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On Motion Planning for Dexterous Manipulation, Part I: The Problem Formulation

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Abstract

In the study of multifingered robot hands, the process of manipulating an object from one grasp configuration to another is called dexterous manipulation. A unique feature of dexterous manipulation is that the object can be held stably by the hand through the course of manipulation. Motion planning for dexterous manipulation amounts to generating a sequence of trajectories of the fingers and possibly of the object so that a final grasp configuration can be reached from an initial grasp configuration. In part I of this paper, we formulate the motion planning problem for dexterous manipulation and in the forthcoming part II we will construct solutions based on this formulation. First, we show that the configuration space of the fingers is the space that we should plan the motion. For this we decompose dexterous manipulation into the following four basic manipulation modes: (i) coordinated manipulation; (ii) rolling motion; (iii) sliding motion and (iv) finger relocation. Then, we develop motion constraints for each of the manipulation modes, and show that for finger motions that satisfy these constraints these exists a well defined map, called the hand map, which maps the finger motion onto the object motion. It is precisely the hand map that enables us to determine completely the state of the hand manipulation system and plan motions in the configuration space of the fingers only. We also classify other types of constraints such as finger kinematics constraints and constraints for collision avoidance. Special to this paper is the incorporation of dynamic constraints into motion planning. Also, the hand map has extreme importance of its own: it gives an intrinsic characterization of the workspace of a multifingered robot hand. This definition of hand workspace depends neither on the initial grasp configuration nor on the manipulation modes. It is an invariant associated with the kinematic structure of the hand and the object. Thus, it provides a criterion for evaluating designs of multifingered robot hands.

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1 Introduction

A fascinating new area of robotics research has been the use of multifingered robot hands for dexterous manipulation. The versatility and dexterity proved by human hands have lured many researchers into constructing computer-controlled multifingered hands to perform functions similar to that of a human. Undoubtedly, a multifingered robot hand can accomplish a much larger class of tasks than a simple parallel-jaw gripper. For example, consider a scribing task. The pencil can be picked up in a stable grasp configuration, and then be manipulated within the hand to a final grasp configuration, which is usually a better grasp for the task than the initial grasp ([9]). The ability to adjust grasp configurations without dropping the object is the key feature associated with a multifingered robot hand, and this feature is absent from a simple parallel-jaw gripper. Moreover, for almost all sophisticated tasks the optimal grasp configuration can not be obtained from the initial grasp with accessibility constraints. *The procedure for adjusting grasps without the risk of dropping the object has become indispensable for accomplishing a sophisticated task, and is called dexterous manipulation by a multifingered robot hand.*

Over the last decade, there have been a great deal of works in building the basic building blocks for this problem ([3], [4], [7], [5], [10], [11], [12], [15]). For example, Salisbury ([12]) and Cutkosky ([4]) have formulated the contact modeling problem between robot fingers and objects; Kerr ([7]) studied hand kinematics; Montana ([11]) developed the kinematics of contact and Fearing ([5]) designed tactile sensors for robot fingers. Recently, Li et al ([10]) proposed a coordinated control algorithm for object manipulation, Cole et al ([3]) showed that this algorithm could be combined with the work of [11] and [7] to give a coordinated control algorithm for pure rolling constraints. Trinkle ([15]) also looked at the problem of sliding fingers across the object while maintaining the object in the hand. Among these control related problems ([3], [10], [11] [15]), people have assumed that trajectories of the object or the fingers are given, and the control algorithms generate the appropriate torque commands to realize the desired trajectories.

If we define phase (I) of dexterous manipulation to be *the autonomous generation of finger/object trajectories that lead from the initial to the final grasp configurations*, and phase (II) to be *the development of appropriate control laws for the fingers to realize the desired trajectories*, then, in our view phase (II) of dexterous manipulation has been accomplished through the combined effort of ([3], [10], [7], [11], [15]), and phase (I) is still unknown. It is the objective of this two part paper to complete phase (I) of dexterous manipulation.

Our strategy for solving this problem consists of the following: First, we formulate the problem into a "generalized" robot motion planning problem. This amounts to (i) identifying the appropriate configuration space and (ii) classifying motion constraints on the configuration space. Then, we solve the generalized motion planning problem by extending and applying known results from robot motion planning ([1], [8], [13]). We feel that the work of Canny ([1]) most closely matches our problem.

We summarize here briefly what robot motion planning, one of the most studied problems in robotics, can provide us: Given an initial and a final configurations in the configuration space of a manipulator, construct a trajectory that connects the two configurations. The trajectory must have the property that the manipulator along it is collision free with the obstacles in the environment. Canny's solution to this problem is as follows: First, a coordinatization of the configuration space is chosen, and the set of collision free configurations is defined in terms of the coordinate variables, which constitutes a semi-algebraic set Q. Second, Q is stratified into manifolds of lower dimensions. Third, using tools from algebraic topology, a one dimensional curve in Q that connects the initial to the final configurations is constructed, along with some complexity bounds. Thus, when the manipulator follows this one dimensional curve from the initial configuration to the final configuration, it is guaranteed to be collision free with the environment obstacles.

To put the current problem into a robot motion planning problem, the first thing is to identify the appropriate configuration space where motion can be planned. For a hand manipulation system, however, we have at least the following candidates (see Table 1 for the notation): (1) The configuration space of the object, $SE_o(3)$; (2) the configuration space of the k-fingers, $SE^k(3)$; (3) the product space $SE^k(3) \times SE_o(3)$; and (4) the space of contact points together with one of the above listed spaces.

We argue that the appropriate space is the configuration space of the fingers. For this, let us assume that both the object and the finger are smooth and convex, and at least one of them is strictly convex. Thus, given configurations of the object and the finger where they are in contact, the points of contact wil be uniquely determined. This eliminates the space of (4) from the candidate list. Also, the space $SE_0(3)$ is clearly not adequate for completely specifying motion of the system, this rules (1) out. For the remaining two spaces, we see that the space of (3) is 6 dimensions larger than the space of (2). Furthermore, motion of the object is not directly controllable. Thus, defining trajectories for the object in $SE_o(3)$ is meaningless unless certain constraints can be satisfied ([10]). This eliminates (3) as well. But, if the configuration space of the fingers, $SE^k(3)$, is where we should plan the motion, it has to have the following property: *under appropriate constraints on $SE^k(3)$, the state of the hand manipulation system should be well defined for a given configuration of the fingers,* $\underline{g}_{f,p} \in SE^k(3)$. By the earlier assumption that both the object and the fingers are convex, the above is true if the configuration of the object is related to that of the fingers.

We will decompose dexterous manipulation into the following manipulation modes: (a) Coordinated manipulation; (b) Rolling motion; (c) Sliding motion and (d) Finger relocation. For each of the manipulation modes appropriate constraints on the finger trajectories will be imposed. We then show that, for finger trajectories which satisfy these constraints the trajectory of the object is well defined and is related to that of the fingers' by a map, called the hand map "H" (Section 5). Thus, if Q denote the set of admissible finger trajectories, then $H(Q) \subset SE_o(3)$ is the corresponding set of object trajectory. Let $(\underline{g}^0_{f,p}, g^0_{o,p})$, $(\underline{g}^f_{f,p}, g^f_{o,p}) \in Q \times H(Q) \subset SE^k(3) \times SE_o(3)$ be the initial and the final grasp configurations, respectively. Then, if a trajectory $\gamma(t) \in Q, t \in [0, t_f]$, with the property that $\gamma(0) = \underline{g}^0_{f,p}$; $\gamma(t_f) = \underline{g}^f_{f,p}$ and $H(\gamma(0)) = g^0_{o,p}$; $H(\gamma(t_f)) = g^f_{o,p}$, can be constructed, the problem of dexterous manipulation is completely solved. This is the theme of this two part paper.

An outline of part I is as follows: In Section 2 we use the building blocks developed in ([7], [10], [11]) to formulate the kinematics of a hand manipulation system. In Section 3 we classify the generic types of motion constraints. These include finger kinematics constraints and constraints for collision avoidance. In Section 4, we define constraints for each of the manipulation modes. These constraints differ from those of Section 3 in the sense that they are dynamic. In Section 5, we define the hand map for each of the manipulation modes. In Section 6, we give a non-trivial example to illustrate the preceding discussions. Finally, in Section 7 we conclude the paper with several important remarks.

2 Kinematics of a Multifingered Robot Hand

In this section, we study the kinematics of a multifingered robot hand system.

Consider the hand manipulation system shown in Figure 1. To describe motion of the system, appropriate coordinate frames have been attached to the respective bodies. For

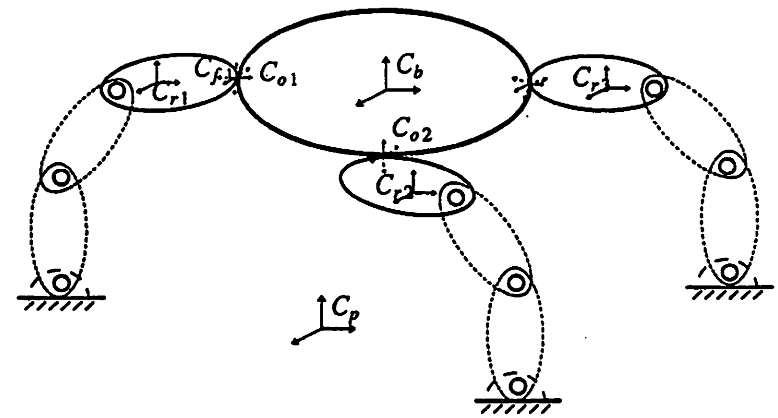

Figure 1: A hand manipulation system

any two coordinate frames C_i, C_j, where i, j are arbitrary subscripts, let $g_{i,j} = (r_{i,j}, R_{i,j}) \in SE(3)$ denote the translation and rotation of C_i relative to C_j. Then, the translational velocity of C_i relative to C_j is given by $v_{i,j} = R^t_{i,j}\dot{r}_{i,j}$, and the rotational velocity by $w_{i,j} = S^{-1}(R^t_{i,j}\dot{R}_{i,j})$, where $S : R^3 \longrightarrow so(3)$ is the operator that identifies $\Re^3$ with the space of 3×3 skew symmetric matrices.

Without loss of generality, a finger will be represented by its last link, and its configuration space will be denoted by $SE_i(3), i = 1, ...k$. Also, we let $SE_o(3)$ denote the configuration space of the object being manipulated.

We make the following assumptions: *(A1) The boundaries of the object and of the fingers are smooth 2-dimensional surfaces in $\Re^3$; and (A2) both the object and the finger are convex and at least one of them is strictly convex.* As a consequence of these assumptions, we have (a) whenever two bodies are in contact the contact points will be unique; and (b), if $S_o \subset \Re^3$ represent the boundary of the object, then S_o can be expressed as the union of $m_o \in \mathcal{N}$ open sets $\{S^j_o\}_{j\in\underline{m}_o}$, where each S^j_o is the image of a diffeomorphism $\phi^j_o : U \subset \Re^2 \longrightarrow S^j_o$. Furthermore, the partial derivatives $(\phi^j_o)_u(\mathbf{u})$, $(\phi^j_o)_v(\mathbf{u})$ will be linearly independent for all $\mathbf{u} = (u, v) \in U$. The pair (ϕ^j_o, U) is called a coordinate chart of S_o, and the coordinates of a point $s \in S^j_o$ is given by $(u, v) = (\phi^j_o)^{-1}(s)$. The set $\{S^j_o\}_{j\in\underline{m}_o}$ is called the atlas of S_o. Similarly, we can define coordinate charts $(\phi^j_i, U)_{j\in\underline{m}_i}, m_i \in \mathcal{N}$ and atlas $(S^j_i)_{j\in\underline{m}_i}$ for $S_i, i \in \underline{k}$.

Following the notation of [11] (See also appendix A), we denote by $K_o(s) \in \Re^{2\times2}$ the curvature form, $T_o(s) \in \Re^{1\times2}$ the torsion form, and $M_o(s) \in \Re^{2\times2}$ the metric of the object at $s \in S_o$. These notions are invariant of the surface and can be computed using the coordinate chart $\{\phi_o^j, U\}_{j \in \underline{m}_o}$. Similar definitions hold for $S_i, i \in \underline{k}$.

To describe the kinematics of contact between finger i and the object, we let $c_{oi}(t) \in S_o \subset \Re^3$ and $c_{fi}(t) \in S_i \in \Re^3$ be the positions at time t of the point of contact relative to C_o and C_{ri}. We will restrict our attention to an interval I such that $c_{oi}(t), c_{fi}(t)$ belong to a single coordinate chart of S_o and S_i, respectively. A set of coordinate frames is defined as follows: The local contact frame C_{oi} of the object has origin at the point of contact (i.e., $r_{oi,o} = c_{oi}$) and z-axis the outward pointing normal to S_o. C_{oi} is fixed relative to C_o. Similarly, the local frame C_{fi} of finger i has origin at the point of contact (i.e., $r_{fi,ri} = c_{fi}$) and z-axis the outward pointing normal to S_i. C_{fi} is fixed relative to C_{ri}. The local frames C_{oi}, C_{fi} share a common origin and have their $x-$, y -axes in the common tangent plane. We define the contact angle ψ_i by the angle between the x-axes of C_{oi} and C_{fi}. We choose the sign of ψ_i so that a rotation of C_{oi} through $-\psi_i$ around its z-axis aligns the x-axes.

We let (v_x^i, v_y^i, v_z^i) and (w_x^i, w_y^i, w_z^i) denote the translational and the rotational velocity of C_{oi} relative to C_{fi}. These are in fact the velocities of the object relative to the finger expressed in their respective local frames. Let $(v_{oi,p}, w_{oi,p})$ be the velocity of the object (relative to C_p) and $(v_{fi,p}, w_{fi,p})$ the velocity of finger i, then the following relation exists ([10]).

$$\begin{bmatrix} v_{oi,p} \\ w_{oi,p} \end{bmatrix} = \begin{bmatrix} \hat{A}_{\psi_i} & 0 \\ 0 & \hat{A}_{\psi_i} \end{bmatrix} \begin{bmatrix} v_{fi,p} \\ w_{fi,p} \end{bmatrix} + \begin{bmatrix} v_x^i \\ v_y^i \\ v_z^i \\ w_x^i \\ w_y^i \\ w_z^i \end{bmatrix}, \tag{1}$$

where

$$\hat{A}_{\psi_i} = \begin{bmatrix} \cos\psi_i & -\sin\psi_i & 0 \\ -\sin\psi_i & -\cos\psi_i & 0 \\ 0 & 0 & -1 \end{bmatrix} \triangleq \begin{bmatrix} A_{\psi_i} & 0 \\ & 0 \\ 0\,0 & -1 \end{bmatrix}.$$

(1) simply says that whenever finger i is in contact with the object its velocity is related to that of the object by an affine transformation, whereas the affine part is given by the relative velocity terms.

Note that some components of the relative velocity in (1) have to be zero in order to satisfy certain contact constraints. For example, for *(a) fixed contact points, we have*

$$\begin{bmatrix} v_x^i \\ v_y^i \\ v_z^i \end{bmatrix} = 0, \textit{and } w_x^i = w_y^i = 0; \tag{2}$$

(b) coordinated manipulation with rolling constraints, we have

$$\begin{bmatrix} v_x^i \\ v_y^i \\ v_z^i \end{bmatrix} = 0, \quad and\ w_z^i = 0; \tag{3}$$

and (c) finger i sliding across the object surface, we have

$$v_z^i = 0\ and \begin{bmatrix} w_x^i \\ w_y^i \\ w_z^i \end{bmatrix} = 0. \tag{4}$$

When (2) or (3) holds, one has from (1) that

$$v_{oi,p} = \hat{A}_{\psi_i} v_{fi,p}. \tag{5}$$

On the other hand, since C_{oi} is fixed relative to C_o, the velocity of C_{oi} is related to the velocity of C_o by a similarity transformation, given by

$$\begin{bmatrix} v_{oi,p} \\ w_{oi,p} \end{bmatrix} = Ad_{g_{oi,o}^{-1}} \begin{bmatrix} v_{o,p} \\ w_{o,p} \end{bmatrix}, \tag{6}$$

where

$$Ad_{g_{oi,o}^{-1}} = \begin{bmatrix} R_{oi,o}^t & -S(r_{oi,o})R_{oi,o}^t \\ 0 & R_{oi,o} \end{bmatrix}, \quad g_{oi,o} = (r_{oi,o}, R_{oi,o}).$$

Similarly, one has for the finger that

$$\begin{bmatrix} v_{fi,p} \\ w_{fi,p} \end{bmatrix} = Ad_{g_{fi,ri}^{-1}} \begin{bmatrix} v_{ri,p} \\ w_{ri,p} \end{bmatrix}. \tag{7}$$

Using (6) and (7) we can rewrite (5) in the form

$$B_i^t Ad_{g_{oi,o}^{-1}} \begin{bmatrix} v_{o,p} \\ w_{o,p} \end{bmatrix} = B_i^t T_{ri,oi} \begin{bmatrix} v_{ri,p} \\ w_{ri,p} \end{bmatrix}, \tag{8}$$

where

$$B_i^t = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } T_{ri,oi} = \begin{bmatrix} \hat{A}_{\psi_i} & 0 \\ 0 & \hat{A}_{\psi_i} \end{bmatrix} Ad_{g_{fi,ri}^{-1}}.$$

Summing (8) up for $i = 1, \ldots k$, yields the well known velocity constraint equation for a hand system.

$$G^t \begin{bmatrix} v_{o,p} \\ w_{o,p} \end{bmatrix} = \hat{J}\ \xi_{f,p}, \tag{9}$$

where

$$\hat{J} = \text{Diag}\{B_1^t, \ldots B_k^t\}\ \text{Diag}\{T_{ri,oi}, \ldots . T_{rk,bk}\} \in \Re^{3k \times 6k}, \quad \xi_{f,p} = \begin{bmatrix} v_{r1,p} \\ w_{r1,p} \\ \cdot \\ \cdot \\ \cdot \\ v_{rk,p} \\ w_{rk,p} \end{bmatrix} \in \Re^{6k}$$

is the vector of finger velocity, and

$$G = \left[Ad_{g^{-1}_{o1,o}}, ... Ad_{g^{-1}_{ok,o}} \right] \mathrm{Diag}\{B_1, ... B_k\} \in \Re^{6 \times 3k}$$

is the grip Jacobian. *Notice that G depends on the contact points $\underline{c}_o = \{c_{oi}\}_{i \in \underline{k}} \in \Re^{3k}$ only, and since $p_{oi} = (\varphi_o^j)^{-1}(c_{oi})$, and $\underline{p}_o \triangleq \{p_{oi}\}_{i \in \underline{k}} \in \Re^{2k}$, we can write $G = G(\underline{c}_o) = G(\underline{p}_o)$.*

The wrenches exerted upon the object by the fingers can be expressed as

$$\begin{bmatrix} f_o \\ \tau_o \end{bmatrix} = G \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_k \end{bmatrix}, \tag{10}$$

where $\tau_o \in \Re^3$ is the torque about the origin of C_o, and f_o is the linear force; x_i is the component vector of contact wrenches of finger i and is constrained to lie in the friction cone K_i, specified by

$$K_i = \{x_i \in \Re^3,\ x_{i,3} \leq 0,\ x_{i,1}^2 + x_{i,2}^2 \leq \mu^2 x_{i,3}^2\}$$

where $\mu \in \Re_+$ is the static Coulomb friction coefficient.

We let

$$K = K_1 \oplus ... \oplus K_k \tag{11}$$

denote the friction cone of the hand.

Definition 1 *(Grasp or Force Closure Condition)*[1] *A set of contact points $\underline{c}_o = \{c_{oi}\}_{i \in \underline{k}} \in \Re^{3k}$ (or $\underline{p}_o = \{p_{oi}\}_{i \in \underline{k}} \in \Re^{2k}$) is said to form a grasp if $G = G(\underline{c}_o)$ satisfies*

$$G(K) = \Re^6. \tag{12}$$

When (12) is true, then G is onto (or the null space of G^t is empty), and $G\ G^t \in \Re^{6 \times 6}$ is nonsingular. Consequently, there exists a well defined map from the velocity space of the fingers to the velocity space of the object. This is given from (9) by

$$\begin{bmatrix} v_{o,p} \\ w_{o,p} \end{bmatrix} = F \xi_{f,p}, \quad \text{where } F = (GG^t)^{-1} G \hat{J} \in \Re^{6 \times 6k}. \tag{13}$$

Clearly, the entries of the matrix F depend on the contact coordinates of the object and of the fingers, and the contact angles. We write $F = F(\underline{p}_o, \underline{p}_f, \underline{\psi})$ to emphasize this dependence. On the other hand, the contact coordinates and the contact angles evolve as a

[1]The grasp condition can be transformed into constraint equations on the configuration variables of the fingers using the hand map of Section 5. See Appendix B.

function of the relative velocity. According to the kinematic equations of contact ([11]), we have

$$\begin{aligned}
\dot{p}_{oi} &= M_{oi}^{-1}(K_{oi}+\hat{K}_{fi})^{-1}\left(\begin{bmatrix} -w_y^i \\ w_x^i \end{bmatrix} - \hat{K}_{fi}\begin{bmatrix} v_x^i \\ v_y^i \end{bmatrix}\right), && (14)\\
\dot{p}_{fi} &= M_{fi}^{-1}A_{\psi_i}(K_{oi}+\hat{K}_{fi})^{-1}\left(\begin{bmatrix} -w_y^i \\ w_x^i \end{bmatrix} + K_{oi}\begin{bmatrix} v_x^i \\ v_y^i \end{bmatrix}\right), && (15)\\
\dot{\psi}_i &= w_z^i + T_{oi}M_{oi}\dot{p}_{oi} + T_{fi}M_{fi}\dot{p}_{fi}, && (16)\\
v_z^i &= 0, && (17)
\end{aligned}$$

where

$$\hat{K}_{fi} = A_{\psi_i}K_{fi}A_{\psi_i}$$

is the curvature form of the finger seen by the object. Note that the curvature forms, the torsion forms as well as the metric forms have been evaluated at the point of contact, i.e., $K_{oi} = K_{oi}(p_{oi})$, and etc.

3 Classification of Motion Constraints

In this section, we classify the set of basic constraints on finger motions. These include *(i) constraints for collision avoidance and (ii) constraints by the kinematic structures of the fingers.* These constraints will be defined on the configuration space of the fingers: $SE^k(3) \triangleq SE_1(3) \times \ldots \times SE_k(3)$.

We assume that geometries of the fingers/object are known, and they satisfy assumptions (A1) and (A2). Furthermore, parameterizations of the fingers/object are given.

A. Constraints for Collision Avoidance

During the course of manipulation, collisions between links of all k-fingers should be prevented. Since each finger is represented by its last link, the constraints can be formulated directly in terms of the finger configuration variables. Consider the hand manipulation system shown in Figure 1. Let "F_i", $i \in \underline{k}$, stand for finger i. Let $d : \Re^3 \times \Re^3 \longrightarrow \Re, d^2(x,y) = \sum_{i=1}^3 |x_i - y_i|^2$, be the Euclidean 2-norm. We define the distance function of "F_i" with "F_j", $j \neq i$ as follows[2]

$$d(F_i, F_j) = \min_{\substack{x \in S_j^r,\, r \in \underline{m}_j \\ y \in S_i^l,\, l \in \underline{m}_i}} d(g_{rj,p}x,\ g_{ri,p}y), \qquad (18)$$

[2]Strictly speaking, we should take the "min" over $x \in$ {the set of $\Re^3$ enclosed by S_j} and $y \in$ { the set of $\Re^3$ enclosed by S_i}. But, since "F_i" and "F_j" are separated to start with, (18) is okay.

where

$$g_{rj,p}x \triangleq R_{rj,p}x + r_{rj,p}$$

According to Canny ([1], Ch. 2), for given features of finger i and j, $d(F_i, F_j)$ defines a function on the configuration variables of finger i and finger j. Without loss of generality, we will write, that

$$d(F_i, F_j) : SE^k(3) \longrightarrow \Re. \tag{19}$$

For computational advantages Canny used quaterion coordinates for the orientation space SO(3). But, from [14] conversions between quaterion coordinates and orientation matrices are rather straightforward.

Definition 2 *Collision between finger i and finger j can be prevented if and only if*

$$d(F_i, F_j) > 0, \tag{20}$$

The subspace of $SE^k(3)$ where finger i is collision free with finger j is denoted by $d(F_i, F_j)^{-1}((0, \infty))$ $\triangleq \{(g_{r1,p}, ...g_{rk,p}) \in SE^k(3) \mid d(F_i, F_j) > 0\}$, and the constraint subspace for collision avoidance of all k-fingers is the intersection:

$$\bigcap_{i<j} d(F_i, F_j)^{-1}((0, \infty)) \subset SE^k(3. \tag{21}$$

Remark: *It is straightforward, using the kinematic functions of the fingers, to formulate the constraints for collision avoidance between links of all k-fingers, where each finger has more than one link.*

B. Constraints by Finger Kinematics

The second type of basic constraint is the constraint due to the finger kinematic structures. Since the last link is connected to the hand palm by n_i links the set of reachable configurations by the finger is a compact submanifold Q_i of $SE_i(3)$. As was shown in [1], Q_i is a semi-algebraic set, and can be expressed by a set of inequalities in terms of the configuration variables, $g_{ri,p}$:

$$Q_i = \{g_{ri,p} \in SE_i(3) : f(g_{ri,p}) \geq 0\}. \tag{22}$$

The subspace of $SE^k(3)$ where finger kinematic constraints is satisfied is the product:

$$(Q_1 \times ... \times Q_k) \subset SE^k(3).$$

Finally, the subspace of $SE^k(3)$, where all the constraints discussed in this section are satisfied is given by

$$Q_s \triangleq \left\{\cap_{i<j} d(F_i, F_j)^{-1}((0,\infty))\right\} \bigcap \{(Q_1 \times ... \times Q_k)\} \subset SE^k(3). \tag{23}$$

4 The Basic Manipulation Modes

By assumptions (A1) and (A2), a state of the hand manipulation system is specified by a point $(\underline{g}_{f,p}, g_{o,p}) \triangleq (g_{r1,p}, ... g_{rk,p}, g_{o,p})$ in the space $SE^k(3) \times SE_o(3)$. We let $(\underline{g}^0_{f,p}, g^0_{o,p}) \in SE^k(3) \times SE_o(3)$ denote the initial grasp configuration, and $(\underline{g}^f_{f,p}, g^f_{o,p}) \in SE^k(3) \times SE_o(3)$ denote the final grasp configuration. The objective of dexterous manipulation is to reach from an initial grasp configuration to a final grasp configuration, by commanding the fingers to follow *a prescribed trajectory in* $SE^k(3)$. Since the object motion is affected only by motion of the fingers, it is necessary that *while the fingers travel in* $SE^k(3)$ *from* $\underline{g}^0_{f,p}$ *to* $\underline{g}^f_{f,p}$, *the object will travel in* $SE_o(3)$ *from* $g^0_{o,p}$ *to* $g^f_{o,p}$.

We will decompose dexterous manipulation into the following *manipulation modes: (A) Coordinated manipulation; (B) Rolling motion; (C) Sliding motion and (D) Finger relocation* . Also, let $[0, t_f]$ be the time interval it takes to reach from the initial state to the final state. $[0, t_f]$ is divided into the union of successive sub-intervals, i.e., $[0, t_f] = \bigcup_{i=0}^{n-1} [t_i, t_{i+1}]$, $0 = t_0 < t_1 < ... < t_n = t_f$, such that at each sub-interval $[t_i, t_{i+1}]$ the finger motion is in one of the manipulation modes.

A. Coordinated Manipulation

Coordinated manipulation by a multifingered robot hand has been studied extensively in [10]. It was shown that the fingers can be controlled to move in a coordinated fashion so that the object can be manipulated from one configuration to another. We see that, in addition to satisfying the generic types of constraints discussed in Section 3, the fingers motion must also guarantee that the points of contact can in fact stay in contact with the body. We shall formulate exactly the constraints on finger trajectories for coordinated manipulation mode.

Consider an initial state of the hand manipulation system, given by $(\underline{g}_{f,p}(0), g_{o,p}(0)) \in SE^k(3) \times SE_o(3)$. Let the initial contact points be $\underline{c}_o \in \Re^{3k}$ and $\underline{c}_f \in \Re^{3k}$, respectively, and assume that $\underline{c}_o$ form a grasp.

Let $g_{f,p}(t) = (g_{r1,p}(t), ... g_{rk,p}(t)) \in SE^k(3)$, $t \in [0, t_1]$, be a set of trajectories of the fingers. Note that the time interval $[0, t_1]$ could represent any of the sub-intervals discussed previously. From (13), let

$$\begin{bmatrix} v_{o,p} \\ w_{o,p} \end{bmatrix} = F(\underline{c}_o, \underline{c}_f, \underline{\psi})\, \xi_{f,p}, \tag{24}$$

where, with $g_{ri,p} = (r_{ri,p}, R_{ri,p}) \in SE_i(3)$,

$$\xi_{f,p} = \begin{bmatrix} v_{r1,p} \\ \cdot \\ \cdot \\ \cdot \\ w_{rk,p} \end{bmatrix}, \quad v_{ri,p} = R^t_{ri,p}\dot{r}_{ri,p}, \text{ and } w_{ri,p} = S^{-1}(R^t_{ri,p}\dot{R}_{ri,p}).$$

This is equivalent to say that the *relative translational velocity* is constrained to be zero. Now, substitute (24) into the second equation of (1) and use (6) and (7) to get an equation expressing the relative rotational velocity in terms of $\xi_{f,p}$:

$$\begin{bmatrix} w^i_x \\ w^i_y \\ w^i_z \end{bmatrix} = -\hat{A}_{\psi_i} A^t_{fi,ri} w_{ri,p} + A^t_{ri,o} w_{o,p}(\xi_{f,p}), \quad i \in \underline{k}. \tag{25}$$

According to the contact equation (17), the contact angle ψ_i is related to w^i_z by

$$\dot{\psi}^i = w^i_z, \; i \in \underline{k}. \tag{26}$$

Notice that (24), (25) and (26) together constitute a system of differential equations with algebraic constraints:

$$\left\{ \begin{array}{l} \begin{bmatrix} v_{o,p} \\ w_{o,p} \end{bmatrix} = F(\underline{c}_o, \underline{c}_f, \underline{\psi})\, \xi_{f,p}, \\ \begin{bmatrix} w^i_x \\ w^i_y \\ w^i_z \end{bmatrix} = -\hat{A}_{\psi_i} A^t_{fi,ri} w_{ri,p} + A^t_{ri,o} w_{o,p}(\xi_{f,p}), \quad i \in \underline{k}, \\ \dot{\psi}^i = w^i_z, \; i \in \underline{k}. \end{array} \right. \tag{27}$$

For a given set of finger trajectories, the relative rotational velocity can be solved from (27) in terms of the velocity of the fingers. (See [6] for further details on systems of differential equations with algebraic constraints). On the other hand, the points of contact (p_{oi}, p_{fi}) evolve as a function of the relative rotational velocity according to (15), (16):

$$\dot{p}_{oi} = M^{-1}_{oi}(K_{oi} + \hat{K}_{fi})^{-1} \begin{bmatrix} -w^i_y \\ w^i_x \end{bmatrix},$$

$$\dot{p}_{fi} = M^{-1}_{fi} A_{\psi_i}(K_{oi} + \hat{K}_{fi})^{-1} \begin{bmatrix} -w^i_y \\ w^i_x \end{bmatrix}.$$

We have

Definition 3 (<u>*Coordinated Manipulation*</u>) *We say that a set of finger trajectories* $\underline{g}_{f,p}(t) \in SE^k(3)$, $t \in [0, t_1]$, *constitutes a set of admissible trajectories for coordinated manipulation if the relative rotational velocity* $(w_x^i, w_y^i), i \in \underline{k}$ *solved from (27) is identically zero, for all* $t \in [0, t_1]$.

Remarks (1) $(w_x^i, w_y^i) = 0$ *implies that* (p_{oi}, p_{fi}) *is constant. This ensures that the contacting fingers will not slip.*

(2) *Note that* w_z^i *is not necessary zero in a coordinated manipulation mode. Each finger is allowed to spin around the contact normal.*

(3) *If the contact coordinates* p_{oi} *and* p_{fi} *stay constant, then,* G *also stays constant and the grasp condition (12) is satisfied during the course of coordinated manipulation. On the other hand,* $\hat{J}$ *is not constant, as* $\dot{\psi}_i = w_z^i$ *is not necessary zero.*

The set of finger trajectories which satisfies Definition 3 will be denoted by $\sum_C$. Clearly, we have $\sum_C \subset SE^k(3)$.

<u>B. Rolling Motion</u>

An efficient manipulation mode for effecting motion of both the object and contact coordinates is *rolling*. Cole et al ([3]) show that when the initial contact points $\underline{c}_o$ is properly chosen then the object can be manipulated with pure rolling constraints. We formulate the constraints on finger trajectories for rolling motion as follows:

Consider an initial state of the hand manipulation system, with initial contact points $\underline{c}_o(0)$ (or $\underline{p}_o(0)$) and $\underline{c}_f(0)$ (or $\underline{p}_f(0)$), respectively. Assume that $\underline{c}_o(0)$ (or $\underline{p}_o(0)$) forms a grasp.

Let $\underline{g}_{f,p}(t) \in SE^k(3)$, $t \in [0, t_1]$, be a set of finger trajectories and consider the following system of differential equations with algebraic constraints ($i \in \underline{k}$):

$$\left\{ \begin{array}{l} \begin{bmatrix} v_{o,p} \\ w_{o,p} \end{bmatrix} = F(\underline{p}_o(t), \underline{p}_f(t), \underline{\psi})\, \xi_{f,p}, \\ \begin{bmatrix} w_x^i \\ w_y^i \\ w_z^i \end{bmatrix} = -\hat{A}_{\psi_i} A^t_{fi,ri} w_{ri,p} + A^t_{oi,o} w_{o,p}(\xi_{f,p}), \\ \dot{p}_{oi} = M_{oi}^{-1}(K_{oi} + \hat{K}_{fi})^{-1} \begin{bmatrix} -w_y^i \\ w_x^i \end{bmatrix}, \\ \dot{p}_{fi} = M_{fi}^{-1} A_{\psi_i}(K_{oi} + \hat{K}_{fi})^{-1} \begin{bmatrix} -w_y^i \\ w_x^i \end{bmatrix}, \\ \dot{\psi}_i = w_z^i + T_{oi} M_{oi} \dot{p}_{oi} + T_{fi} M_{fi} \dot{p}_{fi} \end{array} \right. \qquad (28)$$

Here, the first two equations are algebraic, and $\xi_{f,p}$ is considered as an input term. The initial conditions are given by the initial state of the hand system. Let $p_{oi}(t)$ and $p_{fi}(t), i \in \underline{k}$, be the solutions of (28), and

$$c_{oi}(t) = \varphi^j_{oi}(p_{oi}(t)),\ t \in [0, t_1], i \in \underline{k}.$$

be the contact points.

Definition 4 *(Rolling Motion) We say that a set of finger trajectories $g_{f,p}(t) \in SE^k(3)$, $t \in [0, t_1]$, constitutes a set of admissible trajectories for rolling motion if: (i) $w^i_z, i \in \underline{k}$, from (28) is identically zero for all $t \in [0, t_1]$; and (ii) the set of contact points $\underline{c}_o(t) \in \Re^{3k}$ forms a grasp for all $t \in [0, t_1]$.*

Remark (1) *By the first equation of (28) sliding is not possible, and by the grasp condition (ii) the fingers are able to exert any desired wrenches upon the object through the course of rolling motion.*

The set of finger trajectories which satisfies Definition 4 will be denoted by $\sum_R \subset SE^k(3)$.

C. Sliding Motion

When a finger or a group of m fingers ($1 \le m \le k$) are commanded to slide along the object surface, the remaining (non-sliding) fingers, together with contact wrenches from the sliding fingers constrained to the boundaries of the friction cones, should be able to held the object in the same configuration. The control algorithm presented in [10] can be modified for this purpose. Constraint formulation for sliding motion is given here.

Assume that gravity is the only external force to be balanced during the course of sliding. Let $\hat{g}_p$ denote the gravity vector relative to C_p, and $Ad^t_{g^{-1}_{o,p}}(\hat{g}^t_p, 0)^t$ is then the equivalent wrench on the object relative to C_o.

Let $1 \le m \le k$ be the number of fingers to be slid simultaneously. Let $\pi_m = \{(\pi^i_m)^{i=m}_{i=1}, \pi^i_m \in \underline{k}\}$ define a permutation of m fingers to be slid. For example, if $\pi_3 = \{1, 3, 4\}$, then finger 1, 3 and 4 will be slid simultaneously. Note that for a given m, there are $\begin{pmatrix} k \\ m \end{pmatrix} = \frac{k!}{(k-m)!m!}$ different ways that a total number of m fingers can possibly slide. Thus, we have to perform $(2^k - 1)$ tests for all possible sliding motions.

Consider an initial state of the hand manipulation system given by $(\underline{g}_{f,p}(0), g_{o,p}(0)) \in SE^k(3) \times SE_o(3)$, and assume that the corresponding contact points $\underline{c}_o(0)$ forms a grasp.

Let $g_{f,p}(t) \in SE^k(3)$, $t \in [0, t_1]$, be a set of trajectories of the fingers such that

$$g_{ri,p}(t) = g_{ri,p}(0), \; \forall i \in \underline{k} \backslash \pi_m.$$

In other words, the trajectories of the non-sliding fingers stay constant. Let

$$G(K)\backslash\pi_m \triangleq \sum_{i \in \underline{k}\backslash\pi_m} G_i(K_i) \tag{29}$$

denote the set of contact wrenches from the nonsliding fingers, and

$$\sum_{i \in \pi_m} G_i(\partial K_i)$$

denote the set of contact wrenches from the sliding fingers, where ∂K_i stands for the boundary of K_i. Then, the object can be held stationary under gravity force while simultaneously sliding fingers in π_m if

$$Ad^t_{g^{-1}_{o,p}} \begin{bmatrix} \hat{g}_p \\ 0 \end{bmatrix} \in G(K)\backslash\pi_m + \sum_{j \in \pi_m} G_i(\partial K_i). \tag{30}$$

With the object configuration stays constant, the velocity of the sliding fingers relative to the object is simply the velocity of the fingers, i.e.,

$$\begin{bmatrix} v_x^1 \\ v_y^i \\ v_z^i \\ w_x^i \\ w_y^i \\ w_z^i \end{bmatrix} = -\begin{bmatrix} \hat{A}_{\psi_i} & 0 \\ 0 & \hat{A}_{\psi_i} \end{bmatrix} Ad_{g^{-1}_{fi,ri}} \begin{bmatrix} v_{ri,p} \\ w_{ri,p} \end{bmatrix}, \; \forall i \in \pi_m. \tag{31}$$

If the relative rotational velocity is zero, the contact coordinates for the sliding fingers evolve according to

$$\begin{cases} \dot{p}_{oi} = -M_{oi}^{-1}(K_{oi} + \hat{K}_{fi})^{-1}\hat{K}_{fi} \begin{bmatrix} v_x^i \\ v_y^i \end{bmatrix}, \; \forall i \in \pi_m, \\ \dot{p}_{fi} = M_{fi}^{-1} A_{\psi_i}(K_{oi} + \hat{K}_{fi})^{-1} K_{oi} \begin{bmatrix} v_x^i \\ v_y^i \end{bmatrix}, \; \forall i \in \pi_m, \\ \dot{\psi}_i = T_{oi} M_{oi} \dot{p}_{oi} + T_{fi} M_{fi} \dot{p}_{fi}, \; \forall i \in \pi_m. \end{cases} \tag{32}$$

(31) and (32) together constitute a system of differential equations with algebraic constraints, and is denoted by (*).

Definition 5 *(<u>Sliding Motion</u>) We say that a set of finger trajectories* $\underline{g}_{f,p}(t) \in SE^k(3)$, $t \in [0, t_1]$, *constitutes a set of admissible trajectories for sliding motion if: (i)* (w_x^i, w_y^i, w_z^i) *and* v_z^i *defined by (*) are identically zero for all* $t \in [0, t_1]$ *and* $i \in \pi_m$*; (ii) (30) is satisfied for all* $t \in [0, t_1]$*, and (iii) the set of contact points* $\underline{c}_o(t) \in \Re^{3k}$ *forms a grasp for all* $t \in [0, t_1]$.

Remark (1) *Condition (i) implies that pure sliding for all the fingers in π_m, (ii) implies that the object can be held in a stationary configuration under gravity force; and (iii) implies that the grasp condition will be satisfied whenever sliding motion terminates.*

The set of finger trajectories which satisfies Definition 5 will be denoted by $\sum_S^{\pi_m}$. Furthermore, we denote by $\sum_S = \bigcup_{m \in \underline{k}} \sum_S^{\pi_m}$ the set of all possible sliding trajectories.

D. Finger Relocation

Finally, we conclude this section by defining constraints for finger relocation. In a finger relocation mode, a group of m fingers $(1 \leq m \leq k)$ are allowed to break contacts with the object and they will be positioned at other locations, provided that the set of contact points by the remaining fingers still forms a grasp.

Again, let the initial state of the hand manipulation system be $(\underline{g}_{f,p}(0), g_{o,p}(o)) \in SE^k(3) \times SE_o(3)$ and Let π_m be defined as before.

Definition 6 *(Finger Relocation) We say that a set of finger trajectories $\underline{g}_{f,p}(t) \in SE^k(3)$, $t \in [0, t_1]$, such that $g_{ri,p}(t) = g_{ri,p}(0), \forall i \in \underline{k} \backslash \pi_m$ constitutes a set of admissible trajectories for finger relocation if*

$$G(K) \backslash \pi_m = \Re^6. \tag{33}$$

In other words, the set of contact points by the remaining fingers still forms a grasp.

The set of finger trajectories that satisfies Definition 6 will be denoted by $\sum_F^{\pi_m}$, and we let $\sum_F = \bigcup_{m \in \underline{k}} \sum_S^{\pi_m}$.

5 The Hand Map

In this section, we define the hand map for the set of admissible finger trajectories. Without the hand map, it would not be possible to understand dexterous manipulation. The hand map, H, relates the set of admissible finger trajectories to the corresponding object trajectory. For example, given a set of finger motions that satisfies the requirement of coordinated manipulation, the hand map tells where the object will be through the manipulation mode.

Let

$$Q = \sum_C \cup \sum_R \cup \sum_S \cup \sum_F \subset SE^k(3).$$

Then,

$$H : Q \subset SE^k(3) \longrightarrow SE_o(3) \tag{34}$$

is defined as follows:

(1) $H|_{\sum_C}$:

Let $\underline{g}_{f,p}(t) \in \sum_C, t \in [0, t_1]$. By Definition 4, $\dot{p}_{oi}(t) = \dot{p}_{fi}(t) = 0, \forall i \in \underline{k}$. Thus, $c_{oi} = \varphi_o^j(p_{oi})$ and $c_{fi} = \varphi_i^j(p_{fi})$ are constant. It is clear from Figure 1 that the coordinates of the contact point relative to the hand fixed frame, C_p is the same through either the object or the finger. This is mathematically equivalent to

$$g_{o,p}(t)c_{oi} = g_{ri,p}(t)c_{fi}, \quad \forall t \in [0, t_1], i \in \underline{k}. \tag{35}$$

Since $\underline{c}_o \in \Re^{3k}$ forms a grasp, the object configuration variable $g_{o,p}(t)$ can be solved uniquely from the set of k equations given in (35). The reader should convince herself that there are enough independent equations in (35) because the set of contact points forms a grasp. Denoting the solution by $g_{o,p}^C(t)$, $t \in [0, t_1]$, and we define $H|_{\sum_C}$ by

$$H(\underline{g}_{f,p}(t)) = g_{o,p}^C(t), \quad \forall g_{f,p}(t) \in \sum_C, t \in [0, t_1]. \tag{36}$$

Remark (1). *An alternative procedure for solving $g_{o,p}(t)$ from $\underline{g}_{f,p}(t)$ is as follows: By definition, we can write in the homogeneous representation that*

$$g_{o,p}^{-1}(t)\dot{g}_{o,p}(t) = \begin{bmatrix} S(w_{o,p}) & v_{o,p} \\ 0 & 0 \end{bmatrix} \in \Re^{4\times 4}.$$

Substituting (17) into the above equation and rearranging the terms, yields a differential equation for $g_{o,p}(t)$:

$$\dot{g}_{o,p}(t) = g_{o,p}\hat{F}(\xi_{f,p}, \underline{\psi}(t)), \tag{37}$$

where $\hat{F}$ has been derived from F. Notice that the only variables of $\hat{F}$ are $\xi_{f,p}$ and $\underline{\psi}(t)$. On the other hand, the differential equation governing the contact angles is given from (16) by

$$\dot{\psi}_i(t) = w_z^i(\underline{\psi}, \xi_{f,p}), \quad i \in \underline{k}. \tag{38}$$

(37) and (38) form a system of coupled differential equations and it can be solved using Newton's algorithm to give $g_{o,p}(t)$.

(2) $H|_{\sum_R}$:

Let $\underline{g}_{f,p}(t) \in \sum_R, t \in [0, t_1]$. By Definition 5, $(v_x^i, v_y^i, v_z^i) = 0$, and $w_z^i = 0, \forall i \in \underline{k}$. Moreover, since $\underline{p}_o(t), \underline{p}_f(t) \in \Re^{2k}$, and $\underline{\psi} \in \Re^k$ can be solved from the system of differential equations given in (28)) and $\underline{c}_o(t)$ forms a grasp for all $t \in [0, t_1]$, the following differential equation of $g_{o,p}(t)$ is well defined for all $t \in [0, t_1]$ (see the previous remark).

$$\dot{g}_{o,p}(t) = g_{o,p}(t)\hat{F}(\underline{p}_o(t), \underline{p}_f(t), \underline{\psi}(t), \xi_{f,p}) \tag{39}$$

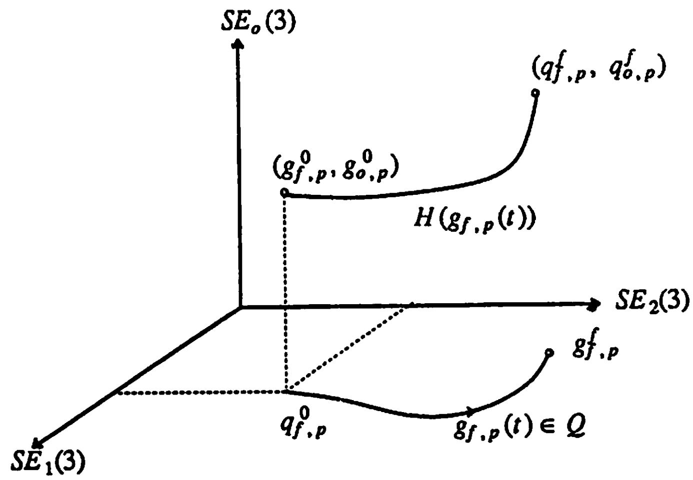

Figure 2: Action of the hand map.

Let the solution to (39) be denoted by $g_{o,p}^R(t) \in SE_o(3), t \in [0, t_1]$, and we define

$$H(g_{f,p}(t)) = g_{o,p}^R(t), \forall t \in [0, t_1], g_{f,p}(t) \in \textstyle\sum_R. \tag{40}$$

(3) $H|_{\sum_S}$:

It is clear from the context of sliding motion that the object configuration is constant, so that we have

$$H(g_{f,p}(t)) = g_{o,p}(0), \ \forall t \in [0, t_1], g_{f,p}(t) \in \textstyle\sum_S \tag{41}$$

(4) $H|_{\sum_F}$:

The Hand map for finger relocation mode is the same as for the sliding mode, we set

$$H(g_{f,p}(t)) = g_{f,p}(0), \ \ \forall t \in [0, t_1], g_{f,p}(t) \in \textstyle\sum_F. \tag{42}$$

This completes the definition of the hand map.

One can visualize the hand map by drawing the finger configuration space in the plane and the object configuration space in the vertical axis. H is then the height function associated with each admissible finger trajectory in Q. Let $(g_{f,p}^0, g_{o,p}^0) \in SE^k(3) \times SE_o(3)$ be the initial state of the hand manipulation system, and $(g_{f,p}^f, g_{o,p}^f)$ the final state. The objective of motion planning is to generate a sequence of finger trajectories, $g_{f,p}(t) \in Q$, $t \in \cup_{i=1}^{i=n}[t_{i-1}, t_i]$, $t_0 < t_1 < ...t_i < t_{i+1} < ... < t_n = t_f$, such that $g_{f,p}(0) = g_{f,p}^0$, $g_{f,p}(t_n) = g_{f,p}^f$, along with $H(g_{f,p}(0)) = g_{o,p}^0$, and $H(g_{f,p}(t_n)) = g_{o,p}^f$.(See Figure 2.)

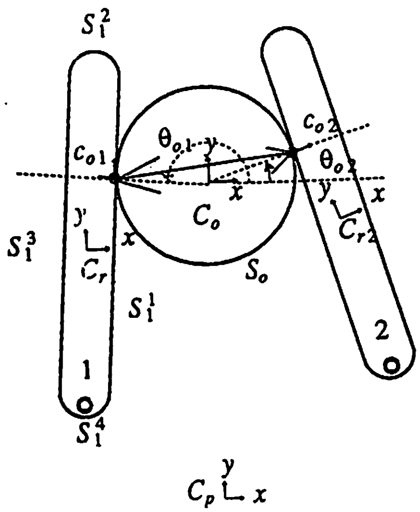

Figure 3: A hand manipulation system in $\Re^2$

Note that the hand workspace relative to an object can be defined as " *the set of of reachable configurations of the object while holding the object stably in the hand*"[3]. Then, from the preceeding discussions we have

Proposition 1 *(Hand workspace) Let $V(g^0_{b,f}, g^0_{o,p}) \subset SE_o(3)$ be the set of reachable configurations of the object, starting from the initial grasp configuration $(\underline{g}^0_{f,p}, g^0_{o,p})$, Then, $V(\underline{g}^0_{f,p}, g^0_{o,p}) = H(Q)$.*

In fact, this notion of hand workspace does not depend on the initial grasp. Let $(\underline{g}^0_{f,p}, g^0_{o,p})$ and $(\underline{g}^1_{f,p}, g^1_{o,p})$ be two initial grasps, such that $(\underline{g}^1_{f,p}, g^1_{o,p}) \in V(\underline{g}^0_{f,p}, g^0_{o,p})$, then it is not difficulty to see that $V(\underline{g}^1_{f,p}, g^1_{o,p}) = V(\underline{g}^0_{f,p}, g^0_{o,p})$.

In other words, $V(\underline{g}^0_{f,p}, g^0_{o,p})$ gives the connected component of the hand workspace.

The computation of the hand workspace is a hope of current research.

6 An Important Example in $\Re^2$

In this section, we present an example to illustrate the preceding discussions. We also provide a solution to the motion planning problem.

Consider the two-fingered planar manipulation system with a unit circle, shown in Figure 3. Each of the cylindrical fingers has 2 unit length, $\frac{1}{4}$ unit width, and its boundary can be represented as the union of four pieces: $S_i = \bigcup_{j=1}^{4} S_i^j, i = 1, 2$.

[3]Such a definition is suggested to us by A.K. Pradeep

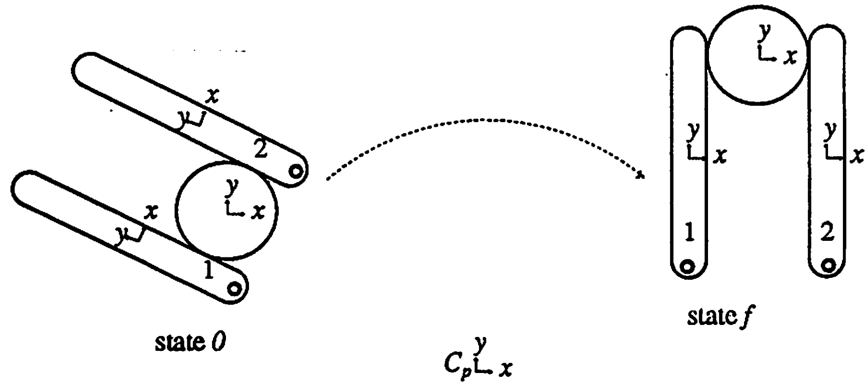

Figure 4: From state 0 to state f by dexterous manipulation

The initial and final grasp configurations are labeled with state 0 and state f(Figure 4). Gravity force is along the y-direction of C_p. For simplicity we will assume that the fingers have no kinematic constraint.

The configuration space of the object/fingers is given by $SE_o(2)/SE_i(2), i = 1,2$, which consists of translations in $\Re^2$ and rotations around the normal that points outward. The homogeneous representation of the configuration variables are

$$g_{o,p} = \begin{bmatrix} \cos\theta_o & -\sin\theta_o & x_o \\ \sin\theta_o & \cos\theta_o & y_o \\ 0 & 0 & 1 \end{bmatrix} \text{ and } g_{ri,p} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & x_i \\ \sin\theta_i & \cos\theta_i & y_i \\ 0 & 0 & 1 \end{bmatrix}, i = 1,2$$

Here, (x_o, y_o) are coordinates of the origin of C_o relative to C_p, and θ_o is the angle between the x−axis of C_o and the x−axis of C_p, measured counter-clock wise. The object/fingers velocities are related to the derivatives of the configuration variables by

$$\begin{bmatrix} v_{o,p} \\ w_{o,p} \end{bmatrix} = \begin{bmatrix} \dot{x}_o \cos\theta_o + \dot{y}_o \sin\theta_o \\ \dot{y}_o \cos\theta_o - \dot{x}_o \sin\theta_o \\ \dot{\theta}_o \end{bmatrix}, \tag{43}$$

and

$$\begin{bmatrix} v_{ri,p} \\ w_{ri,p} \end{bmatrix} = \begin{bmatrix} \dot{x}_i \cos\theta_i + \dot{y}_i \sin\theta_i \\ \dot{y}_i \cos\theta_i - \dot{x}_i \sin\theta_i \\ \dot{\theta}_i \end{bmatrix}, \quad i = 1,2. \tag{44}$$

The initial and final states of the system are represented as points in the space $(g_{f,p}, g_{o,p}) \in SE^2(2) \times SE_o(2)$, where $g_{f,p} = (g_{r1,p}, g_{r2,p}) \in SE^2(2)$.

The goal is to plan a sequence of finger motions so that state f can be reached from state 0. For this, we need to formulate the system constraints and construct the free space discussed in the preceding sections.

<u>1. Constraints for Collision Avoidance.</u>

For a given configuration, $g_{f,p} \in SE^2(2)$, of the fingers, $g_{r1,p}^{-1} g_{r2,p}$ represents the relative configuration of finger 2 to finger 1.

The distance function (18) is expressed in terms of $g_{ri,p}^{-1} g_{r2,p}$:

$$d(F_1, F_2) = \min_{\substack{x \in S_1^r, r \in \underline{m}_1 \\ y \in S_2^l, l \in \underline{m}_2}} \|x - g_{r1,p}^{-1} g_{r2,p} y\|$$

where $\underline{m}_i = \{1,2,3,4\}, i = 1,2$. Substituting the coordinates of $S_i^l, l \in \underline{m}_i$, into the above equation gives the constraint equation on $SE^2(2)$ for collision avoidance.

2. Grasp Condition.

We parameterize the boundary S_o of the object by the angle with respect to the $x-$axis of C_o. Thus, the coordinate of contact point $c_{oi}, i = 1,2$, is the angle θ_{oi} of c_{oi} relative to the $x-$axis of C_o. Note that $0 \leq \theta_{oi} \leq 2\pi$. The configurations of the two contact frames of the object are

$$g_{oi,o} = \begin{bmatrix} \sin\theta_{oi} & \cos\theta_{oi} & \cos\theta_{oi} \\ \cos\theta_{oi} & \sin\theta_{oi} & \sin\theta_{oi} \\ 0 & 0 & 1 \end{bmatrix}, \quad i = 1,2.$$

and the grip Jacobian G is

$$G = \begin{bmatrix} \sin\theta_{o1} & \cos\theta_{o1} & \sin\theta_{o2} & \cos\theta_{o2} \\ -\cos\theta_{o1} & \sin\theta_{o1} & -\cos\theta_{o2} & \sin\theta_{o2} \\ -1 & 0 & -1 & 0 \end{bmatrix}.$$

One can easily verify that (c_{o1}, c_{o2}) forms a grasp if and only if the line joining c_{o1} and c_{o2} lies in the friction cone (see Figure 3). The equivalent mathematical statement is

$$|\pi - |\theta_{o1} - \theta_{o2}|\,| < 2\tan^{-1}\mu, \tag{45}$$

where μ is the coefficient of friction, and is assumed to be 1 here.

(45) implies that

$$|\pi - |\theta_{o1} - \theta_{o2}|\,| < 2\tan^{-1}\mu < 2 \cdot \frac{\pi}{2}$$

which simplifies to

$$0 < |\theta_{o1} - \theta_{o2}| < 2\pi.$$

Since the determinant of GG^t is

$$\det(GG^t) = 4(1 - \cos(\theta_{o1} - \theta_{o2})),$$

the grasp condition implies that G has full rank.

For a given state $(\underline{g}_{f,p}, g_{o,p}) \in SE^2(2) \times SE_o(2)$ of the system, $g_{ri,p}^{-1} g_{o,p}$ represents the configuration of the object relative to finger i. When the object is in contact with finger i, the point of contact c_{oi} is unique and can be expressed as a function of $g_{ri,p}^{-1} g_{o,p}$:

$$c_{oi} = c_{oi}(g_{ri,p}^{-1} g_{o,p})$$

Derivation of this function is rather straightforward (using the hand map constructed below).

3. Coordinated Manipulation.

Because $\dot{p}_{oi} = \dot{p}_{fi} = 0$, the fingers and the object become a single (rigid) connected component for coordinated manipulation. Let $[0, t_1]$ be the time interval of interest, and $g_{r2,r1}(0)$ be the configuration of C_{r2} relative to C_{r1} at time 0. Then, for coordinated manipulation, the trajectory of finger 1 is related to that of finger 2 by a constant transformation, i.e.,

$$g_{r2,p}(t) = g_{r1,p}(t) g_{r2,r1}(0), \ \forall t \in [0, t_1].$$

and the set of admissible finger trajectories can be expressed as

$$\sum_C = \{\underline{g}_{f,p}(t) \in SE^2(2), | g_{r2,p}(t) = g_{r1,p}(t) g_{r2,r1}(0)\}$$

On the other hand, trajectories of the object is also related to that of finger 1 by a constant transformation,

$$g_{o,p}(t) = g_{r1,p}(t) g_{o,r1}(0), \ \forall t \in [0, t_1].$$

where $g_{o,r1}(0)$ is the configuration of the object relative to finger 1 at $t = 0$. This defines the hand map by

$$H(\underline{g}_{f,p}(t)) = g_{r1,p}(t) g_{o,r1}(0), \forall t \in [0, t_1], \underline{g}_{f,p}(t) \in \sum_C.$$

4. Rolling Motion.

Let $[0, t_1]$ be the time interval of interest for rolling motion. As shown in Figure 3, let $y_{pi}, i = 1, 2$, be the $y-$coordinate of the contact point c_{fi} relative to C_{ri}. For rolling motion with rolling velocity $\dot{\theta}_{oi}, i = 1, 2$, y_{pi} can be expressed as

$$\begin{aligned} y_{p1}(t) &= y_{p1}(0) - (\theta_{o1}(t) - \theta_{o1}(0)), \\ y_{p2}(t) &= y_{p2}(0) + (\theta_{o2}(t) - \theta_{o2}(0)). \end{aligned}$$

Then, the configurations of the contact frames, $C_{fi}, i = 1, 2$, relative to C_{ri} are

$$g_{f1,r1} = \begin{bmatrix} 0 & 1 & \frac{1}{8} \\ -1 & 0 & y_{p1} \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } g_{f2,r2} = \begin{bmatrix} 0 & -1 & -\frac{1}{8} \\ 1 & 0 & y_{p2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, the $\hat{J}$ matrix of (9) is

$$\hat{J} = \begin{bmatrix} 0 & -1 & -\frac{1}{8} & 0 & 0 & 0 \\ 1 & 0 & -y_{p1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{8} \\ 0 & 0 & 0 & -1 & 0 & y_{p2} \end{bmatrix} = \hat{J}(\theta_{o1}(t), \theta_{o2}(t)).$$

Combining (9) with (43) and (44), yields a system of differential/algebraic equations that describe constraints for rolling motion:

$$\begin{cases} G^t \begin{bmatrix} \dot{x}_o \cos\theta_o + \dot{y}_o \sin\theta_o \\ \dot{y}_o \cos\theta_o - \dot{x}_o \sin\theta_o \\ \dot{\theta}_o \end{bmatrix} = \hat{J} \begin{bmatrix} \dot{x}_i \cos\theta_i + \dot{y}_i \sin\theta_i \\ \dot{y}_i \cos\theta_i - \dot{x}_i \sin\theta_i \\ \dot{\theta}_i \end{bmatrix}, \\ \dot{\theta}_{o1} = \dot{\theta}_o - \dot{\theta}_1, \\ \dot{\theta}_{o2} = \dot{\theta}_o - \dot{\theta}_2, \\ |\pi - |\theta_{o1} - \theta_{o2}|\,| < 2\tan^{-1}\mu. \end{cases} \tag{46}$$

(46) can be solved to give the set $\sum_R \subset SE^2(2)$ of admissible finger trajectories, and the resulting object trajectories given by the first equation of (46) defines the hand map for rolling motion.

A set of finger trajectories that satisfies (46) is that each finger rolls with equal velocity (see also [11]). Thus, if the fingers started in antipodal positions, they will remain antipodal.

5. Sliding Motion.

In this example, sliding motion implies that the fingers must move in the tangent direction of the unit circle. Thus, $\dot{p}_{oi} = 0, i = 1, 2$, and if the two fingers started in positions parallel with each other, thet will remain parallel with each other during sliding mode.

Gravity force relative to C_p is $\hat{g}_p = \begin{bmatrix} 0 \\ 9.8 \end{bmatrix}$, and it can be expressed relative to C_o, and is given by $Ad^t_{g_{o,p}^{-1}} \begin{bmatrix} \hat{g}_p \\ 0 \end{bmatrix}$, where

$$Ad^t_{g_{o,p}^{-1}} = \begin{bmatrix} \cos\theta_o & -\sin\theta_o & 0 \\ \sin\theta_o & \cos\theta_o & 0 \\ x_o \sin\theta_o - y_o \cos\theta_o & x_o \cos\theta_o + y_o \sin\theta_o & 1 \end{bmatrix}.$$

Let $[0, t_1]$ be the time interval of interest and $(g_{f,p}(0), g_{o,p}(0))$ the initial state for sliding. Since the number k of fingers is 2, we have to exam $2^2 - 1 = 3$ possible cases for sliding. These include: $\pi_1 = \{1\}$, finger 1 slides; $\pi_1 = \{2\}$, finger 2 slides and $\pi_2 = \{1, 2\}$, both fingers slide. The other constraint for sliding motion is

$$Ad^t_{g_{o,p}^{-1}} \begin{bmatrix} \hat{g}_p \\ 0 \end{bmatrix} \in G(K) \backslash \pi_m + \sum_{j \in \pi_m} G_i(\partial K_i). \tag{47}$$

A simple calculation shows that, in order to satisfy (47), the set of finger trajectories for $\pi_m = \{1,2\}$ is empty, i.e., $\sum_S^{\pi_2} = \{0\}$, and for the other cases we have:

(a) $\pi_1 = \{1\}$: Let $g_{o,p}(c_{o1} - c_{o2})$ be the vector (relative to C_p) joining c_{o2} to c_{o1}. Then, slding finger 1 is possible if the vector points upward relative to C_p, i.e.,

$$g_{o,p}(c_{o1} - c_{o2}) \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} > 0. \tag{48}$$

Moreover, the finger must slide upward, i.e.,

$$\dot{r}_{ri,p} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \geq 0.$$

Note that if finger 1 is below finger 2, i.e., (48) fails to hold, then sliding finger 1 is not possible.

(b) $\pi_1 = \{2\}$: We simply interchange the subscript 1 with 2, and obtain similar constraints for sliding finger 2.

Finger Relocation.

It is clear that $\sum_F = 0$.

A sequence of finger motions that brings the system from state 0 to state f and satisfies the constraints of (1) through (6) is shown in Figure 5, where each picture shows the system at the end of a manipulation mode. Here, we have divided the time interval into 7 successive subintervals: $0 = t_0 < t_1 < ... < t_7 = t_f$. The manipulation mode corresponds to each subinterval is: $[t_0, t_1]$: coordinated manipulation; $[t_1, t_2]$: sliding finger 2 up ; $[t_2, t_3]$: rolling motion (to change relative orientation); $[t_3, t_4]$: sliding finger 2 up; $[t_4, t_5]$: coordinated manipulation; $[t_5, t_6]$: sliding finger 1 up; $[t_6, t_7]$: coordinated manipulation. Note that the initial state of a manipulation mode is always the final state of the preceding mode.

It is interesting to note how the relative configurations of the object to finger 2 change as the system goes from state 0 to state f. This is shown in Figure 6, where θ is the orientation angle, and y is the $y-$coordinate of the origin of C_o relative to C_{r2}. Sliding motion changes only the y coordinate, while rolling motion changes both the y and the θ coordinates. In order to reach state f from state 0, the fingers have to follow these lines of motion.

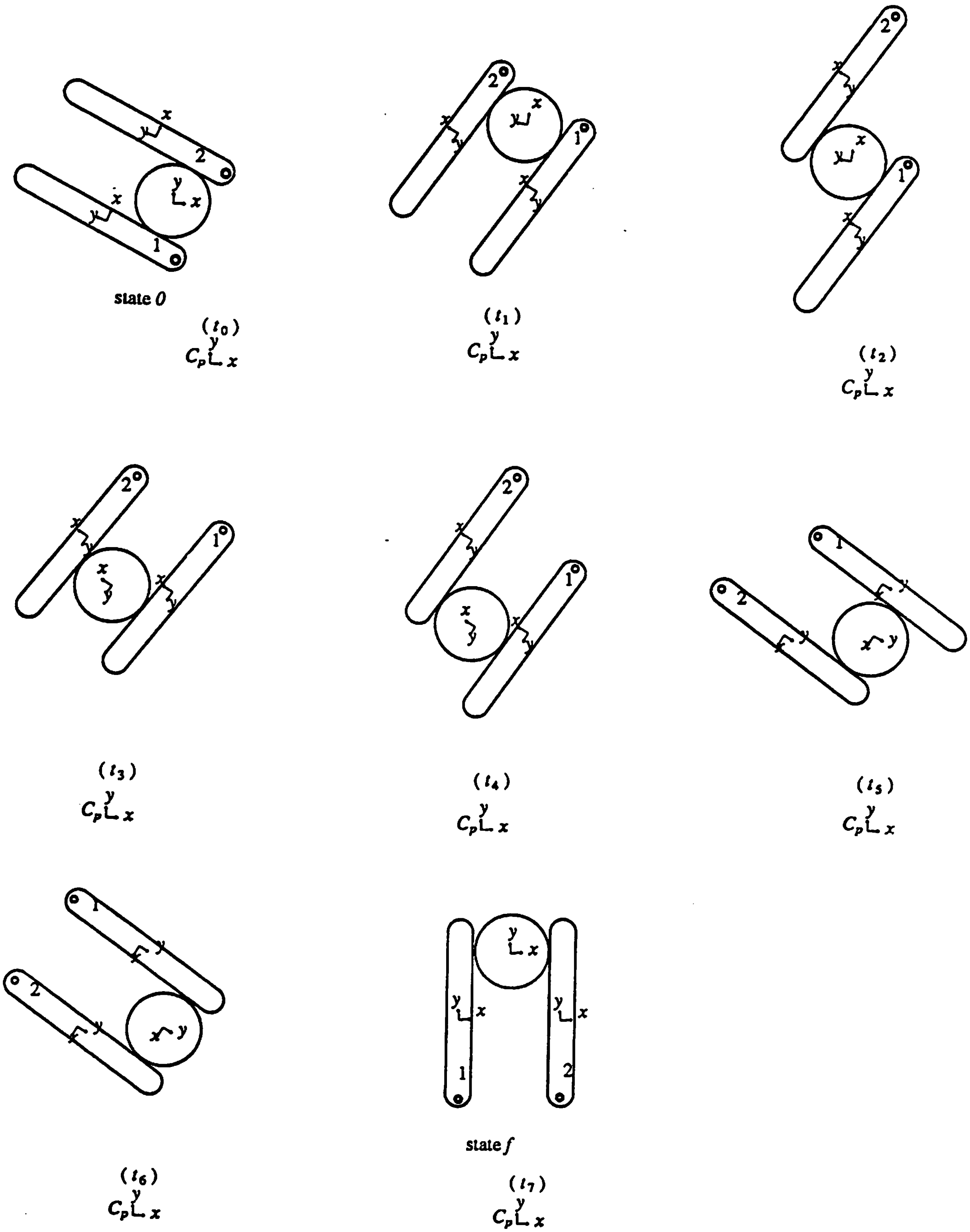

Figure 5: A solution to the dexterous manipulation problem.

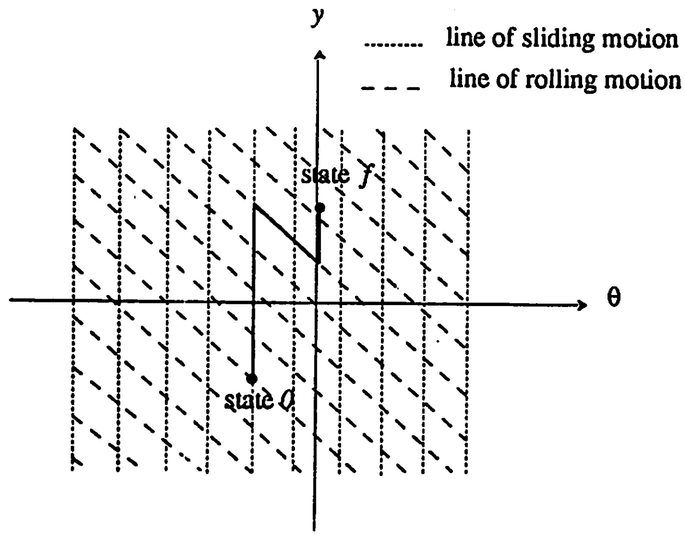

Figure 6: Relative configurations of the object to finger 2 from state 0 to state f.

7 Discussions

In part I of this paper, we have formulated the motion planning problem for dexterous manipulation. By decomposing dexterous manipulation into four basic manipulation modes and introducing the hand map, we have shown that motion can be planned in the configuration space of the fingers.

In deriving the free space for motion planning, we have encountered two types of constraints: A *spatial type of constraints*, which is defined on the configuration variables of the fingers (Section 3), and the other is a *dynamic type of constraints*, which is formulated in terms of the finger trajectories. We showed that for dynamic constraints, solving a set of differential/algebraic equations is needed to obtain the free space Q.

Finally, the free space that satisfies both the spatial constraints and the dynamic constraints is the intersection:

$$Q_s \bigcap Q \subset SE^k(3).$$

Solution curves that lie in $Q_s \bigcap Q$ and connect an initial grasp to a final grasp will be constructed in part II of this paper.

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Appendix A: Geometries of a Surface

To have the paper relatively more self-contained, we provide here the definitions of curvature form, torsion form and metric discussed in Section 2. These notions are geometric invariants of a surface. For more detailed treatment on this subject, see [11] and the references therein.

Consider a surface S. Let (ϕ, U) be the coordinate chart for S. We assume that (ϕ, U) is orthogonal (and right-handed) in the sense that $\phi_u(\mathbf{u}) \cdot \phi_v(\mathbf{u}) = 0$ for all $\mathbf{u} \in U$. The contact frame at a point $\mathbf{u} \in U$ is defined to be the coordinate frame with origin at $\phi(\mathbf{u})$ and coordinate axes

$$\mathbf{x}(\mathbf{u}) = \phi_u(\mathbf{u})/\|\phi_u(\mathbf{u})\|, \ \mathbf{y}(\mathbf{u}) = \phi_v(\mathbf{u})/\|\phi_v(\mathbf{u})\|, \ \mathbf{z}(\mathbf{u}) = \mathbf{x}(\mathbf{u}) \times \mathbf{y}(\mathbf{u}).$$

At the point $s = \phi(\mathbf{u})$, the curvature form $K(s)$ is defined as the 2×2 matrix

$$K(s) = [\mathbf{x}(\mathbf{u}), \mathbf{y}(\mathbf{u})]^t[\mathbf{z}_u(\mathbf{u})/\|\phi_u(\mathbf{u})\|, \mathbf{z}_v(\mathbf{u})/\|\phi_v(\mathbf{u})\|],$$

the torsion form at s is the 1×2 matrix

$$T(s) = \mathbf{y}(\mathbf{u})^t[\mathbf{x}_u(\mathbf{u})/\|\phi_u(\mathbf{u})\|, \mathbf{x}_v(\mathbf{u})/\|\phi_v(\mathbf{u})\|],$$

and the metric $M(s)$ is the 2×2 diagonal matrix

$$M(s) = \mathrm{diag}(\|\phi_u(\mathbf{u})\|, \|\phi_v(\mathbf{u})\|).$$

Example (from [11]). Consider the sphere S of radius R. The following is a coordinate chart for S.

$$U = \{(u, v) | -\pi/2 < u < \pi/2, -\pi < v < \pi\}$$

and the map

$$\phi : U \longrightarrow \Re^3 : (u, v) \longmapsto (-R\cos u \cos v, R\sin u, R\cos u \sin u).$$

The coordinates u and v are known as the latitude and longitude, respectively. The coordinate vectors of the contact frame are given by

$$\mathbf{x}(\mathbf{u}) = \begin{bmatrix} -\sin u \cos v \\ \sin u \sin v \\ \cos u \end{bmatrix}, \ \mathbf{y}(\mathbf{u}) = \begin{bmatrix} \sin v \\ -\cos v \\ 0 \end{bmatrix}, \ \mathbf{z}(\mathbf{u}) = \begin{bmatrix} \cos u \cos v \\ -\cos u \sin v \\ \sin u \end{bmatrix}.$$

The curvature form, torsion form and metric are

$$K = \begin{bmatrix} 1/R & 0 \\ 0 & 1/R \end{bmatrix}, T = [0 \quad -\tan u/R], M = \begin{bmatrix} R & 0 \\ 0 & R\cos u \end{bmatrix}.$$

Appendix B. Grasp Conditions As Constraints on the Finger Configuration Space

In this appenddix, we show how the grasp conditions of Section 3 can be transformed into constraints on the configuration space of the fingers: $SE^k(3)$.

By assumption (A2), the points of contact, $\underline{p}_o$, will be uniquely determined given configurations of the object and the fingers: $(\underline{g}_{f,p}, g_{o,p}) \in SE^k(3) \times SE_o(3)$. On the other hand, the object configuration is related to that of the fingers by the hand map. This enables us to write $G(\underline{p}_o)=G(\underline{p}_o(\underline{g}_{f,p}, H(\underline{g}_{f,p}))) = G(\underline{g}_{f,p})$.

Since K is a convex cone, we have

Lemma 1 $G(K) = \Re^6$ *if and only if G(K) does not have a supporting hyperplane.*

The proof is a direct consequence of the Separation Theorem in convex analysis. A supporting hyperplane has a unique normal through the origin, and the inner product of the normal with the set is negative. This gives

Proposition 2 $G(K) = \Re^6$ *if and only if*

$$\min_{y \in S^5} \max_{i \in \underline{k}} \max_{x_j \in K_i \cap S^2} \langle y, G_i x_j \rangle > 0 \tag{49}$$

(49) defines an inequality with $\underline{g}_{f,p}$ as parameters. Thus, this directly transforms into a constraint equation on the configuration variables of the fingers.

Notations

$\Re^n$	n-dimensional Euclidean space.
k	Total number of fingers in a hand manipulation system.
SO(3)	The special rotational group of $\Re^3$.
SE(3)	($\Re^3 \times SO(3)$) The Euclidean group of $\Re^3$
$SE_i(3)$	A copy of SE(3), designated to represent the configuration space of finger i.
$SE_o(3)$	A copy of SE(3), designated to represent the configuration space of the object.
$SE^k(3)$	$= SE_1(3) \times ... \times SE_k(3)$, configuration space of the fingers.
$\mathcal{N}$	The set of natural numbers.
$\underline{k}$	$= \{1, ...k\}$.
C_{ri}	Reference coordinate frame fixed to the last link of finger i, $i \in \underline{k}$.
C_o	Reference coordinate frame fixed to the object.
C_p	Inertia reference coordinate frame at the hand palm.
G	The grip Jacobian.
H	The hand map.
c_{oi}	($i \in \underline{k}$) contact point of the object with finger i, relative to C_o.
c_{fi}	($i \in \underline{k}$) contact points of finger i, relative to C_{ri}.
p_{oi}	($= (\varphi_o^j)^{-1}(c_{oi})$ Coordinates of the contact point c_{oi}.
p_{fi}	($= (\varphi_i^j)^{-1}(c_{fi})$ Coordinates of the contact point c_{fi}.
ψ_i	Contact angle of $i-th$ contact.
$\underline{c}_o$	$= (c_{o1}^t, ...c_{ok}^t)^t \in \Re^{3k}$.
$\underline{c}_f$	$= (c_{f1}^t, ...c_{fk}^t)^t \in \Re^{3k}$.
$\underline{p}_o$	$= (p_{o1}^t, ...p_{ok}^t)^t \in \Re^{2k}$.
$\underline{p}_f$	$= (p_{f1}^t, ...p_{fk}^t)^t \in \Re^{2k}$.
$\underline{\psi}$	$= (\psi_1, ...\psi_k)^t \in \Re^k$.
$g_{ri,p}$	($\in SE_i(3)$) Configuration variable of finger i relative to C_p.
$g_{o,p}$	($\in SE_o(3)$) Configuration variable of the object relative to C_p.
$\underline{g}_{f,p}$	$= (g_{r1,p}^t, ...g_{rk,p}^t)^t$, configuration variable.
$A \backslash B$	The complement of set B in set A.