y_2) \in V$, then

$$\|y\|^2 = \langle y_1\,, A_1 y_1 \rangle + \langle y_2\,, y_2 \rangle\,, \tag{6.2i}$$

where $\langle \cdot\,,\cdot \rangle$ is the L_2 inner product. For any given $t \in [0\,,1]$, let $v(t) \in V$ be defined by $v(t) \triangleq (w(t\,,x)\,, w(t\,,1)\,, w_t(t\,,x)\,, w_t(t\,,1))$, and let the operator $A_2 : D(A_2) \to V$, where $D(A_2) = D(A_1) \times D(A_1) \subset V$, be defined by

$$A_2 v(t) \triangleq \begin{bmatrix} 0 & I \\ -A_1 & -Q \end{bmatrix} v(t) = \begin{bmatrix} w_t(t\,,x) \\ w_t(t\,,1) \\ -\tau\frac{\mathrm{CI}}{\mathrm{m}} w_{txxxx}(t\,,x) - \tau^2 \frac{\mathrm{EI}}{\mathrm{m}} w_{xxxx}(t\,,x) \\ -\tau \mathrm{CI} w_{txxx}(t\,,1) - \tau^2 \mathrm{EI} w_{xxx}(t\,,1) \end{bmatrix}\,. \tag{6.2j}$$

It is shown in Section 2 in [Gib.1], that there exists a unique maximal dissipative extension of A_2 to A_3 where A_3 is the generator of a contraction semigroup that represents the free response of the system (6.2h). It is shown in [Sho.1] that A_3 generates an analytic semigroup. The standard form (2.2a) is then obtained by defining the state by $z(t) \triangleq (v(t), \Theta, \Omega) \in V \times \mathbb{R}^2$, and

$$A \triangleq \begin{bmatrix} A_3 & 0 & 0 \\ 0 & 0 & \tau \\ 0 & 0 & 0 \end{bmatrix}, \quad F(z(t), u(t)) \triangleq \begin{bmatrix} 0 \\ 0 \\ \Phi(\zeta(t), u(t), \Omega(t)) \\ 0 \\ \tau u(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tau^2\Omega^2(t) w(t, x) - \tau^2 u(t) x/(1+m/3) \\ \tau^2\Omega^2(t) w(t, 1) - \tau^2 u(t) \\ 0 \\ \tau u(t) \end{bmatrix}. \quad (6.2k)$$

It follows that A satisfies Assumption 3.2 and that F satisfies assumption 3.1.

We considered three slewing problems which shared two requirements: *(a)* the tube had to be rotated 45°, from rest[5] to rest, and *(b)* the acceleration produced by the motor was limited to 5 rads/sec^2. The first problem, $\mathbf{P}_1$, was a minimum time problem, subject to the above constraints; the second problem, $\mathbf{P}_2$, was a minimum energy problem, subject to the above constraints and an upper bound on the time allowed; and the last problem, $\mathbf{P}_3$, was a minimum time problem, subject to the above constraints and an upper bound on the potential energy due to deformation of the tube throughout the entire maneuver (i.e., a worst case deformation constraint).

The transcription of the problems P_1, P_2, and P_3 into the form (2.3b) required the introduction of the following functions. With τ denoting the final time, let

$$g^1(u, \tau) \triangleq \tau. \quad (6.3)$$

The energy consumed by the maneuver is given by

$$g^2(u, \tau) \triangleq \int_0^1 u(t)^2 dt. \quad (6.4)$$

The angular error at the final time is measured by

$$g^3(u, \tau) \triangleq (\Theta(1) - \pi/4)^2 \quad (6.5)$$

The rigid body energy at final time is given by

$$g^4(u, \tau) \triangleq \Omega(1)^2, \quad (6.6)$$

The kinetic energy due to vibration of the tube at time τ is given by

$$g^5(u, \tau) \triangleq \frac{m}{2}\int_0^1 w_t(t, x)^2 dx, \quad (6.7)$$

and the potential energy due to deformation of the tube at time τ is given by

[5] We say that *the tube is at rest* when the total energy of the tube is zero. This energy is composed of the energy due to rigid body motion and energy due to vibration and deformation.

$$g^6(u\ ,\tau) \triangleq P(\tau\ ,u) = \frac{EI}{2}\int_0^1 w_{xx}(\tau\ ,x)^2 dx\ . \tag{6.8}$$

we see that the tube is at rest when $g^4(u\ ,\tau) = g^5(u\ ,\tau) = g^6(u\ ,\tau) = 0$.

The deformation constraint for problem $\mathbf{P_3}$ has the form $P(t\ ,u) \le f(t)$ for all $t \in [0\ ,1]$, where $f(\cdot)$ is a given positive bound function. This is a *state-space constraint*. To reduce the computational burden, we replaced it by the equivalent requirement $g^7(u\ ,\tau) \le 0$, where

$$g^7(u\ ,\tau) \triangleq \int_0^1 [\max\{ P(t\ ,u) - f(t)\ ,\ 0\ \}]^2 dt\ . \tag{6.10}$$

Since $P(t\ ,u)$ is continuous, $g^7(u\ ,\tau) = 0$ if and only if $P(t\ ,u) \le f(t)$ for all $t \in [0\ ,\tau]$. Transformations such as (6.10) must be used with great care because for any feasible pair $(u\ ,\tau)$, $g^7(u\ ,\tau) = 0$ and $\nabla g^7(u\ ,\tau) = 0$, and hence $\theta(u\ ,\tau) = 0$, which causes our algorithm to stop up at such a pair. However, the problems caused by this violation can be circumvented by initializing the algorithm with an infeasible point, keeping the parameter γ, in Algorithm 5.11, small, and introducing an ε into the problem statement, as shown below.

It can be shown that all the above functions $g^j : G \times [0\ ,\tau] \to \mathbb{R}$ are continuously differentiable (in the $L_2[0\ ,1] \times \mathbb{R}$ topology) in u and t for all $j \in \{1\ ,2\ ,\ldots\ ,7\}$. To conform with the format of problem (2.3b), we relax each of the equality constraints by a small amount. The three problems now acquire the following mathematical form[6], where $G \triangleq \{ u \in L_2[0\ ,1] \mid |u(t)| \le 1\ \forall t \in [0\ ,1]\}$ and $\mathbf{T} = [\tau_0\ ,\tau_f\ ]$, with $\tau_0 > 0$ very small and $\tau_f < \infty$ very large.

$$\mathbf{P_1}:\ \min\{ g^1(u\ ,\tau) \mid g^3(u\ ,\tau) - \varepsilon \le 0\ ,\ g^4(u\ ,\tau) - \varepsilon \le 0\ ,\ g^5(u\ ,\tau) - \varepsilon \le 0\ , \\ g^6(u\ ,\tau) - \varepsilon \le 0\ ,\ (u\ ,\tau) \in G \times \mathbf{T}\ \}. \tag{6.11a}$$

$$\mathbf{P_2}:\ \min\{ g^2(u\ ,\tau) \mid g^1(u\ ,\tau) - \tau_f \le 0\ ,\ g^3(u\ ,\tau) - \varepsilon \le 0\ ,\ g^4(u\ ,\tau) - \varepsilon \le 0\ , \\ g^5(u\ ,\tau) - \varepsilon \le 0\ ,\ g^6(u\ ,\tau) - \varepsilon \le 0\ ,\ (u\ ,\tau) \in G \times \mathbf{T}\ \} \tag{6.11b}$$

$$\mathbf{P_3}:\ \min\{ g^1(u\ ,\tau) \mid g^3(u\ ,\tau) - \varepsilon \le 0\ ,\ g^4(u\ ,\tau) - \varepsilon \le 0\ ,\ g^5(u\ ,\tau) - \varepsilon \le 0\ , \\ g^6(u\ ,\tau) - \varepsilon \le 0\ ,\ g^7(u\ ,\tau) - \varepsilon \le 0\ ,\ (u\ ,\tau) \in G \times \mathbf{T}\ \}. \tag{6.11c}$$

In our experiments, we set $\varepsilon = 10^{-4}$. Thus, with this relaxation, we are requiring that the final value of the angle Θ be in the interval $[45^\circ - 0.5^\circ\ ,45^\circ + 0.5^\circ]$. We assume that because of model simplifications and other inevitable modeling errors, a linear feedback system would be used to assure final pointing accuracy.

---

[6] Note that we find it convenient at this point to abandon the convention that the cost function is $g^0(\cdot\ ,\cdot)$ as well as the linear numbering of the constraints.

In the computational experiments reported in this paper, the term $\Omega^2(t)$ was neglected in equation (6.1a) - (6.1c). Similar results were obtained in computational experiments in which the term $\Omega^2(t)$ was kept. We used a cubic Hermit spline implementation of the Finite Element Method for spatial discretization and Newmark's β–method, with $\beta = 0$, for temporal discretization of both responses and sensitivities[7]. This approach is quite stable and gives accurate simulations. The results of our computational experiments are shown in Figs. 2 -11.

Problem P_1: For simplicity, we chose the zero function as the initial control and 2 for an initial value for the maneuver time. The initial discretization consisted of 32 time steps and 6 finite elements. The discretization was refined at iterations 67, 99, and 123. Figure 2 is a graph of the control after 150 iterations. At this point, the number of time steps was 256 and the number of finite elements 48. Figure 3a is a graph of $\psi_{q_s, q_t}(u, \tau)$ as a function of the iteration number. Figure 3b shows $\psi_{q_s, q_t}(u, \tau)$ for the first 15 iterations. After precision refinement, the algorithm finds a control $u \in G_{q_t}$ and final time $\tau \in \mathbf{T}$ such that $\psi_{q_s, q_t}(u, \tau) < 0$ in only a few additional iterations. Note that each time precision of discretization was increased, the value of $\psi_{q_s, q_t}(u_i, \tau_i)$ increases. This is due to improvement in the accuracy of the evaluation of the partial differential equation. This increase in constraint violation $\psi_{q_s, q_t(u_i, \tau_i)}$ decreases each time the discretization is increased and we see that in the limit the increase is zero. Figure 4 is the graph of the cost as a function of iteration number. Figure 5 is the graph of $w(t, 1)$, the displacement of the tip of the tube, from the *shadow tube*, as a function of time. The maximum displacement of the tip of about 5 mm and is within the range of validity of the Euler-Bernoulli model. The tip displacement is large between 0.36 seconds and 0.437 seconds. Figure 6 is a profile of the tube deformation, $w(t, x)$ (see Figure 1), during this interval. The total time for the entire maneuver is 0.7886 seconds.

Problem P_2: Figure 7 is the graph of the control produced by minimizing the total input energy while constraining the final time to be less than 0.800 seconds, i.e., only 1.4 percent longer than the minimum time computed for $\mathbf{P_1}$. The resulting final time is 0.800. The control is much smoother than the minimum time control, and the total energy consumption is reduced by 18%, from 19.15 to 15.72. Figure 8 is the graph of the control when the bound on the final time is extended to 1.00 second, 27% over the minimum time for the maneuver. The result is a total energy is reduction by 62%, to 7.27.

Problem P_3: In problem $\mathbf{P_3}$, we have the additional requirement to keep the potential energy, which is a measure of the total tube deformation, below the parabola (B) for all time. Figure 9 shows the optimal control for problem $\mathbf{P_3}$. The optimal final time for this case is 0.8177 seconds, an increase of 3.7 percent over the solution of problem $\mathbf{P_1}$. Figure 10 shows the potential energy curve for this case, which was

[7] See [Bak.1, Chap. 8] for implementation details, that are based on the results in [Fuj.1, Fuj.2, Fuj.3, Ode.1].

constrained to lie below a parabola (B). For comparison In Figure 11, curve A is the graph of the potential energy of the tube as a function of time for the control generated in solving the minimum time problem $\mathbf{P_1}$.

7. CONCLUSION

We have presented an approximation theory for the numerical solution of optimal control problems with dynamics in evolution equation form, with control and state space constraints. It should be obvious that the theory can be trivially adapted to deal with problems with constraints on the initial state, as well as with unconstrained problems. Although not included in this paper, we have results (reported in [Bak.1, Bak.2]) which show that our theory can be used in conjunction with finite element techniques to produce reasonably efficient numerical procedures which have the property that all the accumulation points of the control sequences that they produce satisfy the problem constraints as well as an optimality condition either for the original or the relaxed problem, depending on whether the accumulation point is in the $L_2^m[0, 1]$ topology or in the relaxed controls topology.

8. ACKNOWLEDGEMENT

The authors are grateful to Prof. Jack Warga for his comments on a draft of this paper and to the referees for helpful suggestions.

9. APPENDIX: DIFFERENTIABILITY OF MILD SOLUTIONS

We will now establish the Frechet differentiability of solutions of (2.2f) with respect to the control $u \in L_2^m[0, 1]$ and the scaling parameter τ.

Let $\tilde{M}$, $\omega \in (0, \infty)$ be such that $\|T(t)\| \le \tilde{M} e^{\omega t}$ for all $t \in [0, 1]$, and let $M \triangleq \tilde{M} e^{\omega \tau_{max}}$.

Lemma A.1. (Lipschitz Continuity of $z(t, u, \tau)$ in (u, τ).)
There exists $b_3 \in (0, \infty)$ such that for all $u', u \in L_2^m[0, 1]$, $t \in [0, 1]$, $\tau \in [\tau_{min}, \tau_{max}]$,

$$\|z(t, u', \tau') - z(t, u, \tau)\|_X \le b_3(\|u' - u\|_2^2 + |\tau' - \tau|^2)^{1/2}. \tag{A.1}$$

Proof. For any $u, u' \in L_2^m[0, 1]$ and $t \in [0, 1]$,

$$z(t, u', \tau') - z(t, u, \tau) = T(\tau' t)z_0 + \int_0^t \tau' T(\tau'(t-s))F(z(s, u', \tau'), u'(s))ds$$

$$- T(\tau t)z_0 - \int_0^t \tau T(\tau(t-s))F(z(s, u, \tau), u(s))ds$$

$$= [T(\tau' t) - T(\tau t)]z_0 + \int_0^t \Bigg\{ \tau' T(\tau'(t-s))[F(z(s,u',\tau'),u'(s)) - F(z(s,u,\tau),u(s))]$$

$$- [\tau T(\tau(t-s)) - \tau' T(\tau'(t-s))]F(z(s,u,\tau),u(s)) \Bigg\} ds \,. \tag{A.2a}$$

Since $\{z(t,u,\tau) \in S \triangleq \{z \in X \mid \|z\|_X \le b_1\}$, by Assumption 3.1*(ii)*, we conclude from Assumption 3.1*(iii)* and Lemma 3.4 that there exists constants K_S, $L \in (0,\infty)$, such that, with $y(t) \triangleq \|z(t,u',\tau) - z(t,u,\tau)\|_X$, for $t \in [0,1]$,

$$y(t) \le \tau_{\max} M K_S \int_0^t [y(s) + \|u'(s) - u(s)\|_2]\, ds + L\,|\tau' - \tau| \,. \tag{A.2b}$$

Applying the Bellman-Gronwall Inequality, and making use of the fact that by the Schwartz Inequality, $\|u\|_1 \le \|u\|_2$, we obtain that

$$y(t) \le e^{\tau_{\max} M K_S} \{ \tau_{\max} M K_S \|u' - u\|_1 + L\,|\tau' - \tau| \} \le b_3(\|u' - u\|_2^2 + |\tau' - \tau|^2)^{1/2} \,, \tag{A.2c}$$

where $b_3 \triangleq \sqrt{2}\max\{\tau_{\max} M K_S, L\} e^{\tau_{\max} M K_S}$. □

Next, for $u', u \in L_2^m[0,1]$ and $\tau', \tau \in [\tau_{\min}, \tau_{\max}]$, we define $\delta u = u' - u$, $\delta\tau = \tau' - \tau$, and $\delta z(\cdot, u, \tau, \delta u, \delta\tau) \in C([0,1], X)$ to be the solution to the equation

$$\delta z(t) = \int_0^t \Bigg\{ T(\tau(t-s))\tau\Big(\frac{\partial F}{\partial z}(z(s,u,\tau),u(s))\delta z(s) + \frac{\partial F}{\partial u}(z(s),u(s))\delta u(s)\Big)$$

$$+ (T(\tau(t-s)) + \tau(t-s)AT(\tau(t-s)))F(z(s,u,\tau),u(s))\delta\tau \Bigg\} ds + tAT(\tau t)z_0\delta\tau \,. \tag{A.3}$$

Note that (A.3) is the first variation with respect to (u,τ) of equation (2.2f).

Theorem A.4. (Frechet Differentiability of $z(t,u,\tau)$ with respect to (u,τ).)
For all $u', u \in L_2^m[0,1]$, $\tau', \tau \in [\tau_{\min}, \tau_{\max}]$

$$\|z(t,u',\tau') - z(t,u,\tau) - \delta z(t,u,\tau,u'-u,\tau'-\tau)\|_X \le o(u'-u,\tau'-\tau) \,, \tag{A.4}$$

where $o(\delta u, \delta\tau)/(\|\delta u\|_2^2 + |\delta\tau|^2)^{1/2} \to 0$ as $(\delta u, \delta\tau) \to 0$.

Proof. To simplify notation, we define $\Delta z(t) \triangleq z(t,u',\tau') - z(t,u,\tau)$, $\delta z(t) \triangleq \delta z(t,u,\tau,\delta u,\delta\tau)$, $\delta u \triangleq u' - u$, $\delta\tau \triangleq \tau' - \tau$, and we remove obvious arguments by setting $F(t) \triangleq F(z(t,u,\tau),u(t))$, $F'(t) \triangleq F(z(t,u',\tau'),u'(t))$, $F_z(t) \triangleq \frac{\partial F}{\partial z}(z(t,u,\tau),u(t))$, $F_u(t) \triangleq \frac{\partial F}{\partial u}(z(t,u,\tau),u(t))$.

First, in terms of this simplified notation, we have that

$$\delta z(t) = \int_0^t \left\{ \tau T(\tau(t-s)[F_z(s)\delta z(s) + F_u(s)\delta u(s)] + \right.$$

$$\left. + [T(\tau(t-s)) + \tau(t-s)AT(\tau(t-s)]F(s))\delta\tau \right\} ds + tAT(\tau t)z_0\delta\tau \ , \tag{A.5}$$

$$\Delta z(t) = [T(\tau' t) - T(\tau t)]z_0 + \int_0^T [\tau' T(\tau'(t-s))F'(s) - \tau T(\tau(t-s))F(s)]ds$$

$$= [T(\tau' t) - T(\tau t)]z_0 + \tau\int_0^t T(\tau(t-s))[F'(s) - F(s)]ds$$

$$+ \int_0^t [\tau' T(\tau'(t-s)) - \tau T(\tau(t-s))]F'(s)ds \ . \tag{A.6}$$

Hence,

$$[\Delta z(t) - \delta z(t)] = [T(\tau' t) - T(\tau t) - \delta\tau t AT(\tau t)]z_0$$

$$+ \int_0^t \left\{ [\tau' T(\tau'(t-s)) - \tau T(\tau(t-s))]F'(s) - \delta\tau[\tau(t-s)AT(\tau(t-s)) + T(\tau(t-s))]F(s) \right\} ds$$

$$+ \tau\int_0^t T(\tau(t-s)) \left\{ F_z(s)[\Delta z(s) - \delta z(s)] + [F'(s) - F(s) - F_z(s)\Delta z(s) - F_u(s)\delta u(s) \right\} ds \ .$$

(A.7a)

We will deal with the three groups of terms in the right hand side of (A.7a) one at a time. We will give full details for the last group only, since the calculations are quite laborious. First, since by Lemma 3.3, $(d/dt)T(t) = AT(t)$,

$$\| T(s+\delta s) - T(s) - AT(t)\delta s \| = o_1(\delta s) \ ,$$

where $o_1(\delta s)/\delta s \to 0$ as $\delta s \to 0$. Now let $s = t\tau$ and $s + \delta s = t\tau'$. Hence $\delta s = t(\tau' - \tau) = t\,\delta\tau$ where $\delta\tau = \tau' - \tau$. Therefore,

$$\| T(t\tau') - T(t\tau) - AT(t)\delta\tau t \| \le o_1(\delta\tau t) \ ,$$

and hence

$$\| [T(\tau' t) - T(\tau t) - \delta\tau t AT(\tau t)]z_0 \| \le \| z_0 \|_X \, o_1(\delta\tau) \ . \tag{A.7b}$$

Next, making use of Lemmas 3.3 and 3.4, one can show that

$$\left\| \int_0^t [\tau' T(\tau'(t-s)) - \tau T(\tau(t-s))]F'(s) - \delta\tau[\tau(t-s)AT(\tau(t-s)) + T(\tau(t-s))]F(s)ds \right\|_X = o_2(\delta u\ , \delta\tau) \ ,$$

(A.7c)

where $o_2((\delta u, \delta\tau))/(\|\delta u\|_2^2 + |\delta\tau|^2)^{1/2} \to 0$ as $(\delta u, \delta\tau) \to 0$.

Finally, making use of Assumption 3.1 and Lemma 3.4 and Lemma A.1, we obtain that

$$\tau\| \int_0^t T(\tau(t-s))\left\{ F_z(s)[\Delta z(s) - \delta z(s)] + [F'(s) - F(s) - F_z(s)\Delta z(s) - F_u(s)\delta u(s)\right\} ds\,\|_X$$

$$\leq \tau_{\max} M b_2 \int_0^t \|\Delta z(s) - \delta z(s)\|_X\, ds + \tau_{\max} M \int_0^t \|\{ F'(s) - F(s) - F_z(s)\Delta z(s) - F_u(s)\delta u(s)\}\|_X\, ds$$

$$\leq \tau_{\max} M \int_0^t \left\{ b_2 \|\Delta z(s) - \delta z(s)\|_X + \int_0^1 \|\frac{\partial F}{\partial z}(z(s) + r\Delta z(s), u(s) + r\delta u(s)) - F_z(s)\| dr\, \|\Delta z(s)\|_X \right.$$

$$\left. + \int_0^1 \|\frac{\partial F}{\partial u}(z(s) + r\Delta z(s), u(s) + r\delta u(s)) - F_u(s)\| dr\, \|\delta u(s)\| \right\} ds$$

$$\leq \tau_{\max} M \int_0^t \left\{ b_2 \|\Delta z(s) - \delta z(s)\|_X + \int_0^1 K_S\, r(\|\Delta z(s)\|_X + \|\delta u(s)\|) dr\, \|\Delta z(s)\|_X \right.$$

$$\left. + \int_0^1 K_S\, r(\|\Delta z(s)\|_X + \|\delta u(s)\|) dr\, \|\delta u(s)\| \right\} ds$$

$$\leq \tau_{\max} M \int_0^t \{ b_2 \|\Delta z(s) - \delta z(s)\|_X + K_S [\|\Delta z(s)\|_X + \|\delta u(s)\|]^2 \}\, ds\,. \tag{A.7d}$$

Since by Lemma A.1, $\|\Delta z(s)\|_X \leq b_3(\|\delta u\|_2^2 + |\delta\tau|^2)^{1/2}$, we obtain, combining (A.7b) -(A.7d) that

$$\|\Delta z(t) - \delta z(t)\|_X \leq \tau_{\max} M \int_0^t \{ b_2 \|\Delta z(s) - \delta z(s)\|_X \}\, ds + \tau_{\max} M K_S [b_3 \|\delta u\|_2 + o_3((\delta u, \delta\tau))]\,, \tag{A.7e}$$

where $o_3((\delta u, \delta\tau))/(\|\delta u\|_2^2 + |\delta\tau|^2)^{1/2} \to 0$ as $(\delta u, \delta\tau) \to 0$. Applying the Bellman-Gronwall Lemma, we obtain that

$$\|\Delta z(t) - \delta z(t)\|_X \leq o((\delta u, \delta\tau))\,, \tag{A.7f}$$

where $o((\delta u, \delta\tau))/(\|\delta u\|_2^2 + |\delta\tau|^2)^{1/2} \to 0$ as $(\delta u, \delta\tau) \to 0$, which completes our proof. □

Proceeding by analogy with the proof of Lemma A.1, it is easy to establish the following result:

Lemma A.5. The solution $\delta z(t, u, \tau, \delta u, \delta\tau)$, of (A.3), is linear in $(\delta u, \delta\tau)$ for each $t \in [0, 1]$, $u \in L_2^m[0, 1]$, and $\tau \in [\tau_{\min}, \tau_{\max}]$, and it is Lipschitz continuous in $(u, \tau) \in G \times [\tau_{\min}, \tau_{\max}]$, i.e., there exists $b_4 < \infty$ such that for all $u', u \in L_2^m[0, 1]$, $t \in [0, 1]$, $\tau \in [\tau_{\min}, \tau_{\max}]$,

$$\|\delta z(t, u', \tau', \delta u, \delta\tau) - \delta z(t, u, \tau, \delta u, \delta\tau)\|_X \le b_4(\|u' - u\|_2^2 + |\tau' - \tau|^2)^{1/2} . \tag{A.8}$$

□

If we denote by $z_{u,\tau}(t, u, \tau)$ the linear map $\delta u \to \delta z(t, u, \tau, \delta u, \delta\tau)$ and make use of Assumption 3.1*(v)* and Theorem A.4, we obtain the following theorem:

Theorem A.6. For all $u \in L_2^m[0, 1]$, $\tau \in [\tau_{\min}, \tau_{\max}]$, and $t \in [0, 1]$, $z(t, u, \tau)$ admits a Lipschitz continuous Frechet derivative. That is, there exists a Lipschitz continuous linear operator $Dz(t, u, \tau) = (D_u z(t, u, \tau), D_\tau z(t, u, \tau)) \in \mathbf{B}(L_2^m[0, 1], X)$ such that for all $\delta u \in L_2^m[0, 1]$ and $\delta\tau \in \mathbb{R}$,

$$\lim_{\substack{\|\delta u\|_2 \to 0 \\ |\delta\tau| \to 0}} \frac{z(t, u + \delta u, \tau + \delta\tau) - z(t, u, \tau) - D_u z(t, u, \tau)\delta u - D_u z(t, u, \tau)\delta\tau}{(\|\delta u\|_2^2 + |\delta\tau|^2)^{1/2}} = 0 . \tag{A.9}$$

□

10. REFERENCES

[Ahm.1] N. H . AHMED, *Properties of relaxed trajectories for a class of nonlinear evolution equations on a Banach space*, SIAM J. Control and Optimization, Vol.21, No.6, pp 953-967, 1983.

[Ale.1] B. M. ALEKSEEV, V. M. TIKHOMIROV, and S. V. FOMIN, *Optimalnoye Upravleniye (Optimal Control)*, Nauka, Moscow, 1979.

[Arm.1] L. ARMIJO, *Minimization of functions having Lipschitz continuous first partial derivatives*, *Pacific Journal of Mathematics* Vol 16, pp 1-3, 1966.

[Bak.1] T. E. BAKER, *Algorithms for optimal control of systems described by partial and ordinary differential equations*, (Ph. D. Dissertation, University of California, Berkeley, June 1988) Memo No. UCB/ERL M88/45, University of California, Berkeley, Electronics Research Laboratory 21 June, 1988.

[Bak.2] T. E. BAKER and E. POLAK, *An Algorithm for optimal slewing of flexible Structures*, University of California, Electronics Research Laboratory, Memo UCB/ERL M89/37, 11 April 1989.

[Ben.1] J. BEN-ASHER, J. A. BURNS and E. M. CLIFF, *Time optimal slewing of flexible spacecraft*, Proc. 26th IEEE Conference on Decision and Control, 1987, pp. 524-528.

[Bur.1] J. A. BURNS, R. E. MILLER, and E. M. CLIFF, *Control of a viscoelastic shaft with attached tip mass*, Proc. 26th Conference on Decision and Control, pp. 997-999, 1987.

[Ber.1] C. BERGE, *Topological Spaces*, Macmillan Co., N.Y., 1963.

[Chu.1] H. M. CHUN, *Large-angle slewing maneuvers for flexible spacecraft*, Ph. D. Dissertation, M.I.T., 1986.

[Cul.1] J. CULLUM, *Discrete approximations to continuous optimal control problems*, Siam J. Control, Vol. 7, No. 1, February, 1969.

[Cul.2] J. CULLUM, *An Explicit Procedure for Discretizing Continuous Optimal Control Problems*, JOTA, Vol. 8, pp. 15-34, 1971.

[Dun.1] J. C. DUNN, *Diagonally modified conditional gradient methods for input constrained optimal control problems*, SIAM J. Control and Optimization, Vol. 24, 1986, pp. 1177-1191.

[Dun.2] J. C. DUNN and E. SACHS, *The effect of perturbations on the convergence rates of optimization algorithms*, Appl. Math. Optim., Vol. 10, 1983, pp. 143-157. [Ode.1] 8 J. T. ODEN and R. B. FOST, *Convergence, Accuracy and Stability of Finite Element Approximations of a Class of Non-linear Hyperbolic Equations* Intl. Journal for Numerical Methods in Engineering, Vol. 6, 1973, pp. 357-365.

[Flo.1] M. A. FLOYD, M. E. BROWN, J. D. TURNER and W. E. VANDERWELDE, *Implementation* *of a minimum time and fuel on/off thruster control system for flexible spacecraft*, Journal of Astronautical Sciences, To Appear.

[Fuj.1] H. FUJII, *Finite Element schemes: stability and convergence*, in Advances in Computational Methods in Structural Mechanics and Design, J. T. Oden, R. W. Clough, and Y. Yamamoto, eds, pp 201-218, Papers at Second U.S. Japan Seminar, University of Huntsville, Alabama press, 1972

[Fuj.2] H. FUJII, *A note on finite element approximation for evolution equations*, Kokyuroku, RIMS, Kyoto Univ., No. 202, pp 96-117, 1974.

[Fuj.3] H. FUJITA and T. SUZUKI, *Evolution Problems*, in HANDBOOOK OF NUMERICAL ANALYSIS Vol. II, Finite Element Methods (Part 1), P. G. Ciarlet, J. L. Lions, eds., Elsevier Science Pub., 1991.

[Gib.1] J. S. GIBSON, *An Analysis of optimal modal regulation: convergence and stability*, SIAM J. Control and Optimization, Vol. 19, No. 5, 1981.

[Gib.2] J. S. GIBSON, D. L. MINGORI, A. ADAMIAN, and F. JABBARI, *Approximation of optimal infinite dimensional compensators for flexible structures*, Proc. Workshop on Identification and Control of Flexible Space Structures, Vol. II, April 1985, pp. 201-218.

[Gib.3] J. S. GIBSON, *The Riccati integral equations for optimal control problems on Hilbert spaces*, SIAM J. Control and Optimization, Vol. 17, No. 4, 1979.

[Jun.1] J. L. JUNKINS and J. D. TURNER, *Optimal Spacecraft Rotational Maneuvers*, Studies in Astronautics, 3, Elsevier, Amsterdam, 1986.

[Hua.1] P. HUARD, *Programmation Mathematique Convexe*, Rev. Francaise Inf. Rech. Oper., Vol. 7,1968, pp. 43-59.

[Kle.1] R. KLESSIG and E. POLAK, *An Adaptive Algorithm for Unconstrained Optimization with Applications to Optimal Control*, SIAM J. Control, Vol. 11, No. 1, 1973, pp. 80-94.

[Lan.1] S. LANG, *Real Analysis*, Second Edition, Addison Wesley Co, Reading, Mass, 1983.

[May.1] D. Q. MAYNE and E. POLAK, *First order, strong variations algorithms for optimal control*, JOTA, Vol. 16, No. 3/4, 1975, pp. 277-301.

[May.3] D. Q. MAYNE and E. POLAK, *A feasible directions algorithm for optimal control problems with terminal inequality constraints*, IEEE Transactions on Automatic Control, Vol. AC-22, No. 5, 1977, pp. 741-751.

[May.4] D. Q. MAYNE and E. POLAK, *An Exact penalty function algorithm for optimal control problems with control and terminal equality constraints, Part 1*, JOTA, Vol. 32 No. 2, 1980, pp. 211-246.

[May.5] D. Q. MAYNE and E. POLAK, *An exact penalty function algorithm for optimal control problems with control and terminal equality constraints, Part 2*, JOTA, Vol. 32 No. 3, 1980, pp. 345-363.

[Pap.1] N. S. PAPAGEORGIOU, *Properites of relaxed trajectories of evolution equations and optimal control*, SIAM J. Control Optim., Vol. 27, No. 2, pp. 267-288, 1989.

[Paz.1] A. PAZY, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[Pir.1] O. PIRONNEAU and E. POLAK, *On the Rate of Convergence of Certain Methods of Centers*, Mathematical Programming, Vol. 2, No. 2, 1972, pp. 230-258.

[Pir.2] O. PIRONNEAU and E. POLAK, *A Dual Method for Optimal Control Problems with Initial and Final Boundary Constraints*, SIAM J. Control, Vol. 11, No. 3, 1973, pp. 534-549.

[May.2] E. POLAK, and D. Q. MAYNE, *First order, strong variations algorithms for optimal control problems with terminal inequality constraints*, JOTA, Vol. 16, No. 3/4, 1975, pp. 303-325.

[Pol.1] E. POLAK, *On the Mathematical Foundations of Nondifferentiable Optimization in Engineering Design*, SIAM Review, 1987, pp. 21-91.

[Pol.2] E. POLAK and L. HE, *A Unified Phase I Phase II Method of Feasible Directions for Semi-infinite Optimization*, University of California, Electronics Research Laboratory, Memo UCB/ERL M89/7, 3 February 1989. To appear in JOTA.

[Pol.3] E. POLAK, *Computational Methods in Optimization: A unified Approach*, Academic Press, New York, 1972.

[Psh.1] B. N. PSHENICHNYI and Yu. M. DANILIN, *Numerical Methods in Extremal Problems*, Nauka, Moscow, 1975.

[Sho.1] R. E. SHOWALTER, *Hilbert Space Methods for Partial Differential Equations*, Pitman, London, 1977.

[Sin.1] G. SING, P. T. KABAMBA, and N. H. MCLAMROCH, *Planar, time-optimal rest to rest slewing maneuvers of flexible spacecraft*, J. Guidance, Control, Dyn., Vol.12, 1989, pp. 71-81.

[Sla.1] M. SLATER, *Lagrange Multipliers revisited: a contribution to nonlinear programming*, Cowles Commission Discussion Paper, Mathematics 403, November 1950.

[Teo.1] K. L. TEO and Z. S. WU, *Computational Methods for Optimizing Distributed Systems*, Academic Press, New York, 1984.

[Teo.2] K. L. TEO, K. H. WONG, and D. J. CLEMENTS, *A Feasible directions algorithm for time-lag optimal control problems with control and terminal inequality constraints*, JOTA, Vol. 46, 1985, pp. 295-318.

[War.1] J. WARGA, *Optimal Control of Differential Equations and Functional Equations*, Academic Press, New York, 1972.

[War.2] J. WARGA, *Steepest Descent with Relaxed Controls*, SIAM J. Control, Vol. 15, 1977, pp. 674-682.

[War.3] J. WARGA, *Iterative procedures for constrained and unilateral optimization problems*, SIAM J. Control, Vol. 20, 1982, pp. 360-367.

[War.4] J. WARGA, *Iterative optimization with equality constraints*, *Math. of Oper. Res.*, Vol. 9, 1984, pp. 592-605.

[Wil.1] L. J. WILLIAMSON and E. POLAK, *Relaxed controls and the convergence of optimal control algorithms*, SIAM J. Control Vol. 14, No. 4, 1976, pp. 737-757.

[Won.1] Wong, K. H. and Teo, K. L., *A Conditional Gradient Method for a Class of Time-Lag Optimal Control Problems*, Journal of Australian Mathematical Society, Vol. 25, No. Series B, 1984, pp. 518-537.

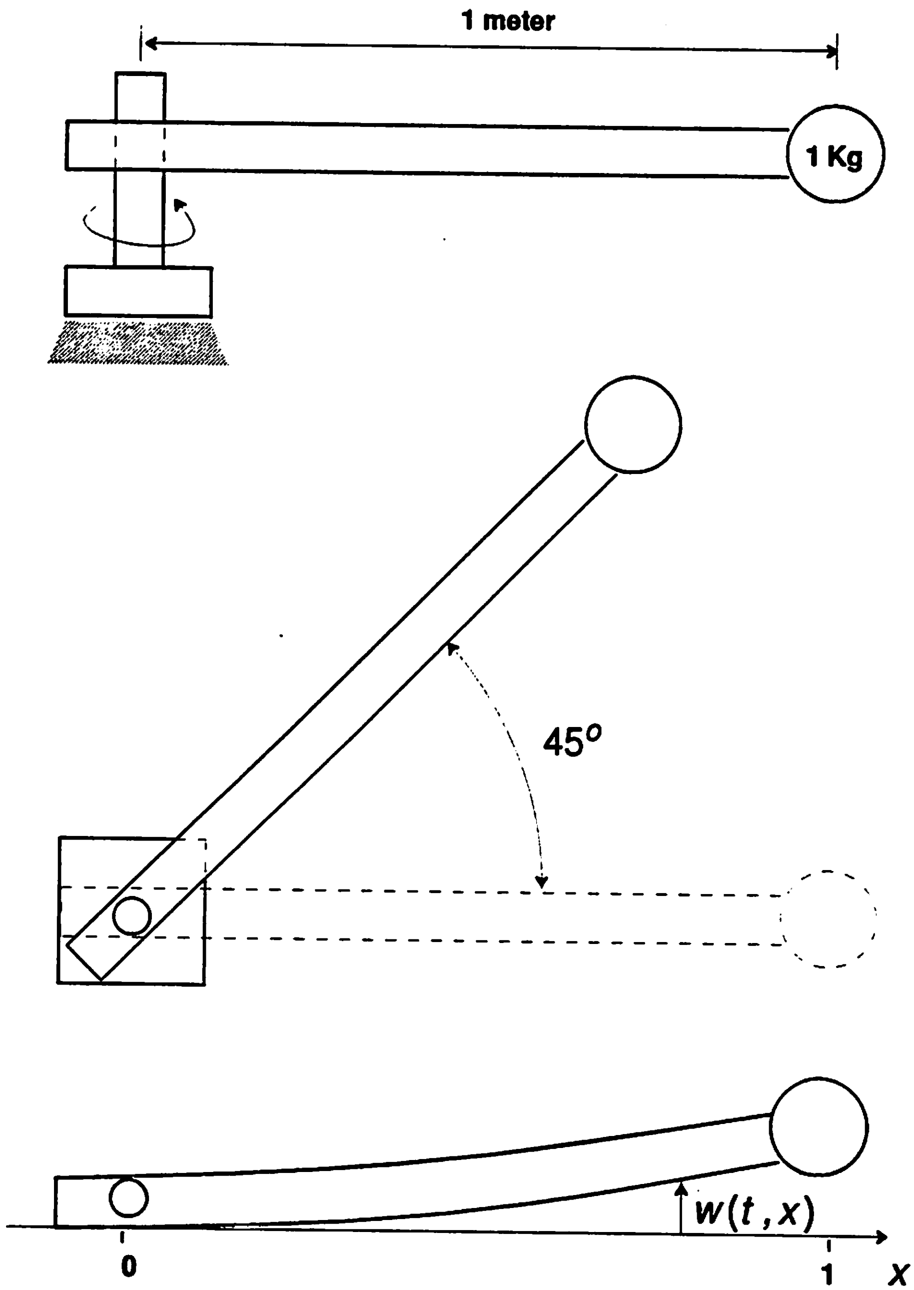

Figure 1. Configuration of Slewing Experiment

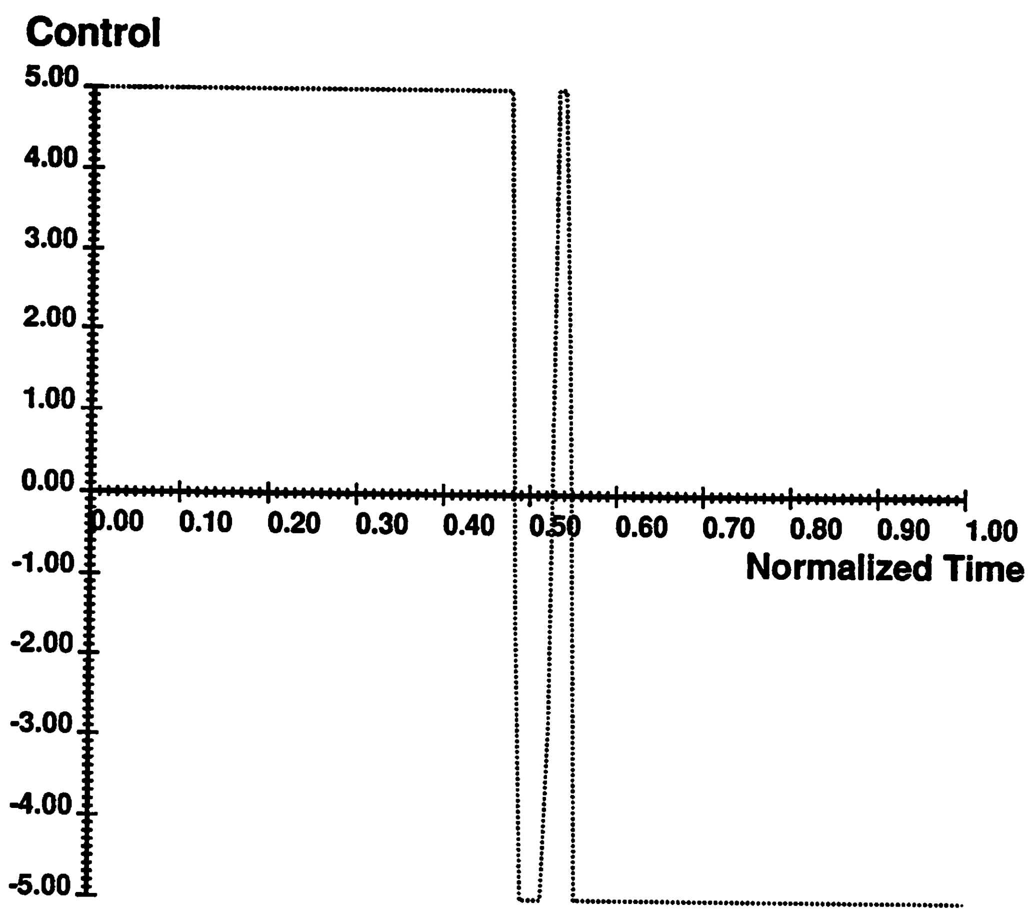

Figure 2. Final Control for Problem 1

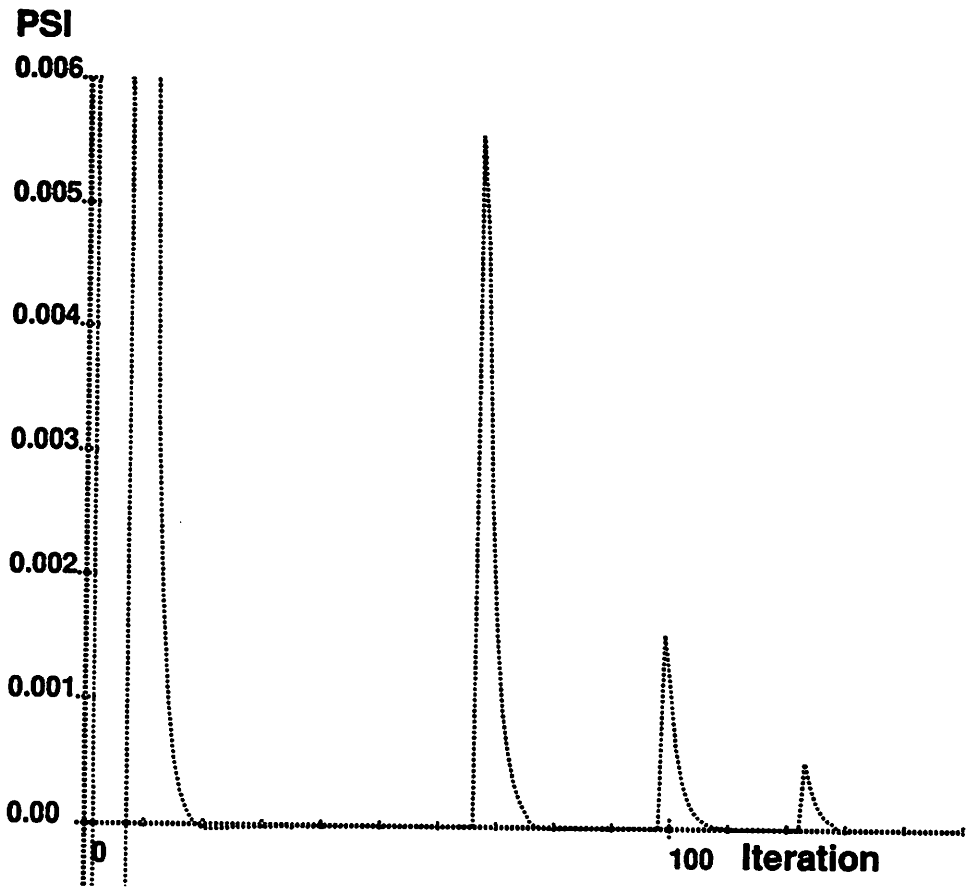

Figure 3a. Constraint Violation in Problem 1: 150 Iterations

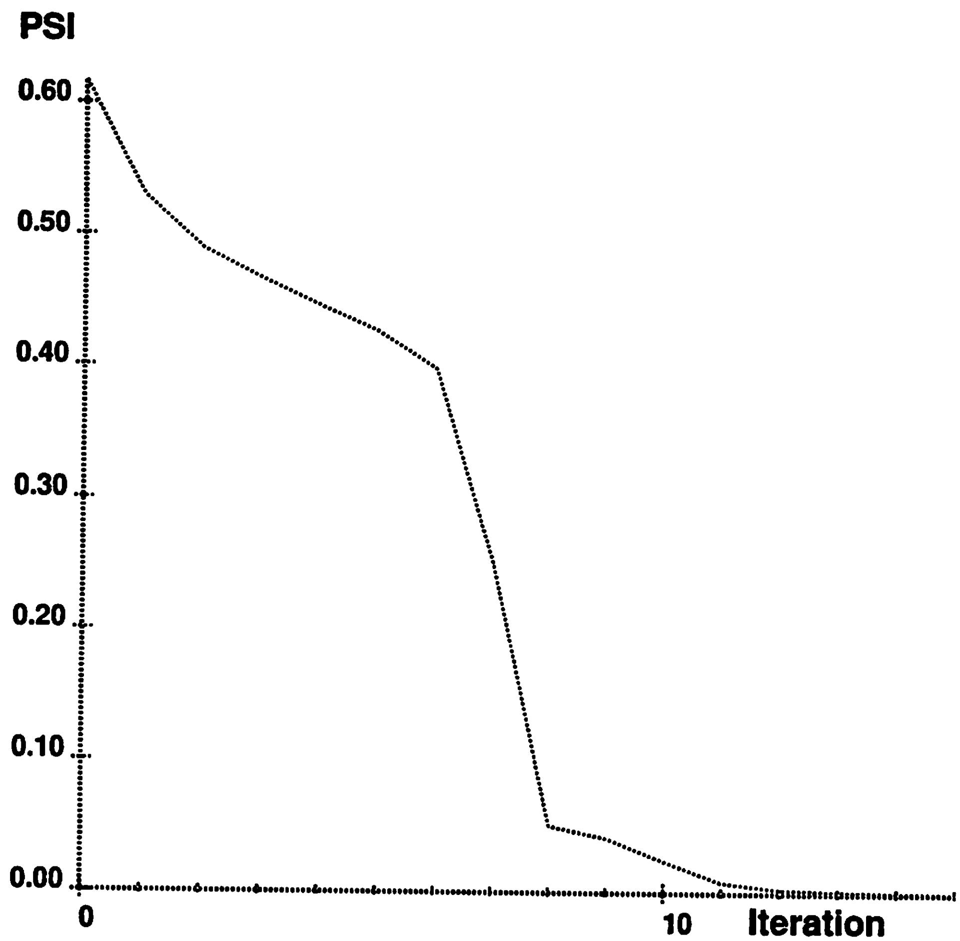

Figure 3b. Constraint Violation in Problem 1: first 15 Iterations

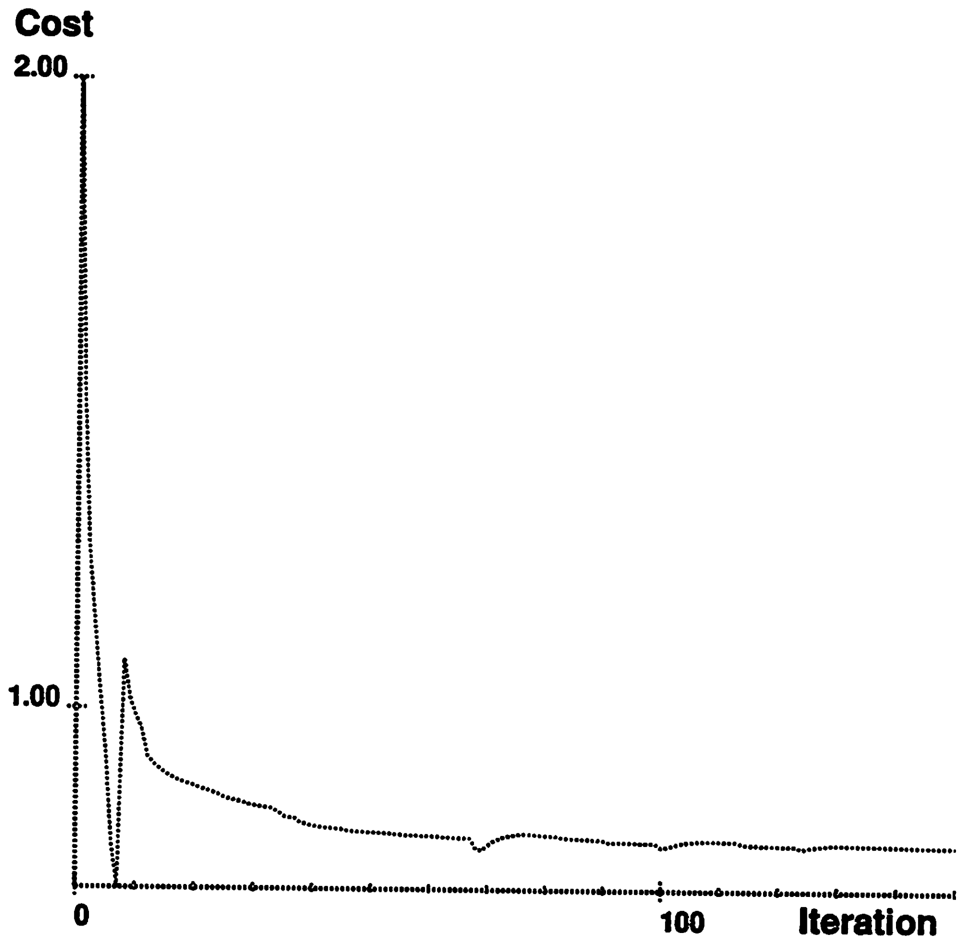

Figure 4. Cost v/s Iteration number for Problem 1

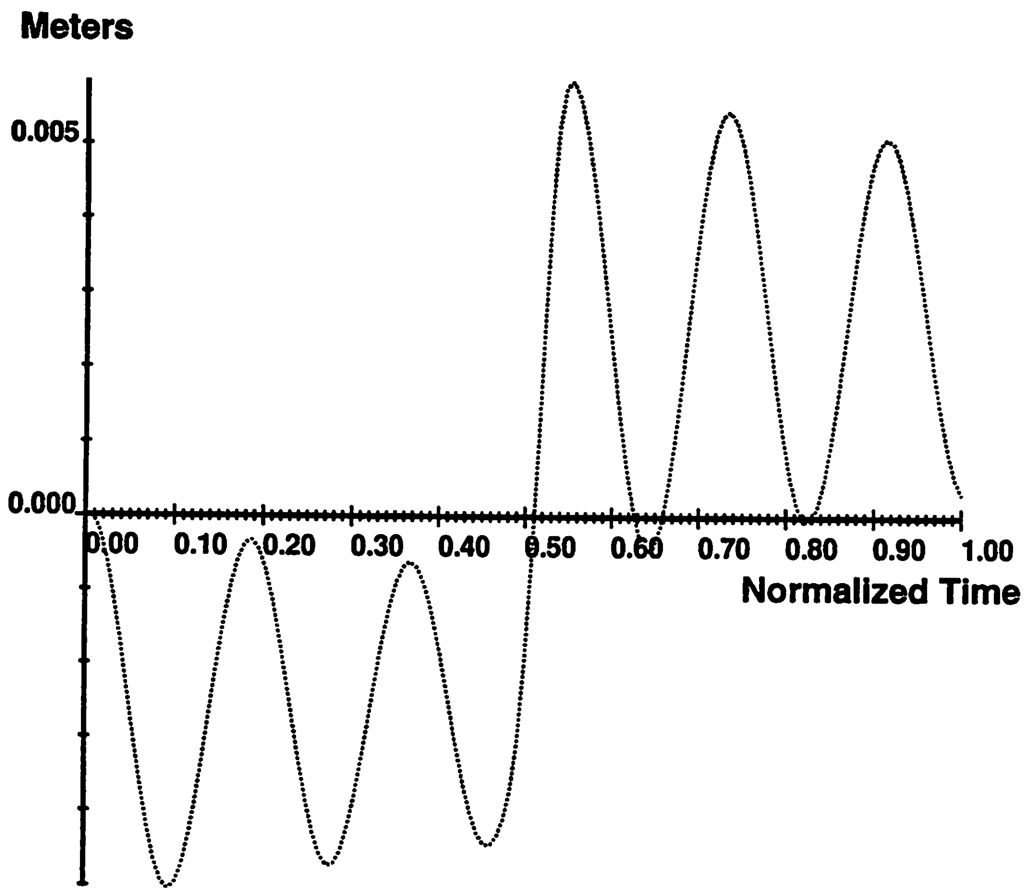

Figure 5. Displacement of Tip of Tube, Problem 1

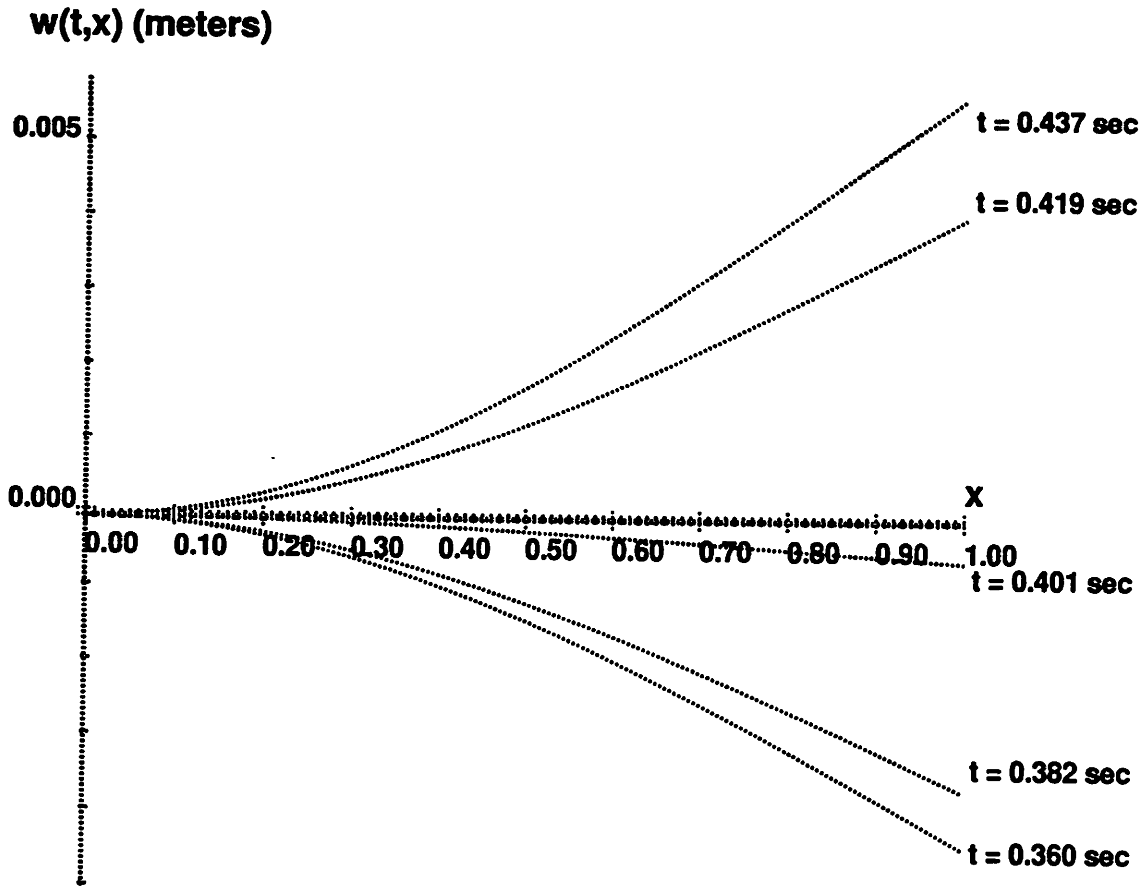

Figure 6. Beam Profile

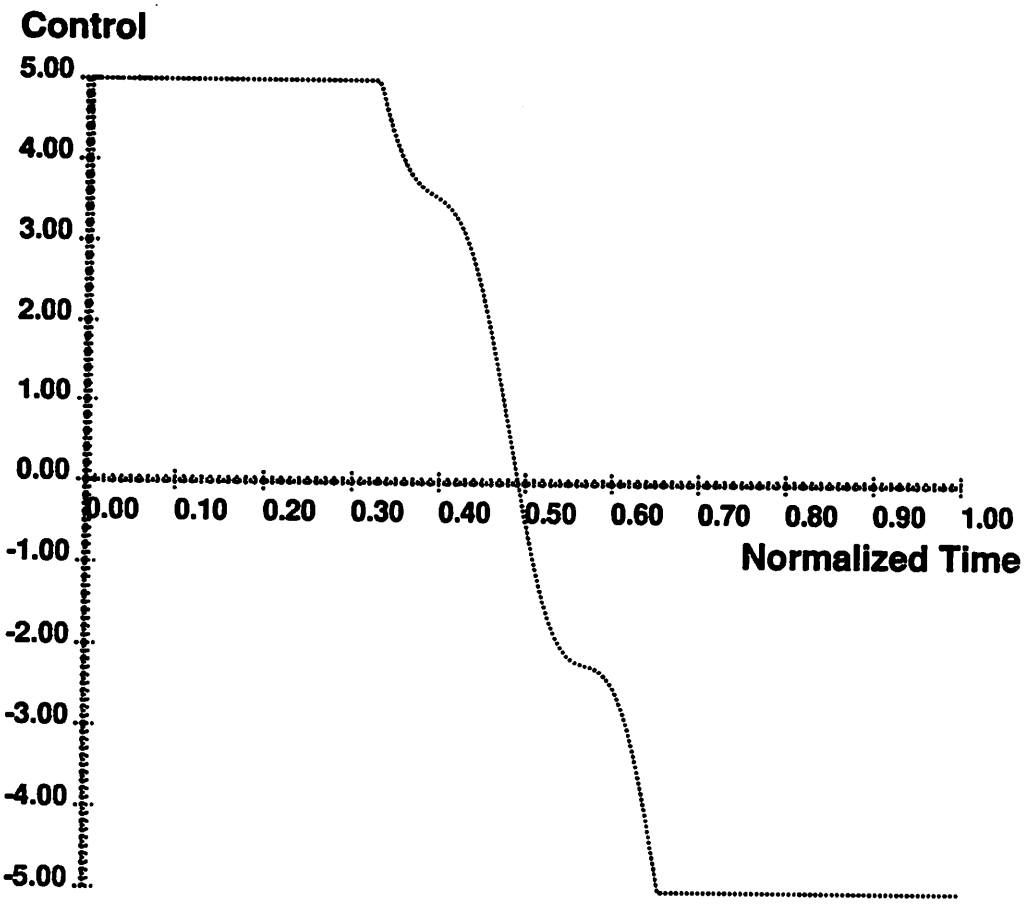

Figure 8a. Final Control for Problem 2: Time of Maneuver = 0.900 seconds

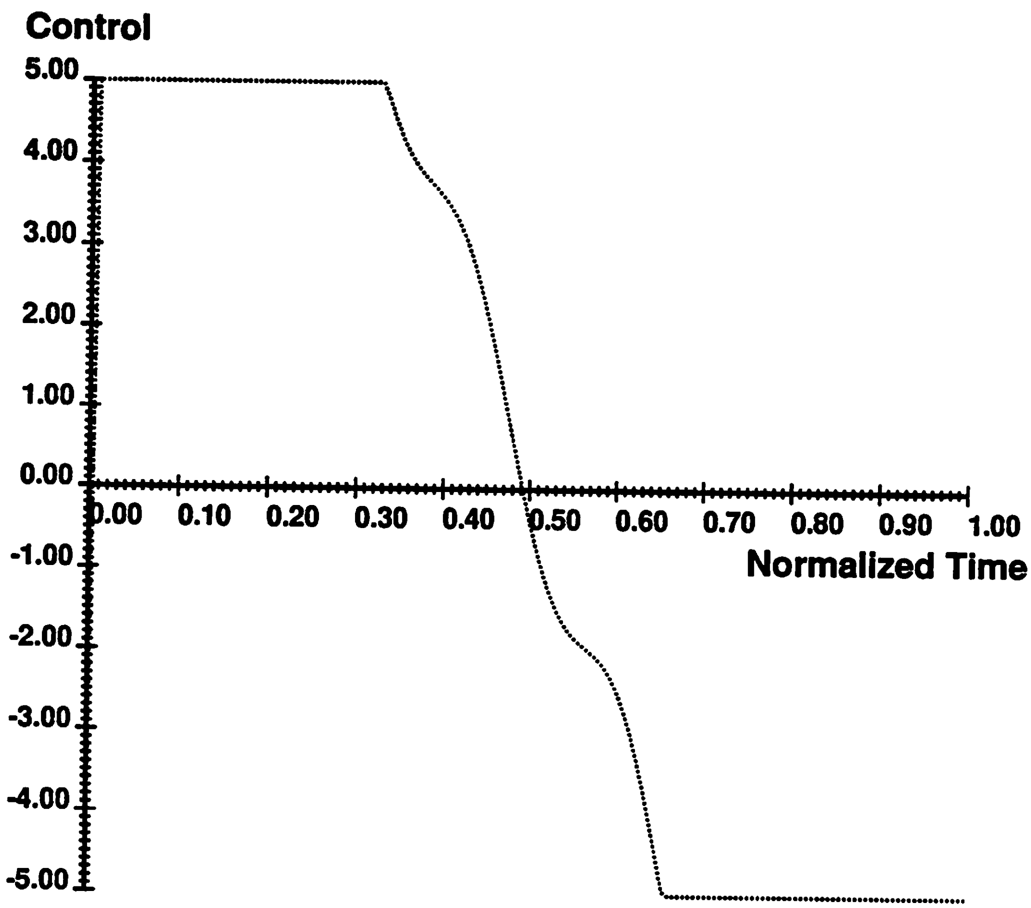

Figure 7. Final Control for Problem 2: Time of Maneuver = 0.800 seconds

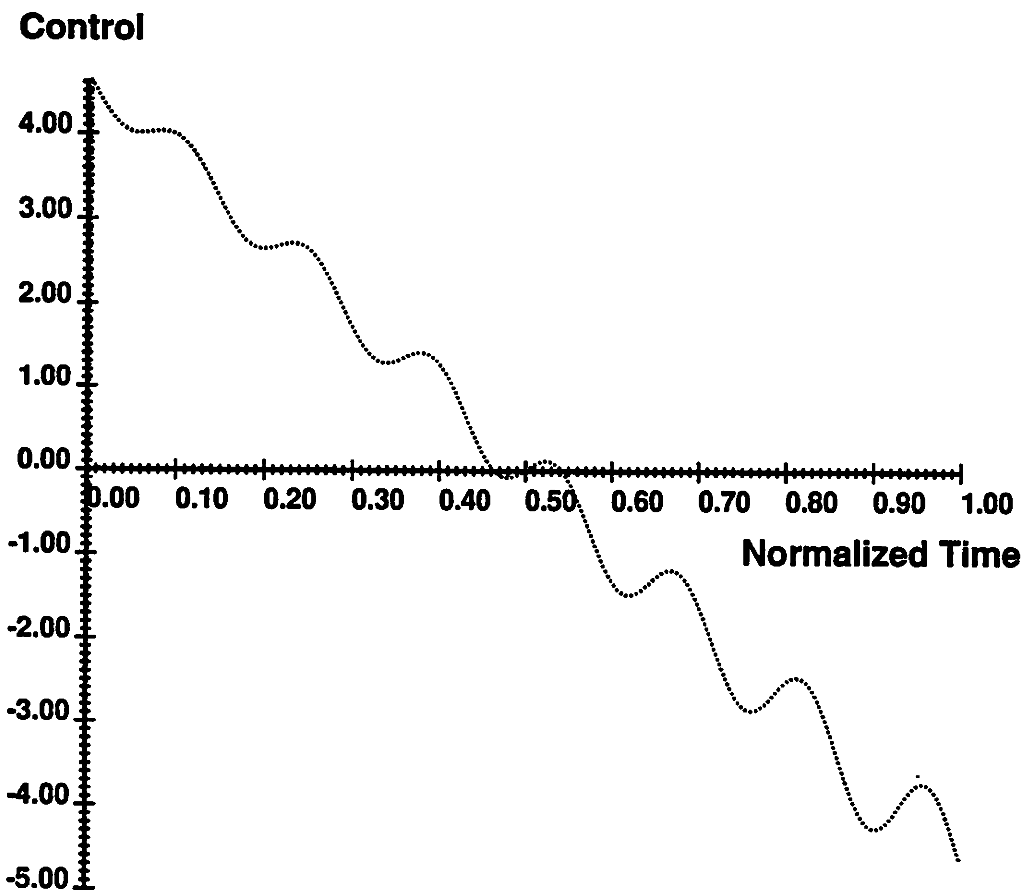

Figure 8b. Final Control for Problem 2: Time of Maneuver = 1.000 seconds

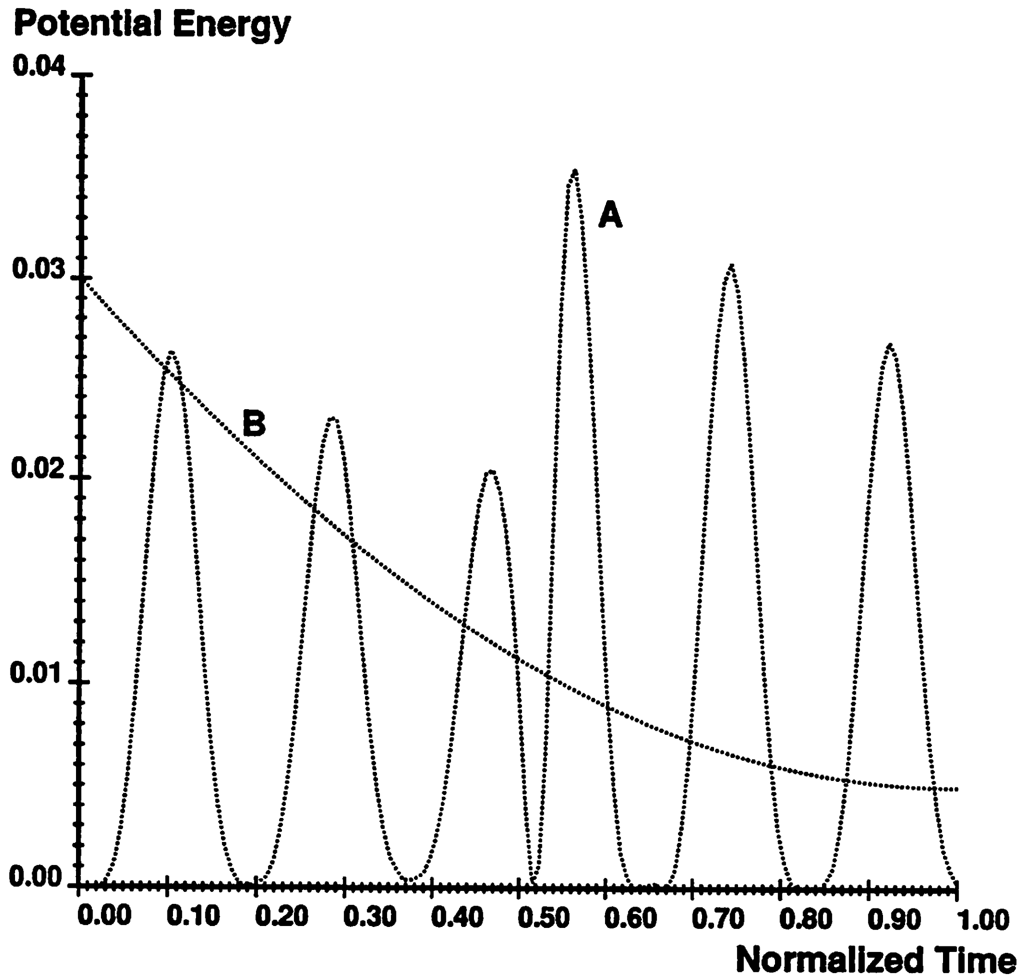

Figure 9. Problem 1: Potential Energy

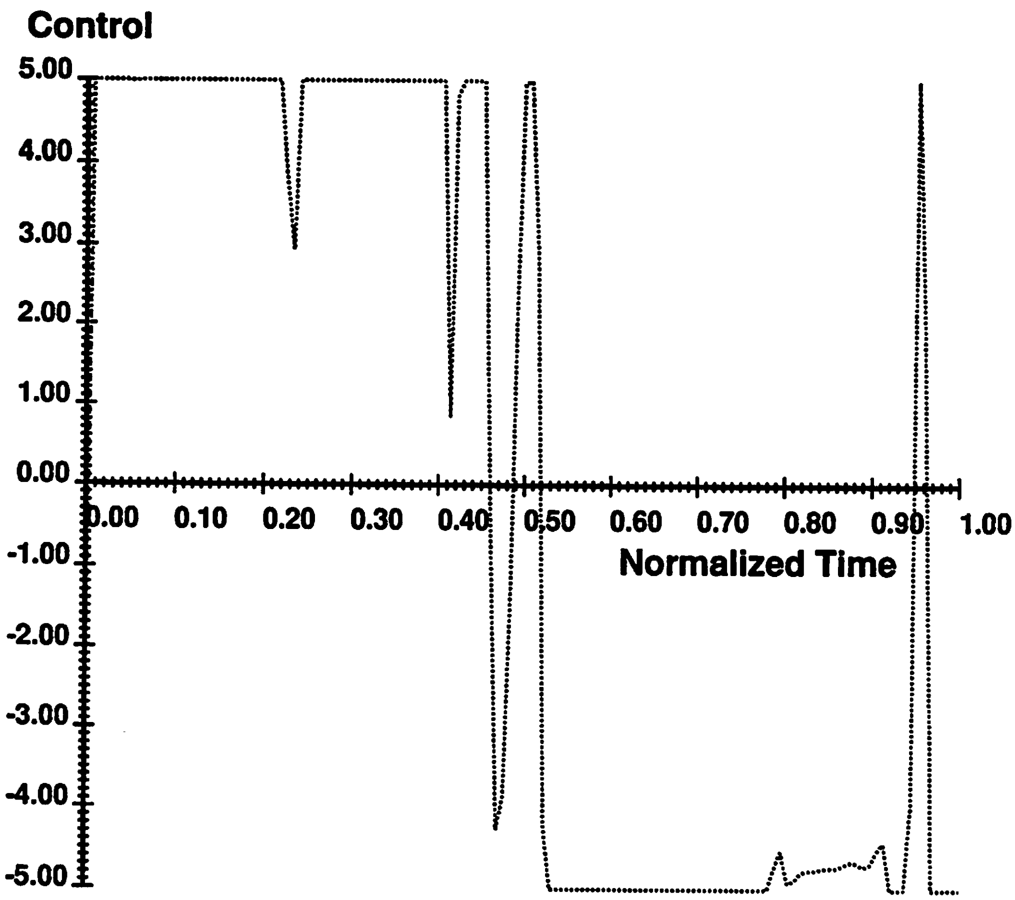

Figure 10. Final Control for Problem 3

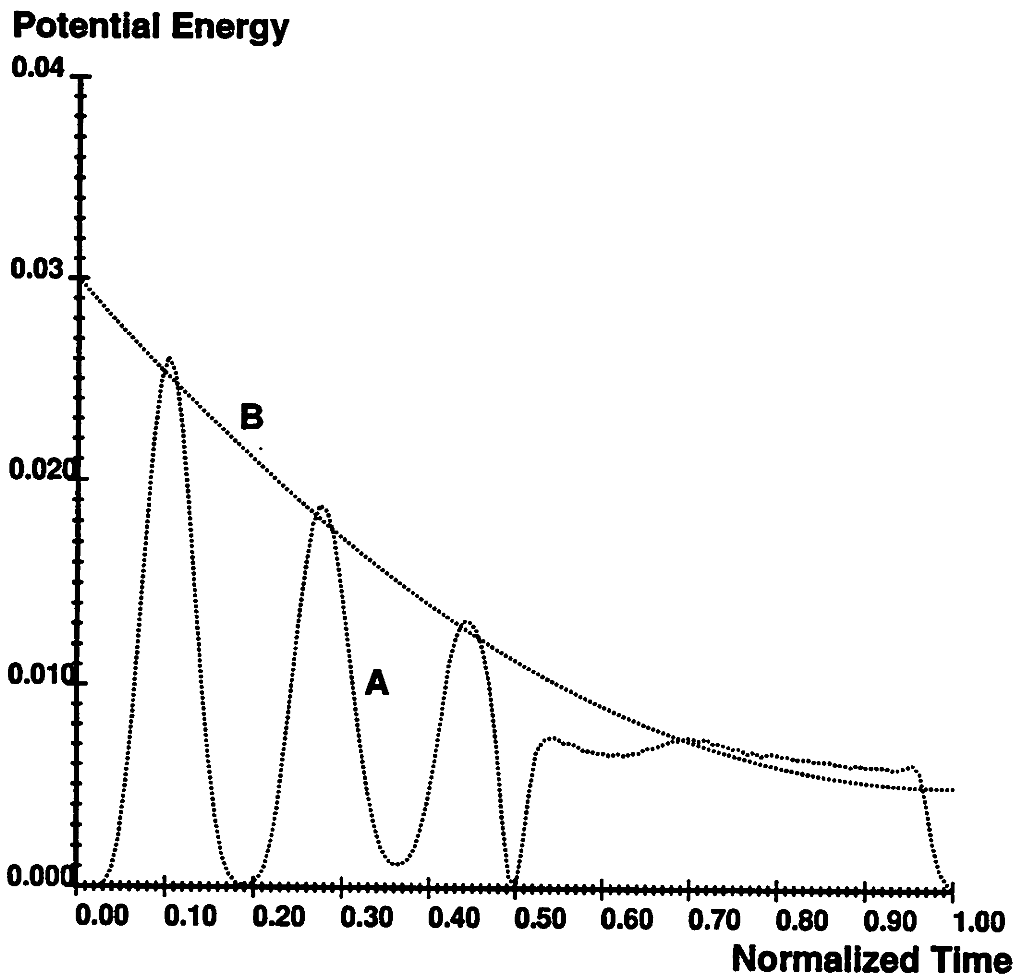

Figure 11. Problem 3: Potential Energy